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Volume Minimizing Hypersurfaces in Manifolds of Nonnegative Scalar Curvature

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Abstract.

We prove that if a manifold of nonnegative scalar curvature contains a two-sided hypersurface which is locally of least area and admits no metric of positive scalar curvature, then it splits isometrically in a neighborhood of the hypersurface.

We report here on joint work with G. Galloway concerning the study of rigidity of manifolds with nonnegative scalar curvature. Let us first recall the following theorem of Schoen and Yau.

Theorem 1. *Let (M, g) be a smooth n-manifold with positive scalar curvature, $S > 0$. If Σ is a compact immersed two-sided stable minimal hypersurface in M, then Σ admits a metric of positive scalar curvature.*

The above theorem follows from the proof of Theorem 1 in [SY]. If M is merely assumed to have nonnegative scalar curvature, the conclusion of the above theorem may not hold. Consider, for example, $T^{n-1} \times S^1$, where T^{n-1} is an $n-1$ torus. It is known that T^{n-1} does not admit a metric of positive scalar curvature ([GL], [SY]). However, in this direction one has the following theorem (cf. [SY], [FCS]).

Theorem 2. *Let (M, g) be a smooth n-manifold with nonnegative scalar curvature, $S \geq 0$. Let Σ be a compact manifold which does not admit a metric of positive scalar curvature. If Σ is immersed as a two-sided stable minimal hypersurface in M, then Σ is totally geodesic. Furthermore, the ambient scalar curvature S, the intrinsic scalar curvature $\tilde{S}$ and the Ricci curvature in the normal direction Ric_{nn} along Σ all vanish.*

We outline here the proof of Theorem 2 for $n > 3$ (for $n = 3$, see [FCS]). Denote by Π the second fundamental form. The minimality and

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stability conditions of Σ imply that, for all smooth functions ϕ on Σ,

$$\int_\Sigma |\nabla\phi|^2 - \int_\Sigma (\mathrm{Ric}_{nn} + |\Pi|^2)\,\phi^2 \geq 0. \tag{1}$$

Using the Gauss curvature equation and relating the ambient and intrinsic scalar curvatures along Σ, one gets the following "rearrangement"

$$S = \tilde{S} + 2\,\mathrm{Ric}_{nn} + |\Pi|^2. \tag{2}$$

Substituting (2) into (1), we have

$$\int_\Sigma |\nabla\phi|^2 + \frac{1}{2}\int_\Sigma (\tilde{S} - S - |\Pi|^2)\,\phi^2 \geq 0. \tag{3}$$

Since $S \geq 0$, we conclude that

$$-\frac{1}{2}\int_\Sigma \tilde{S}|\phi|^2 \leq \int_\Sigma |\nabla\phi|^2 \tag{4}$$

for any smooth function ϕ on Σ.

Now, consider the operator

$$L = \Delta - \frac{n-3}{4(n-2)}\tilde{S}.$$

We claim that all the eigenvalues of L are nonnegative. Suppose the contrary and let ϕ be a nonzero solution of

$$L\phi = -\lambda\phi$$

for some $\lambda < 0$. Multiplying the above equation by ϕ and integrating, we obtain

$$\frac{2(n-2)}{n-3}\int_\Sigma |\nabla\phi|^2 = -\frac{1}{2}\int_\Sigma \tilde{S}\phi^2 + \frac{2\lambda(n-2)}{n-3}\int_\Sigma \phi^2 < \int_\Sigma |\nabla\phi|^2 \tag{5}$$

where the inequality follows from (4). But this is not possible as $2(n-2)/(n-3) > 1$.

Now we show that the first eigenvalue, λ_0, is zero. We argue again by contradiction. Suppose the first eigenvalue $\lambda_0 > 0$ and let u be a first eigenfunction. It is well-known that the first eigenfunctions for operators of the form of L do not change sign, hence we may assume that $u > 0$. If we multiply the metric of Σ by $u^{4/(n-3)}$, the scalar curvature of Σ is transformed to

$$u^{-\frac{n+1}{n-3}}(\tilde{S}u - \frac{4(n-2)}{n-3}\Delta u) = \frac{4(n-2)}{n-3}u^{-\frac{n+1}{n-3}}\lambda_0 u > 0.$$

This contradicts our assumption that Σ does not admit a metric of positive scalar curvature.

