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Some Results on Modular Forms — Subgroups of the Modular Group Whose Ring of Modular Forms is a Polynomial Ring

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§1. Introduction

This paper is the first of the sequel of papers on the joint work of these authors on modular forms. We consider the problem of determining finite index subgroups of the modular group $\mathrm{SL}(2,\mathbb{Z})$ whose ring of modular forms is isomorphic to a polynomial ring. First, in this paper, we consider this question for modular forms of integral weights. In subsequent papers, we will consider the problem for modular forms of half-integral weights, and more generally, of $1/l$-integral weights. It turns out that the case of $l = 5$ is particularly interesting in connection with the classical work of F. Klein [9], as well as its analogy with the other two cases of $l = 1$ and $l = 2$, which are related to ternary and binary self-dual codes, respectively. In this first paper, we explain our overall motivation, and we prove the results only for the integral weight case. We remark that some preliminary announcements of some of the results given in the present paper have been made in two unofficial publications [2] and [17] written in Japanese.

§2. Statement of Results

Let Γ be a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$. We denote by $\mathfrak{M}(\Gamma)$ the ring of modular forms of integral weights on the group Γ. It is well known that

$$\mathfrak{M}(\mathrm{SL}(2,\mathbb{Z})) = \mathbb{C}[E_4, E_6], \tag{1}$$

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where E_4 and E_6 are the Eisenstein series of weights 4 and 6, respectively. Since E_4 and E_6 are algebraically independent, $\mathfrak{M}(\mathrm{SL}(2,\mathbb{Z}))$ is isomorphic to the polynomial ring in two variables. There are proper subgroups Γ of $\mathrm{SL}(2,\mathbb{Z})$ whose rings of modular forms of integral weights are isomorphic to polynomial rings. Note that, if a subgroup Γ has this property, then its ring of modular forms of integral weights is isomorphic to the polynomial ring in two variables. It is the purpose of the present paper to give a classification of such subgroups up to conjugacy in $\mathrm{SL}(2,\mathbb{Z})$.

Theorem 1. *Let $\mathfrak{M}(\Gamma)$ be the ring of modular forms on a finite index subgroup Γ of the modular group $\mathrm{SL}(2,\mathbb{Z})$. Suppose that $\mathfrak{M}(\Gamma) = \mathbb{C}[\phi_1,\phi_2]$ where ϕ_1 and ϕ_2 are algebraically independent modular forms of integral weights. Then Γ is conjugate in $\mathrm{SL}(2,\mathbb{Z})$ to one of the seventeen subgroups listed in Table 1.*

	wt	μ	ν_2	ν_3	$u+v$ $=\nu_\infty$	index Γ	index $C(\Gamma)$	Γ	No.
(a)	4,6	1	1	1	1	1	1	$\mathrm{SL}(2,\mathbb{Z})$	1
(b)	2,4	3	1	0	2	3	6	$\Gamma_0(2)$	2
(c)	2,2	6	0	0	3	6	6	$\Gamma(2)$	3
							24	$\Gamma_0(4)$	4
(d)	1,3	4	0	1	2+0	8	24	$\Gamma_1(3)$	5
(e)	1,2	6	0	0	2+1	12	48	$\sigma_0^{-1}\Gamma_1(4)\sigma_0$	6
							48	$\sigma_0^{-1}\sigma_1^{-1}\Gamma_1(4)\sigma_1\sigma_0$	7
							48	$\sigma_0^{-1}\sigma_2^{-1}\Gamma_1(4)\sigma_2\sigma_0$	8
							48	$\Gamma_1(4)$	9
							48	$\sigma_1^{-1}\Gamma_1(4)\sigma_1$	10
							192	$\sigma_2^{-1}\Gamma_1(4)\sigma_2$	11
(f)	1,1	12	0	0	4+0	24	24	$\Gamma(3)$	12
							48	$\Gamma_1(4)\cap\Gamma(2)$	13
							120	$\Gamma_1(5)$	14
							144	$\Gamma_1(6)$	15
							192	$\Gamma_0(8)\cap\Gamma_1(4)$	16
							648	$\Gamma_0(9)\cap\Gamma_1(3)$	17

Table 1. List of Subgroups

In Table 1, The column labeled as "wt" gives the weights of the modular forms ϕ_1,ϕ_2 in Theorem 1. The parameters $\mu,\nu_2,\nu_3,\nu_\infty,u,v$ will be defined in Section 3. The intersection of all conjugates of Γ in

