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Generic Initial Ideals and Graded Betti Numbers

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§ Introduction

The purpose of this article is to give the algebraic background of the shifting theory developed by Kalai [26], [27]. The reader who is interested in the combinatorial aspects of the theory should consult Kalai's survey paper [26] and his article in this volume.

In the present article we are mainly interested in the behaviour of graded Betti numbers under the operation of algebraic shifting. Algebraic shifting is intimately related to the theory of generic initial ideals. In Section 1 we recall some of the basic facts of this theory. The next section is devoted to the study of stable and strongly stable ideals since generic initial ideals are of this kind, provided the base field is of characteristic 0. In Section 3 we describe the Betti numbers of stable and squarefree stable ideals, and in Section 4 the Cartan complex which provides the graded minimal free resolution of the residue class field of the exterior algebra. For the theory of squarefree monomial ideals, which is significant for combinatorial applications, it is necessary to study graded ideals, graded modules and their resolutions over the exterior algebra. In Section 5 we explain how the graded Betti numbers of squarefree monomial ideals over the exterior and symmetric algebra are related. The following two sections are devoted to the proof of a theorem on extremal Betti numbers by Bayer, Charalambous and S. Popescu [12], as well as to the corresponding theorem in the squarefree case by Aramova and the author [4]. In Section 8 we describe various shifting operators and apply the homological theory of the previous sections. Symmetric algebraic shifting and a theorem of Björner and Kalai [15] are applied in Section 9 in order to deduce a theorem on superextremal Betti numbers. In the final section extremality properties of lexsegment ideals are briefly sketched.

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Not all proofs could be included. But in most cases an outline of the proofs or precise references to the original papers are given.

Several unsolved problems and conjectures are included. The author hopes that this survey inspires the readers to study and solve some of the open problems.

§1. Generic initial ideals

Most of the content of this section can be found in the book of Eisenbud [16] or the lecture notes by M. Green [20]. We will therefore omit almost all of the proofs.

Let K be an infinite field, and $S = K[x_1, \dots, x_n]$ the polynomial ring over K. The set of monomials of degree d in S will be denoted by M_d.

We will fix a term order $<$ satisfying $x_1 > x_2 > \dots > x_n$. Let $I \subset S$ be an ideal. Then we denote by $\operatorname{in}_<(I)$ (or simply by $\operatorname{in}(I)$) the initial ideal of I, that is, the ideal which is generated by all initial terms of I.

Let $GL(n)$ denote the general linear group with coefficients in K. Any $\varphi = (a_{ij}) \in GL(n)$ induces an automorphism of the graded K-algebra S, again denoted by φ, namely

$$\varphi(f(x_1, \dots, x_n)) = f(\sum_{i=1}^{n} a_{i1}x_i, \dots, \sum_{i=1}^{n} a_{in}x_i) \quad \text{for all} \quad f \in S.$$

One basic fact in the theory of generic initial ideals is the following

Theorem 1.1 (Galligo, Bayer and Stillman). *Let $I \subset S$ be a graded ideal. Then there is a nonempty Zariski open set $U \subseteq GL(n)$ such that* $\operatorname{in}(\varphi(I))$ *does not depend on $\varphi \in U$. Moreover, U meets non trivially the Borel subgroup of $GL(n)$ consisting of all upper triangular invertible matrices.*

For $\varphi \in U$ the monomial ideal $\operatorname{in}(\varphi(I))$ is called the *generic initial ideal of I*, and will be denoted $\operatorname{Gin}(I)$.

For the details of the proof of Theorem 1.1 we refer to [16, Theorem 15.18]. Each homogeneous component $\operatorname{Gin}(I)_d$ of $\operatorname{Gin}(I)$ may be computed as follows: consider a transcendental field extension L/K, where L has the transcendental basis $\{a_{ij} \colon i, j = 1, \dots, n, \quad i \leq j\}$. Let $S' = L[x_1, \dots, x_n]$, $I' = \varphi(I)S'$ where $\varphi(x_j) = \sum_{i=1}^{j} a_{ij}x_i$ for $j, \dots, n$. Choose an L-basis $f_1, \dots, f_m$ of I'_d. Each of the f_i is a linear combination of monomials $u \in M_d$ whose coefficients are (homogeneous) polynomials in $K[a_{ij} \colon i, j = 1, \dots, n]$ (of degree d), say, $f_i = \sum_{u \in M_d} c_{iu}u$. Now form the $m \times |M_d|$-matrix $C = (c_{iu})$ where the columns are ordered according to the given term order, and view C as a matrix with coefficients in L. Notice that C has rank m since the polynomials $f_1, \dots, f_m$ are linearly independent over L. For $i = 1, \dots, m$, let u_i be the largest monomial such that $c_{iu_i} \neq 0$; then $u_i = \operatorname{in}(f_i)$.

After elementary row operations (which amounts to choose another L-basis of I'_d), we may assume that $u_1 > u_2 > \dots > u_m$. Then $\operatorname{Gin}(I)_d = Ku_1 + \dots + Ku_m$.

We order the m-tuples $(v_1, \dots, v_m)$ of monomials of M_d lexicographically. This means that $(v_1, \dots, v_m) > (w_1, \dots, w_m)$ if for some i one has $v_j = w_j$ for $j < i$, and $v_i > w_i$. Then our discussion shows that $\operatorname{Gin}(I)_d$ is the span of largest m-tupel $(u_1, \dots, u_m)$ of monomials such that $\det(c_{iu_i})_{i=1,\dots,m} \neq 0$.

Another basic result on generic ideals is

Theorem 1.2 (Galligo, Bayer-Stillman). *Let $I \subset S$ be a graded ideal. Then $\operatorname{Gin}(I)$ is Borel fixed, that is, $\varphi(\operatorname{Gin}(I)) = \operatorname{Gin}(I)$ for all φ which belong to the Borel group of invertible upper triangular matrices.*

Generic initial ideals behave especially well when one uses the reverse lexicographic order. We will discuss this in Section 4. Let $u, v \in M_d$, $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$. Then $u > v$ in reverse lexicographic order, if $\deg u > \deg v$ or $\deg u = \deg v$ and for some i one has $a_j = b_j$ for $j > i$, and $a_i < b_i$.

The following example demonstrates the difference between the lexicographic and the reverse lexicographic order. We order the monomials in three variables of degree 2 first lexicographically, and then reverse lexicographically:

(1) $x_1^2 > x_1x_2 > x_1x_3 > x_2^2 > x_2x_3 > x_3^2$
(2) $x_1^2 > x_1x_2 > x_2^2 > x_1x_3 > x_2x_3 > x_3^2$

The nice behaviour of the reverse lexicographic order is a consequence of the easy to prove

Property 1.3. *Let $<$ be the reverse lexicographic order. If $f \in S$ is a homogeneous polynomial with $\operatorname{in}_<(f) \in (x_i, \dots, x_n)$ for some i, then $f \in (x_i, \dots, x_n)$.*

This property immediately implies (cf. [16, Proposition 15.12])

Proposition 1.4. *Let $I \subset S$ be a graded ideal. Then with respect to the reverse lexicographic order one has*

(a) $\operatorname{in}(I) + x_nS = \operatorname{in}(I + x_nS)$;
(b) $\operatorname{in}(I) : x_n = \operatorname{in}(I : x_n)$.

A monomial $u \in S$, $u = x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}$ is called *squarefree*, if $a_i \leq 1$ for $i = 1, \dots, n$, and a monomial ideal in S is called *squarefree* if it is generated by squarefree monomials. In combinatorial contexts, squarefree monomial ideals are interesting since they appear as the defining ideals of Stanley-Reisner rings. Unfortunately $\mathrm{Gin}(I)$ of a squarefree monomial ideal I is never squarefree, unless I is generated by a subset of the variables. Thus for combinatorial applications one has to find an analogue of the operation Gin which yields a squarefree monomial ideal. The most natural way to define such an operation, is to work in the exterior algebra instead of the symmetric algebra.

Let V be an n-dimensional K-vector space with basis $e_1 \dots e_n$. The exterior algebra $E = \bigwedge^{\bullet}(V)$ is a finite dimensional graded K-algebra. The ith graded component $\bigwedge^{i}(V)$ has the K-basis

$$e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i} \quad \text{with} \quad j_1 < j_2 < \ldots < j_i.$$

Let $[n] = \{1, \dots, n\}$; for a subset $\sigma \subset [n]$, $\sigma = \{j_1 < j_2 < \ldots < j_i\}$, we set $e_\sigma = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i}$. The elements e_σ are called the monomials of E. Term orders, initial terms and initial ideals are defined as in the polynomial ring. For example, the lexicographic or the reverse lexicographic order is defined by restriction to squarefree monomials.

In the following example we list all monomials in 4 variables of degree 2 in the exterior algebra in lexicographic and reverse lexicographic order:

(1) $e_1 \wedge e_2 > e_1 \wedge e_3 > e_1 \wedge e_4 > e_2 \wedge e_3 > e_2 \wedge e_4 > e_3 \wedge e_4$.
(2) $e_1 \wedge e_2 > e_1 \wedge e_3 > e_2 \wedge e_3 > e_1 \wedge e_4 > e_2 \wedge e_4 > e_3 \wedge e_4$.

In the exterior algebra the generic initial ideal $\mathrm{Gin}(I)$ of a graded ideal $I \subset E$ is defined similarly as in the case of the polynomial ring. In other words, $\mathrm{Gin}(I) = \mathrm{in}(\varphi(I))$ where φ is a linear automorphism of E. Of course, $\mathrm{Gin}(I)$ is a monomial ideal in E (which is automatically squarefree). The analogues of the theorems of Galligo, Bayer and Stillman, as well as Proposition 1.4, hold and are proved similarly in the exterior case. We refer the reader to [6] about some general facts on Gröbner basis theory in exterior algebras.

§2. Special monomial ideals

Let p be a prime number, and k and l be non-negative integers with p-adic expansion $k = \sum_i k_i p^i$ and $l = \sum_i l_i p^i$. We set $k \leq_p l$ if $k_i \leq l_i$ for all i. In order to have a consistent notation, we also set $k \leq_0 l$ if $k \leq l$ (in the usual sense).

Definition 2.1. Let p be a prime number, or $p = 0$. A monomial ideal $I \subset S$ is p-Borel, if the following condition holds: for each monomial $u \in I$, $u = \prod_i x_i^{\mu_i}$, one has $(x_i/x_j)^\nu u \in I$ for all i, j with $1 \leq i < j \leq n$ and all $\nu \leq_p \mu_j$.

The significance of p-Borel ideals follows from

Proposition 2.2. *Suppose* $\operatorname{char} K = p \geq 0$*, and let* $I \subset S$ *be a monomial ideal. Then* I *is Borel-fixed if and only if* I *is* p*-Borel.*

For the proof of Proposition 2.2 we refer to [16, Theorem 15.23].

For $p > 0$, the p-Borel ideals have a rather complicated combinatorial structure. The reader who is interested in more details about such ideals may consult [31], [3] and [23]. In these notes we will concentrate on 0-Borel ideals, which henceforth will be called *strongly stable ideals.*

For a monomial $u \in S$ we set $m(u) = \max\{i\colon x_i \quad \text{divides} \quad u\}$.

Definition 2.3. A subset $B \subset S$ of monomials is called *strongly stable*, if $x_i(u/x_j) \in B$ for all $u \in B$, all x_j that divides u, and all $i < j$. The set B is called *stable*, if $x_i(u/x_{m(u)}) \in B$ for all $u \in B$, and all $i < m(u)$.

It follows from Definition 2.1 that a strongly stable ideal is a monomial ideal I for which the set of monomials in I is a strongly stable monomial set. If the set of monomials in I is a stable set, then I is called a *stable monomial ideal.* Stable monomial ideals were introduced by Eliahou and Kervaire [17].

Examples 2.4. (a) Let $u_1, \ldots, u_m$ be monomials. There is a unique smallest strongly stable ideal I with $u_j \in I$ for $j = 1, \ldots, m$. The monomials $u_1, \ldots, u_m$ are called *Borel generators* of I, and we write $I = \langle u_1, \ldots, u_m \rangle$. I is called *principal Borel* if $I = \langle u \rangle$ for some monomial u. For example the ideal

$$I = (x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2^2, x_2x_3, x_2x_4)$$

is principal Borel with Borel generator x_2x_4.

(b) A set L of monomials is called a *lexsegment*, if for all $u \in L$ and all $v \geq_{lex} u$ with $\deg v = \deg u$, it follows that $v \in L$. An monomial ideal $I \subset S$ is called a *lexsegment ideal* if the set monomials in I a lexsegment. It is obvious that lexsegment ideals are strongly stable.

(c) Replacing in (b) everywhere the word 'lex' by 'revlex', one obtains the definition of a revlexsegment and a revlexsegment ideal. It is obvious that revlexsegment ideals are strongly stable.

Remark 2.5. (a) Let I be a monomial ideal. We denote by $G(I)$ the unique minimal set of monomial generators of I. It is easily seen that I is strongly stable, if for all monomial generators u of I one has $(x_i/x_j)u \in I$ for all x_j that divide u, and all $i < j$.

Let $N \subset M_d$. The set $\{x_iu\colon u \in N, i = 1, \ldots, n\} \subset M_d$ is called the *shadow of* N, and is denoted $\operatorname{Shad}(N)$. The simple proof of the following lemma is left to the reader.

Lemma 2.6. *Let $N \subset M_d$. If N is a (strongly) stable set (resp. a lexsegment), then* $\operatorname{Shad}(N)$ *is a (strongly) stable set (resp. a lexsegment).*

Notice that the shadow of a revlexsegment is in general not a revlexsegment. For example consider the revlexsegment $\{x_1^2\}$ in $K[x_1, x_2, x_3]$. Then $x_1^2x_3$ is in the shadow of this set, but x_2^3 is not.

Let $N \subset M_d$. Then there is a unique lexsegment, denoted N^{lex} such that $|N^{lex}| = |N|$. The following important result holds

Theorem 2.7. *For any subset $N \subset M_d$ one has $|\operatorname{Shad}(N^{lex})| \leq |\operatorname{Shad}(N)|$. In other words, lexsegments have the smallest possible shadow.*

Before we indicate the proof of Theorem 2.7 we note the following consequence

Corollary 2.8. *Let $I \subset S$ be graded ideal. Then there exists a unique lexsegment ideal, denoted $I^{lex} \subset S$, such that S/I and S/I^{lex} have the same Hilbert function.*

Proof. Since S/I and $S/\operatorname{in}(I)$ have the same Hilbert function, we may replace I by $\operatorname{in}(I)$, and hence may assume that I is a monomial ideal. Let I_d be spanned by the set of monomials N_d, and I_d^{lex} the subspace of S_d spanned by N_d^{lex}. We set $I^{lex} = \bigoplus_{d\geq 0} I_d^{lex}$, and only need to show that I^{lex} is an ideal. In other words, we have to show that $\{x_1, \dots, x_n\}I_d^{lex} \subset I_{d+1}^{lex}$ for all d. By Theorem 2.7 we have $|\operatorname{Shad}(N_d^{lex})| \leq |\operatorname{Shad}(N_d)| \leq |N_{d+1}| = |N_{d+1}^{lex}|$. Since $\operatorname{Shad}(N_d^{lex})$ and N_{d+1}^{lex} are both lexsegments, this inequality implies $\operatorname{Shad}(N_d^{lex}) \subset N_{d+1}^{lex}$, as desired. Q.E.D.

For the proof of Theorem 2.7 we have to introduce some notation: let $B \subset M_d$ be a set of monomials. We let $m_i(B)$ be the number of $u \in B$ with $m(u) = i$, and set $m_{\leq i}(B) = \sum_{j=1}^{i} m_j(B)$.

Lemma 2.9. *Let $B \subset M_d$ be a stable set of monomials. Then*

(a) $m_i(\operatorname{Shad}(B)) = m_{\leq i}(B)$;

(b) $|\operatorname{Shad}(B)| = \sum_{i=1}^{n} m_{\leq i}(B)$.

Proof. (b) is of course a consequence of (a). For the proof of (a) we note that the map

$$\varphi\colon \{u \in B\colon m(u) \leq i\} \to \{u \in \operatorname{Shad}(B)\colon m(u) = i\}, \quad u \mapsto ux_i$$

is a bijection. In fact, φ is clearly injective. To see that φ is surjective, we let $v \in \operatorname{Shad}(B)$ with $m(v) = i$. Since $v \in \operatorname{Shad}(B)$, there exists $w \in B$ with $v = x_jw$ for some $j \leq i$. It follows that $m(w) \leq i$. If $j = i$, then we are done. Otherwise, $j < i$ and $m(w) = i$. Hence, since B is stable it follows that $u = (x_j/x_i)w \in B$. The assertion follows, since $v = ux_i$. Q.E.D.