Inequality (4) together with the equation in (5) implies that the eigenfunctions corresponding to the eigenvalue 0 must be constants and that $\tilde{S} = 0$. Substituting $\tilde{S} = 0$ and $\phi = 1$ into (3), we see that both S and Π vanish. Theorem 2 is thus proved.

Theorem 2 may be loosely paraphrased as: if Σ does not admit a metric of positive scalar curvature and if $\Sigma \subset M$ is infinitesimally of least area, then M infinitesimally splits along Σ. The aim of this paper is to establish a noninfinitesimal version of this result. Our main theorem is the following

Theorem 3. *Let (M, g) be a smooth n-manifold with nonnegative scalar curvature, $S \geq 0$. Let Σ be a compact manifold which does not admit a metric of positive scalar curvature. If Σ is immersed as a two-sided hypersurface in M which is locally of least area, then Σ has zero scalar curvature and a neighborhood of Σ in M splits isometrically as a product.*

By definition, a compact two-sided hypersurface Σ in a manifold M is locally of least area provided in some normal neighborhood V of Σ, $A(\Sigma) \leq A(\Sigma')$ for all Σ' isotopic to Σ in V, where A is the area functional. If the inequality is strict for $\Sigma' \neq \Sigma$, we say that Σ is locally strictly of least area. Note that "locally of least area" in the theorem cannot be replaced by "stable minimal". Take, for example, $S^2 \times S^1$, where S^2 is a modified sphere with an infinitesimally flattened equator E. Then $E \times S^1$ is a torus which does not admit a metric of positive curvature and which is stable minimal in $S^2 \times S^1$.

Theorem 3 was proved in [CG] for $n = 3$. We thank an anonymous referee for pointing out to us that ideas there also apply to higher dimensions.

The idea of the proof of Theorem 3 is as follows. We first show that Σ cannot be locally *strictly* of least area. If it were, then under a sufficiently small perturbation of the metric to a metric of (strictly) positive scalar curvature, Σ would be perturbed to a minimal hypersurface which would admit a metric of positive scalar curvature. But this would contradict our assumption. We then show that on each side of Σ there is a hypersurface which has the same volume as Σ. This implies that a neighborhood of Σ is foliated by local minimizers, which in turn implies that the neighborhood is a product.

The folowing lemma is proved in [CG] which shows that locally any metric of nonnegative scalar curvature can be perturbed to a nearby metric of positive scalar curvature.

Lemma 1. *Suppose Σ is a compact two-sided hypersurface in an n-manifold (M, g) with nonnegative scalar curvature, $S(g) \geq 0$. Then there exists a neighborhood U of Σ and a sequence of metrics $\{g_n\}$ on U such that $g_n \to g$ in C^∞ topology on U, and each g_n has strictly positive scalar curvature, $S(g_n) > 0$.*

The next lemma is proved in [CG] for $n = 3$. The arguments there probably do not extend beyond dimension 7. We adopt here an alternative approach suggested by the anonymous referee.

Lemma 2. *Let Σ be as in Theorem 3. Σ cannot be locally strictly of least area.*

Proof. Denote by X the set of C^∞ sections of the normal bundle of Σ with sufficiently small C^1 norm. For $u \in X$, let $H(u)$ be the mean curvature of $\text{graph}_\Sigma u$ in normal coordinates. H is a Fredholm operator and has the linerization