$\mathrm{SL}(2,\mathbb{Z})$ is denoted by $C(\Gamma)$, so that the index in $\mathrm{SL}(2,\mathbb{Z})$ of $C(\Gamma)$ is the order of the permutation group induced by the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathrm{SL}(2,\mathbb{Z})/\Gamma$. The columns labeled as "index" give the indices of Γ and $C(\Gamma)$ in $\mathrm{SL}(2,\mathbb{Z})$. The elements $\sigma_0, \sigma_1, \sigma_2$ appearing in case (e) will be defined in Section 4, where we give a proof of Theorem 1. In Section 5, we show that for each of the seventeen subgroups Γ, the ring of modular forms of integral weights on Γ is indeed the polynomial ring in two modular forms.

§3. Preliminaries

We assume that the reader is familiar with basic concepts of modular forms of integral weights on finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$, as they are available in [13] and [16]. For $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$, let us set $\overline{\Gamma} = \Gamma \cdot \{\pm 1\}/\{\pm 1\} \subset \mathrm{PSL}(2,\mathbb{Z})$. The following parameters of a finite index subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$ are commonly used:

$$\begin{aligned}
\mu &= |\,\mathrm{PSL}(2,\mathbb{Z}) : \overline{\Gamma}|, \\
\nu_2 &= \text{ the number of inequivalent elliptic points of order 2,} \\
\nu_3 &= \text{ the number of inequivalent elliptic points of order 3,} \\
\nu_\infty &= \text{ the number of inequivalent cusps,} \\
g &= \text{ the genus of } \Gamma \\
&= 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}.
\end{aligned}$$

Furthermore, if $-1 \notin \Gamma$, then we distinguish two types of cusps, called regular and irregular. Namely, suppose that x is a cusp of Γ, $\sigma(x) = \infty$, $\sigma \in \mathrm{SL}(2,\mathbb{Z})$. Then we have $\sigma\Gamma_x\sigma^{-1} = \langle \psi^h \rangle$ or $\langle -\psi^h \rangle$ for some positive integer h, where

$$\psi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In the former case the cusp x is called regular, otherwise it is called irregular. Let us denote by u (resp. v) the number of inequivalent regular (resp. irregular) cusps. Then, obviously $\nu_\infty = u + v$ holds. Let $\mathfrak{M}_k(\Gamma)$ denote the space of modular form of weight k on Γ. Then $\dim \mathfrak{M}_k(\Gamma)$ ($k \geq 2$) can be calculated by using just the above parameters. Namely, we have

$$\dim \mathfrak{M}_2(\Gamma) = \begin{cases} g + \nu_\infty - 1 & \text{if } \nu_\infty > 0, \\ g & \text{if } \nu_\infty = 0, \end{cases}$$

and

$$\dim \mathfrak{M}_k(\Gamma) = (k-1)(g-1) + \nu_2[\frac{k}{4}] + \nu_3[\frac{k}{3}] + \frac{k}{2}\nu_\infty,$$

if k is even and $k \geq 4$,

$$\dim \mathfrak{M}_k(\Gamma) = (k-1)(g-1) + \nu_2[\frac{k}{4}] + \nu_3[\frac{k}{3}] + \frac{k}{2}u + \frac{k-1}{2}v,$$

if k is odd, $k \geq 3$, and $-1 \notin \Gamma$. If $-1 \in \Gamma$, then $\mathfrak{M}_k(\Gamma) = 0$ for odd k. Note that the formula for $\dim \mathfrak{M}_1(\Gamma)$ is not known in general.

To conclude this section, we explain the notation used to describe the subgroups in Table 1. Recall the standard notation for certain subgroups of $\mathrm{SL}(2,\mathbb{Z})$:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; \begin{array}{l} b \equiv c \equiv 0 \pmod N \\ a \equiv d \equiv 1 \pmod N \end{array} \right\},$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod N \right\},$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; \begin{array}{l} c \equiv 0 \pmod N \\ a \equiv d \equiv 1 \pmod N \end{array} \right\},$$

The groups No. 6–11 are pairwise conjugate in $\mathrm{GL}(2,\mathbb{Q})$. The elements $\sigma_0, \sigma_1, \sigma_2$ are defined by