Now Theorem 2.7 follows immediately from Lemma 2.9 and the next theorem which is due to Bayer [10]. We will give below the proof of the similar theorem in the squarefree case.

Theorem 2.10. *Let $L \subset M_d$ be a lexsegment, and $B \subset M_d$ be a stable set of monomials with $|L| \leq |B|$. Then $m_{\leq i}(L) \leq m_{\leq i}(B)$ for $i = 1, \dots, n$.*

The length of the shadow of a lexsegment can be computed. Let i be a positive integer. Then $a \in \mathbb{N}$ has a unique expansion

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

with $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$; see [14] or [21].

We define

$$a^{\langle i \rangle} = \binom{a_i + 1}{i+1} + \binom{a_{i-1}+1}{i} + \cdots + \binom{a_j + 1}{j+1},$$

and

$$a^{(i)} = \binom{a_i}{i+1} + \binom{a_{i-1}}{i} + \cdots + \binom{a_j}{j+1}.$$

Lemma 2.11. *Let $L \subset M_d$ be a lexsegment with $a = |M_d \setminus L|$. Then*

$$|M_{d+1} \setminus \operatorname{Shad}(L)| = a^{\langle d \rangle}.$$

For the proof of this lemma we refer the reader to [14, Prop.4.2.8].

As a consequence of Corollary 2.8 and Lemma 2.11 we now obtain

Theorem 2.12 (Macaulay). *Let $h\colon \mathbb{N} \to \mathbb{N}$ be a numerical function. The following conditions are equivalent:*

(a) *h is the Hilbert function of a standard graded K-algebra;*

(b) *$h(0) = 1$, and $h(d+1) \leq h(d)^{\langle d \rangle}$ for all $d \geq 0$.*

We close this section with a discussion of the analogue theorems in the squarefree case. Let $B \subset E$ be a set of monomials in the exterior algebra. Then B is called (*strongly*) *stable* if B satisfies conditions analogue to those of Definition 2.3. Thus, for example, B is stable, if for all monomials $u \in B$, $u = e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_j}$ with $i_1 < i_2 < \ldots i_j$ it follows that $e_i \wedge e_{i_1} \wedge \ldots \wedge e_{i_{j-1}} \in B$ for all $i < i_j$ and $i \notin \{i_1, \ldots, i_j\}$.

Let $u = x_{i_1} x_{i_2} \ldots x_{i_j} \in S$ be a squarefree monomial. Then we call $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_j}$ the monomial in E corresponding to u. Let $I \subset S$ be squarefree monomial ideal, $B \subset I$ the set of squarefree monomials in I, and B' be the corresponding set of monomials in E. Notice that the K-subspace J of E spanned by B' is an ideal in E. We call it the monomial ideal in E corresponding to I, and I is called a *squarefree (strongly) stable* monomial ideal, resp. a *squarefree lexsegment* ideal, if J is a (strongly) stable resp. lexsegment ideal in E.

Corresponding to Proposition 2.2 one has

Proposition 2.13. *A Borel-fixed ideal $J \subset E$ is strongly stable. In particular, the generic initial ideal of any graded ideal in E is strongly stable.*

For the shadow of a stable set of monomials in E one has

Lemma 2.14. *Let $B \subset E_d$ be a strongly stable set of monomials. Then* $\operatorname{Shad}(B)$ *is again stable and* $|\operatorname{Shad}(B)| = \sum_{i=1}^{n-1} m_{\leq i}(B)$.

We leave the proof of Lemma 2.14 to the reader.

We now prove the squarefree version of Bayer's Theorem 2.10.

Theorem 2.15. *Let $L \subset E_d$ be a lexsegment of monomials, and $B \subset E_d$ a stable set of monomials with $|L| \leq |B|$. Then $m_{\leq i}(L) \leq m_{\leq i}(B)$ for $i = 1, \ldots, n$.*

For the proof of the theorem we need some preparation. Let $d < n$ and write N_d for the set of all (squarefree) monomials of degree d in E. If $N \subset N_d$ we denote by $\min(N)$ the smallest monomial $u \in N$ (with respect to the lexicographic order). Furthermore we define a map $\alpha\colon N_d \to N_d$ by setting $\alpha(u) = u$, if $n \notin \operatorname{supp}(u)$, and $\alpha(u) = (e_j \wedge u)/e_n$ if $n \in \operatorname{supp}(u)$, where j is the largest integer $< n$ which does not belong to $\operatorname{supp}(u)$. Here $\operatorname{supp}(u)$ is the set of elements $i \in [n]$ such that $e_i | u$.

Lemma 2.16. *With the notation introduced we have:*

(a) *The map $\alpha\colon N_d \to N_d$ is order preserving, that is, for $u, u' \in N_d$, $u \leq_{lex} u'$, one has $\alpha(u) \leq_{lex} \alpha(u')$.*

(b) *Let $B = B' + B'' \wedge e_n$ be a strongly stable set of monomials of degree d, where B' and B'' are sets of monomials in the elements $e_1, e_2, \ldots, e_{n-1}$. Then $\alpha(\min(B)) = \min(G(B'))$.*

Proof. (a) Let u and u' be two monomials of degree d with $u \leq_{lex} u'$ and $m(u) = m(u') = n$, say $u = e_{i_1} \wedge \cdots \wedge e_{i_{d-1}} \wedge e_n$ and $u' = e_{i'_1} \wedge \cdots \wedge e_{i'_{d-1}} \wedge e_n$ with $1 \leq i_1 < i_2 < \cdots < i_{d-1} < n$ and $1 \leq i'_1 < i'_2 <$

$\cdots < i'_{d-1} < n$. Then there exists an integer t with $1 \leq t \leq d-1$ such that $i_1 = i'_1, \ldots, i_{t-1} = i'_{t-1}$ and $i_t > i'_t$. Let j be the largest integer $< d$ which is not in supp(u), and define j' similarly for u'. Since $i_t > i'_t$, there is at least one 'gap' in the sequence $i'_t, \ldots, i'_{d-1}, n$. Thus $j' > i'_t$. Hence if $j \geq i_t$, then the first indices of the factors of $\alpha(u)$ and $\alpha(u')$ in which they differ are again i_t and i'_t, and the inequality is preserved. On the other hand, if $j < i_t$, then we must have

$$u = e_{i_1} \wedge \cdots \wedge e_{i_{t-1}} \wedge e_{n-d+t} \wedge e_{n-d+t+1} \wedge \cdots \wedge e_{n-1} \wedge e_n,$$

and $j = i_t - 1 = n - d + t - 1$ since $i_{t-1} = i'_{t-1} < i'_t < i_t$. That is, the factors 'after' $e_{i_{t-1}}$ have the highest possible indices. It is then obvious that $\alpha(u) \leq_{lex} \alpha(u')$ as desired. By the similar way one treats the case $m(u') < m(u) = n$, while if $m(u) < m(u') = n$ one has $\alpha(u) = u \leq_{lex} u' \leq_{lex} \alpha(u')$.

(b) It follows from the above result (a) that $\alpha(\min(B)) \leq_{lex} \alpha(\min (B')) = \min(B')$ since $\min(B) \leq_{lex} \min(B')$. On the other hand, since B is strongly stable, $\alpha(\min(B)) \in B'$, which implies the reverse inequality. Q.E.D.

Proof of Theorem 2.15. We proceed by induction on n, the number of variables. The inequality $m_{\leq n}(L) \leq m_{\leq n}(B)$ is just our hypothesis. In order to prove it for $i < n$, we write $L = L' + L'' \wedge e_n$ and $B = B' + B'' \wedge e_n$ with L', L'', B' and B'' sets of monomials in $e_1, e_2, \ldots, e_{n-1}$. It is clear that L' is lexsegment, and that B' is strongly stable. Hence if we show that $|L'| \leq |B'|$, we may apply our induction hypothesis, and the assertion follows immediately.

It may be assumed that B' and B'' are lexsegments. In fact, let B^* (resp. B^{**}) be the lexsegments in $e_1, e_2, \ldots, e_{n-1}$ of degree d (resp. $d-1$) such that $|B^*| = |B'|$ (resp. $|B^{**}| = |B''|$) and set $\tilde{B} = B^* + B^{**} \wedge e_n$. Then it is not hard to see that $\tilde{B}$ is again strongly stable.

Now we are in the following situation: $L = L' + L'' \wedge e_n$ is lexsegment, and $B = B' + B'' \wedge e_n$ strongly stable as before, but in addition B' and B'' are lexsegments. Assuming $|L| \leq |B|$, we want to show that $|L'| \leq |B'|$. Thanks to Lemma 2.16 we have

$$\min(B') = \alpha(\min(B)) \leq_{lex} \alpha(\min(L)) = \min(L').$$

Since L' and B' are lexsegments, the required inequality follows. Q.E.D.

As a consequence one obtains similarly as in Corollary 2.8 that for any graded ideal $J \subset E$ there exists a unique lexsegment ideal $J^{lex} \subset E$ such that E/J and E/J^{lex} have the same Hilbert function. Detailed proofs of these statements can be found in [6].

Corollary 2.17. *Let $I \subset S$ be a squarefree monomial ideal. Then there exists a unique squarefree lexsegment ideal, denoted I^{sqlex} such that S/I and S/I^{sqlex} have the same Hilbert function.*

Proof. Let J be the corresponding ideal of I in E, B' the set of monomials of J^{lex}, and B the set of squarefree monomials in S corresponding to B'. The ideal $L \subset S$ spanned by B is clearly a squarefree lexsegment ideal. It follows from the next lemma that S/I and S/L have the same Hilbert function. Q.E.D.

Lemma 2.18. *Let $I \subset S$ be a squarefree monomial ideal, and $J \subset E$ the corresponding monomial ideal in E. Let $H_{E/J}(t) = \sum_{i=0}^n a_i t^i$ be the Hilbert function of E/J. Then the Hilbert function of S/I is given by*

$$H_{S/I}(t) = \sum_{i=0}^n a_i \frac{t^i}{(1-t)^i}.$$

This lemma implies in particular that the Hilbert function of E/J and that of S/I determine each other. A proof of this simple result can be found for example in [14, Theorem 5.1.7].

The exterior version of Lemma 2.11 is the following (cf. [6, Theorem 4.2])

Lemma 2.19. *Let $L \subset EM_d$ be a lexsegment of monomials, where EM_d denotes the set of monomials of degree d in E. Suppose that $a = |EM_d \setminus L|$. Then*

$$|EM_d \setminus \mathrm{Shad}(L)| = a^{(d)}.$$

As in the case of the polynomial rings one now deduces (cf. [6, Theorem 4.1])

Theorem 2.20 (Kruskal-Katona). *Let $(h_1, \dots, h_n)$ be a sequence of integers. Then the following conditions are equivalent:*

(a) $1 + \sum_{d=1}^n h_d t^d$ *is the Hilbert series of a graded K-algebra E/J;*

(b) $0 \le h_{d+1} \le h_d^{(d)}$ *for all i with $0 \le d < n$.*

§3. Graded Betti numbers of initial ideals

Let M be a finitely generated graded S-module. Then M has a graded free S-resolution of the form

$$\dots \to \bigoplus_j S(-j)^{\beta_{ij}} \to \dots \to \bigoplus_j S(-j)^{\beta_{1j}} \to \bigoplus_j S(-j)^{\beta_{0j}} \to M \to 0.$$

The numbers β_{ij} are called the *graded Betti numbers of* M. Note that the Tor-groups $\mathrm{Tor}_i(K, M)$ are finitely generated, graded K-vector spaces, and that

$$\beta_{ij}(M) = \dim_K \mathrm{Tor}_i(K, M)_j \quad \text{for all} \quad i, j.$$

The following basic result holds:

Theorem 3.1. *Let $I \subset S$ be a graded ideal. Then for any term order $<$ one has*

$$\beta_{ij}(S/I) \leq \beta_{ij}(S/\operatorname{in}_<(I)) \quad \textit{for all} \quad i, j.$$

Proof. Let $\tilde{S}$ be the $K[t]$-algebra $S[t]$, where t is an indeterminate of degree 0. By [16, Theorem 15.17] there exists a graded ideal $\tilde{I} \subset \tilde{S}$ such that the $K[t]$-algebra $\tilde{S}/\tilde{I}$ is free $K[t]$-module (and thus flat over $K[t]$), and such that

(1) $$(\tilde{S}/\tilde{I})/t(\tilde{S}/\tilde{I}) \cong S/\operatorname{in}(I),$$

and

(2) $$(\tilde{S}/\tilde{I})_t \cong (S/I) \otimes_K K[t, t^{-1}],$$

as graded K-algebras.

Let $F_{\bullet}$ be the minimal graded free $\tilde{S}$-resolution of $\tilde{S}/\tilde{I}$. Then (1) implies that $F_{\bullet}/tF_{\bullet}$ is a graded minimal free S-resolution of S/I, so that $\beta_{ij}(\tilde{S}/\tilde{I}) = \beta_{ij}(S/\operatorname{in}(I))$ for all i and j, and (2) implies that $(F_{\bullet})_t$ is a graded (not necessarily minimal) free $S \otimes_K K[t, t^{-1}]$ resolution of $(S/I) \otimes_K K[t, t^{-1}]$. Thus, $\beta_{ij}(S/I) = \beta_{ij}((S/I) \otimes_K K[t, t^{-1}]) \leq \beta_{ij}(\tilde{S}/\tilde{I})$, as desired. Q.E.D.

Let M be a finitely generated graded S-module. The *regularity* of M is defined to be the number $\operatorname{reg}(M) = \max\{j - i \colon \beta_{ij}(M) \neq 0\}$. As an immediate consequence of Theorem 3.1 we have

Corollary 3.2. *Let $I \subset S$ be a graded ideal. Then for any term order $<$ one has*:

(a) $\operatorname{proj\,dim} S/I \leq \operatorname{proj\,dim} S/\operatorname{in}_<(I)$.
(b) $\operatorname{depth} S/I \geq \operatorname{depth} S/\operatorname{in}_<(I)$.
(c) *If $S/\operatorname{in}_<(I)$ is Cohen-Macaulay (Gorenstein), then so is S/I.*
(d) $\operatorname{reg} S/I \leq \operatorname{reg} S/\operatorname{in}_<(I)$.

We shall see in the next section that all inequalities of Corollary 3.2 become equalities, if $\operatorname{in}_<(I)$ is replaced by $\operatorname{Gin}(I)$ with respect to the reverse lexicographic order. Since by Proposition 2.2, at least in characteristic 0, the generic initial ideal is strongly stable, it is of interest to compute the graded Betti numbers of stable ideals. Eliahou and Kervaire described explicitly the resolution of such ideals. Here we are only interested in its graded Betti numbers, so that we only need to compute the graded K-vector spaces $\mathrm{Tor}_i(K, S/I)$.

Let $K_{\bullet}(x; S/I)$ be the Koszul complex of S/I with respect to $x_1, ..., x_n$. We denote by $H_{\bullet}(x; S/I)$ the Koszul homology. Since there is a graded isomorphism $\mathrm{Tor}_{\bullet}(K, S/I) \cong H_{\bullet}(x; S/I)$, we may as well compute $H_{\bullet}(x; S/I)$ in order to determine the graded Betti numbers. Recall that $K_i(x_1, ..., x_n) = K_{\bullet}(x; S/I)$ is a free S/I- module with basis e_σ, $\sigma \subset \{1, \dots, n\}$, $|\sigma| = i$, where $e_\sigma = e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_i}$ for $\sigma = \{j_1, \dots, j_i\}$, $j_1 < j_2 < \dots < j_i$. The differential ∂ of $K_{\bullet}$ is given by $\partial(e_\sigma) = \sum_{t \in \sigma} (-1)^{\alpha(\sigma, t)} x_t e_{\sigma \setminus t}$. Here $\alpha(\sigma, t) = |\{r \in \sigma : r < t\}|$.

For a monomial ideal I we denote by $G(I)$ the unique set of monomial generators of I. We let $\varepsilon \colon S \to S/I$ be the canonical epimorphism, and set $u' = u/x_{m(u)}$ for all $u \in G(I)$.

Theorem 3.3. *Let $I \subset S$ be a stable ideal. For all $j = 1, \dots, n$ and $i > 0$, the Koszul homology $H_i(x_j, \dots, x_n)$ is annihilated by $\mathfrak{m} = (x_1, \dots, x_n)$. In other words, all these homology modules are K-vector spaces. A basis of $H_i(x_j, \dots, x_n)$ is given by the homology classes of the cycles*

$$\varepsilon(u') e_\sigma \wedge e_{m(u)}, u \in G(I), |\sigma| = i - 1, j \leq \min(\sigma), \max(\sigma) < m(u).$$

Proof. We proceed by induction on $n - j$. For $j = n$, we only have to consider $H_1(x_n)$ which is obviously minimally generated by the homology classes of the elements $\varepsilon(u') e_n$ with $u \in G(I)$ such that $m(u) = n$. Since by the definition of stable ideals $x_i u' \in I$ for all i, we see that $H_1(x_n)$ is a k-vector space.

