Advanced Studies in Pure Mathematics 31, 2001
Taniguchi Conference on Mathematics Nara '98
pp. 93–125

Very Singular Diffusion Equations

Mi-Ho Giga, Yoshikazu Giga[1] and Ryo Kobayashi[2]

§1. Introduction

In the modeling of nonequilibrium phase transition it is often interesting to consider motion of phase-boundaries driven by singular surface energy. This topic was initiated by J. Taylor [T] and independently by S. Angenent and M. Gurtin [AG] who formulated motion of faceted curves moved by 'crystalline energy'. The governing equation is formally written in a quasilinear diffusion equation. However, because of singularity of energy, the diffusion effect is so strong that it may not be local. Even the notion of solution is not clear in general. There are two ways to handle such very singular diffusion equations systematically as a limit of diffusion equation with smooth energy. The first one is variational approach or the theory of nonlinear semigroups initiated by Y. Kōmura [Ko] and developed by many mathematicians for many years. It provides mathematical formulation of various important problems including the Stefan problem and the Hele-Shaw problem as explained in a book of A. Visintin [V]. The application of this theory to motion with facets is found in [FG] and is further developed by [EGS]; the theory developed in [HZ] is in the line of this approach. The second one is an approach by extending the theory of viscosity solutions initiated by M.-H. Giga and Y. Giga [GG1], [GG3], [GG4]. The first method applies to problem for arbitrary dimensions but the method needs the divergence structure of equations. The second method is so far limited in one space dimension and spatially homogeneous problem. However, it does not require divergence structure of the equation so that it applies equations

Received May 27, 1999.
[1] Partly supported by Nissan Science Foundation and the Grant-in-Aid for Science Research No. 10304010, No. 08874005, the Japan Ministry of Education, Science and Culture.
[2] Partly supported by the Grant-in-Aid for Science Research No. 09650074, the Japan Ministry of Education, Science and Culture.

describing the motion of curves moved by singular surface energy [GG4]. The bibliography of [GG1] and [GG3] includes many references to recent work on the motion by crystalline energy and singular energy. The reader is referred to [GG1] and [GG3] for related references as well as the background of problems.

In this paper, as an example of very singular diffusion equations, we consider a quasilinear equation

$$\frac{\partial u}{\partial t} = \frac{1}{b} \operatorname{div}\left(a \frac{\nabla u}{|\nabla u|} \right), \tag{1.1}$$

where a and b are a given positive function. This equation is interpreted as the gradient system by taking energy

$$E(u) = \int_\Omega a(x) |\nabla u(x)| \, dx \tag{1.2}$$

with respect to the norm $\|f\| = \left(\int_\Omega b(x) |f(x)|^2 \, dx \right)^{1/2}$, where Ω is a domain in $\mathbf{R}^n$. In other words (1.1) is written as a gradient system

$$\frac{\partial u}{\partial t} = -(\operatorname{grad}_{\|\cdot\|} E)(u),$$

where $\operatorname{grad}_{\|\cdot\|}$ denotes the gradient with respect to $\|\cdot\|$. When a and b are identically equal to one, $E(u)$ is called a coarea functional and (1.1) is called the coarea gradient flow equation. Its Dirichlet problem is studied in [HZ], where it is proved that the asymptotic limit of solution as time tends to infinity exists and enjoys the relvent minimizing properties for time-independent boundary data. As we observe from (1.2) the energy density $a(x)|p|$ in (1.2) is not differentiable at $p = 0 \in \mathbf{R}^n$ so that the equation has singularity at $|\nabla u| = 0$ in (1.1). The equation (1.1) is also important to describe motion of multi-grain problem studied in [KWC], where b is proportional to a.

The goal of this paper is to review the variational approach to (1.1) to define correct notion of a solution and to give several examples of solutions which develops "facet" or "plateau" (flat portion of the graph of solutions) as an effect of nonlocal diffusion. There is another review paper [KG] on this subject for physicists and material scientists. The present paper describes underlying mathematical basis on this subject for mathematicians.