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \ \sigma_1 = \sigma_0 \psi \phi \sigma_0^{-1}, \ \sigma_2 = \sigma_0 \psi \sigma_0^{-1}, \text{ where } \phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The group No. 16 is conjugate in $\mathrm{GL}(2,\mathbb{Q})$ to the group No. 13:

$$\Gamma_0(8) \cap \Gamma_1(4) = \sigma_0 \left(\Gamma_1(4) \cap \Gamma(2)\right) \sigma_0^{-1}. \tag{2}$$

Also, the group No. 17 is conjugate in $\mathrm{GL}(2,\mathbb{Q})$ to the group No. 12:

$$\Gamma_0(9) \cap \Gamma_1(3) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \Gamma(3) \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{-1}. \tag{3}$$

§4. Proof of Theorem 1

Suppose that $\mathfrak{M}(\Gamma) = \mathbb{C}[\phi_1, \phi_2]$, where ϕ_1, ϕ_2 are algebraically independent modular forms of weight a_1, a_2, respectively, on Γ. Then we have, as formal power series,

$$\Phi(\Gamma) := \sum_{k=0}^{\infty} \dim \mathfrak{M}_k(\Gamma) \cdot t^k = \frac{1}{(1-t^{a_1})(1-t^{a_2})}. \tag{4}$$

Without loss of generality we may assume $a_1 \leq a_2$. Since $\mathfrak{M}(\Gamma) \supset \mathfrak{M}(\mathrm{SL}(2,\mathbb{Z}))$, (1) implies $a_1 \leq 4$ and $a_2 \leq 6$.

First consider the case where $-1 \notin \Gamma$. For fixed a_1, a_2, comparing the coefficients in (4), we obtain a system of linear equations with unknowns g, ν_2, ν_3, u, v. Taking the conditions $\mu > 0$, $\nu_2 \geq 0$, $\nu_3 \geq 0$ into account, the list of solutions consists of the cases (d)–(f) in Table 1 and the case

$$\text{(g)} \qquad (a_1, a_2, g, \mu, \nu_2, \nu_3, u, v) = (1, 4, 0, 3, 1, 0, 2, 0).$$

In order to classify subgroups Γ of $\mathrm{SL}(2,\mathbb{Z})$ of given parameters, we use a modification of the technique used in Millington [12]. If we put $\lambda = \phi^{-1}\psi$, then $\mathrm{SL}(2,\mathbb{Z})$ has a presentation $\langle \phi, \lambda \mid \phi^4 = \lambda^3 = 1,\ [\lambda, \phi^2] = 1 \rangle$. Let Γ be a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$, $X = \mathrm{SL}(2,\mathbb{Z})/\Gamma$, $\overline{X} = \mathrm{SL}(2,\mathbb{Z})/\langle \Gamma, -1 \rangle$. Then $\mathrm{SL}(2,\mathbb{Z})$ acts on X, $\overline{X}$, and we have $|\overline{X}| = \mu$.

Lemma 2. *ϕ fixes ν_2 elements of $\overline{X}$, λ fixes ν_3 elements of $\overline{X}$, and ψ has ν_∞ cycles on $\overline{X}$. If $-1 \notin \Gamma$, then a cusp x is regular if and only if $\langle \psi \rangle$ has two orbits on $\langle \psi, -1 \rangle \sigma\Gamma/\Gamma$, where $\sigma(x) = \infty$, $\sigma \in \mathrm{SL}(2,\mathbb{Z})$. In particular, ψ has $2u + v$ cycles on X.*

Proof. The statement on the action on $\overline{X}$ has been proved in [12]. As for the regularity of a cusp x, it suffices to prove that x is irregular if and only if $\langle \psi \rangle$ acts transitively on $\langle \psi, -1 \rangle \sigma\Gamma/\Gamma$. The latter condition is equivalent to the existence of a positive integer h satisfying $\psi^h \sigma\Gamma = -\sigma\Gamma$. This implies $-\psi^h \in \sigma\Gamma_x\sigma^{-1}$, hence the cusp x is irregular. The proof of the converse is similar. Q.E.D.

We now describe how to obtain the list of subgroups in the cases (d)–(f), and how to prove the nonexistence of a subgroup in the case (g). First, we enumerate all subgroups $\overline{\Gamma}$ of index μ in $\mathrm{PSL}(2,\mathbb{Z})$. This can be done by GAP [6], using the command `LowIndexSubgroupsFpGroup`, if one defines $\mathrm{PSL}(2,\mathbb{Z})$ as $\langle \overline{\phi}, \overline{\lambda} \mid \overline{\phi}^2 = \overline{\lambda}^3 = 1 \rangle$. Since $\overline{X}$ can be identified naturally with $\mathrm{PSL}(2,\mathbb{Z})/\overline{\Gamma}$, the parameters ν_2, ν_3 and ν_∞ make sense for $\overline{\Gamma}$. Thus we can extract only those subgroups $\overline{\Gamma}$ of index μ having the parameters ν_2, ν_3, ν_∞ as prescribed in the cases (d)–(g).