$$H'(0) = -\Delta - (|\Pi|^2 + \text{Ric}_{nn}).$$

Since both Π and Ric_{nn} vanish by Theorem 2, $H'(0) = -\Delta$ and hence the cokernel, as well as the kernel, of $H'(0)$ consists of constant functions on Σ. Denote by p the projection from $C^\infty(\Sigma)$ to Y, where $Y = \{u \mid \int_\Sigma u = 0\}$. The composition $p \circ H$ is then a submersion from X to Y (some shrinkage of the domain may be necessary) and $(p \circ H)^{-1}(0)$ is a one-dimensional submanifold of X whose graphs constitute a family of constant mean curvature hypersurfaces. The area functional A_g restricted to this submanifold has a strict minimum at the zero. Let $\tilde{g}$ be a small pertubation of g with positive scalar curvature, $\tilde{S} > 0$, and let $\tilde{H}$ be the corresponding mean curvature operator. The existence of $\tilde{g}$ is guaranteed by Lemma 1. When the perturbation is sufficiently small, $(p \circ \tilde{H})^{-1}(0)$ will be a one-dimensional submanifold whose graphs will be a family of constant mean curvature hypersurfaces in the metric $\tilde{g}$, and the area function $A_{\tilde{g}}$ has a local minimum in it close to 0. We first show that this local minimum is a minimal hypersurface.

To this end, let $u(t)$ be a parametrization of $(p \circ H)^{-1}(0)$ with $u(0) = 0$. Since $u'(0)$ is in the kernel of $p \circ H'(0)$, $u'(0)$ is a (nonzero) constant function. Without loss of generality, we assume $u'(0)$ is a positive constant. We then parametrize $(p \circ \tilde{H})^{-1}(0)$ by $\tilde{u}(t)$ in such a way that $\tilde{u}(t)$ is close to $u(t)$, $\tilde{u}(0)$ is the local minimum of $A_{\tilde{g}} \circ \tilde{u}$ and $\tilde{u}'(0)$ is a positive function.

For simplicity, denote $A_{\tilde{g}} \circ \tilde{u}$ by $\tilde{A}$, $\tilde{u}'(0)$ by ϕ and the graph corresponding to $\tilde{u}(0)$ by $\tilde{\Sigma}$.

Since 0 is an extremum of $\tilde{A}$, $\tilde{A}'(0) = 0$. On the other hand, the first variational formula shows that

$$\tilde{A}'(0) = \int_{\tilde{\Sigma}} \tilde{H}(0)\phi \,,$$

where $\tilde{H}(0)$ is the mean curvature of $\tilde{\Sigma}$. Since $\tilde{H}(0)$ is constant and ϕ is positive, the above shows that $\tilde{H}(0) = 0$, i.e., $\tilde{\Sigma}$ is an minimal hypersurface. Now we show that $\tilde{\Sigma}$ admits a metric of positive scalar curvature, contradicting our assumption on Σ as $\tilde{\Sigma}$ is diffeomorphic to Σ.

Since $(p \circ \tilde{H} \circ \tilde{u})'(0) = 0$, we have

$$-\tilde{\Delta}\phi - (\tilde{\Pi}^2 + \tilde{\mathrm{Ric}}_{nn})\phi = c \,, \tag{6}$$

where c is in the kernel of p and hence is a constant. We claim that $c \geq 0$. In fact, since $\tilde{A}''(0) \geq 0$ and

$$\tilde{A}''(0) = \int_{\tilde{\Sigma}} (-\tilde{\Delta}\phi - (\tilde{\Pi}^2 + \tilde{\mathrm{Ric}}_{nn})\phi)\phi \,,$$

it follows that $\int_{\tilde{\Sigma}} c\phi \geq 0$. This together with $\phi > 0$ implies that $c \geq 0$.

Applying the "rearrangement" to (6), we get

$$-\tilde{\Delta}\phi + \frac{1}{2}(\tilde{\tilde{S}} - \tilde{S} - \Pi^2)\phi = c \geq 0,$$

where $\tilde{\tilde{S}}$ is the intrinsic scalar curvature of $\tilde{\Sigma}$.