For one dimensional version of (1.1) (with Dirichlet boundary condition) we derive an interesting necessary and suffieicnt condition so that "plateau" of a solution is preserved when a and b are not necessarily constant. For spatially homogeneous problem, i.e., for constant

a and b, plateau does not break. For piecewise constant initial data we give an explicit way to represent solution even after the time when some plateau merges. We also prove that the number of peaks does not increase during evolution; similar property is well-known for usual diffusion equations ([A], [M], ...). These results seem to be new. Roughly speaking, one of sufficient conditions for nonbreaking of plateau is the concavity of a with respect to 'metric' induced by b at each maximal interval where the solution is constant. This condition is fulfilled for the system proposed by [KWC] where the equation (1.1) is coupled except the fact that a and b now depend on time. We note that a different type of spatially inhomogeneous problem has been studied in [GG2]. At the end of this paper we also note that solution may become discontinuous instanteneously if a is not spatially homogeneous because of strong diffusion (when $b \equiv 1$).

§2. Variational formulation

We recall an abstract formulation for a gradient flow equation for a convex energy. Let H be a real Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$. Let φ be a real-valued convex function defined on a convex subset $D(\varphi)$ of H. For technical convenience we extend φ outside $D(\varphi)$ by setting its values as $+\infty$ (which is larger than any real number). The extended function is still denoted φ and $D(\varphi)$ is called the domain of φ.

For application H is taken a space of functions and φ is its energy. It is important to analyse variation or gradient of φ. However, unfortunately φ need not be differentiable in H. For a convex function φ the notion of subdifferential substitutes the notion of gradient. A *subdifferential* of φ at $v \in D(\varphi)$ is the set of all $f \in H$ that satisfies

$$\varphi(v+h) - \varphi(v) \geq \langle f, h \rangle \tag{2.1}$$

for all $h \in H$. The subdifferential of φ at v is denoted $\partial\varphi(v)$. For example, for $\varphi(v) = |v|$ for $v \in \mathbf{R}$ we have $\partial\varphi(v) = \{1\}$ for $v > 0$; $\partial\varphi(v) = [-1, 1]$ for $v = 0$; $\partial\varphi(v) = \{-1\}$ for $v < 0$, if the set $\mathbf{R}$ of all real numbers is regarded as a Hilbert space equipped with the standard inner product. Of course if φ is differentiable at v, $\partial\varphi(v)$ is a singleton and consists of the gradient of φ at v.

We now recall a fundamental theorem for unique existence of solutions for the gradient system

$$\frac{du}{dt} \in -\partial\varphi(u), \quad u|_{t=0} = u_0. \tag{2.2}$$

Unique Existence Theorem. *Assume that φ is convex with nonempty domain $D(\varphi)$ in H and that φ is lower semicontinuous in H, i.e.,*

$$\varphi(v) \leq \liminf_{w \to v} \varphi(w) \quad \text{for all } v \in H,$$

where $w \to v$ denotes the convergence in the norm in H. For each $u_0 \in H$ there is a unique solution u of (2.2) in the sense that

(i) *u is continuous from the time interval $[0, \infty)$ to H and u is absolutely continuous with values in H on each compact set in $(0, \infty)$.*

(ii)

$$\frac{du}{dt}(t) \in -\partial\varphi(u(t)) \quad \text{for almost all } t \geq 0, \tag{2.3}$$

$$u(0) = u_0. \tag{2.4}$$

Here, the time derivative $f = \frac{du}{dt}(t)$ is defined by the unique element of H that satisfies

$$\lim_{s \to 0} \left\| \frac{u(t+s) - u(t)}{s} - f \right\| = 0,$$

where $\|v\| = \langle v, v \rangle^{1/2}$ is the norm of v in H.

For the proof the reader is referred to a book [Ba]. To apply this theorem to our problem we need to interpret our energy E in (1.2) as a lower semicontinuous convex function on a suitable Hilbert space. We consider the Dirichlet problem in this paper.