The next step is to find subgroups Γ of index 2μ in $\mathrm{SL}(2,\mathbb{Z})$ whose images are one of the $\overline{\Gamma}$ found in the previous step. We need to check whether Γ satisfies the condition on the parameters u, v described in Lemma 2. This step can also be done easily by GAP, and we obtain the subgroups No. 5–17. We remark that the six subgroups in the case (f) appeared in [3].

Next consider the case where $-1 \in \Gamma$. The method is similar to the previous case, and the computation is far simpler. Comparing the coefficients in (4), we see that the list of possible parameters is as described in the cases (a)–(c) in Table 1. Then we enumerate all subgroups $\overline{\Gamma}$ of index μ in $\mathrm{PSL}(2,\mathbb{Z})$ having the parameters as in (a)–(c). The subgroup Γ is the full inverse image of $\overline{\Gamma}$ in $\mathrm{SL}(2,\mathbb{Z})$.

§5. Generators of the rings of modular forms

In this section, we show that for each of the seventeen subgroups Γ in Table 1, its ring of modular forms is isomorphic to a polynomial ring. We have seen that this is the case for $\mathrm{SL}(2,\mathbb{Z})$. Indeed, for the cases (a)–(c) in Table 1, since the weights are even, it is sufficient to check (4) using the dimension formula; it follows from (4) that there exist algebraic independent modular forms of weight a_1, a_2. To be more precise, let $\theta_3(\tau), \theta_2(\tau)$ be Jacobi's theta functions. It is well known and easy to see that $\mathfrak{M}(\Gamma(2)) = \mathbb{C}[\theta_3(2\tau)^4, \theta_2(2\tau)^4]$ and that $\Gamma_0(4) = \sigma_0\Gamma(2)\sigma_0^{-1}$. So, we have the assertions for the groups No. 3 and No. 4. As for cases (b) and (d), more explicit information can be found in [11, p.52, Corollary] for $\Gamma_0(2)$, [11, p.53, Theorem 2] for $\Gamma_1(3)$. We note that the notation of subgroups in [11] is different from ours. The groups No. 6–11 are pairwise conjugate in $\mathrm{GL}(2,\mathbb{Q})$, so it suffices to give generators for No. 6 only. The result for the group No. 6 is given in [8, p.186] as $\mathfrak{M}(\sigma_0^{-1}\Gamma_1(4)\sigma_0) = \mathbb{C}[\theta_3(2\tau)^2, \theta_2(2\tau)^4]$.

Let Γ be one of the subgroups No. 12–17. Suppose that there exist modular forms ϕ_1, ϕ_2 of weight 1 on Γ such that the leading terms of their Fourier expansion with respect to $q = e^{2\pi i\tau}$ are 1, q, respectively. Considering the leading terms of $\phi_1^n, \phi_1^{n-1}\phi_2, \dots, \phi_2^n$, we can prove that $\phi_1^n, \phi_1^{n-1}\phi_2, \dots, \phi_2^n$ are linearly independent. Hence ϕ_1, ϕ_2 are algebraically independent. Since $\dim \mathfrak{M}_2(\Gamma) = 3$, we have $\dim \mathfrak{M}_1(\Gamma) \le 2$. Therefore, to prove the claim, we have only to find modular forms ϕ_1, ϕ_2 of weight 1 on Γ such that the leading terms of their Fourier expansion are 1, q, respectively. This means that, we only need to find two linearly independent modular forms of weight 1 on Γ.

Let N be a positive integer, χ a primitive Dirichlet character mod N such that $\chi(-1) = -1$. Then the Eisenstein series

$$E_\chi(\tau) = \frac{1}{2}L(0,\chi) + \sum_{n=1}^{\infty}\Big(\sum_{d|n,d>0} \chi(d)\Big)q^n$$

is a modular form of type $(1,\chi)$ on $\Gamma_0(N)$ (see Hecke [7]).

