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Representations of finite Chevalley groups

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§1. Introduction

This note is a brief exposition of the representation theory of finite Chevalley groups. The main problem we are concerned here is the classification of irreducible ordinary representations of such groups, and giving a general algorithm of computing character tables. Lusztig succeeded, in 1980's, in classifying all the irreducible representations of finite reductive groups $G(\mathbf{F}_q)$ and in determining their degrees ([L1]). So the remaining problem is the determination of character values. In order to approach this problem from a general point of view, Lusztig founded the theory of character sheaves ([L2]), and showed that certain class functions arising from character sheaves are computable, and form a basis of the space of class functions of $G(\mathbf{F}_q)$. Under these circumstances he proposed a conjecture connecting such class functions with irreducible characters. Lusztig's conjecture provides us a general algorithm of computing irreducible characters. In the case where the center of G is connected, Lusztig's conjecture was solved by the author, by using the theory of Shintani descent developed by Shintani, Kawanaka and Asai, (see e.g., [K]).

In this note, we review the classification of irreducible characters. We formulate Lusztig's conjecture, and summarize related results in the case where the center is connected. In the case of disconnected center, Lusztig's conjecture is not yet established. We discuss this case, in connection with recent results on Shintani descent, the Mackey formula and generalized Gelfand-Graev representations.

§2. The classification of irreducible representations

Let G be a connected reductive algebraic group defined over $\mathbf{F}_q$, a finite field of q elements with $ch\mathbf{F}_q = p$. We denote by $F : G \to G$

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the corresponding Frobenius map on G. The finite group $G(\mathbf{F}_q)$ of $\mathbf{F}_q$-rational points in G coincides with the subgroup G^F of fixed points by F in G. Let $\bar{\mathbf{Q}}_l$ be the algebraic closure of the l-adic number field $\mathbf{Q}_l$, for $l \neq p$. Then $\bar{\mathbf{Q}}_l \simeq \mathbf{C}$, and we consider the representations of G^F over $\bar{\mathbf{Q}}_l$ so that the l-adic cohomology theory can be applied. We are interested in the following problem.

Problem. Classify all the irreducible representations of G^F, and give a general algorithm of computing irreducible characters.

The fundamental tool for the classification is the virtual G^F-module $R_T^G(\theta)$ introduced by Deligne and Lusztig in 1976. For a pair (T, θ), where T is an F-stable maximal torus of G and θ is a linear character of T^F, $R_T^G(\theta)$ is constructed as an alternating sum of certain l-adic cohomology groups on which G^F acts naturally. Let G^F_{uni} be the set of unipotent elements in G^F. We define a function $Q_T^G : G^F_{\text{uni}} \to \bar{\mathbf{Q}}_l$ by

$$Q_T^G(u) = \operatorname{Tr}(u, R_T^G(\theta)).$$

Q_T^G is called the **Green function** of G^F, which does not depend on the choice of θ. The computation of character values of $R_T^G(\theta)$ is reduced, by a simple character formula, to the determination of Green functions of various reductive subgroups of G. More generally, one can define a virtual G^F-module $R_{L\subset P}^G(\pi)$ for a representation π of L^F, where L is an F-stable Levi subgroup of (not necessarily F-stable) parabolic subgroup P of G. The assignment $\pi \mapsto R_{L\subset P}^G(\pi)$ is extended to the Lusztig induction $R_{L\subset P}^G$ from virtual L^F-modules to virtual G^F-modules.

In what follows, we denote by $\widehat{G}^F$ the set of irreducible characters of G^F. The first step for the classification is the partition of $\widehat{G}^F$ into certain subsets. Let G^* be the dual group of G, i.e., G^* is a connected reductive group over $\mathbf{F}_q$, with Frobenius map F, and its root system is dual to the original one. For each F-stable maximal torus T in G, there corresponds an F-stable maximal torus T^* in G^* which is dual to T, (unique up to G^{*F}-conjugate). Then the set of pairs (T, θ) (up to G^F-conjugate) is in bijection with the set of pairs (T^*, s) for $s \in T^{*F}$ (up to G^{*F}-conjugate). For each F-stable semisimple class $\{s\}$ in G^*, we define a subset of $\widehat{G}^F$ by

$$\mathcal{E}(G^F, \{s\}) = \bigcup_{(T_1,\theta_1)} \{\rho \in \widehat{G}^F \mid \langle \rho, R_{T_1}^G(\theta_1)\rangle_{G^F} \neq 0\},$$

where (T_1, θ_1) runs over all the pairs such that (T_1, θ_1) corresponds to (T_1^*, s_1) with $s_1 \in T_1^{*F} \cap \{s\}$ under the above correspondence. In the case where the center of G is connected, the set $\{s\}^F$ consists of a single G^{*F}-class, and we may choose $s_1 = s$. Moreover in this case, the centralizer $Z_{G^*}(s)$ of s is connected.