Lemma (Lower semicontinuous interpretation). *Let a be a positive continuous function defined in a smoothly bounded domain Ω in $\mathbf{R}^n$. Assume that both a and $1/a$ are bounded in Ω. For a given (Lipschitz) continuous boundary data g on $\partial\Omega$ let $\tilde{g}$ denote a Lipschitz extension of g to $\mathbf{R}^n$. For $v \in H = L^2(\Omega)$ let $\tilde{v}$ be its extension to $\mathbf{R}^n$ such that $\tilde{v}(x) = \tilde{g}(x)$ for $\mathbf{R}^n \backslash \Omega$. Then the functional*

$$\varphi(v) = \int_{\overline{\Omega}} a(x) |\nabla \tilde{v}(x)| \, dx \tag{2.5}$$

with $D(\varphi) = BV(\Omega)$ is convex and lower semicontinuous in the Hilbert space $H = L^2(\Omega)$, where $BV(\Omega)$ denotes the space of functions with bounded variation in Ω. The functional $\varphi(v)$ is independent of the way of extension $\tilde{g}$ of g; it depends on g.

The convexity is easy to check. Note that if one assigns $\varphi \equiv \infty$ outside the set $\{v \in BV(\Omega) \; ; \; v|_{\partial\Omega} = g\}$, then φ is not lower semicontinuous in $L^2(\Omega)$. The proof of the lower semicontinuity is standard in the theory of BV [Giu] so we omit it. Note that we have to give a meaning of (2.5) when ∇v is merely a Radon measure before starting the proof. Note also that (2.5) also measures $v|_{\partial\Omega} - g$ on $\partial\Omega$.

For the functional φ defined by (2.5) we derive an explicit form of the gradient form at least in the formal level. The subdifferential depends on the metric of H and there are various inner products of $L^2(\Omega)$. For a given positive continuous function b in Ω we set an inner product.

$$\langle v, w\rangle_b = \int_\Omega v(x)w(x)b(x)\,dx, \quad v, w \in L^2(\Omega).$$

If b identically equals one, this inner product is nothing but the standard inner product of $L^2(\Omega)$. The norm induced by $\langle\,\cdot\,,\,\cdot\,\rangle_b$ is equivalent to the standard one provided that both b and $1/b$ are bounded on Ω.

It is not easy to express the subdifferential when ∇v is a measure as presented in [Te] for a different but related problem. We give a simpler version.

Lemma (Subdifferentials). *Assume the same hypotheses of the preceeding lemma concerning Ω, a, g and φ. Assume that v is Lipschitz continuous (so that ∇v is bounded on Ω) and that $v|_{\partial\Omega} = g$. Let $\partial\varphi(v)$ denote the subdifferential of φ at v with respect to the inner product $\langle\,\cdot\,,\,\cdot\,\rangle_b$ where both $b > 0$ and $1/b$ are assumed to be bounded continuous. Then $f \in \partial\varphi(v)$ if and only if there is a locally integrable function ξ on Ω that satisfies*

$$f = -\frac{1}{b}\operatorname{div}(a\xi), \quad \xi(x) \in \partial j(\nabla v(x)) \tag{2.6}$$

for almost all $x \in \Omega$, where $j(p) = |p|$ for $p \in \mathbf{R}^n$ and ∂j is the subdifferential of j with respect to the standard inner product of $\mathbf{R}^n$.

This is an easy corollary of the result in [AD] by modifying $j(p)$ for large p so that $j(p)/|p| \to \infty$ as $|p| \to \infty$ as in [FG]. The proof that (2.6) implies $f \in \partial\varphi(v)$ directly follows from the definition of subdifferential while the converse is nontrivial. This lemma asserts that the equation (2.2) (with φ given by (2.5)) is formally written as

$$\frac{\partial u}{\partial t} = \frac{1}{b}\operatorname{div}\left(a\frac{\nabla u}{|\nabla u|}\right), \quad u|_{t=0} = u_0, \tag{2.7}$$