The subgroup No. 12, 17. For the group $\Gamma(3)$, the result is well known (see [5, Theorem 5.4]). Namely, $\mathfrak{M}(\Gamma(3)) = \mathbb{C}[\varphi_1, \varphi_2]$ with

$$(5) \qquad \varphi_1 = \sum_{(x,y)\in\mathbb{Z}^2} q^{x^2-xy+y^2}, \qquad \varphi_2 = q^{\frac{1}{3}} \sum_{(x,y)\in\mathbb{Z}^2} q^{x^2-xy+y^2+x-y}.$$

Interestingly enough, this fact was known in connection with the weight enumerators of ternary self-dual codes. For a future use, we remark that $\mathrm{SL}(2,\mathbb{Z})$ acts on the 2-dimensional space spanned by φ_1, φ_2 as the unitary reflection group (No. 4 in [15])

$$\langle \frac{1}{i\sqrt{3}}\begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & e^{\frac{2\pi i}{3}}\end{pmatrix}\rangle \cong \mathrm{SL}(2,\mathbb{Z}/3\mathbb{Z}) \cong \mathrm{SL}(2,\mathbb{Z})/\Gamma(3).$$

The ring of polynomial invariants of this group is the polynomial ring in f, g, where

$$f(x,y) = x^4 + 8xy^3, \qquad g(x,y) = x^6 - 20x^3y^3 - 8y^6.$$

Moreover, $f(\varphi_1, \varphi_2) = E_4$ and $g(\varphi_1, \varphi_2) = E_6$ hold. The ring of invariants $\mathbb{C}[f,g]$ contains the ring of weight enumerators of ternary self-dual codes (see [4]). In view of (3), the ring $\mathfrak{M}(\Gamma_0(9) \cap \Gamma_1(3))$ is generated by $\varphi_1(3\tau), \varphi_2(3\tau)$. However, we also give different generators of this ring as follows.

Let χ_1 be the non-trivial Dirichlet character mod 3. Then the Eisenstein series $E_{\chi_1}(\tau)$ is a modular form of type $(1,\chi_1)$ on $\Gamma_0(3)$. This implies that $E_{\chi_1}(\tau)$ and $E_{\chi_1}(3\tau)$ are linearly independent modular forms of type $(1,\chi_1)$ on $\Gamma_0(9)$. Hence they are modular forms of weight 1 on $\Gamma_0(9) \cap \Gamma_1(3)$.

Let $\eta(\tau)$ be the Dedekind eta-function. Then it is shown in [10] that $\eta(9\tau)^3/\eta(3\tau)$ and $\eta(\tau)^3/\eta(3\tau)$ also are modular forms of type $(1,\chi_1)$ on $\Gamma_0(9)$. The relations between these forms are:

$$\begin{aligned}\eta(9\tau)^3/\eta(3\tau) &= E_{\chi_1}(\tau) - E_{\chi_1}(3\tau),\\ \eta(\tau)^3/\eta(3\tau) &= -3(E_{\chi_1}(\tau) - 3E_{\chi_1}(3\tau)).\end{aligned}$$

Moreover, we have

$$\begin{aligned}\varphi_1(3\tau) &= 6E_{\chi_1}(\tau),\\ \varphi_2(3\tau) &= E_{\chi_1}(\tau) - E_{\chi_1}(3\tau).\end{aligned}$$

The subgroups No. 13, 16. Let χ_2 be the non-trivial Dirichlet character mod 4. Then the Eisenstein series $E_{\chi_2}(\tau)$ is a modular form of type $(1,\chi_2)$ on $\Gamma_0(4)$. This implies that $E_{\chi_2}(\tau)$ and $E_{\chi_2}(2\tau)$ are linearly independent modular forms of type $(1,\chi_2)$ on $\Gamma_0(8)$. Hence they are modular forms of weight 1 on the subgroup No. 16: $\Gamma_0(8) \cap \Gamma_1(4)$. Alternatively, it is shown in [10] that $\eta(8\tau)^4/\eta(4\tau)^2$ and $\eta(\tau)^4/\eta(2\tau)^2$ also are modular forms of type $(1,\chi_2)$ on $\Gamma_0(8)$. The relations between these forms are:

$$\begin{aligned}\eta(8\tau)^4/\eta(4\tau)^2 &= E_{\chi_2}(\tau) - E_{\chi_2}(2\tau),\\ \eta(\tau)^4/\eta(2\tau)^2 &= -4(E_{\chi_2}(\tau) - 2E_{\chi_2}(2\tau)).\end{aligned}$$