Now assume that $j < n$, and that the assertion is proved for $j + 1$. Then $x_j H_i(x_{j+1}, \dots, x_n) = 0$ for all $i > 0$, so that the long exact sequence (cf. [14, Cor.1.6.13])

$$\cdots \xrightarrow{x_j} H_i(x_{j+1}, \dots, x_n) \longrightarrow H_i(x_j, \dots, x_n) \longrightarrow H_{i-1}(x_{j+1}, \dots, x_n)$$
$$\xrightarrow{x_j} H_{i-1}(x_{j+1}, \dots, x_n) \longrightarrow H_{i-1}(x_j, \dots, x_n) \longrightarrow \cdots$$

splits into the exact sequences

$$(3) \quad 0 \longrightarrow H_1(x_{j+1}, \dots, x_n) \longrightarrow H_1(x_j \dots, x_n) \longrightarrow S_j/I_j \xrightarrow{x_j} S_j/I_j$$

and

$$(4) \quad 0 \longrightarrow H_i(x_{j+1}, ..., x_n) \longrightarrow H_i(x_j, ..., x_n)$$
$$\longrightarrow H_{i-1}(x_{j+1}, ..., x_n) \longrightarrow 0.$$

for $i > 0$. Here S_j is the polynomial ring $K[x_1, \dots, x_j]$, I_j the ideal in S_j generated by the monomials $u \in G(I)$ which are not divisible by any x_i with $i > j$, in other words, $I_j = I \cap S_j$.

In sequence (3), $\operatorname{Ker} x_j$ is minimally generated by the residues of the monomials u' with $u \in G(I)$ and $m(u) = j$. Note that the sets $\{u \in G(I) \colon m(u) = j\}$ and $\{u \in G(I_j) \colon m(u) = j\}$ are equal, and that I_j is a stable ideal in S_j. Therefore $\operatorname{Ker} x_j$ is a K-vector space.

We now consider the short exact sequence

$$(5) \quad 0 \longrightarrow H_1(x_{j+1}, \dots, x_n) \longrightarrow H_1(x_j, \dots, x_n) \longrightarrow \operatorname{Ker} x_j \longrightarrow 0.$$

It is clear that the elements $\varepsilon(u')e_j$, $u' \in G(I)$, $m(u) = j$ are cycles in $K_1(x_j, \dots, x_n)$ such that $\delta([\varepsilon(u')e_j]) = u' + I_j$. Therefore, by (5) and our induction hypothesis, it follows that the set $\mathcal{S} = \{[\varepsilon(u')e_i] \colon u \in G(I), m(u) = i \geq j\}$ generates $H_1(x_j, \dots, x_n)$. Since I is a stable ideal we see that $x_j[\varepsilon(u')e_i] = 0$ for all $j = 1, \dots, n$ and all $[\varepsilon(u')e_i] \in \mathcal{S}$. In other words, $H_1(x_j, \dots, x_n)$ is a K-vector space. Finally, since the number of elements of $\mathcal{S}$ equals $\dim_k H_1(x_{j+1}, \dots, x_n) + \dim \operatorname{Ker} x_j$, we conclude that $\mathcal{S}$ is a basis of $H_1(x_j, \dots, x_n)$.

In order to prove our assertion for $i > 1$ we consider the exact sequences (4). By induction hypothesis the homology module $H_{i-1}(x_{j+1}, \dots, x_n)$ is a K-vector space with basis

$$[\varepsilon(u')e_\sigma \wedge e_{m(u)}], u \in G(I), |\sigma| = i-2, j+1 \leq \min(\sigma), \max(\sigma) < m(u).$$

Given such a homology class, consider the element $\varepsilon(u')e_j \wedge e_\sigma \wedge e_{m(u)}$. It is clear that this element is a cycle in $K_i(x_j, \dots, x_n)$, and that

$$\delta([\varepsilon(u')e_j \wedge e_\sigma \wedge e_{m(u)}] = \pm[\varepsilon(u')e_\sigma \wedge e_{m(u)}].$$

Thus from the exact sequence (4) and our induction hypothesis it follows that the homology classes of the cycles described in the theorem generate $H_i(x_j, \dots, x_n)$. Again the stability of the ideal I implies that $\mathfrak{m}$ annihilates all these homology classes, so that $H_i(x_j, \dots, x_n)$ is a K-vector space. Finally, just as for $i = 1$, a dimension argument shows that these homology classes form a basis of $H_i(x_j, \dots, x_n)$. Q.E.D.

Let I be a monomial ideal. We denote by $G(I)_j$ the set of monomial generators of degree j. The following result of Eliahou and Kervaire [17] follows immediately from Theorem 3.3.

Corollary 3.4. *Let $I \subset S$ be a stable ideal. Then*
(a) $\beta_{ii+j}(I) = \sum_{u \in G(I)_j} \binom{m(u)-1}{i}$;
(b) $\operatorname{proj\,dim} S/I = \max\{m(u) \colon u \in G(I)\}$;
(c) $\operatorname{reg}(I) = \max\{\deg(u) \colon u \in G(I)\}$.

With similar methods one can compute the graded Betti numbers of a squarefree stable ideal. For a monomial $u \in S$ we let $\operatorname{supp}(u) = \{i\colon x_i \text{ divides } u\}$.

Theorem 3.5. *Let $I \subset S$ be a squarefree stable ideal. Then for every $i > 0$, a basis of the homology classes of $H_i(x_1, x_2, \ldots, x_n)$ is given by the homology classes of the cycles*

$$u' e_\sigma \wedge e_{m(u)}, u \in G(I), |\sigma| = i - 1, \max(\sigma) < m(u), \sigma \cap \operatorname{supp}(u) = \emptyset.$$

Proof. A minimal free S-resolution of S/I is multigraded; in other words, the differentials are homogeneous homomorphisms and, for each i, we have $F_i = \bigoplus_j S(-a_{ij})$ with $a_{ij} \in \mathbb{Z}^n$. Moreover, by virtue of [24, Theorem (5.1)], all shifts a_{ij} are squarefree, i.e., $a_{ij} \in \mathbb{Z}^n$ is of the form $\sum_{t \in \tau} \varepsilon_t$, where τ is a subset of $\{1, 2, \ldots, n\}$, and where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ is the canonical basis of $\mathbb{Z}^n$. Thus it follows that $H_i(x_1, x_2, \ldots, x_n)$ is multigraded K-vector space with $H_i(x_1, x_2, \ldots, x_n)_a = 0$, if $a \in \mathbb{Z}^n$ is not squarefree. Hence, if we want to compute the homology module $H_i(x_1, x_2, \ldots, x_n)$, it suffices to consider its squarefree multigraded components.

For each $0 < j < n$, there is an exact sequence whose graded part for each $a \in \mathbb{Z}^n$ yields the long exact sequence of vector spaces

$$\begin{aligned} \cdots \xrightarrow{x_j} H_i(x_{j+1}, \ldots, x_n)_a \longrightarrow H_i(x_j, \ldots, x_n)_a \longrightarrow H_{i-1}(x_{j+1}, \ldots, x_n)_{a-\varepsilon_j} \\ \xrightarrow{x_j} H_{i-1}(x_{j+1}, \ldots, x_n)_a \longrightarrow H_{i-1}(x_j, \ldots, x_n)_a \longrightarrow \cdots . \end{aligned}$$

We now show the following more precise result: For all $i > 0$, all $0 < j \leq n$ and all squarefree $a \in \mathbb{Z}^n$, $H_i(x_j, \ldots, x_n)_a$ is generated by the homology classes of the cycles

$$u' e_\sigma \wedge e_{m(u)}, \ldots u \in G(I), \ldots |\sigma| = i - 1$$

with

$$j \leq \min(\sigma), \max(\sigma) < m(u), \sigma \cap \operatorname{supp}(u) = \emptyset \text{ and } \sigma \cup \operatorname{supp}(u) = a.$$

The proof is achieved by induction on $n - j$. The assertion is obvious for $j = n$. We now suppose that $j < n$. For such j, but $i = 1$, the assertion is again obvious. Hence we assume in addition that $i > 1$. We first claim that

$$H_{i-1}(x_{j+1}, \ldots, x_n)_{a-\varepsilon_j} \xrightarrow{x_j} H_{i-1}(x_{j+1}, \ldots, x_n)_a$$

is the zero map. Since $a \in \mathbb{Z}^n$ is squarefree, the components of a are either 0 or 1. If the j-th component of a is 0, then $a - \varepsilon_j$ has a negative component; hence $H_{i-1}(x_{j+1},\dots,x_n)_{a-\varepsilon_j}=0$. Thus we may assume the j-th component of a is 1. Then $a-\varepsilon_j$ is squarefree and, by induction hypothesis, $H_{i-1}(x_{j+1},\dots,x_n)_{a-\varepsilon_j}$ is generated by the homology classes of cycles of the form $u'e_\sigma\wedge e_{m(u)}$ with $j\notin \operatorname{supp}(u)$. Such an element is mapped to the homology class of $u'x_je_\sigma\wedge e_{m(u)}$ in $H_{i-1}(x_{j+1},\dots,x_n)_a$. However, since I is stable, we have $u'x_j=0$ as desired.

From these observations we deduce that we have short exact sequences

$$0\longrightarrow H_i(x_{j+1},\dots,x_n)_a\longrightarrow H_i(x_j,\dots,x_n)_a$$
$$\longrightarrow H_{i-1}(x_{j+1},\dots,x_n)_{a-\varepsilon_j}\longrightarrow 0$$

for all $i>1$. The first map $H_i(x_{j+1},\dots,x_n)_a\to H_i(x_j,\dots,x_n)_a$ of the above exact sequence is simply induced by the natural inclusion map of the corresponding Koszul complexes, while the second map $H_i(x_j,\dots,x_n)_a\to H_{i-1}(x_{j+1},\dots,x_n)_{a-\varepsilon_j}$ is a connecting homomorphism. Given the homology class of a cycle $z=u'e_\sigma\wedge e_{m(u)}$ in $H_{i-1}(x_{j+1},\dots,x_n)_{a-\varepsilon_j}$, it is easy to see that, up to a sign, the homology class of the cycle $u'e_j\wedge e_\sigma\wedge e_{m(u)}$ in $H_i(x_j,\dots,x_n)_a$ is mapped to $[z]$. This guarantees all of our assertions as required. Q.E.D.

Corollary 3.6. *Let $I\subset S$ be a squarefree stable ideal. Then*

(a) $\beta_{ii+j}(I)=\sum_{u\in G(I)_j}\binom{m(u)-j}{i}$;
(b) $\operatorname{proj\,dim} S/I=\max\{m(u)-\deg(u)+1\colon u\in G(I)\}$;
(c) $\operatorname{reg}(I)=\max\{\deg(u)\colon u\in G(I)\}$.

Remark 3.7. It follows immediately from Corollary 3.4(a) and Corollary 3.6(a) that a (squarefree) stable ideal which is generated in one degree, has a linear resolution. Very recently Römer has shown (unpublished) that among all (squarefree) ideals with linear resolution the ideals generated by (squarefree) revlexsegments have minimal Betti numbers.

§4. The Cartan complex

Let $\mathcal{M}_l$ (resp. $\mathcal{M}_r$) denote the category of finitely generated graded left (right) E-modules, and $\mathcal{M}$ the category of finitely generated graded left and right E-modules, satisfying $ax=(-1)^{\deg a\deg x}xa$ for all homogeneous elements $a\in E$ and $x\in M$. For example, any graded ideal $I\subset E$ belongs to $\mathcal{M}$.

A module $M\in\mathcal{M}_l$ has a minimal, graded free E-resolution (as a left E-module), which is always infinite, unless M is free. The ijth Betti number $\beta_{ij}(M)$ is the K-dimension of $\mathrm{Tor}_i^E(K,M)_j$. These dimensions may be computed by using the graded free E-resolution of the residue class field K. This resolution is called the Cartan complex. We will briefly describe this complex.

Let $\mathbf{v} = v_1, \cdots, v_m$ be a sequence of elements of degree 1 in E. The Cartan complex $C_{\bullet}(\mathbf{v}; E)$ of the sequence $\mathbf{v}$ with values in E is defined as the complex whose i-chains $C_i(\mathbf{v}; E)$ are the elements of degree i of the free divided power algebra $C_{\bullet}(\mathbf{v}; E) = E\langle x_1, \dots, x_m\rangle$. Recall that $C_{\bullet}(\mathbf{v}; E)$ is the polynomial ring over E in the set of variables

$$x_i^{(j)}, \quad i = 1, \dots, m, \quad j = 1, 2, \dots$$

modulo the relations

$$x_i^{(j)} x_i^{(k)} = \frac{(j+k)!}{j!k!} x_i^{(j+k)}.$$

We set $x_i^{(0)} = 1$ and $x_i^{(1)} = x_i$ for $i = 1, \dots, m$. The algebra $C_{\bullet}(\mathbf{v}; E)$ is a free E-module with basis $x^{(a)} = x_1^{(a_1)} x_2^{(a_2)} \dots x_m^{(a_m)}$, $a \in \mathbb{N}^m$. We say that $x^{(a)}$ has degree i if $|a| = i$ where $|a| = a_1 + \dots + a_m$. Thus $C_i(\mathbf{v}; E) = \bigoplus_{|a|=i} E x^{(a)}$.

The E-linear differential on $C_{\bullet}(\mathbf{v}; E)$ is defined as follows: for $x^{(a)} = x_1^{(a_1)} \cdots x_m^{(a_m)}$ we set

$$\partial(x^{(a)}) = \sum_{a_i > 0} v_i x_1^{(a_1)} \cdots x_i^{(a_i - 1)} \cdots x_m^{(a_m)}.$$

It is easily verified that $\partial \circ \partial = 0$, so that $(C_{\bullet}(\mathbf{v}; E), \partial)$ is indeed a complex. Moreover, ∂ is an E-derivation, that is, ∂ is E-linear and

$$\partial(g_1 g_2) = g_1 \partial(g_2) + \partial(g_1) g_2$$

for any two homogeneous elements g_1 and g_2 in $C_{\bullet}(\mathbf{v}; E)$.

These rules imply that the cycles $Z_{\bullet}(\mathbf{v}; E)$ of $C_{\bullet}(\mathbf{v}; E)$ form a divided power algebra, and that the boundaries $B_{\bullet}(\mathbf{v}; E)$ form an ideal in $Z_{\bullet}(\mathbf{v}; E)$, so that the homology $H_{\bullet}(\mathbf{v}; E)$ of $C_{\bullet}(\mathbf{v}; E)$ inherits a natural structure of a divided power algebra. Let M be left E-module; then $C_{\bullet}(\mathbf{v}; M) = C_{\bullet}(\mathbf{v}; E) \otimes_E M$ is called the *Cartan complex of M with respect to the sequence* $\mathbf{v}$. The homology of $C_{\bullet}(\mathbf{v}; M)$ will be denoted by $H_{\bullet}(\mathbf{v}; M)$. Note that $H_{\bullet}(\mathbf{v}; M)$ has a natural left $H_{\bullet}(\mathbf{v}; E)$-module structure.

For each $j = 1, \dots, m-1$ there exists an exact sequence of complexes $0 \to C_{\bullet}(v_1, \dots, v_j; M) \xrightarrow{\iota} C_{\bullet}(v_1, \dots, v_{j+1}; M)$

$\xrightarrow{\tau} C_\bullet(v_1, \ldots, v_{j+1}; M)(-1) \to 0,$

where ι is a natural inclusion map, and where τ is given by

$$\tau(g_0 + g_1 x_{j+1} + \cdots + g_k x_{j+1}^{(k)}) = g_1 + g_2 x_{j+1} + \cdots + g_k x_{j+1}^{(k-1)},$$

with $g_i \in C_{k-i}(v_1, \ldots, v_j; M)$.

From this exact sequence one obtains immediately the following long exact sequences for the Cartan homology.

Proposition 4.1. *Let $M \in \mathcal{M}_l$; then for all $j = 1, \ldots, m-1$ there exists a long exact sequence of graded left E-modules*

$$\begin{aligned}\cdots &\longrightarrow H_i(v_1, ..., v_j; M) \xrightarrow{\alpha_i} H_i(v_1, ..., v_{j+1}; M)\\ &\xrightarrow{\beta_i} H_{i-1}(v_1, ..., v_{j+1}; M)(-1) \xrightarrow{\delta_{i-1}} H_{i-1}(v_1, ..., v_j; M)\\ &\longrightarrow H_{i-1}(v_1, ..., v_{j+1}; M) \longrightarrow \cdots .\end{aligned}$$

Here α_i is induced by the inclusion map ι, β_i by τ, and δ_{i-1} is the connecting homomorphism, which acts as follows: if $z = g_0 + g_1 x_{j+1} + \cdots + g_{i-1} x_{j+1}^{(i-1)}$ is a cycle in $C_{i-1}(l_1, \ldots, l_{j+1}; M)$, then $\delta_{i-1}([z]) = [g_0 v_{j+1}]$.