with the Dirichlet boundary condition $u|_{\partial\Omega} = g$, since

$$\partial j(p) = \begin{cases} \{p/|p|\}, & \text{for } p \neq 0, \\ B_1, & \text{for } p = 0, \end{cases} \tag{2.8}$$

where B_1 is a closed unit ball centered at the origin in $\mathbf{R}^n$. By a solution of (2.7) with $u|_{\partial\Omega} = g$ we mean that it is a solution of (2.2) with φ given by (2.5) defined in $H = L^2(\Omega)$ equipped with the inner product $\langle \cdot\,, \cdot \rangle_b$. Note that the boundary condition is hidden in φ. The unique existence theorem for (2.2) asserts that the equation (2.7) with $u|_{\partial\Omega} = g$ is uniquely solvable. If $a \equiv b \equiv 1$, (2.7) is called the coarea gradient flow equation in [HZ] which is qualitatively different from the level set mean curvature flow equation

$$\frac{\partial u}{\partial t} = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$

analysed in [CGG], [ES] since (2.7) turns to have a nonlocal effects while the last equation does not have such an effert; see [G] for review on the level set equations. If a and b depend on time, (2.7) is not written in the form of (2.2) unless we use time dependent inner product and energy. Fortunately, an abstract theory including such situation has been developed by A. Damlamian [D]. We thank Professor N. Kenmochi for pointing out this reference.

We come back to the abstract setting in the unique existence theorem. The condition (2.3) can be interpreted as a variational inequality. By definition (2.1) the condition (2.3) is equivalent to a variational inequality:

$$\varphi(v) - \varphi(u(t)) \geq \left\langle v - u(t), -\frac{du}{dt}(t) \right\rangle \quad \text{for all } v \in H \text{ and almost all } t \geq 0. \tag{2.9}$$

It is not difficult to see the uniqueness of solution of (2.2). Indeed, let w fulfill (2.9), i.e.,

$$\varphi(v) - \varphi(w(t)) \geq \left\langle v - w(t), -\frac{dw}{dt}(t) \right\rangle \quad \text{for all } v \in H \text{ and almost all } t \geq 0. \tag{2.9$'$}$$

Setting $v = w(t)$ in (2.9) and $v = u(t)$ in (2.9)$'$ and adding (2.9) and (2.9)$'$ yields

$$\begin{aligned} 0 &\geq \left\langle w(t) - u(t), \frac{dw}{dt}(t) - \frac{du}{dt}(t) \right\rangle \\ &= \frac{1}{2}\frac{d}{dt}\|w(t) - u(t)\|^2 \quad \text{for almost all } t \geq 0. \end{aligned}$$

This implies the *contraction property*:

$$\|w(t) - u(t)\| \le \|w_0 - u_0\| \quad \text{for all } t \ge 0,$$

where w_0 is the initial data $w(0)$ of w. The contraction property immediately implies the *uniqueness* of solution of (2.2).

The evolution law (2.2) looks ambiguous since $\partial\varphi$ is multivalued. But by the uniqueness of solution the solution knows how to evolve. Does du/dt choose a special element of subdifferential $\partial\varphi(u(t))$? The next theorem which is well-known gives an answer. For the proof see e.g. [Ba].

Theorem on characterization of the speed. *Let u be the solution of (2.2) with $u_0 \in H$. Then u is right differentiable for all $t > 0$. Let d^+u/dt denote the right derivative. Then $f = -\frac{d^+u}{dt}(t)$ is the canonical restriction (or minimal section) of $\partial\varphi(u(t))$, i.e., $f \in \partial\varphi(u(t))$ and*

$$\|f\| = \min\{\|q\| \ ; \ q \in \partial\varphi(u(t))\}.$$

Conversely, if a continuous function u from $[0,\infty)$ to H is right differentiable at all $t > 0$ and $-\frac{d^+u}{dt}(t)$ is the canonical restriction of $\partial\varphi(u(t))$, then u is the solution of (2.2) with initial data $u_0 = u(0)$.