Note that $\mathfrak{M}(\Gamma_1(4) \cap \Gamma(2)) = \mathbb{C}[\theta_3(\tau)^2, \theta_4(\tau)^2]$ (see [8, p.186]) follows from (2). More explicitly, we have

$$\begin{aligned}\theta_3(2\tau)^2 &= 4E_{\chi_2}(\tau),\\ \theta_4(2\tau)^2 &= -4(E_{\chi_2}(\tau) - 2E_{\chi_2}(2\tau)).\end{aligned}$$

The subgroup No. 14. Let χ_3 be the Dirichlet character mod 5 such that $\chi_3(2) = \sqrt{-1}$. Then the Eisenstein series $E_{\chi_3}(\tau)$ and $E_{\overline{\chi_3}}(\tau)$ are modular forms of type $(1, \chi_3), (1, \overline{\chi_3})$, respectively on $\Gamma_0(5)$. Hence they are linearly independent modular forms of weight 1 on $\Gamma_1(5)$.

The subgroup No. 15. Recall that $E_{\chi_1}(\tau)$ is a modular form of type $(1, \chi_1)$ on $\Gamma_0(3)$, where χ_1 is the non-trivial Dirichlet character mod 3. This implies that $E_{\chi_1}(\tau)$ and $E_{\chi_1}(2\tau)$ are linearly independent modular forms of type $(1, \chi_1)$ on $\Gamma_0(6)$. Hence they are modular forms of weight 1 on $\Gamma_1(6)$. It is shown in [10] that $\eta(\tau)\eta(6\tau)^6/\eta(2\tau)^2\eta(3\tau)^3$ and $\eta(6\tau)\eta(\tau)^6/\eta(3\tau)^2\eta(2\tau)^3$ also are modular forms of type $(1, \chi_1)$ on $\Gamma_0(6)$. The relations between these forms are:

$$\begin{aligned}\eta(\tau)\eta(6\tau)^6/\eta(2\tau)^2\eta(3\tau)^3 &= E_{\chi_1}(\tau) - E_{\chi_1}(2\tau),\\ \eta(6\tau)\eta(\tau)^6/\eta(3\tau)^2\eta(2\tau)^3 &= -6(E_{\chi_1}(\tau) - 2E_{\chi_1}(2\tau)).\end{aligned}$$

§6. Concluding remarks

We note that the classification of the subgroups $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ whose ring of modular forms is isomorphic to a polynomial ring is regarded as an analogue of the classification of the finite unitary reflection groups of dimension 2. We expect that higher dimensional analogue for this is the classification of the subgroups in the Siegel modular groups whose ring of Siegel modular forms is isomorphic to a polynomial ring. There are many possible generalizations of the ideas and the motivations presented in this paper. We will discuss some of the generalizations in subsequent papers, we briefly mention some of them below.

(1) Classify discrete subgroups of $\mathrm{SL}(2, \mathbb{R})$, not necessarily contained in $\mathrm{SL}(2, \mathbb{Z})$, whose ring of modular forms is isomorphic to a polynomial ring.

(2) Classify subgroups of $\mathrm{SL}(2, \mathbb{Z})$ whose ring of modular forms of half-integral weights is isomorphic to a polynomial ring.

Furthermore, we can consider a similar problem for $1/l$-integral weights (see Rankin [14] for the definition of modular forms of fractional weights). In general, if the ring of modular forms of $1/l$-integral weights on Γ is isomorphic to the polynomial ring generated by two modular forms of weight $1/l$, then we see that Γ must be a subgroup of index $24l$ in $\mathrm{SL}(2,\mathbb{Z})$. A recent work of A. Sebbar on the classification of genus zero congruence subgroups with no elliptic points implies that they are noncongruence subgroups except for finitely many exceptions. The complete classifications of such subgroups of index $24l$ seems very difficult in general. In the case $l = 2$, using the method described in Section 4, we can see that there are 191 possible such subgroups Γ of index 48 in $\mathrm{SL}(2,\mathbb{Z})$ up to the conjugacy. Some of them are congruence subgroups and others are noncongruence subgroups. We expect that many of them, hopefully all of them, satisfy the property mentioned in (2). Note that some results on modular forms on noncongruence subgroups are given in [1].

As we remarked in Section 1, the case $l = 5$ is interesting, and this will be treated in a subsequent paper. We also mention that, partly motivated by our present research, T. Ibukiyama is recently developing a theory of modular forms of fractional weights from a more general viewpoint, which will be published in due course.

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