Let $e_1, \ldots, e_n$ be a K-basis of E_1. Using Proposition 4.1 it follows easily by induction on i that $C_\bullet(e_1, \ldots, e_i; E)$ is acyclic for $i = 1, \ldots, n$. In particular, $C_\bullet(e_1, \ldots, e_n; E)$ is a minimal, graded free E-resolution of K.

Corollary 4.2. *Let $M \in \mathcal{M}_l$. Then*

(a) *for all $i \geq 0$ there are graded isomorphisms* $\mathrm{Tor}_i^E(K, M) \cong H_i(e_1, ..., e_n; M)$ *of K-vector spaces;*
(b) *for all $i \geq 0$ one has $\beta_{ii}(K) = \binom{n-1+i}{i}$ and $\beta_{ij}(K) = 0$ for $j \neq i$;*
(c) $\mathrm{reg}(M) \leq \max\{j \colon M_j \neq 0\}$.

Proof. The statements (a) and (b) are clear by the discussions preceding this corollary. Since $C_i(e_1, \ldots, e_n; E) \cong \bigoplus E(-i)$, it follows from (a) that $\mathrm{Tor}_i^E(K, M)$ is a subquotient of $\bigoplus M(-i)$. This implies (c). Q.E.D.

For any finitely generated left E-module M, the Cartan cohomology with respect to the sequence $\mathbf{v} = v_1, \ldots, v_m$ is defined to be the homology of the cocomplex $C^\bullet(\mathbf{v}; M) = \mathrm{Hom}_E(C_\bullet(\mathbf{v}; E), M)$. Explicitly, we have

$$C^\bullet(\mathbf{v}; M) : 0 \xrightarrow{\partial^0} C^0(M) \xrightarrow{\partial^1} C^1(M) \longrightarrow \ldots ,$$

where the cochains $C^\bullet(\mathbf{v}; M)$ and the cochain maps $\partial^\bullet$ can be described as follows: the elements of $C^i(\mathbf{v}; M)$ may be identified with all homogeneous polynomials $\sum_a m_a y^a$ of degree i in the variables $y_1, \dots, y_m$ with coefficients $m_a \in M$, and where as usual for $a \in \mathbb{N}^n$, y^a denotes the monomial $y_1^{a_1} y_2^{a_2} \dots y_n^{a_n}$. The element $m_a y^a \in C^\bullet(\mathbf{v}; M)$ is defined by the mapping property

$$m_a y^a(x^{(b)}) = \begin{cases} m_a & \text{for} \quad b = a, \\ 0 & \text{for} \quad b \neq a. \end{cases}$$

After this identification the cochain maps are simply multiplication by the element $y_\mathbf{v} = \sum_{i=1}^n v_i y_i$. In other words, we have

$$\partial^i \colon C^i(\mathbf{v}; M) \longrightarrow C^{i+1}(\mathbf{v}; M), \quad f \mapsto y_\mathbf{v} f.$$

In particular we see that $C^\bullet(\mathbf{v}; E)$ may be identified with the polynomial ring $E[y_1, \dots, y_m]$, and that $C^\bullet(\mathbf{v}; M)$ is a finitely generated $C^\bullet(\mathbf{v}; E)$-module. It is obvious that cocycles and coboundaries of $C^\bullet(\mathbf{v}; M)$ are $E[y_1, \dots, y_m]$-submodules of $C^\bullet(\mathbf{v}; M)$. As $E[y_1, \dots, y_m]$ is Noetherian, it follows that the Cartan cohomology $H^\bullet(\mathbf{v}; M)$ of M is a finitely generated (graded) $E[y_1, \dots, y_m]$-module.

We set $M^* = \operatorname{Hom}_E(M, E)$. Cartan homology and cohomology are related as follows:

Proposition 4.3. *Let $M \in \mathcal{M}$. Then*

$$H_i(\mathbf{v}; M)^* \cong H^i(\mathbf{v}; M^*) \quad \textit{for all} \quad i.$$

Proof. Since E is injective, the functor $(-)^*$ commutes with homology and we obtain

$$\begin{aligned} &H_i(\mathbf{v}; M)^* \cong H^i(\operatorname{Hom}_E(C_i(\mathbf{v}; M), E)) \\ &\cong H^i(\operatorname{Hom}_E(C_i(\mathbf{v}; E), M^*) \cong H^i(\mathbf{v}; M^*). \end{aligned}$$

Q.E.D.

Proposition 4.4. *Let $M \in \mathcal{M}_l$. Then for all $j = 1, \dots, m-1$ there exists a long exact sequence of graded left E-modules*

$$\begin{aligned} &\cdots \longrightarrow H^{i-1}(v_1, \dots, v_{j+1}; M) \longrightarrow H^{i-1}(v_1, \dots, v_j; M) \\ &\longrightarrow H^{i-1}(v_1, \dots, v_{j+1}; M)(-1) \xrightarrow{y_{j+1}} H^i(v_1, \dots, v_{j+1}; M) \\ &\longrightarrow H^i(v_1, \dots, v_j; M) \longrightarrow \cdots. \end{aligned}$$

Proof. It is immediate that such a sequence exists. We only show that the map

$$H^{i-1}(v_1, \dots, v_{j+1}; M)(-1) \to H^i(v_1, \dots, v_{j+1}; M)$$

is indeed multiplication by y_{j+1}. We show this on the level of cochains. In order to simplify notation we set $C_i = C_i(v_1, \dots, v_{j+1}; E)$ for all i, and let

$$\gamma\colon \operatorname{Hom}_E(C_{i-1}, M) \to \operatorname{Hom}_E(C_i, M)$$

be the map induced by $\tau\colon C_i \to C_{i-1}$, where

$$\tau(x^{(b)}) = \begin{cases} x_1^{(b_1)} \cdots x_{j+1}^{(b_{j+1}-1)} & \text{if} \quad b_{j+1} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Our assertion is that γ is multiplication by y_{j+1}.

For all $x^{(b)} \in C_i$ we have $\gamma(my^a)(x^{(b)}) = my^a(\tau(x^{(b)}))$. This implies that

$$\gamma(my^a)(x^{(b)}) = \begin{cases} m & \text{if} \quad (b_1, \dots, b_{j+1}) = (a_1, \dots, a_{j+1} + 1), \\ 0 & \text{otherwise.} \end{cases}$$

Hence we see that $\gamma(my^a) = my^a y_{j+1}$, as desired. Q.E.D.

§5. Simplicial cohomology

Besides Cartan cohomology, there is another natural cohomology attached to any graded E-module: let $v \in E$ be a homogeneous element of degree 1, and let $M \in \mathcal{M}_l$. Since $v^2 = 0$, we obtain a finite complex of finitely generated K-vector spaces

$$(M, v)\colon \qquad \cdots \longrightarrow M_{i-1} \xrightarrow{l_v} M_i \xrightarrow{l_v} M_{i+1} \longrightarrow \cdots$$

where l_v denotes left multiplication by v. We denote the ith cohomology of this complex by $H^i(M, v)$. Notice that $H^\bullet(M, v) = \bigoplus_i H^i(M, v)$ is again an object in $\mathcal{M}_l$. Indeed,

$$H^\bullet(M, v) = \frac{0 :_M v}{vM},$$

where $0 :_M v = \{a \in M \colon va = 0\}$.

It is clear that a short exact sequence

$$0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0,$$

of finitely generated graded E-modules induces the long exact cohomology sequence

$$\cdots \to H^i(U, v) \to H^i(M, v) \to H^i(N, v) \to H^{i+1}(U, v) \to \cdots .$$

Definition 5.1. Let $M \in \mathcal{M}_l$. An element $v \in E_1$ is called *generic for* M if $\dim_K H^i(M,v) \leq \dim_K H^i(M,u)$ for all i and all $u \in E_1$.

The property of being generic for M is an open condition, that is, there exists a non-empty Zariski open subset $G \subset E_1$, such that $v \in E_1$ is generic if and only if $v \in G$.

Let Δ be simplicial complex on the vertex set $[n] = \{1, \dots, n\}$. One denotes by $I_\Delta \subset S$ the squarefree monomial ideal generated by all monomials $x_{i_1} \cdots x_{i_k}$ such that $\{i_1, \dots, i_k\} \notin \Delta$. The K-algebra $K[\Delta] = S/I_\Delta$ is called the *Stanley-Reisner ring of* Δ. Detailed information about this well studied ring can be found in [33], [14] and [21].

We denote by J_Δ the monomial ideal in E corresponding to I_Δ. The K-algebra $K\{\Delta\} = E/J_\Delta$ is called the *exterior face ring of* Δ. This algebra was first studied in a systematic way by Kalai [26] in connection with algebraic shifting. Notice that the Hilbert series of $K\{\Delta\}$ is given by

$$H_{K\{\Delta\}}(t) = \sum_{i \geq 0} f_{i-1} t^i,$$

where f_i is the number of i-dimensional faces of Δ.

Lemma 5.2. *Let Δ be a simplicial complex, and $v \in E_1$ a generic element for $K\{\Delta\}$. Then for all i we have*

$$H^i(K\{\Delta\}, v) \cong \tilde{H}^{i-1}(\Delta; K).$$

Proof. Let $e = \sum_i^n e_i$. It follows immediately from the definition of simplicial cohomology that $\tilde{H}^{i-1}(\Delta; K) \cong H^i(K\{\Delta\}, e)$. Thus it remains to be shown that e is generic for $K\{\Delta\}$. Let $\bar{K}$ be an algebraic closure of K. Then $H^i(K\{\Delta\}, v) \otimes_K \bar{K} \cong H^i(\bar{K}\{\Delta\})$, and $\tilde{H}^{i-1}(\Delta; K) \otimes_K \bar{K} \cong \tilde{H}^{i-1}(\Delta; \bar{K})$. Thus we may as well assume that K is algebraically closed. The set L of elements $v = \sum_i^n a_i e_i \in E_1$ with $\prod_i^n a_i \neq 0$ is open. Moreover, the complexes $(K\{\Delta\}, v)$ and $(K\{\Delta\}, e)$ are isomorphic for all $v \in L$. In fact, the isomorphism of complexes is induced by the algebra automorphism $\varphi\colon K\{\Delta\} \to K\{\Delta\}$ with $\varphi(e_i) = a_i e_i$ for $i = 1, \dots, n$. Let $G \subset E_1$ be the subset of generic elements for $K\{\Delta\}$. Since K is algebraically closed and since L and G are non-empty open subsets of the irreducible space E_1, their intersection is non-empty. Let v be an element of this intersection. Then v is general, and $\dim_K H^i(K\{\Delta\}, v) = \dim_K H^i(K\{\Delta\}, e)$ for all i. This proves the assertion. Q.E.D.

For the rest of the section we discuss the following question: Let $I \subset S$ be a squarefree monomial ideal, $J \subset E$ the corresponding monomial ideal in the exterior algebra. Is there a relation between the S-resolution of I and the E-resolution of J? We will show that this is indeed the case. In order to describe this relation it is convenient to consider the attached simplicial complexes.

Both algebras, $K[\Delta]$ as well as $K\{\Delta\}$, are $\mathbb{Z}^n$-graded, and hence have $\mathbb{Z}^n$-graded resolutions. A formula for the $\mathbb{Z}^n$-graded Betti numbers $\beta_{ia}(K[\Delta])$ is given by Hochster [24] in terms of reduced simplicial homology.

Let $a \in \mathbb{Z}^n$. We set $\operatorname{supp}(a) = \{i \in [n] \colon a_i \neq 0\}$ and $|a| = \sum_i a_i$. The element $a \in \mathbb{Z}^n$ is called *squarefree*, if $a_i \in \{0,1\}$ for $i = 1, \dots, n$.

Let $\sigma \subset [n]$. The *restriction of* Δ *to* σ is the simplicial complex $\Delta_\sigma = \{\tau \in \Delta \colon \tau \subset \sigma\}$.

Theorem 5.3. *Let Δ be a simplicial complex with vertex set $[n]$, and $a \in \mathbb{N}^n$. Then for all $i \geq 0$, we have*

(a) $\beta^S_{ia}(K[\Delta]) = 0$, *if a is not squarefree;*
(b) $\beta^S_{ia}(K[\Delta]) = \dim_K \tilde{H}_{|a|-i-1}(\Delta_{\operatorname{supp}(a)}; K)$, *if a is squarefree.*

For the proof we refer to Hochster's original paper [24], or to [14].

There is a similar kind of formula for the $\mathbb{Z}^n$-graded Betti numbers of $K\{\Delta\}$ given in [6].

Theorem 5.4. *Let Δ be a simplicial complex with vertex set $[n]$, and $a \in \mathbb{Z}^n$. Then for all $i \geq 0$, we have*

$$\beta^E_{ia}(K\{\Delta\}) = \dim \tilde{H}^{|a|-i-1}(\Delta_{\operatorname{supp}(a)}; K).$$

Proof. Set $\alpha = \operatorname{supp}(a)$, and let $\tilde{\mathcal{C}}(\Delta_\alpha)$ be the augmented oriented chain complex of Δ_α. The module $\mathcal{C}_i$ of i-chains of $\tilde{\mathcal{C}}(\Delta_\alpha)$ is the free $\mathbb{Z}$-module with basis $\sigma \in \Delta_\alpha$, $|\sigma| = i+1$. Thus the module of i-cochains $\mathcal{C}^i(K)$ is a K-vector space with basis σ^*, $\sigma \in \Delta_\alpha$, $|\sigma| = i+1$ where $\sigma^* \colon \mathcal{C}_i \to K$ is the $\mathbb{Z}$-linear map with $\sigma^*(\tau) = 0$ for $\tau \neq \sigma$ and $\sigma^*(\tau) = 1$ for $\tau = \sigma$.

On the other hand, $\operatorname{Tor}^E(K\{\Delta\}, K)_a$ may be identified with the homology of the ath graded piece $C(e_1, \dots, e_n; K\{\Delta\})_a$ of the Cartan complex. In degree i this complex has the following K-basis

$$e_\sigma x^{(a_\sigma)}, \qquad \sigma \in \Delta_\alpha, \qquad |a_\sigma| = i.$$

Here $a_\sigma = (a'_1, \dots, a'_n)$ where $a'_j = a_j$ for $j \notin \sigma$ and $a'_j = a_j - 1$ for $j \in \sigma$.

We define a K-linear map

$$\varphi_i \colon C_i(e_1, \dots, e_n; K\{\Delta\})_a \longrightarrow \mathcal{C}^{d-i-1}(K)$$

by setting $\varphi_i(e_\sigma x^{(a_\sigma)}) = \sigma^*$. One easily checks that φ is an isomorphism of complexes. Q.E.D.

A comparison of the formulas in Theorem 5.3 and Theorem 5.4 leads to

Corollary 5.5. *Let Δ be a simplicial complex with vertex set $[n]$. Then*

$$\sum_i \sum_{a \in \mathbb{N}^n} \beta^E_{ia}(K\{\Delta\}) t^i s^a = \sum_i \sum_{a \in \mathbb{N}^n} \beta^S_{ia}(K[\Delta]) \frac{t^i s^a}{\prod_{j \in \operatorname{supp} a}(1 - ts_j)}.$$

There is actually an explicit construction for the E-resolution of $K\{\Delta\}$ in terms of the S-resolution of $K[\Delta]$. This construction is described in [9].

§6. Regularity and extremal Betti numbers

In this section we present the theorem of Bayer, Charalambous and S. Popescu [12] which asserts that the extremal Betti numbers of a graded ideal and its generic ideal coincide.

Throughout this section we assume that the base field K is infinite. Let M be a finitely generated graded S-module. A Betti number of M is called *extremal* if $\beta_{ii+j} = 0$ for all $(i,j) \neq (k,m)$ with $i \geq k$ and $j \geq m$. The corollary of the next theorem provides a characterization of extremal Betti numbers in terms of annihilators of almost regular sequences.

An element $x \in S_1$ is called *almost M-regular*, if the colon module $0 :_M x = \{c \in M \colon xc = 0\}$ is of finite length. The set of almost M-regular elements is a nonempty open subset of S_1. Indeed, $M/H^0_{\mathfrak{m}}(M)$ is a module of positive depth, so that the Zariski open set $\mathcal{S} \subset S_1$ of regular elements of $M/H^0_{\mathfrak{m}}(M)$ in S_1 is not empty. For any element $x \in \mathcal{S}$ we have that $0 :_M x$ is a finite length module.