Since φ is convex and lower semicontinuous, the set $\partial\varphi(v)$ is always a closed convex set (which may be empty), so the canonical restriction is unique which is denoted $\partial\varphi^0(v)$ $(\in \partial\varphi(v))$.

We only give a formal proof for the characterization of the speed at $t_0 > 0$ by assuming that $\frac{du}{dt}(t_0)$ exists with the property that $\frac{du}{dt}(t_0) \in -\partial\varphi(u(t_0))$ and du/dt is right continuous at $t = t_0$. For the detailed proof see e.g. [Ba]. We set $t = t_0 + s$, $s > 0$ in (2.9) and $v = u(t_0)$ to get

$$\varphi(u(t_0)) - \varphi(u(t_0 + s)) \ge \left\langle u(t_0) - u(t_0 + s), -\frac{du}{dt}(t_0 + s)\right\rangle$$

For any $\zeta \in \partial\varphi(u(t_0))$ by definition

$$\langle \zeta, u(t_0) - u(t_0 + s)\rangle \ge \varphi(u(t_0)) - \varphi(u(t_0 + s)).$$

These two inequalities yield

$$\left\langle \frac{du}{dt}(t_0 + s), u(t_0 + s) - u(t_0)\right\rangle \le \langle -\zeta, u(t_0 + s) - u(t_0)\rangle.$$

Dividing both sides by s and sending s to zero yields

$$\left\|\frac{du}{dt}(t_0)\right\|^2 \le \left\langle -\zeta, \frac{du}{dt}(t_0)\right\rangle \le \|\zeta\| \left\|\frac{du}{dt}(t_0)\right\|$$

by the Schwarz inequality since du/dt is assumed to be right continuous at $t = t_0$. Thus for any $\zeta \in \partial\varphi(u(t_0))$ we have

$$\left\| \frac{du}{dt}(t_0) \right\| \leq \|\zeta\|.$$

Since $-\frac{du}{dt}(t_0) \in \partial\varphi(u(t_0))$, this minimality implies that $-\frac{du}{dt}(t_0) = \partial\varphi^0(u(t_0))$.

The solution of (2.2) we obtain is a nice stability property for perturbation of energy φ.

Stability Theorem. *Assume that φ_ε converges to φ in the sense of Mosco as $\varepsilon \to 0$, i.e., for any $v \in H$, $\varphi(v) \leq \liminf_{\varepsilon\to 0} \varphi_\varepsilon(v_\varepsilon)$ for any v_ε with $\|v_\varepsilon - v\| \to 0$ and there is a sequence $\tilde{v}_\varepsilon$ converges weakly in H that $\varphi(v) = \lim_{\varepsilon\to 0} \varphi_\varepsilon(\tilde{v}_\varepsilon)$. Assume that $\|u_0^\varepsilon - u_0\| \to 0$. Let u_0^ε be the solution of*

$$\frac{du}{dt} \in -\partial\varphi_\varepsilon(u), \quad u|_{t=0} = u_0^\varepsilon,$$

where φ_ε is a lower semicontinuous, convex function. Then for every $T > 0$

$$\lim_{\varepsilon\to 0} \sup_{0\leq t\leq T} \|u^\varepsilon(t) - u(t)\| = 0.$$

The result is due to J. Watanabe [W] based on a result of H. Brezis-A. Pazy [BP]. It asserts that our solution (2.2) can be obtained as a limit of approximate problems if the energy is approximated in a proper way. In practice it gives a way to calculate numerically a solution approximately by approximating φ by a smoother energy. We approximate φ in (2.5) by a smoother energy so that the approximate equation enjoys a comparison principle. By the stability theorem the comparison principle is inherited to (2.7).

Comparison principle. *Assume that φ is defined by (2.5) and $H = L^2(\Omega)$ equipped with inner product $\langle\ ,\ \rangle_b$. For solutions u and v of $du/dt \in -\partial\varphi(u)$, $u \leq v$ for all $t \geq 0$ if $u \leq v$ at $t = 0$.*

We are curious whether u^ε convergerges to u locally uniformly in space-time domain. So far such a convergence results is proved only for problem for one space dimension based on the theory of viscosity solutions [GG1], [GG3], when the equation is spatially homogeneous. In our problem (2.7) is spatially homogeneous if a and b are constants. We only give a simplest version of a general convergence result including nondivergence type equations proved in [GG3].