Lusztig has proved the following result.

Theorem 2.1 (Lusztig [L1]). *Assume that the center of G is connected. Then*

(i) *$\widehat{G}^F$ is partitioned as $\widehat{G}^F = \coprod_{\{s\}} \mathcal{E}(G^F, \{s\})$, where $\{s\}$ runs over all semisimple classes in G^{*F}.*

(ii) *There exists a natural bijection $\mathcal{E}(G^F, \{s\}) \simeq \mathcal{E}(Z_{G^*}(s)^*, \{1\})$.*

An irreducible character ρ is called a unipotent character if ρ belongs to the set $\mathcal{E}(G^F, \{1\})$, i.e., if $\langle \rho, R_T^G(1)\rangle_{G^F} \neq 0$ for some T. In view of (ii) in the theorem, the classification of $\widehat{G}^F$ is reduced to that of unipotent characters whenever the center of G is connected.

• The classification of unipotent characters.

In order to explain the parameterization of unipotent characters due to Lusztig, we prepare some notation. Let T_0 be an F-stable maximal torus contained in an F-stable Borel subgroup B of G. Such a pair (B, T_0) is unique up to G^F-conjugate. Let $W = N_G(T_0)/T_0$ be the Weyl group of G, on which F acts naturally. We assume, for simplicity, that F acts trivially on W, i.e., G^F is of split type (or G^F is a finite Chevalley group). Then the G^F-conjugacy classes of F-stable maximal tori in G are in one to one correspondence with the conjugacy classes in W. We denote by T_w an F-stable maximal torus corresponding to $w \in W$. The torus T_0 coincides with T_w with $w = 1$, and in this case we have $R_{T_0}^G(1) = \mathrm{Ind}_{B^F}^{G^F} 1$. It is known that

$$\mathrm{End}_{G^F}(\mathrm{Ind}_{B^F}^{G^F} 1) \simeq H_q(W) \simeq \bar{\mathbf{Q}}_l[W],$$

where $H_q(W)$ is the Iwahori-Hecke algebra of W with parameter q. Hence $\mathrm{Ind}_{B^F}^{G^F} 1$ is decomposed, as $H_q(W) \times G^F$-module,

$$\mathrm{Ind}_{B^F}^{G^F} 1 \simeq \bigoplus_{E \in W^\wedge} V_E \otimes \rho_E,$$

where V_E is an irreducible character of $H_q(W)$ and ρ_E the corresponding irreducible character of G^F. In particular, we obtain (a part of) unipotent characters $\{\rho_E \mid E \in W^\wedge\}$ parametrized by the set of irreducible characters of W. We now define, for $E \in W^\wedge$,

$$R_E = |W|^{-1} \sum_{w \in W} E(w) R_{T_w}^G(1) \in \mathcal{V}_G.$$

Here $\mathcal{V}_G$ denotes the $\bar{\mathbf{Q}}_l$-space of class functions of G^F endowed with the usual inner product. Now it follows from the orthogonality relations for $R_T^G(\theta)$ that $\{R_E \mid E \in W^\wedge\}$ gives rise to an orthonormal system in $\mathcal{V}_G$.

According to Lusztig, the set $\mathcal{E}(G^F, \{1\})$ is parametrized by a set $X(W)$, which is completely described in terms of the data coming from two sided cells of $H_q(W)$. In particular, the parameterization depends only on the Coxeter diagram of W, and independent of p. He also showed the existence of a certain non-degenerate pairing $\{\ ,\ \} : X(W) \times X(W) \to \bar{\mathbf{Q}}_l$. We express the unipotent character corresponding to $x \in X(W)$ by ρ_x. By the previous argument, there exists an injection $W^\wedge \hookrightarrow X(W)$ via $E \mapsto x_E$ with $\rho_{x_E} = \rho_E$. The following formula gives the decomposition of R_E into irreducible characters of G^F.

$$R_E = \sum_{y \in X(W)} \{y, x_E\}\rho_y.$$

Note that in certain $E \in W^\wedge$ for type E_7 or E_8 (exceptional characters of W), some modification is needed for the above formula. We also note that except the above case, unipotent characters are characterized by the multiplicities for various R_E. This is the leading principle of the parameterization by Lusztig.