Let $\mathbf{l} = l_1, \dots, l_m$ be a sequence of linear forms in S. In order to simplify notation we set $M\langle j \rangle = M/(l_1, \dots, l_j)M$, and for $i \geq 1$ we let $H_i(j)$ be the ith Koszul homology $H_i(l_1, \dots, l_j; M)$ of M with respect to the sequence $l_1, \dots, l_j$. We further set $H_i(0) = 0$ for $i > 0$ and for $j \geq 1$ we let $H_0(j-1)$ be the colon ideal $0 :_{M\langle j-1 \rangle} l_j$. Observe that, in our notation, $H_0(j)$ is *not* the 0th Koszul homology.

The sequence $\mathbf{l} = l_1, \dots , l_m$ is called an *almost regular M-sequence* if for all $j = 1, \dots , m$, the linear form l_j is almost $M_{\langle j-1\rangle}$-regular. If all permutations of the sequence $\mathbf{l}$ are almost M-regular, then we call $\mathbf{l}$ an *unconditioned almost regular M-sequence* .

Suppose $\mathbf{l} = l_1, \dots , l_m$ is almost M-regular, then all $H_i(j)$ are modules of finite length and since M is a graded S-module, all $H_i(j)$ are naturally graded. Now suppose in addition that $\mathbf{l}$ is a basis of S_1. Then there are graded isomorphisms $H_i(n)_j \cong \operatorname{Tor}_i(K, M)_j$ for all i and j. In particular, the graded ijth Betti numbers β_{ij} of M coincide with $\dim_K H_i(n)_j$.

Let N be an Artinian graded module. We set $s(N) = \max\{s\colon N_s \neq 0\}$ if $N \neq 0$ and $s(0) = -\infty$. Now we introduce the following numbers attached to M and the basis $\mathbf{l} = l_1, \dots , l_n$. We set

$$r_j = \max\{s(H_i(j)) - i\colon i \geq 1\} \text{ and } s_j = s(H_0(j-1)) \text{ for } j = 1, \dots , n,$$

and put $r_0 = 0$. We observe that $\operatorname{reg}(M) = \max\{r_n, s(M/\mathfrak{m}M)\}$.

Theorem 6.1. *Suppose that the basis $\mathbf{l} = l_1, \dots , l_n$ of S_1 is an almost regular M-sequence. Then*

(a) $r_j = \max\{s_1, \dots , s_j\}$ *for $j = 1, \dots , n$. In particular,* $r_1 \leq r_2 \leq \dots \leq r_n$.

(b) *Let $\mathcal{J} = \{j_1, \dots , j_l\}$, $1 \leq j_1 < j_2 < \dots < j_l \leq n$, be the set of elements $j \in [n]$ such that $r_j - r_{j-1} \neq 0$. Then for all t with $1 \leq t \leq l$ and all j with $j_t \leq j$ we have*
   - (i) $H_i(j)_{i+s} = 0$ *for $s > r_{j_{t-1}}$ and $i > j - j_t + 1$;*
   - (ii) $H_{j-j_t+1}(j)_{j-j_t+1+r_{j_t}} \cong H_0(j_t - 1)_{r_{j_t}}$;
   - (iii) $H_{j-j_t+1}(j)_{j-j_t+1+s}$ *is isomorphic to a submodule of* $H_0(j_t - 1)_s$ *for all* $s > r_{j_{t-1}}$;
   - (iv) $H_0(j-i)_{r_{j_t}}$ *is isomorphic to a factor module of* $H_i(j)_{i+r_{j_t}}$ *for all i with* $i > j - j_{t+1} + 1$.

For the proof of this theorem we refer to [4].

Corollary 6.2. *Let the numbers j_t be defined as in Theorem 6.1, and set $k_t = n - j_t + 1$ and $m_t = r_{j_t}$. Then*

(a) *the Betti number β_{ii+j} of M is extremal if and only if*

$$(i, j) \in \{(k_t, m_t)\colon t = 1, \dots , l\}.$$

Moreover, $\beta_{k_t,k_t+m_t} = \dim_K(0 : l_{j_t})_{s_{j_t}}$ *for* $t = 1, \dots , l$,

(b) *for all $t = 1, \dots , l$ we have*
   - (1) $\beta_{k_t,k_t+s} \leq \dim_K(0 : l_{j_t})_s$ *for all* $s > m_{t-1}$,
   - (2) $\beta_{i,i+m_t} \geq \dim_K(0 : l_{n-i+1})_{m_t}$ *for all* $i > k_{t+1}$.

Now we are ready to prove the main theorem of this section.

Theorem 6.3 (Bayer-Charalambous-S. Popescu). *Let $I \subset S$ be a graded ideal, and let* $\operatorname{Gin}(I)$ *be the generic initial ideal of I with respect to the reverse lexicographic order. Then for any two integers $i, j \in \mathbb{N}$ one has*

(a) *the ijth Betti number of S/I is extremal if and only if the ijth Betti number of $S/\operatorname{Gin}(I)$ is extremal;*
(b) *the corresponding extremal Betti numbers of S/I and $S/\operatorname{Gin}(I)$ are equal.*

Proof. After a generic choice of coordinates we may assume that $\operatorname{Gin}(I) = \operatorname{in}(I)$, and since the condition of being an almost regular sequence is an open condition, we may as well assume that $x_n, \ldots, x_1$ is an almost regular S/I-sequence. Since $\operatorname{Gin}(I)$ is Borel fixed it follows for example from [16, Corollary 15.25] that $x_n, \ldots, x_1$ is an almost regular $S/\operatorname{Gin}(I)$-sequence, too. Set $R\langle j\rangle = (S/I)/(x_n, \ldots, x_j)(S/I)$ and $\bar{R}\langle j\rangle = (S/\operatorname{Gin}(I))/(x_n, \ldots, x_j)(S/\operatorname{Gin}(I))$, then it follows that $0 :_{R\langle n-i+1\rangle} x_{n-i}$ as well as $0 :_{\bar{R}\langle n-i+1\rangle} x_{n-i}$ have finite length for all i. Now since the chosen term order is reverse lexicographic it follows from Proposition 1.4 that $0 :_{R\langle n-i+1\rangle} x_{n-i}$ and $0 :_{\bar{R}\langle n-i+1\rangle} x_{n-i}$ have the same Hilbert function. In particular,

$$s(0 :_{R\langle n-i+1\rangle} x_{n-i}) = s(0 :_{\bar{R}\langle n-i+1\rangle} x_{n-i}) \quad \text{for all} \quad i.$$

Thus Corollary 6.2(a) concludes the proof. Q.E.D.

Corollary 6.4. *Let $I \subset S$ be a graded ideal,* $\operatorname{Gin}(I)$ *the generic initial ideal of I with respect to the reverse lexicographic order. Then*

(a) (Bayer-Stillman) $\operatorname{reg}(I) = \operatorname{reg}(\operatorname{Gin}(I))$;
(b) $\operatorname{proj\,dim} S/I = \operatorname{proj\,dim} S/\operatorname{Gin}(I)$;
(c) *S/I is Cohen-Macaulay, if and only if $S/\operatorname{Gin}(I)$ is Cohen-Macaulay.*

§7. Extremal Betti numbers for squarefree monomial ideals

Let Δ be a simplicial complex, and $J_\Delta \subset E$ the Stanley-Reisner ideal of Δ in the exterior algebra. The *exterior algebraic shifted complex* of Δ is the simplicial complex Δ^e with

$$J_{\Delta^e} = \operatorname{Gin}(J_\Delta).$$

We will study algebraic shifting more systematically in the next section. Here we are interested in the comparison of the graded Betti numbers of $I_\Delta \subset S$ with those of I_{Δ^e}. Though it is not yet known whether or not the graded Betti numbers increase when passing from I_Δ to I_{Δ^e}, it can be shown (see [4]) that I_Δ and I_{Δ^e} have the same extremal Betti numbers. In fact one has

Theorem 7.1. *Let Δ be a simplicial complex. Then for any two integers $i, j \in \mathbb{N}$ one has*

(a) *the ijth Betti number of S/I_Δ is extremal if and only if the ijth Betti number of S/I_{Δ^e} is extremal;*
(b) *the corresponding extremal Betti numbers of S/I_Δ and S/I_{Δ^e} are equal.*

We will describe the main steps of the proof. For simplicity we set $J = J_\Delta$ $I = I_\Delta$. Set $P_j(t) = \sum_{i\geq 0} \beta^E_{ii+j}(E/J)t^i$, then Corollary 5.5 yields

$$P_j(t) = \sum_{i\geq 0}(\sum_{k=0}^{i} \binom{i+j-1}{j+k-1}\beta^S_{kk+j})t^i.$$

Setting $k(j) = \max\{k\colon \beta^S_{kk+j}(S/I) \neq 0\}$, we see that

$$P_j(t) = \frac{\sum_{k=0}^{k(j)} \beta^S_{kk+j}(S/I)t^k(1-t)^{k(j)-k} + R(t)(1-t)^{k(j)+j}}{(1-t)^{k(j)+j}},$$

with a certain polynomial $R(t)$.

We set $d_j(E/J) = k(j) + j$ and $e_j(E/J) = \beta^S_{k(j),k(j)+j}(S/I)$.

Corollary 7.2. *The following conditions are equivalent:*

(a) *$\beta^S_{ii+j}(S/I)$ is an extremal Betti number of S/I;*
(b) *$i = k(j)$, and $d_{j'}(E/J) - d_j(E/J) < j' - j$ for all $j' > j$.*

For the further discussion we need (see [4, Corollary 4.6]) a different interpretation of the numbers d_j and e_j.

Proposition 7.3. *Let $M \in \mathcal{M}$, and let $v_1, \dots, v_n$ be a generic basis of E_1. Then the natural maps*

$$H_i(v_1, \dots, v_{j+1}; M) \xrightarrow{\beta_i} H_i(v_1, \dots, v_{j+1}; M)$$

of Cartan homology attached with the sequence $v_1, \dots, v_n$ (cf. Proposition 4.1) are surjective for all $j = 0, \dots, n-1$ and all $j \gg 0$.

We now fix $M \in \mathcal{M}$ and a sequence $\mathbf{v} = v_1, \dots, v_n$ in E_1. Similarly as in Section 1 we set $M\langle j-1\rangle = M/(v_1, \dots, v_{j-1})M$ and put $H_i(j) = H_i(v_1, \dots, v_j; M)$ for $i > 0$ and $H_0(j) = H^\bullet(M\langle j-1\rangle, v_j)$ for

$j = 1, \dots, n$. Furthermore we set $H_i(0) = 0$ for all i. Notice that $H_0(j)$ is *not* the 0th Cartan homology of M with respect to $v_1, \dots, v_j$, but is the cohomology of $M\langle j-1\rangle$ with respect to v_j as defined in Section 5. From Proposition 4.1 we obtain immediately the following long exact sequence of graded E-modules

$$
\begin{aligned}
(6)\quad & H_2(j) \to H_1(j)(-1) \to H_1(j-1) \to H_1(j) \to H_0(j)(-1) \to 0 \\
& \cdots \to H_i(j-1) \to H_i(j) \to H_{i-1}(j)(-1) \to H_{i-1}(j-1) \to \cdots .
\end{aligned}
$$

We fix an integer j. By Proposition 7.3 there exists an integer i_0 such that for all $i \geq i_0$ and all $k = 1, \dots, n$ the sequences

$$
(7)\quad 0 \longrightarrow H_{i+1}(k-1)_{(i+1)+j} \longrightarrow H_{i+1}(k)_{(i+1)+j} \longrightarrow H_i(k)_{i+j} \longrightarrow 0
$$

are exact.

Set $h_i^k = \dim_K H_i(k)_{i+j}$, and $c_k = h_{i_0}^k$ for $k = 1, \dots, n$. The exact sequences (7) yield the equations

$$
(8)\qquad h_{i+1}^k = h_{i+1}^{k-1} + h_i^k
$$

for all $i \geq i_0$, and $k = 1, \dots, n$. It follows from (8) that

$$
h_{i_0+i}^n = \binom{i+n-2}{n-1} c_1 + \binom{i}{n-3} c_2 + \cdots + \binom{i}{1} c_{n-1} + c_n \text{for all} i \geq 0.
$$

Since $\beta_{ii+j}^E(M) = h_i^n$ for all i, we see that

$$
\sum_{i\geq 0} \beta_{ii+j}^E(M) = t^{i_0+1} \sum_{i=1}^{n} \frac{c_i}{(1-t)^{n-i+1}} + Q(t),
$$

where $Q(t)$ is a polynomial. Thus we obtain:

Proposition 7.4. *Let d_j and e_j be defined as above. Then*

$$
d_j(E/J) = n + 1 - \min\{i\colon c_i \neq 0\} \quad \text{and} \quad e_j(E/J) = c_{n-d_j+1}.
$$

In order to relate the invariants d_j and e_j to the generalized simplicial homology modules $H_0(k)$ we need the following

Lemma 7.5. *Let $1 \leq l \leq n$ and j be integers. The following conditions are equivalent:*

(a) (1) $H_0(k)_j = 0$ *for* $k < l$, *and* $H_0(l)_j \neq 0$
  (2) $H_0(k)_{j'} = 0$ *for all* $j' > j$ *and all* $k \leq l + j - j'$.

(b) *For all* $i \geq 0$ *we have*
  (1) $H_i(k)_{i+j} = 0$ *for* $k < l$, *and* $H_i(l)_{i+j} \neq 0$
  (2) $H_i(k)_{i+j'} = 0$ *for all* $j' > j$ *and all* $k \leq l + j - j'$.

(c) *Condition* (b) *is satisfied for some* i.

Moreover, if the equivalent conditions hold, then $H_i(l)_{i+j} \cong H_0(l)_j$ *for all* $i \geq 0$.

Proof. In our proof we will use the following exact sequence

(9) $$H_i(k-1)_{i+j'} \to H_i(k)_{i+j'} \to H_{i-1}(k)_{(i-1)+j'} \to H_{i-1}(k-1)_{(i-1)+(j'+1)}$$

(a) $\Rightarrow$ (b): We prove (b) by induction on i. For $i = 0$, there is nothing to show. So now let $i > 0$ and assume that (1) and (2) hold for $i-1$. By (9) we have the exact sequence

$$H_i(l)_{i+j} \longrightarrow H_{i-1}(l)_{(i-1)+j} \longrightarrow H_{i-1}(l-1)_{(i-1)+(j+1)}.$$

Since $l-1 \leq l+j-(j+1)$, we have $H_{i-1}(l-1)_{(i-1)+(j+1)} = 0$ by induction hypothesis. Also by induction hypothesis, $H_{i-1}(l)_{(i-1)+j} \neq 0$; therefore, $H_i(l)_{i+j} \neq 0$.

Now let $k < l$. Then (9) yields the exact sequence

$$H_i(k-1)_{i+j} \longrightarrow H_i(k)_{i+j} \longrightarrow H_{i-1}(k)_{(i-1)+j}.$$

By induction hypothesis we have $H_{i-1}(k)_{(i-1)+j} = 0$. Now by induction on k we may assume that $H_i(k-1)_{i+j} = 0$. Therefore, $H_i(k)_{i+j} = 0$, and this shows (1).

In order to prove (b)(2), we let $j' > j$ and $k \leq l + (j - j')$, and consider the exact sequence

$$H_i(k-1)_{i+j'} \longrightarrow H_i(k)_{i+j'} \longrightarrow H_{i-1}(k-1)_{(i-1)+j'},$$

from which the assertion follows by induction on i and k.

(c) $\Rightarrow$ (a): We show that if the conditions (1) and (2) hold for $i > 0$, then they also hold for $i-1$. Therefore backwards induction yields the desired conclusion.

We begin with the proof of (2) for $i-1$ by induction on k. For $k = 0$, there is nothing to show. Now let $j' > j$, and $0 < k \leq l + (j - j')$, and consider the exact sequence

$$H_i(k)_{i+j'} \longrightarrow H_{i-1}(k)_{(i-1)+j'} \longrightarrow H_{i-1}(k-1)_{(i-1)+(j'+1)}.$$

Since $k-1 \leq l+j-(j'+1)$ it follows by our induction hypothesis that $H_{i-1}(k-1)_{(i-1)+(j'+1)} = 0$. On the other hand, by assumption we have $H_i(k)_{i+j'} = 0$, and hence $H_{i-1}(k)_{(i-1)+j'} = 0$.

In order to prove (1) for $i-1$ we consider the exact sequence

$$H_i(l-1)_{i+j} \longrightarrow H_i(l)_{i+j} \longrightarrow H_{i-1}(l)_{(i-1)+j} \longrightarrow H_{i-1}(l-1)_{(i-1)+(j+1)}.$$

Since $l-1 \leq l+j-(j+1)$, we know from (2) (which we have already shown for $i-1$) that $H_{i-1}(l-1)_{(i-1)+(j+1)} = 0$. By our assumption we have $H_i(l-1)_{i+j} = 0$, and hence

$$H_{i-1}(l)_{(i-1)+j} \cong H_i(l)_{i+j} \neq 0.$$

That $H_{i-1}(k)_{(i-1)+j} = 0$ for $k < l$ is proved similarly. This concludes the proof of the implication (c) $\Rightarrow$ (a).