Uniform convergence. *Assume that $W_\varepsilon(p)$ converves to $|p|$ as $\varepsilon \to 0$ locally uniformly and that the second derivative $W_\varepsilon'' > 0$ and W_ε is smooth. Assume that u_0^ε and u_0 is continuous in a closed bounded interval Ω in $\mathbf{R}$ with $u_0^\varepsilon = u_0 = 0$ on the boundary $\partial\Omega$ of Ω. Assume that u_0^ε converges to u_0 uniformly in $\overline{\Omega}$. Let u^ε be the solution of*

$$\frac{\partial u}{\partial t} = (W_\varepsilon'(u_x))_x, \qquad u|_{t=0} = u_0^\varepsilon,\ u|_{\partial\Omega} = 0.$$

Then u^ε converges to a unique solution u of

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{u_x}{|u_x|}\right), \qquad u|_{t=0} = u_0,\ u|_{\partial\Omega} = 0$$

as $\varepsilon \to 0$ and the convergence is locally uniform in $\overline{\Omega} \times [0,\infty)$, where $u_x = \partial_x u = \partial u/\partial x$. Both u^ε and u are at least continuous in $\overline{\Omega} \times [0,\infty)$.

§3. Examples of solutions

We consider one-dimensional version of the equation (2.7). The equation is of form

$$\frac{\partial u}{\partial t} = \frac{1}{b}\frac{\partial}{\partial x}\left(a\frac{u_x}{|u_x|}\right). \tag{3.1}$$

3.1. Spatially homogeneous equation

To see the nonlocal effect of strong diffusion we first consider the spatially homogeneous equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{u_x}{|u_x|}\right), \tag{3.2}$$

which is of course an example of (3.1). We consider (3.2) for $x \in \Omega$ and $t > 0$ where $\Omega = (x_0, x_1)$ is an bounded open interval. For boundary condition we impose zero Dirichlet data but this is just to fix our problem. The problem (3.2) with zero Dirichlet data and initial data is formulated by (2.2) by taking

$$\begin{aligned} &H = L^2(\Omega), \quad \varphi(v) = \int_\Omega |v_x|\,dx + |v(x_0)| + |v(x_1)|, \\ &D(\varphi) = BV(\Omega), \end{aligned} \tag{3.3}$$

where H is equipped with the inner product

$$\langle f, g\rangle_1 = \int_\Omega f(x)g(x)\,dx.$$

We shall calculate subdifferential of φ at v when $v \in D(\varphi)$ is a Lipschitz continuous function with $v(x_0) = v(x_1) = 0$ having the property

$$\begin{cases} \text{monotone increasing on} & [x_0, \alpha], \\ \text{constant on} & [\alpha, \beta], \\ \text{monotone decreasing} & [\beta, x_1], \end{cases}$$

where $\Omega = (x_0, x_1)$ and $x_0 < \alpha < \beta < x_1$. By Lemma on subdifferentials we see $f \in \partial\varphi(v) \subset H$ if and only if there exists ξ satisfying

$$f = -\xi_x,$$
$$\xi(x) \begin{cases} = 1, & \text{if } v \text{ is increasing near } x, \\ = -1, & \text{if } v \text{ is decreasing near } x, \\ \in [-1, 1], & \text{otherwise.} \end{cases}$$

Since f is in $L^2(\Omega)$, ξ must be continuous on Ω. Thus the graph of ξ is as in Figure 1 (a). If $f = -\xi_x$ is the canonical restriction of $\partial\varphi(v)$, ξ must minimize

$$\int_\alpha^\beta |\xi_x|^2 \, dx$$

under constraint $|\xi| \leq 1$ on (α, β) with $\xi(\alpha) = 1$, $\xi(\beta) = -1$. The minimizer is an affine function on (α, β) so that $\xi_x = -2/(\beta - \alpha)$ as shown in Figure 1 (b). Thus

$$\partial\varphi^0(v)(x) = \begin{cases} 0, & x \in (x_0, \alpha) \text{ or } x \in (\beta, x_1), \\ +\frac{2}{\beta-\alpha}, & x \in (\alpha, \beta). \end{cases}$$

Note that $\partial\varphi^0(v)(x)$ is a nonlocal quantity determined by v for $x \in (\alpha, \beta)$.