Now the above decomposition of R_E suggests to define formally a class function R_x on G^F for any $x \in X(W)$ by

$$R_x = \sum_{y \in X(W)} \{y, x\}\rho_y.$$

Then the orthogonality property holds also for such R_x, and we see that $\{R_x \mid x \in X(W)\}$ gives rise to an orthonormal basis of the subspace of $\mathcal{V}_G$ generated by unipotent characters.

Remark 2.2. (i) More generally, the set $\mathcal{E}(G^F, \{s\})$ is described in a similar way (cf. (ii) of Theorem 2.1), and we get the total parameter set $X(G^F)$ for G^F. Then one can define functions R_x for $x \in X(G^F)$ similar to the previous case. The set $\{R_x \mid x \in X(G^F)\}$ gives rise to an orthonormal basis of $\mathcal{V}_G$, and R_x's are called **almost characters** of G^F.

(ii) In the case where the center of G is disconnected, the classification of $\widehat{G}^F$ is done by reducing it to the case of connected center. However, the construction of almost characters in this case is not so clear.

§3. Character sheaves and Lusztig's conjecture

Character sheaves are certain G-equivariant simple perverse sheaves on G. In general, for an F-stable perverse sheaf K on G, by fixing an isomorphism $\varphi_K : F^*K \xrightarrow{\sim} K$, one can associate to K a characteristic function χ_{K,φ_K}. If K is G-equivariant (with respect to the adjoint action of G), χ_{K,φ_K} turns out to be a class function on G^F. In this way, a lot of useful class functions of G^F are produced from the geometric setting, though they are not virtual characters in general.

Before stating Lusztig's results on character sheaves, we prepare some notation. A prime $p = ch\mathbf{F}_q$ is called almost good for G if p satisfies the following conditions;

$$\begin{cases} p \neq 2,3 & \text{if } G \text{ has factors of type } E_7, F_4, G_2, \\ p \neq 3 & \text{if } G \text{ has a factor of type } E_6, \\ p \neq 2,3,5 & \text{if } G \text{ has a factor of type } E_8, \end{cases}$$

and no conditions for factors of classical type. We denote by $(\widehat{G})^F$ the set of F-stable character sheaves on G. (Do not confuse this with $\widehat{G}^F$.) For each $A \in (\widehat{G})^F$, we choose $\varphi_A : F^*A \xrightarrow{\sim} A$, and consider the characteristic function χ_{A,φ_A} on G^F. Note that since A is simple, φ_A is unique up to scalar multiple.

Theorem 3.1 (Lusztig [L2]). *Assume that p is almost good for G. Then*

(i) *Under a certain choice of φ_A, $\{\chi_{A,\varphi_A} \mid A \in (\widehat{G})^F\}$ gives rise to an orthonormal basis of $\mathcal{V}_G$.*

(ii) *There exists a general algorithm of computing χ_{A,φ_A}.*

Based on his results, Lusztig proposed the following conjecture.

Conjecture 3.2 (Lusztig). *There exists a natural parameterization $X(G^F) \simeq (\widehat{G})^F$, (which we denote by $x \leftrightarrow A_x$, and write as $\varphi_{A_x} = \varphi_x : F^*A_x \xrightarrow{\sim} A_x$), such that*

$$\chi_{A_x,\varphi_x} = c_x R_x \qquad (c_x \in \bar{\mathbf{Q}}_l^*).$$

Lusztig's conjecture asserts that characteristic functions χ_{A,φ_A} coincide with almost characters up to scalar. Since we know the decomposition of almost characters into irreducible characters (especially in the case of connected center), Lusztig's conjecture provides us an algorithm of computing irreducible characters once we know the scalar constants c_x.

The following result gives a partial answer to Lusztig's conjecture.