In the proof of this implication we have just seen that $H_i(l)_{i+j} \cong H_{i-1}(l)_{(i-1)+j}$. By induction hypothesis we may assume that $H_{i-1}(l)_{(i-1)+j} \cong H_0(l)_j$, and hence $H_i(l)_{i+j} \cong H_0(l)_j$, as desired.

Q.E.D.

A pair of numbers (l, j) satisfying the equivalent conditions of Lemma 7.5 will be a called a *distinguished pair* (*for* M).

Now we may characterize the extremal Betti numbers of S/I as follows:

Corollary 7.6. *The Betti number $\beta_{ii+j}(S/I)$ is extremal if and only if $(n+1-i-j, j)$ is a distinguished pair. Moreover, if the equivalent conditions of Lemma* 7.5 *hold, then $\beta_{ii+j}(S/I) = \dim_K H_0(n+1-i-j)_j$.*

Proof. We know from Corollary 7.2 that $\beta_{ii+j}(S/I)$ is an extremal Betti number if and only if $d_{j'}(E/J) - d_j(E/J) < j' - j$ for all $j' > j$. By Proposition 7.4 this condition is equivalent to

$$\min\{k\colon H_{i_0}(k)_{i_0+j'} \neq 0\} > l + (j - j'),$$

where $l = \min\{k\colon H_{i_0}(k)_{i_0+j} \neq 0\}$. This in turn is equivalent to

$$H_{i_0}(k)_{i_0+j'} = 0 \quad \text{for} \quad k \leq l + (j - j'),$$

which means that (l, j) is a distinguished pair.

From Corollary 7.2 and Proposition 7.4 it follows that $l = n+1-i-j$. Finally, Corollary 7.2, Proposition 7.4 and Lemma 7.5 imply that

$$\beta_{ii+j}(S/I) = e_j(S/I) = c_l = \dim_K H_0(l)_j.$$

Q.E.D.

We we are ready for

Proof of Theorem 7.1. After a generic change of bases we may assume, that $\mathrm{in}(J) = \mathrm{Gin}(J)$, and that $e_n, \ldots, e_1$ is a generic basis for E/J, and since $\mathrm{in}(J)$ is Borel fixed it follows easily that $e_n, \ldots, e_1$ is a generic basis for $E/\operatorname{in}(J)$, too. We let $H_0(k) = H((E/J)/(e_n,\dots, e_{n-k+1}, e_{n-k})(E/J), e_{n-k})$. The corresponding homology modules for $E/\operatorname{in}(J)$ will be denoted by $H_0^e(k)$. It follows from the exterior analogue of Proposition 1.4 that for all $k = 1, \dots, n$ the homology modules $H_0(k)$ and $H_0^e(k)$ have the same Hilbert function. Since the Hilbert functions of these modules determine uniquely the distinguished pairs (l, j), all assertions of the theorem follow from Corollary 7.6. Q.E.D.

§8. Shifting operations

In this section we study shifting operations. They assign to each simplicial complex a shifted simplicial complex which shares basic properties with the original simplicial complex but is combinatorially simpler.

Let Δ be a simplicial complex on the vertex set $[n]$.

Definition 8.1. The simplicial complex Δ is *shifted*, if I_Δ is strongly stable. In other words, Δ is shifted if it satisfies the following property: if $\sigma \in \Delta$, $i \in \sigma$ and $j > i$, then $(\sigma \setminus \{i\}) \cup \{j\} \in \Delta$.

Following Kalai [27] we define a shifting operation by list of properties.

Definition 8.2. A map which assigns to each simplicial complex Δ on the vertex set $[n]$ a simplicial complex $\operatorname{Shift}(\Delta)$ on the same vertex set $[n]$ is called a *shifting operation*, if it satisfies the following conditions:

(S_1) $\operatorname{Shift}(\Delta)$ is shifted;
(S_2) $\operatorname{Shift}(\Delta) = \Delta$, if Δ itself is shifted;
(S_3) the simplicial complexes Δ and $\operatorname{Shift}(\Delta)$ have the same f-vector;
(S_4) if Γ is a subcomplex of Δ, then $\operatorname{Shift}(\Gamma) \subset \operatorname{Shift}(\Delta)$.

Shifting operations were first considered by Erdös, Ko, and Rado (see [1]), while algebraic shifting was introduced by Kalai [26], [27]. In this section we will present and compare the most important shifting operations.

Let us begin with

Combinatorial shifting: In the combinatorics of finite sets one considers the following operation (cf. [1]): Let $\mathcal{A}$ be a collection of subsets of $[n]$. For given integers $1 \le i < j \le n$, and for all $\sigma \in \mathcal{A}$ one defines:

$$S_{ij}(\sigma) = \begin{cases} (\sigma \setminus \{j\}) \cup \{i\}, & \text{if} \quad j \in \sigma, \quad i \notin \sigma, \quad (\sigma \setminus \{j\}) \cup \{i\} \notin \mathcal{A}, \\ \sigma, & \text{otherwise.} \end{cases}$$

For $1 \le i < j \le n$ and $a \in K$ we define an elementary automorphism $\varphi_{ij}^a \colon V \to V$ as follows: $\varphi_{ij}^a(e_k) = e_k$ if $k \ne j$, and $\varphi_{ij}^a(e_j) = ae_i + e_j$.

The following fact is easily checked

Lemma 8.3. *Let $J \subset E$ be a monomial ideal, and let $a \in K$, $a \neq 0$. Then $\operatorname{in}(\varphi_{ij}^a(J))$ has the K-basis $\{e_{S_{ij}(\sigma)} : \sigma \in \mathcal{A}\}$, where $\mathcal{A} = \{\sigma \subset [n] : e_\sigma \in J\}$.*

It follows in particular that the ideal $\operatorname{in}(\varphi_{ij}^a(J))$ does not depend on the choice of a. If Δ is a simplicial complex then $\operatorname{Shift}_{ij}(\Delta)$ is the simplicial complex defined by

$$J_{\operatorname{Shift}_{ij}(\Delta)} = \operatorname{in}(\varphi_{ij}^a(J_\Delta)).$$

Lemma 8.4. *The operator* $\operatorname{Shift}_{ij}$ *satisfies the conditions* (S_2), (S_3) *and* (S_4).

Proof. Suppose Δ is shifted, then I_Δ squarefree strongly stable, and so $J_\Delta \subset E$ is strongly stable. By Lemma 8.3, $\operatorname{in}(\varphi_{ij}^a(J_\Delta))$ has the K-basis $\{e_{S_{ij}(\sigma)} : \sigma \in \mathcal{A}\}$. As J_Δ is strongly stable it follows that $e_{S_{ij}(\sigma)} \in J_\Delta$ for all $e_\sigma \in J_\Delta$. This proves (S_2).

For the proof of (S_3) we note that J_Δ and $\operatorname{in}(\varphi_{ij}^a(J_\Delta))$ have the same Hilbert function. Condition (S_4) follows from Lemma 8.3. Q.E.D.

Simple examples show that (S_1) is in general not satisfied for $\operatorname{Shift}_{ij}$. We will see however that a suitable sequence of these operators yield a shifted simplicial complex.

For a monomial $u \in E$ of degree d, $u = e_{j_1} \wedge \cdots \wedge e_{j_d}$, we set $c_d(u) = \sum_{k=1}^d j_k$. Moreover, if $J \subset E$ is a monomial ideal, we set $c_d(J) = \sum c_d(u)$ where the sum is taken over all monomials of degree d in J.

The following result was shown in [5]

Proposition 8.5. *Let Δ be a simplicial complex. Then*

(a) *$c_d(J_{\operatorname{Shift}_{ij}(\Delta)}) \leq c_d(J_\Delta)$ for all d;*
(b) *if Δ is not shifted, then there exist i and j with $i < j$ such that $c_d(J_{\operatorname{Shift}_{ij}(\Delta)}) < c_d(J_\Delta)$ for some d.*

Proof. Assertion (a) follows from the fact that $c_d(e_{S_{ij}(\sigma)}) \leq c_d(e_\sigma)$ for all monomials of degree d.

Suppose now that J_Δ is not strongly stable. Then there exists a squarefree monomial $e_\sigma \in J_\Delta$ (of degree d) and integers i and j with $i < j$ such that $e_{S_{ij}(\sigma)} \notin J_\Delta$. Since $c_d(e_{S_{ij}(\sigma)}) < c_d(e_\sigma)$, it follows that $c_d(J_{\operatorname{Shift}_{ij}(\Delta)}) < c_d(J_\Delta)$, as desired. Q.E.D.

Corollary 8.6. *Let Δ be a simplicial complex. Then there exists a sequence of pairs of integers $(i_1, j_1), \ldots, (i_r, j_r)$ with $i_k < j_k$ for $k =$*

$1, \ldots, r$ *such that*

$$\mathrm{Shift}_{i_r j_r}(\mathrm{Shift}_{i_{r-1} j_{r-1}}(\ldots(\mathrm{Shift}_{i_1 j_1}(\Delta)\ldots)))$$

is shifted.

Any simplicial complex which is obtained from Δ by a sequence of operations as described in Corollary 8.6 will be denoted by Δ^c. It follows from our discussions that $\Delta \mapsto \Delta^c$ is a shifting operator. We call this operator *combinatorial shifting*. Combinatorial shifting is not very natural. In fact, Δ is not even uniquely defined. The only advantage of this operator is that it is easily computable.

Conjecture 8.7. *For all simplicial complexes Δ on the vertex set $[n]$ and all integer k and l with $1 \leq k < l \leq n$ one has $\beta_{ij}(I_\Delta) \leq \beta_{ij}(I_{\mathrm{Shift}_{kl}(\Delta)})$ for all i and j. In particular, $\beta_{ij}(I_\Delta) \leq \beta_{ij}(I_{\Delta^c})$ for all i and j.*

It is only known that $\beta_{0j}(I_\Delta) \leq \beta_{0j}(I_{\mathrm{Shift}_{kl}(\Delta)})$ for all j.

Exterior algebraic shifting: Let Δ be simpicial complex, $J_\Delta \subset E$ its Stanley-Reisner ideal in the exterior algebra. Recall from Section 7 that the exterior algebraic shifted complex Δ^e of Δ is defined by the equation $J_{\Delta^e} = \mathrm{Gin}(J_\Delta)$.

Proposition 8.8. *Exterior algebraic shifting is in fact a shifting operator, that is, it satisfies the conditions (S_1) - (S_4).*

Proof. Condition (S_1) follows from Proposition 2.13, and (S_3) and (S_4) follow as for combinatorial shifting. In order to prove (S_2) we notice that for any strongly stable ideal $J \subset E$ and any invertible upper triangular matrix φ one has $\varphi(J) = J$. The assertion is clear for elementary upper triangular matrices, as well as for invertible diagonal matrices. Since these matrices generate all invertible upper triangular matrices, we get the desired conclusion. Therefore, if $J \subset E$ is strongly stable, then $\mathrm{Gin}(J) = \mathrm{in}(\varphi(J)) = \mathrm{in}(J) = J$. Q.E.D.

Conjecture 8.9. *Let Δ be simplicial complex. Then*

$$\beta_{ij}(I_\Delta) \leq \beta_{ij}(I_{\Delta^e}).$$

Note that a result similar to Theorem 3.1 holds for ideals in the exterior algebra, so that in particular one has $\beta_{ij}(J) \leq \beta_{ij}(\mathrm{Gin}^E(J))$ for all i and j. Unfortunately this does not imply the conjecture, even if one uses Corollary 5.5.

As a consequence of the fact that I_Δ and I_{Δ^e} have the same extremal Betti numbers we now derive further properties of exterior algebraic shifting which were discovered first (with different methods) by Kalai [26].

Proposition 8.10 (Kalai). *For all i one has*

$$\tilde{H}^i(\Delta; K) \cong \tilde{H}^i(\Delta^e; K).$$

Proof. Hochster's formulas (cf. Theorem 5.3) imply that

$$\beta_{in}(K[\Delta]) = \dim_K \tilde{H}_{n-i-1}(\Delta; K) \quad \text{for all } i, \tag{10}$$

and $\beta_{ij}(K[\Delta]) = 0$ for all i and all $j > n$. In particular we see that the Betti numbers $\beta_{in}(K[\Delta])$ are extremal. Thus $\beta_{in}(K[\Delta]) = \beta_{in}(K[\Delta^e])$, by Theorem 7.1. Since K is a field it follows that $\tilde{H}_{n-i-1}(\Delta; K) = \tilde{H}^{n-i-1}(\Delta; K)$ for all i, and the assertion follows. Q.E.D.

Remark 8.11. Let $J \subset E$ be a graded ideal. Using the exterior version of Proposition 1.4 one easily shows that $\dim_K H^i(E/J) = \dim_K H^i(E/\operatorname{Gin}(J))$ for all i, where $\operatorname{Gin}(J)$ is the generic initial ideal of J with respect to the reverse lexicographic order, and where $H^\bullet(M)$ denotes generalized cohomology of a graded E-module, as defined in Section 5. Note that this observation yields another proof of Proposition 8.10.

The *Alexander dual* of the simplicial complex Δ (on the vertex set $[n]$) is the simplicial complex

$$\Delta^* = \{\sigma \subset [n] \colon [n] \setminus \sigma \notin \Delta\}.$$

We shall need the following result ([18])

Theorem 8.12 (Eagon-Reiner). *Let Δ be a simplicial complex. Then the following conditions are equivalent:*

(a) *I_Δ has a linear resolution;*
(b) *the dual simplicial complex Δ^* is Cohen-Macaulay over K.*

Theorem 8.13 (Kalai). *The following conditions are equivalent:*

(a) *Δ is Cohen-Macaulay over K;*
(b) *Δ^e is Cohen-Macaulay over K;*
(c) *Δ^e is pure.*

Proof. We observe the simple fact (see for example [22, Lemma 1.1]) that $(\Delta^*)^e = (\Delta^e)^*$.

(a) $\Longleftrightarrow$ (b): By Theorem 8.12, Δ is Cohen-Macaulay over K if and only if I_{Δ^*} has linear resolution. Since, by Theorem 7.1, the regularity of $K[\Delta^*]$ and $K[(\Delta^*)^e]$ is the same, it follows that $K[\Delta^*]$ has a linear resolution if and only if $K[(\Delta^*)^e] = K[(\Delta^e)^*]$ has a linear resolution.

This in turn, again by Theorem 8.12, is the case if and only if Δ^e is Cohen-Macaulay over K.

The implication (b) $\Rightarrow$ (c) is true for any simplicial complex.

(c) $\Rightarrow$ (b): Since the maximal faces of Δ^e correspond to the minimal non-faces of $(\Delta^e)^*$, the purity of Δ^e implies that the minimal generators of the defining ideal of $(\Delta^e)^*$ all have the same degree. As $(\Delta^e)^* = (\Delta^*)^e$, we see that $(\Delta^e)^*$ is shifted, and hence its defining ideal is strongly stable. The resolution of a strongly stable ideal which is generated in one degree is linear, as follows from Corollary 3.4. This concludes the proof. Q.E.D.

Theorem 7.1 which says that extremal Betti numbers are preserved under exterior algebraic shifting can be translated into a theorem about the behaviour of links under shifting. Recall that the *link* of a face $\sigma \in \Delta$ is the simplicial complex

$$\mathrm{lk}_\Delta(\sigma) = \{\tau \in \Delta \colon \tau \cap \sigma = \emptyset \quad \text{and} \quad \tau \cup \sigma \in \Delta\}.$$

For the translation one needs the formula (see [18])

$$\tilde{H}_{i-2}(\mathrm{lk}\, \Delta^*(\sigma); K) \cong \tilde{H}^{|T|-i-1}(\Delta_T; K) \quad \text{for all } i \text{ and } \sigma \in \Delta^*,$$

where $T = [n] \setminus \sigma$. This is a slight generalization of the so-called Alexander duality formula:

$$\tilde{H}_{i-2}(\Delta^*; K) \cong \tilde{H}^{n-i-1}(\Delta; K) \quad \text{for all } i.$$

The numbers $\tilde{\beta}_i(\Gamma) = \dim_K \tilde{H}_i(\Gamma; K)$ are called the *reduced Betti numbers (with values in K)* of the simplicial complex Γ. Now we get

Theorem 8.14. *Let i and j be non-negative integers. Suppose $\tilde{\beta}_l(\mathrm{lk}_\Delta(\sigma)) = 0$ for all faces σ with $|\sigma| < j$, and all l with $i \leq l \leq i+t$, where $t = j - |\sigma|$. Then $\sum_{\sigma, |\sigma|=j} \tilde{\beta}_i(\mathrm{lk}_\Delta\, \sigma)$ is preserved under exterior algebraic shifting.*

Symmetric algebraic shifting: Let $I \subset S = K[x_1, \ldots, x_n]$ be a squarefree ideal, where K is field of characteristic 0. We let $\mathrm{Gin}(I)$ be the generic initial ideal of I with respect to the reverse lexicographic term order. We know from Proposition 2.2 that $\mathrm{Gin}(I)$ is a strongly stable ideal in S. But of course it is no longer squarefree.