Based on the calculation of $\partial\varphi^0(v)$ we seek the solution of (3.2) (i.e., the solution of (2.2) with (3.3)) for single peak initial data. Let u be a continuous function in $\overline{\Omega}$ of form

$$u_0(x) = \begin{cases} A(x), & x_0 \leq x \leq \alpha_0, \\ h_0, & \alpha_0 \leq x \leq \beta_0, \\ B(x), & \beta_0 \leq x \leq x_1, \end{cases} \tag{3.4}$$

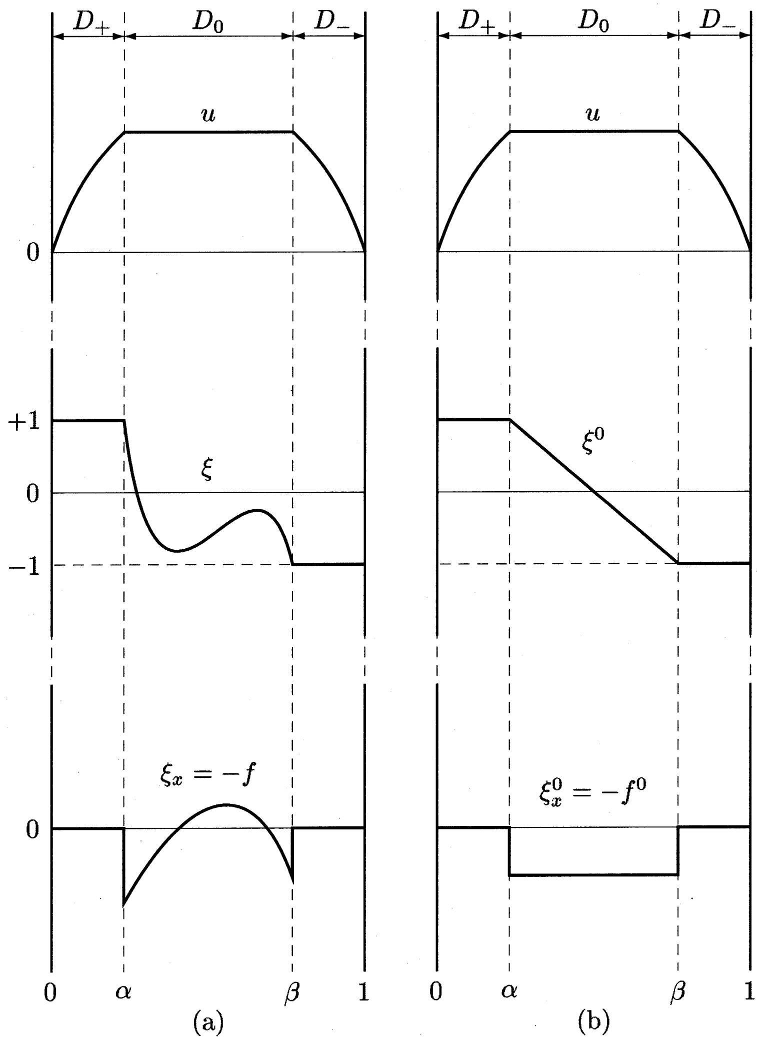

Fig. 1. The interval $[0, 1]$ is divided into three intervals $D_+ = [0, \alpha)$, $D_0 = [\alpha, \beta]$ and $D_- = (\beta, 1]$. In D_+, $\xi \equiv +1$ holds, and $\xi \equiv -1$ in D_-. (a) One example of ξ and $\xi_x = -f$ where f belongs to $\partial\varphi(u)$. (b) ξ^0 and $\xi^0_x = -f^0$ where f^0 is a canonical restriction of $\partial\varphi(u)$.