Similar to the proof of Theorem 2, we now multiply the metric on $\tilde{\Sigma}$ by $\phi^{2/(n-2)}$, the scalar curvature of the new conformed metric is then equal to

$$\phi^{-\frac{n}{n-2}}(-2\tilde{\Delta}\phi + \tilde{\tilde{S}}\phi + \frac{n-1}{n-2}\frac{|\nabla\phi|^2}{\phi})$$
$$= \phi^{-\frac{n}{n-2}}(2c + (\tilde{S} + \Pi^2)\phi + \frac{n-1}{n-2}\frac{|\nabla\phi|^2}{\phi}).$$

Since $c \geq 0$, $\phi > 0$ and $\tilde{S} > 0$, the above is positive. This is a contradiction and Lemma 2 is thus proved. □

Remark 1. It is clear from the proof that Lemma 2 holds for manifolds with $C^{2,\alpha}$ metrics, a fact which will be used later.

Remark 2. Since $u(0) = 0$ and $u'(0)$ is a positive constant, we know that $u(t)$ and t have the same sign when t is sufficiently small. This shows that when a constant mean curvature hypersurface is sufficiently close to Σ, it lies to one side of Σ and does not intersect with Σ unless it coincides with Σ.

We are now in a position to prove Theorem 3.

For simplicity, we assume that Σ is embedded. The general case can be reduced to this one by working in the normal bundle of Σ.

We denote by $\mathcal{F}$ the collection of minimal hypersurfaces which are C^1 close to Σ and have the same volume as Σ. Lemma 2 implies that each element in $\mathcal{F}$ is an accumulation point in $\mathcal{F}$. In fact, we can show that each element is a *two-sided* accumulation point. To see this, let us look at one of the two components of $M\backslash\Sigma$, say U. Taking two copies of U and gluing them along $\partial U = \Sigma$, we get a new manifold, N. Since Σ is totally geodesic, the induced metric on N is of class $C^{2,1}$. Moreover, Σ is locally of least area in the new metric. Applying Lemma 2 (see also Remark 1) to N, we obtain a sequence of mutually distinct hypersurfaces Σ_n in N such that Σ_n has the same volume as Σ and $\Sigma_n \to \Sigma$. It follows from Remark 2 that when n is sufficiently large, Σ_n lies to one side of Σ and does not intersect with Σ. This shows that U contains a sequence of hypersurfaces in $\mathcal{F}$ that is convergent to Σ. Since the choice of U is arbitrary, we conclude that Σ is a two-sided accumulation point in $\mathcal{F}$. The argument certainly applies to every element in $\mathcal{F}$.

We now show that when $|t|$ is sufficiently small, $\text{graph}_\Sigma\, u(t)$ is an element in $\mathcal{F}$, where $u(t)$ is as in the proof of Lemma 2. To do this, let us fix a point x_0 in Σ and consider $r(t) = \exp_{x_0} u(t)N$, where N is the normal vector to Σ. Since every element in $\mathcal{F}$ is a two-sided accumulation point, a continuity argument shows that for each t there is an element Σ_t in $\mathcal{F}$ passing through $r(t)$. Note that $(p \circ H)^{-1}(0)$ consists of all constant mean curvature hypersurfaces that are close to Σ and that Σ_t is a minimal hypersurface, hence, there is t' such that $\Sigma_t = \text{graph}_\Sigma\, u(t')$. Clearly, t' is uniquely determined, and thus we get a map $t \mapsto t'$. It is easy to see that this map is continuous and $0 \mapsto 0$. This implies that at least when $|t|$ is sufficiently small, $\text{graph}_\Sigma\, u(t)$ is a minimizer for the area functional. It then follows from the proof of Lemma 2 that $u(t)$ is a constant section for each t. We thus have obtained a smooth foliation of a neighborhood of Σ by totally geodesic hypersurfaces which are level surfaces of the distance function to Σ. A standard argument shows that the neighborhood is a product of Σ with an interval. This completes the proof of Theorem 3.

Remark 3. It would be interesting to extend Theorem 3 to non-compact hypersurfaces. In dimension 3, Fischer-Colbrie and Schoen ([FCS]) proved that a complete stable minimal surface in an orientable

3-manifold with nonnegative scalar curvature must be conformal to the complex plane or the cylinder A. In the latter case one can show that A is flat and totally geodesic (See [FCS] and [CM]). It seems reasonable to conjecture that if the cylinder A is actually area minimizing (in a suitable sense), then M is a product. (cf. Remark 4 in [CG]).

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