We will transform $\mathrm{Gin}(I)$ into a squarefree monomial ideal by applying a certain operator: for a monomial $u \in S$, $u = x_{i_1} x_{i_2} \cdots x_{i_j} \cdots x_{i_d}$ with $i_1 \leq i_2 \leq \cdots \leq i_j \leq \cdots \leq i_d$, we set

$$u^\sigma = x_{i_1} x_{i_2+1} \cdots x_{i_j+(j-1)} \cdots x_{i_d+(d-1)}.$$

It then follows immediately

$$m(u^\sigma) - \deg u^\sigma = m(u) - 1. \tag{11}$$

If L is a monomial ideal with $G(L) = \{u_1, \dots , u_s\}$, then we write L^σ for the squarefree monomial ideal generated by $u_1^\sigma, \dots , u_s^\sigma$ in $K[x_1, \dots , x_m]$, where $m = \max\{m(u) + \deg u - 1 \colon u \in G(L)\}$.

Let Δ be a simplicial complex on the vertex set $[n]$. The *symmetric algebraic shifted complex* of Δ is defined to be the simplicial complex Δ^s with

$$I_{\Delta^s} = (\mathrm{Gin}(I_\Delta))^\sigma.$$

The definition of symmetric algebraic shifting presented here is formally different from that of Kalai [27]. However it is an easy exercise to see that both notions coincide.

A priori it is not clear from the definition of symmetric algebraic shifting that Δ^s has the same vertex set $[n]$. The next lemma shows that this indeed is the case.

Lemma 8.15. *If I is a squarefree monomial ideal of $S = K[x_1, \dots , x_n]$, then $m(u) + \deg u \leq n + 1$ for all $u \in G(\mathrm{Gin}(I))$.*

Proof. Recall from Examples 2.4 that the graded Betti numbers of a strongly stable ideal I are given by Eliahou–Kervaire :

$$\beta_{i,i+j}(I) = \textstyle\sum_{u\in G(I)_j} \binom{m(u)-1}{i} \tag{12}$$

for all i and j.

Since $\mathrm{Gin}(I)$ is strongly stable, formula (12) implies that $\max\{m(u)+ \deg u - 1 \colon u \in G(\mathrm{Gin}(I))\}$ is the highest shift in the resolution of $\mathrm{Gin}(I)$. The monomial ideal I being squarefree, Hochster's formula (see Theorem 5.3) guarantees that the highest shift in the resolution of I is less than or equal to n. Since the Betti numbers with highest shift in the resolution of I are extremal it follows from Theorem 6.3 that the highest shift in the resolution of I and that of $\mathrm{Gin}(I)$ coincide (see also [22]). This yields the desired inequalities. Q.E.D.

We want to point out that we defined symmetric algebraic shifting only in a polynomial rings whose base field is of characteristic 0, because otherwise we do not know if $\mathrm{Gin}(I)$ is strongly stable. It may be possible that symmetric algebraic shifting can be defined in any characteristic, provided the following question can be answered affirmatively.

Problem 8.16. Let $I \subset S = K[x_1, \dots , x_n]$ be a squarefree monomial ideal. Is it true that $\mathrm{Gin}(I)$ is strongly stable in any characteristic?

More generally one may even ask whether $\operatorname{Gin}(I)$ of a monomial ideal is strongly stable, if the characteristic of the field is larger than all exponents appearing in the monomial generators of I.

On the other hand, if I is squarefree and $\operatorname{char} K > n$, then $\operatorname{Gin}(I)$ is strongly stable. In fact, the highest degree of a generator of $\operatorname{Gin}(I)$ is less than or equal to $\operatorname{reg}(\operatorname{Gin}(I))$, and $\operatorname{reg}\operatorname{Gin}(I) = \operatorname{reg} I$ by Corollary 6.4. Since $\operatorname{reg}(I) \leq n$ by Theorem 5.3, we conclude that the highest degree of a generator of $\operatorname{Gin}(I)$ is $\leq n$. Hence the assertion follows from Proposition 2.2.

Note that condition (S_1) is satisfied since we have

Lemma 8.17. *Let I be a strongly stable ideal with $G(I) = \{u_1, \ldots, u_s\}$. Then the squarefree monomial ideal I^σ is squarefree strongly stable with $G(I^\sigma) = \{u_1^\sigma, \ldots, u_s^\sigma\}$.*

Proof. Suppose that, for some $u \in G(I)$, we have $u^\sigma \notin G(I^\sigma)$. Let $u = x_{i_1} \cdots x_{i_d}$ with $i_1 \leq \cdots \leq i_d$. Then, for some proper subset N of $\{1, 2, \ldots, d\}$ and for some $1 \leq q \leq s$, we have $u_q^\sigma = \prod_{j \in N} x_{i_j + (j-1)}$. Hence $u_q = \prod_{j \in N} x_{i_j + h_j}$, where h_j is the number of integers $1 \leq k < j$ with $k \notin N$. Since I is strongly stable, $\prod_{j \in N} x_{i_j}$ must belong to I. This contradicts $u \in G(I)$. Thus we have $G(I^\sigma) = \{u_1^\sigma, \ldots, u_s^\sigma\}$.

Next, to see why I^σ is squarefree strongly stable, let $u = x_{i_1} \cdots x_{i_d} \in G(I)$ and consider the monomial $(x_b u^\sigma)/x_{i_a + (a-1)}$ with $b \notin \operatorname{supp}(u^\sigma)$ and $b < i_a + (a-1)$. Let $i_p + (p-1) < b < i_{p+1} + p$ for some $p < a$ and set

$$v = (\prod_{j=1}^{p} x_{i_j}) x_{b-p} (\prod_{j=p+1}^{a-1} x_{i_j - 1})(\prod_{j=a+1}^{d} x_{i_j}).$$

Then, since $b - p < i_{p+1} \leq i_a$ and since I is strongly stable, the monomial v belongs to I. Note that $v^\sigma = (x_b u^\sigma)/x_{i_a + (a-1)}$. Say, $v = x_{\ell_1} \cdots x_{\ell_d}$ with $\ell_1 \leq \cdots \leq \ell_d$. Again, since I is strongly stable, it follows that $w = x_{\ell_1} \cdots x_{\ell_c} \in G(I)$ for some $c \leq d$. Since w^σ divides v^σ, we have $(x_b u^\sigma)/x_{i_a + (a-1)} \in I^\sigma$, as desired. Q.E.D.

The operator $I \mapsto I^\sigma$ behaves well with respect to graded Betti numbers.

Lemma 8.18. *If I is a strongly stable monomial ideal, then $\beta_{ii+j}(I) = \beta_{ii+j}(I^\sigma)$ for all i and j.*

Proof. The result follows from (11), Corollary 3.4(a) and Corollary 3.6(a). Q.E.D.

Next we indicate the proof of condition (S_2) for symmetric algebraic shifting.

Theorem 8.19. *Let $I \subset S$ be a squarefree strongly stable ideal of S. Then $I^s = I$.*

For the proof we introduce the operation τ which is inverse to σ: For a squarefree monomial $u = x_{i_1}x_{i_2}\cdots x_{i_j}\cdots x_{i_d}$ with $i_1 < i_2 < \cdots < i_j < \cdots < i_d$, we set

$$u^\tau = x_{i_1}x_{i_2-1}\cdots x_{i_j-(j-1)}\cdots x_{i_d-(d-1)}.$$

If $I \subset S$ is a squarefree monomial ideal with $G(I) = \{u_1, \ldots, u_s\}$, then we write I^τ for the monomial ideal generated by $u_1^\tau, \ldots, u_s^\tau$ in S.

Similarly to Lemma 8.17, we show:

Lemma 8.20. *Let I be a squarefree strongly stable ideal with $G(I) = \{u_1, \ldots, u_s\}$. Then the ideal I^τ is strongly stable with $G(I^\tau) = \{u_1^\tau, \ldots, u_s^\tau\}$.*

Proof. Assume that for some $u \in G(I)$, we have $u^\tau \notin G(I^\tau)$. Let $u = x_{i_1}\cdots x_{i_d}$ with $i_1 < \cdots < i_d$. Then for some proper subset $\{j_1, \ldots, j_t\}$ of $\{1, 2, \ldots, d\}$, where $j_1 < \cdots < j_t$, and for some $1 \le q \le s$, we have $u_q^\tau = \prod_{k=1}^t x_{i_{j_k}-(j_k-1)}$. Hence $u_q = \prod_{k=1}^t x_{i_{j_k}-(j_k-k)}$. Since $i_k \le i_{j_k} - (j_k - k)$ for $1 \le k \le t$ and I is squarefree strongly stable, we get $x_{i_1}\cdots x_{i_t} \in I$ which contradicts $u \in G(I)$.

Now, we show that I^τ is strongly stable. Let $u = x_{i_1}\cdots x_{i_d} \in G(I)$ with $i_1 < \cdots < i_d$, and consider the monomial $v = (x_b u^\tau)/x_{i_k-(k-1)}$ with $b < i_k - (k-1)$. Let $i_p - (p-1) \le b < i_{p+1} - p$ for some $p < k$. Then

$$v^\sigma = (\prod_{j=1}^{p} x_{i_j})x_{b+p}(\prod_{j=p+1}^{k-1} x_{i_j+1})(\prod_{j=k+1}^{d} x_{i_j}).$$

Since $b+p < i_{p+1}$ and $i_j + 1 \le i_{j+1}$ for $p+1 \le j \le k-1$, and since I is squarefree strongly stable, we obtain that $v^\sigma \in I$. Say, $v^\sigma = x_{\ell_1}\cdots x_{\ell_d}$ with $\ell_1 < \cdots < \ell_d$. Again, since I is squarefree strongly stable, it follows that $w = x_{\ell_1}\cdots x_{\ell_c} \in G(I)$ for some $c \le d$. Since w^τ divides $(v^\sigma)^\tau = v$, we have $v \in I^\tau$. Q.E.D.

The proof of Theorem 8.19 is based on the following lemma. We refer the reader to the original paper [7] for the somewhat tedious proof of the lemma.

Lemma 8.21. *Let $I \subset S$ be a squarefree strongly stable ideal generated in degree d. Let $G(I) = \{u_1, \ldots, u_s\}$ where $u_1 > u_2 > \ldots > u_s$.*

Let $g = (a_{ij})_{1\le i,j\le n}$ be a generic upper triangular matrix acting on S by $g(x_i) = \sum_{j=1}^{i} a_{ji}x_j$ for $1 \le i \le n$. Let c_{kj} denote the coefficient of u_j^τ in the polynomial $g(u_k)$ for $1 \le k,j \le s$. Then the determinant of the matrix $(c_{kj})_{1\le k,j\le s}$ is different from zero.

Proof of Theorem 8.19. Since the ideal I is squarefree strongly stable, I is componentwise linear [5]. Therefore by [8, Theorem 1.1], for the graded Betti numbers of I and $\mathrm{Gin}(I)$ it holds: $\beta_{ii+j}(I) = \beta_{ii+j}(\mathrm{Gin}(I))$ for all i and j. On the other hand, the ideal $\mathrm{Gin}(I)$ being strongly stable, it follows from Lemma 8.18 that $\beta_{ii+j}(\mathrm{Gin}(I)) = \beta_{ii+j}((\mathrm{Gin}(I))^\sigma)$ Thus, we obtain the equalities

$$(13) \qquad \beta_{i,i+j}(I) = \beta_{i,i+j}((\mathrm{Gin}(I))^\sigma) \quad \text{for all} \quad i,j,$$

which imply that I and $(\mathrm{Gin}(I))^\sigma$ have the same Hilbert function. Hence it is enough to prove that $I \subseteq (\mathrm{Gin}(I))^\sigma$. By Lemma 8.17 and Lemma 8.20 this inclusion is equivalent to $I^\tau \subseteq \mathrm{Gin}(I)$. So, we will show that $u^\tau \in \mathrm{Gin}(I)$ for every $u \in G(I)$.

We denote by $\langle u\rangle$ the smallest squarefree strongly stable ideal containing u. Since $I = \sum_{u\in G(I)}\langle u\rangle$, and $\mathrm{Gin}(\langle u\rangle) \subseteq \mathrm{Gin}(I)$ for every $u \in G(I)$, it is enough to show that the claim is true for squarefree Borel principal ideals. So, we may assume that $I = \langle u\rangle$. Set $d = \deg u$.

Let $G(I) = \{u_1,\dots,u_s\}$ where $u_1 > u_2 > \cdots > u_s$. Then $u_s = u$. We may assume that the claim is true for all u_k, $1 \le k \le s-1$. Then $(u_1^\tau, u_2^\tau,\dots,u_{s-1}^\tau) \subset \mathrm{Gin}(I)$, and since I^τ and $\mathrm{Gin}(I)$ have the same number of minimal monomial generators, one has $G(\mathrm{Gin}(I)) = \{u_1^\tau, u_2^\tau,\dots,u_{s-1}^\tau, v\}$, where v is a monomial of degree d. We have to prove that $v = u^\tau$.

Assume $v > u^\tau$. We will see that this is impossible. First, we show that $m(v) = m(u^\tau)$. It follows from formula Corollary 3.6(a) that

$$\beta_{ii+d}((\mathrm{Gin}(I))^\sigma) = \sum_{j=1}^{s-1}\binom{m(u_j)-d}{i} + \binom{m(v^\sigma)-d}{i};$$

$$\beta_{ii+d}(I) = \sum_{j=1}^{s-1}\binom{m(u_j)-d}{i} + \binom{m(u)-d}{i}.$$

Therefore, according to (13), we obtain $\binom{m(v^\sigma)-d}{i} = \binom{m(u)-d}{i}$ which implies $m(v^\sigma) = m(u)$, so that $m(v) = m(u^\tau)$.

We fix the following notation: $u = x_{s_1}\cdots x_{s_d}$ where $s_1 < \cdots < s_d$, and $v = x_{j_1}\cdots x_{j_d}$ where $j_1 \le \cdots \le j_d$. Since $v > u^\tau$, there exits a k such that $j_i = s_i - (i-1)$ for $k+1 \le i \le d$ and $j_k < s_k - (k-1)$. As $j_d = m(v) = m(u^\tau) = s_d - (d-1)$, one has $k < d$. If $j_i + (i-1) \le s_i$

for $1 \leq i \leq k$, then $I = \langle u \rangle$ being squarefree strongly stable, one obtains that $v^\sigma \in I$ which implies $v^\sigma = u_t$ for some $1 \leq t \leq s-1$ and the contradiction $v = u_t^\tau$. Thus, there exits an ℓ, $1 \leq \ell < k$, such that $j_\ell + (\ell - 1) > s_\ell$. Then $j_\ell \leq j_k < s_k - (k-1) \leq s_d - (d-1) = m(v)$, therefore $x_{j_\ell} v / x_{m(v)} \in \mathrm{Gin}(I)$, because $\mathrm{Gin}(I)$ is strongly stable. Since $x_{j_\ell} v / x_{m(v)} > v$, we get $x_{j_\ell} v / x_{m(v)} = u_t^\tau$ for some $1 \leq t \leq s-1$. Say $u_t = x_{t_1} \cdots x_{t_d}$ where $t_1 < \cdots < t_d$. As $I = \langle u \rangle$ is a squarefree Borel principal ideal, we have $t_i \leq s_i$ for $1 \leq i \leq d$, therefore $t_i - (i-1) \leq s_i - (i-1)$ for $1 \leq i \leq d$. This contradicts $j_\ell > s_\ell - (\ell - 1)$.

Hence, $v \leq u^\tau$. Now, we apply Lemma 8.21 using same notation. We have $\mathrm{Gin}(I) = \mathrm{in}(g(I))$ and $u_j^\tau \in \mathrm{Gin}(I)$ for $1 \leq j \leq s-1$. Since the rank of the matrix $(c_{kj})_{1 \leq k,j \leq s}$ is maximal, it follows that $v \geq u^\tau$, and so $v = u^\tau$. Q.E.D.

For symmetric algebraic shifting we can prove the inequality of graded Betti numbers which we conjecture for exterior algebraic shifting.

Theorem 8.22. *Let Δ be a simplicial complex. Then*

$$\beta_{ii+j}(I_\Delta) \leq \beta_{ii+j}(I_{\Delta^s}) \quad \text{for all} \quad i \quad \text{and} \quad j.$$

Proof. The desired inequalities follow from Theorem 3.1 and Lemma 8.18. Q.E.D.