Theorem 3.3 ([S2]). *Assume that the center of G is connected, and assume that p is almost good. Then Lusztig's conjecture holds for G^F.*

Remark 3.4. Here we give a remark on the computation of χ_{A,φ_A}. In the theory of character sheaves, there is a notion of cuspidal character sheaves, and induction from them. For example, the constant sheaf $\bar{\mathbf{Q}}_l$ gives, up to shift, a cuspidal character sheaf A_0 on a maximal torus T_0, and the induction $K = \operatorname{ind}_B^G A_0$ is a semisimple perverse sheaf on G, whose simple factors are character sheaves A_E parametrized by $E \in W^\wedge$. One can choose an isomorphism $\varphi_w : F^*K \tilde{\to} K$ for each $w \in W^\wedge$, and we have

$$\chi_{K,\varphi_w} = \sum_{E \in W^\wedge} E(w)\chi_{A_E,\varphi_{A_E}}.$$

Thus the computation of χ_{A_E} is reduced to that of χ_{K,φ_w} for various w. Lusztig defined a Green function $\tilde{Q}^G_{T_w}$ associated to the character sheaves, and showed that the computation of χ_{K,φ_w} is reduced to that of Green functions. More generally, arbitrary χ_{A,φ_A} are computed by making use of generalized Green functions. He showed that there is a simple algorithm of computing generalized Green functions.

In [L4], Lusztig proved that $\tilde{Q}^G_T$ coincides with Q^G_T when q is large enough (for any p). This result was extended in [S2] for arbitrary q.

Concerning the Lusztig's conjecture, Lusztig has proved the following result for arbitrary G, under some restrictions on p and q.

Theorem 3.5 (Lusztig [L6]). *Let G be an arbitrary reductive group. Assume that p and q are large enough. Then for each cuspidal character sheaf A_x, the formula in the conjecture holds.*

Note that if the decomposition of the Lusztig induction $R^G_{L\subset P}$ is known, the above result implies the conjecture (for $p \gg 0, q \gg 0$). However such a decomposition is known, at present, only for the case of connected center (see, e.g., [S1]).

Once Lusztig's conjecture is established (for example, in the case of connected center), the next step is the determination of scalars c_x appearing in the conjecture. In this direction, Lusztig has proved the following.

Theorem 3.6 (Lusztig [L3]). *Let $G = SO_{2n+1}$ and assume that p is odd. Then for almost characters R_x which do not vanish on G^F_{uni}, the scalars c_x are determined.*

He also announced that similar results hold for other groups under some restriction on q.

We can determine the scalar c_x in some special cases. In the following, R_x is called a unipotent almost character if it is a linear combination of unipotent characters.

Theorem 3.7 ([S3]). *Assume that G^F is a Chevalley group of classical type with connected center. Assume further that p is odd. Then the scalar c_x is determined for a unipotent almost character R_x.*

This can be generalized to the case of exceptional groups.

Theorem 3.8 (Lübeck, Shinoda). *Assume that G^F is an exceptional group of adjoint type. Assume further that p is good. Then the scalar c_x is determined for a unipotent almost character R_x.*

Remark 3.9. The above results provide an algorithm of computing unipotent characters. In fact, Lübeck "computed" all the character values of unipotent characters for F_4 and E_6 by making use of the computer algebra system **CHEVIE** ([GPH]). His program will work also for E_7 and E_8. However in applying Lusztig's algorithm in practice, still there remains an ambiguity in choosing rational unipotent classes in a given geometric unipotent class. In order to justify Lübeck's computation, we need to determine some parameters related to the choice of representatives.

§4. The case of disconnected center

In the case where the center of G is disconnected, the main problem is the proof of Lusztig's conjecture. For this we need to know the decomposition of the Lusztig induction $R^G_{L\subset P}$. In the case of connected center, this decomposition was achieved by making use of the theory of Shintani descent ([S1]), which is a theory connecting characters of G^F and F-stable characters of G^{F^m} for some power F^m. This theory was also used in verifying Lusztig's conjecture in [S2]. Hence it is expected that it plays an important role also for the case of disconnected center. The typical example for such a group is $G^F = SL_n(\mathbf{F}_q)$, and the Shintani descent for this group was described in [S4].

In the remainder of this section we assume that G is an arbitrary reductive group.