Theorem 8.22 leads us to conjecture the following inequalities:

Conjecture 8.23. *Let Δ be a simplicial complex. Then for all i and j one has*

$$\beta_{ii+j}(I_{\Delta^s}) \leq \beta_{ii+j}(I_{\Delta^e}) \leq \beta_{ii+j}(I_{\Delta^c}).$$

In virtue of Theorem 8.22 the conjecture implies the yet open inequalities

$$\beta_{ii+j}(I_\Delta) \leq \beta_{ii+j}(I_{\Delta^e})$$

for all i and j. One should expect that there is direct proof of this inequality, avoiding a comparison with the symmmetric shifted ideal. The next result shows that the extremal Betti numbers of the symmetric algebraic shifted ideals behave as expected.

Theorem 8.24. *Let Δ be a simplicial complex. Then for all i and j*

(a) *the following conditions are equivalent:*
   (i) *the ijth Betti number of I_Δ is extremal,*
   (ii) *the ijth Betti number of I_{Δ^s} is extremal.*

(b) *the corresponding extremal Betti numbers of* I_Δ *and* I_{Δ^s} *are equal.*

Proof. The corresponding statements for I and $\mathrm{Gin}^S(I)$ are proved in Theorem 6.3. Hence, since $\beta_{ij}(\mathrm{Gin}^S(I_\Delta)) = \beta_{ij}(I_{\Delta^s})$ by Lemma 8.18, we obtain the assertions for I_Δ and I_{Δ^s}, too. Q.E.D.

The invariance of the extremal Betti numbers for combinatorial shifting is unknown. To prove it, it would suffice to show that I_Δ and $I_{\mathrm{Shift}_{ij}(\Delta)}$ have the same extremal Betti numbers.

As in the case of exterior algebraic shifting we get from Theorem 8.24

Corollary 8.25. *Let* K *be a field of characteristic* 0. *Then*

$$\tilde{H}_i(\Delta; K) \cong \tilde{H}_i(\Delta^s; K) \quad \textit{for all} \quad i.$$

The usefulness of Proposition 8.10 and Corollary 8.25 is partially explained by the fact that $\tilde{H}_\bullet(\Delta^e; K)$ and $\tilde{H}_\bullet(\Delta^s; K)$ can be computed combinatorially in a simple way. In fact, as noted in [27] (in a different terminology), one has

Lemma 8.26. *Let* Δ *be a simplicial complex on the vertex set* $[n]$ *such that* I_Δ *is squarefree strongly stable. Then*

$$\begin{aligned} \dim_K \tilde{H}_i(\Delta; K) &= |\{u \in G(I_\Delta)_{i+2} \colon m(u) = n\}| \\ &= |\{\sigma \in \Delta \colon \dim\sigma = i, \quad \sigma \cup \{n\} \notin \Delta\}|. \end{aligned}$$

Proof. The first equation follows from (10) and Corollary 3.6, while the second equation follows trivially from the definitions. Q.E.D.

§9. Superextremal Betti numbers

As an application of Theorem 8.24 we prove a non-squarefree version of a theorem of Björner and Kalai [15]. We first give a more algebraic proof of their theorem, which applies to any graded ideal in the exterior algebra, and not just to monomial ideals, but nevertheless follows closely the arguments of the original proof of Björner and Kalai.

So let $J \subset E$ be a graded ideal. We set $f_{i-1} = \dim_K(E/J)_i$ for all $i \geq 0$, and call $f = (f_0, f_1, \dots)$ the f-vector of E/J. As in Section we denote by $H^i(E/J)$ the generalized simplicial cohomology of E/J. We let $\beta_{i-1} = \dim_K H^i((E/J))$, and call $\beta = (\beta_{-1}, \beta_0, \beta_1 \dots)$ the Betti sequence of E/J. Jn case $J = J_\Delta$ for some simplicial complex Δ, the β_i are the ordinary Betti numbers of Δ.

A pair of sequences $(f, \beta) \in \mathbb{N}_0^\infty$ is called *compatible* if there exists a graded K-algebra E/J such that f is the f-sequence and β the Betti sequence of E/J.

Theorem 9.1 (Björner and Kalai). *Let K be a field. The following conditions are equivalent:*

(a) *The pair of sequences (f, β) is compatible.*

(b) *Set $\chi_i = (-1)^i \sum_{j=-1}^{i} (-1)^j (f_j - \beta_j)$ for all i. Then*

  (i) *$\chi_{-1} = 1$ and $\chi_i \geq 0$ for all i,*

  (ii) *$\beta_i \leq \chi_{i-1}^{(i)} - \chi_i$ for all i.*

Proof. (a) $\Rightarrow$ (b): The f-vectors of E/J and $E/\operatorname{Gin}(J)$ coincide, since they have the same Hilbert function. By Remark 8.11 we have $H^i(E/J) \cong H^i(E/\operatorname{Gin}(J))$ for all i. Hence also the Betti sequences of E/J and $E/\operatorname{Gin}(J)$ coincide. Thus we may replace J by $\operatorname{Gin}(J)$, and hence may as well assume that J is strongly stable.

Let J' be the ideal generated by all $u \in G(J)$ with $m(u) < n$ and all monomials $u \in E$ such that $u \wedge e_n \in G(J)$. Then J' is again strongly stable and $E_1 J' \subset J$. By Lemma 8.26, the last property implies that

$$\dim_K (J'/J)_i = |\{u \in G(J)_{i+1} \colon m(u) = n\}| = \beta_{i-1}(E/J).$$

It follows that $\dim_K (E/J')_i = f_{i-1} - \beta_{i-1}$ for all i. Now we notice that e_n is regular on E/J', in the sense that the complex

$$E/J' \xrightarrow{e_n} E/J' \xrightarrow{e_n} E/J'$$

is exact. Therefore, for each i we obtain an exact sequence of K-vector spaces

$$(14) \quad \to (E/J')_{i-1} \to (E/J')_i \to (E/J')_{i+1} \to (E/(J' + e_n E))_{i+1} \to 0,$$

and hence $\chi_i = \dim_K (E/(J' + e_n E))_{i+1}$.

Next we observe that $J'/J \cong (J' + e_n E)/(J + e_n E)$ and $E_1(J' + e_n E) \subset J + e_n E$, so that together with the Kruskal-Katona theorem (cf. Section 10) we obtain

$$\begin{aligned} \chi_i + \beta_i &= \dim_K E_{i+1} - \dim_K (J + e_n E)_{i+1} \\ &\leq \dim_K E_{i+1} - \dim_K E_1 (J' + e_n E)_i \leq \chi_{i-1}^{(i)}, \end{aligned}$$

as required.

(b) $\Rightarrow$ (a): The hypotheses imply that $\chi_i \leq \chi_{i-1}^{(i)}$ and $\chi_i + \beta_i \leq (\chi_{i-1} + \beta_{i-1})^{(i)}$. Thus the Kruskal-Katona theorem yields an integer m, and lexsegment ideals $L \subset N$ in the exterior algebra $E' = K\langle e_1, \ldots, e_{m-1}\rangle$ such that $\dim_K (E/N)_{i+1} = \chi_i$ and $\dim_K (E/L)_{i+1} = \chi_i + \beta_i$ that for all i.

Now let $J \subset E = K\langle e_1, \ldots, e_m\rangle$ be the ideal generated by the elements in $G(L)$ and all elements $u \wedge e_m$ with $u \in G(N)$. Moreover we set $J' = NE$. Then $J'/J \cong N/L$, and so

(15) $\dim_K(E/J)_{i+1} = \dim_K(N/L)_{i+1} + \dim_K(E/J')_{i+1} = \beta_i + \dim_K(E/J')_{i+1}.$

On the other hand, e_m is regular on E/J', and so (14) yields

(16) $\dim_K(E/(J' + e_mE))_{i+1} = (-1)^{i+1} \sum_{j=0}^{i+1}(-1)^j \dim_K(E/J')_j$

for all i. Thus, since $E/(J' + e_mE) \cong E'/N$, it follows from (16) that

$\dim_K(E/J')_{i+1} = \dim_K(E'/N)_{i+1} + \dim_K(E'/N)_i = \chi_i + \chi_{i-1} = f_i - \beta_i.$

This together with (15) implies that $\dim_K(E/J)_{i+1} = f_i$.

Finally it is clear from the construction of J that $|\{u \in G(J)_{i+2}\colon m(u) = m\}|$ equals $\dim_K(N/L)_{i+1}$ which is β_i. Thus, by Lemma 8.26, the assertion follows. Q.E.D.

The Björner-Kalai Theorem can be translated into a theorem on super extremal Betti numbers. Let $I \subset S$ be a graded ideal. We let m be the maximal integer j such that $\beta_{ij}(S/I) \neq 0$ for some i. In other words, m is the largest shift in the graded minimal free S-resolution of S/I. It is clear that $\beta_{im}(S/I)$ is an extremal Betti number for all i with $\beta_{im}(S/I) \neq 0$, and that there is at least one such i. These Betti numbers are distinguished by the fact that they are positioned on the diagonal $\{(i, m-i)\colon i = 0, \ldots, m\}$ on the Betti diagram, and that all Betti numbers on the right lower side of the diagonal are zero. The ring S/I may of course have other extremal Betti numbers, not sitting on this diagonal. We call the Betti numbers β_{im}, $i = 0, \ldots, m$, *super extremal*, regardless whether they are zero or not, and ask the question which sequences of numbers $(b_0, b_1, \ldots, b_m)$ appear as sequences of super extremal Betti numbers for graded rings with given Hilbert function.

Before answering the question we have to encode the Hilbert function $H_{S/I}(t)$ of S/I in a suitable way. Using the additivity of the Hilbert function, the graded minimal free resolution of S/I yields the following formula:

$$H_{S/I}(t) = \frac{a_0 + a_1t + a_2t^2 + \cdots + a_mt^m}{(1-t)^n}$$

with $a_i \in \mathbb{Z}$; see for example [14]. It follows that

$$(1-t)^{n-m}H_{S/I}(t) = \frac{a_0 + a_1t + a_2t^2 + \cdots + a_mt^m}{(1-t)^m}.$$

Notice that $n - m$ may take positive or negative values. At any rate, the rational function $(1-t)^{n-m}H_{S/I}(t)$ has degree ≤ 0. One easily verifies that there is a unique expansion

$$(1-t)^{n-m}H_{S/I}(t) = \sum_{i=0}^{m} f_{i-1}\frac{t^i}{(1-t)^i}$$

with $f_i \in \mathbb{Z}$. It is clear that $f_{-1} = 1$, and we shall see later that all $f_i \geq 0$. We call $f = (f_{-1}, f_0, f_1, \ldots, f_{m-1})$ the *f-vector of S/I*. Given the highest shift in the resolution, the f-vector of S/I determines the Hilbert function of S/I, and vice versa.

We set $b_i = \beta_{m-i-1,m}$ and call $b = (b_{-1}, \ldots, b_{m-1})$ the *super extremal sequence of S/I*. Finally we set $\chi_i = (-1)^i \sum_{j=-1}^{i}(-1)^j(f_j - b_j)$ for $i = -1, 0. \ldots, m-1$. The Björner-Kalai theorem has the following counterpart.

Theorem 9.2. *Let K be a field of characteristic 0. Let $f = (f_{-1}, f_0, \ldots, f_{m-1})$ and $b = (b_{-1}, b_0, \ldots, b_{m-1})$ be sequences of non-negative integers. The following conditions are equivalent:*

(a) *there exists a homogeneous K-algebra S/I such that f is the f-vector, and b the super extremal sequence of S/I;*

(b) (i) *$\chi_{-1} = 1$ and $\chi_i \geq 0$ for all i,*
(ii) *$b_i \leq \chi_{i-1}^{(i)} - \chi_i$ for all i.*

Proof. (a) $\Rightarrow$ (b) Since the extremal Betti numbers are preserved when we pass from I to $\operatorname{Gin}^S(I)$, it follows that I and $\operatorname{Gin}^S(I)$ have the same highest shift m, and hence the same b-vector. Since S/I and $S/\operatorname{Gin}^S(I)$ have the same Hilbert function, it also follows that the f-vectors of S/I and $S/\operatorname{Gin}^S(I)$ coincide. Thus, since $\operatorname{char}(K) = 0$, we may assume that I is a strongly stable monomial ideal.

The ideal I^σ is defined in $S' = K[x_1, \ldots, x_m]$ and $\beta_{ii+j}(I) = \beta_{ii+j}(I^\sigma)$ by 8.18. This implies that

$$H_{S'/I^\sigma}(t) = (1-t)^{n-m}H_{S/I}(t).$$

Hence, if we let Δ be the simplicial complex with $I_\Delta = I^\sigma$, then Δ and S/I have the same f-vector, and one has $b_i = \dim_K \tilde{H}_i(\Delta; K)$; see (5.3). Therefore, the conclusion follows from Björner-Kalai Theorem.

(b) $\Rightarrow$ (a): Given an f- and b-sequence satisfying conditions (b), there exists by 9.1 an integer m and a simplicial complex Δ on the vertex set $[m]$ whose f-vector is f and whose β-sequence is b. Then $K[x_1, \ldots, x_m]/I_\Delta$ is a homogeneous K-algebra satisfying (a). Q.E.D.

§10. Extremality properties of Lexsegment ideals

Let $I \subset S$ be a graded ideal. Then $\beta_{ij}(I) \leq \beta_{ij}(\mathrm{Gin}(I))$ for all i and j, by Theorem 3.1. Moreover it follows from Theorem 2.7 that $\beta_{0j}(\mathrm{Gin}(I)) \leq \beta_{0j}(\mathrm{Gin}(I)^{lex})$ for all j. Since $I^{lex} = \mathrm{Gin}(I)^{lex}$ we conclude that

$$\beta_{0j}(I) \leq \beta_{0j}(I^{lex}) \quad \text{for all } j.$$

Similar reasonings show that for all graded ideals $J \subset E$ one has

$$\beta_{0j}(J) \leq \beta_{0j}(J^{lex}) \quad \text{for all } j.$$

The question is whether such inequalities are valid also for the higher graded Betti numbers. In case of the polynomial ring this is known.

Theorem 10.1 (Bigatti, Hulett, Pardue). *Let $I \subset S$ be a graded ideal. Then*

$$\beta_{ij}(I) \leq \beta_{ij}(I^{lex}) \quad \textit{for all } i \textit{ and } j.$$

Bigatti [13] and Hulett [25] proved this theorem independently for base fields of characteristic 0. A proof in arbitrary characteristic was later given by Pardue [31] using some polarization trick.

In the exterior case we have (cf. [6, Theorem 4.4])

Theorem 10.2. *Let $J \subset E$ be a graded ideal. Then*

$$\beta_{ij}(J) \leq \beta_{ij}(J^{lex}) \quad \textit{for all } i \textit{ and } j.$$

Conjecture 10.3. *Let $I \subset S$ be a squarefree monomial ideal. Then*

$$\beta_{ij}(I) \leq \beta_{ij}(I^{sqlex}) \quad \textit{for all } i \textit{ and } j.$$

Theorem 10.4. *Conjecture* 10.3 *is true if* $\mathrm{char}\, K = 0$.

Proof. By Theorem 8.22 we have $\beta_{ij}(I) \leq \beta_{ij}(\mathrm{Gin}(I)^{\sigma})$. Now we use the result (see [5, Theorem 4.4]) that for any squarefree strongly stable ideal L one has $\beta_{ij}(L) \leq \beta_{ij}(L^{lex})$. Applying this result to $L = \mathrm{Gin}(I)^{\sigma}$ and observing that $I^{lex} = (\mathrm{Gin}(I))^{\sigma})^{lex}$, we get the desired inequalities. Q.E.D.

Theorem 10.4 was used by E. Sbarra to prove in his thesis [32] part (a) of the following theorem, while for part (b) he uses the polarization argument of Pardue. Let M be a graded S-module, and $\mathfrak{m}$ the graded maximal ideal of S. Then $H^{\bullet}_{\mathfrak{m}}(M)$ denotes the local cohomology of M. Recall that $H^{\bullet}_{\mathfrak{m}}(M)$ is naturally graded.

Theorem 10.5. *Let* $S = K[x_1, \dots, x_n]$.

(a) *If* $I \subset S$ *is a squarefree monomial ideal, and* $\operatorname{char} K = 0$, *then*

$$\dim_K H^i_{\mathfrak{m}}(S/I)_j \le \dim_K H^i_{\mathfrak{m}}(S/I^{sqlex})_j \quad \textit{for all } i \textit{ and } j.$$

(b) *If* $I \subset S$ *is a graded ideal, then*

$$\dim_K H^i_{\mathfrak{m}}(S/I)_j \le \dim_K H^i_{\mathfrak{m}}(S/I^{lex})_j \quad \textit{for all } i \textit{ and } j.$$

in any characteristic.

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