• The Mackey formula

Another approach for getting the information on the Lusztig induction is the following Mackey formula for Lusztig induction which is an analogue of the usual Mackey formula of finite groups. We define a linear map ${}^*R^G_{L\subset P} : \mathcal{V}_G \to \mathcal{V}_L$, called the Lusztig restriction, as the adjoint functor of the Lusztig induction $R^G_{L\subset P}$. Let M be an F-stable Levi subgroup of another parabolic subgroup Q of G. Put

$$\mathcal{E}(L,M) = \{x \in G \mid L \cap {}^xM \text{ contains a maximal torus of } G\}.$$

The Mackey formula is formulated as follows.

$$ {}^*R^G_{L\subset P} \circ R^G_{M\subset Q} = \sum_{x \in L^F \backslash \mathcal{E}(L,M)^F/M^F} R^L_{L\cap {}^xM \subset L\cap {}^xQ} \circ {}^*R^{{}^xM}_{L\cap {}^xM\subset P\cap {}^xM}.$$

It is not yet known whether the Mackey formula holds in a full generality. It has been verified in the special case where (a) P and Q are F-stable parabolic subgroups, or (b) L or M is a maximal torus of G. We note here that the Mackey formula implies that the Lusztig induction $R^G_{L\subset P}$ depends only on L and not on P.

Recently C. Bonnafé proved the following result.

Theorem 4.1 (Bonnafé [B1]). *Assume that q is large enough (but no assumption on p). Then the Mackey formula holds for any F-stable Levi subgroups L and M.*

He also showed in [B2] that if G is of type A_n, then the Mackey formula holds without restriction on q.

• **Generalized Gelfand-Graev representations.**

The concept of generalized Gelfand-Graev representations (by abbreviation GGGR) was introduced by Kawanaka, by generalizing the usual Gelfand-Graev representations. In the case of disconnected center, contrast to the case of connected center, Deligne-Lusztig theory does not give enough information for describing irreducible characters, and it is expected that GGGR provide us additional informations. In fact, in the case of $SL_n(\mathbf{F}_q)$, GGGR allows us to parameterize irreducible characters in a more precise way than Lusztig's one. Now to each unipotent element $u \in G^F$, one can associate an F-stable parabolic subgroup P with unipotent radical U_P, together with a certain irreducible representation Λ_u of U_P^F. Then $\Gamma_u = \mathrm{Ind}_{U_P^F}^{G^F} \Lambda_u$ depends only on the G^F-conjugacy class of u, and is called the **generalized Gelfand-Graev representation** of G^F associated to the class of u. Note that if u is a regular unipotent element, then Γ_u coincides with the usual Gelfand-Graev representations.

Kawanaka decomposed Γ_u into irreducible characters in the case of GL_n for arbitrary p and q, and also treated the exceptional groups of adjoint type (see, e.g., [K]). On the other hand, under the assumption that p and q are large enough, Lusztig described the decomposition of Γ_u in terms of various χ_{A,φ_A}. Using this, he showed the following result.

Theorem 4.2 (Lusztig [L5]). *Assume that p and q are large enough. Then for any $\rho \in \widehat{G}^F$, there exists a unique unipotent class C in G such that $\sum_{g \in C^F} \rho(g) \neq 0$, and having maximal dimension among the classes with this property.*

The class C attached to ρ is called the **unipotent support** of ρ. Recently, M. Geck succeeded in removing the assumption on q of Lusztig's result in the case where p is good, and then extended it with G. Malle to the case where p is bad.

Theorem 4.3 (Geck [G], Geck-Malle [GM]). *The statement of Theorem 4.2 holds without any restrictions on p and q.*

We close this note by stating the following result, which discusses the Lusztig restriction of Gelfand-Graev characters.

Theorem 4.4 (Digne-Lehrer-Michel [DLM]). *Assume that p is good and that q is large enough. Let Γ_u be the Gelfand-Graev character of G^F associated to a regular unipotent element $u \in G^F$. Let L be an F-stable Levi subgroup of a (not necessarily F-stable) parabolic subgroup P of G. Then there exists a regular unipotent element $v \in L^F$ such that*

$$^*R^G_{L \subset P}(\Gamma_u) = \varepsilon_G \varepsilon_L \Gamma_{L,v},$$

where $\Gamma_{L,v}$ is the Gelfand -Graev character of L^F associated to v, and ε_G (resp. ε_L) is the split rank of G (resp. L).

Note that in the case of disconnected center, the theorem implies that a rational regular unipotent class in G^F determines a rational regular unipotent class in each F-stable Levi subgroup L. However, the explicit correspondence is not yet known.

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