with $\alpha_0 \leq \beta_0$, $A' > 0$, $B' < 0$, $A(\alpha_0) = B(\beta_0) = h_0$ and $A(x_0) = B(x_1) = 0$. We expect the solution of (3.2) is of form

$$u(x,t) = \begin{cases} A(x), & x_0 \leq x \leq \alpha(t), \\ h(t), & \alpha(t) \leq x \leq \beta(t), \\ B(x), & \beta(t) \leq x \leq x_1, \end{cases} \tag{3.5}$$

with $A(\alpha(t)) = B(\beta(t)) = h(t)$ until the time T such that $h(T) = 0$. By the characterization of the speed if

$$h_t(t) = -\partial\varphi^0(u(\,\cdot\,,t)) = -\frac{2}{\beta(t)-\alpha(t)}, \quad 0 < t < T, \tag{3.6}$$

then $u(x,t)$ solves (3.2). Fortunately there is continuous α, β on $[0,T]$ that satisfies (3.6). In fact

$$\begin{aligned} &\alpha(t) = A^{-1}(h(t)), \ \ \beta(t) = B^{-1}(h(t)), \ \ \alpha(0) = \alpha_0, \ \ \beta(0) = \beta_0, \\ &h(t) = S^{-1}(2t), \\ &S(k) = \int_k^{h_0} (B^{-1}(\eta) - A^{-1}(\eta))\, d\eta, \end{aligned}$$

where -1 represents the inverse of a function. By this choice of α, β, h, (3.5) is now the unique solution of (3.2) with initial data u_0 given by (3.4); after the time T we set $u(x,t) \equiv 0$. It is not difficult to study evolution of multi peak function by (3.2). The important feature of the shape is local maximum and minimum is flattened by nonlocal diffusion effects and it may merge. See [KG, Section 3] for such examples as well as numerical simulations.

Solution starting from u_0 of (3.4) is given in [HZ]; a similar example with initial data $-u_0$ is given in [GG1]. One way of proving (3.6) is based on observation given in [FG], where the speed $-2/(\beta-\alpha)$ of evolution equals the canonical restriction of φ; see also [EGS].

3.2. Equations with inhomogeneous diffusion

We now consider (3.1) for $x \in \Omega = (z_0, z_1)$ and $t > 0$ with inhomogeneous Dirichlet boundary condition. We consider a piecewise constant initial data

$$u_0(x) = h_i^0 \quad \text{on } (x_i, x_{i+1}), \ \ i = 0, \dots, m-1, \ m \geq 2 \tag{3.7}$$

where $z_0 = x_0 < x_1 < x_2 < \cdots < x_m = z_1$. The values h_i^0 may be the same as h_{i+1}^0. The boundary condition we impose is $u = h_0^0$ at z_0, $u = h_{m-1}^0$ at z_1. We interpret (3.1) as (2.2) with (2.5) where $g = h_0^0$

at z_0 and $g = h^0_{m-1}$ at z_1 in the definition of $\varphi(v)$; the inner product of $H = L^2(\Omega)$ is chosen by $\langle\ ,\ \rangle_b$ as in §2. In general the solution u with initial data (3.7) may not be locally constant in x because of inhomogeneity of a and b. We shall seek conditions on a and b so that $u(\,\cdot\,,t)$ is piecewise constant and its jump discontinuities are included in $\{x_i\}_{i=1}^{m-1}$. If $h^0_i = h^0_{i+1} = \cdots = h^0_{i+k-1}$, $h^0_{i-1} \neq h^0_i$, $h^0_{i+k-1} \neq h^0_{i+k}$ we say the segment

$$(x_i, x_{i+k}) \times \{h^0_i\}, \quad k \geq 1$$

is a *plateau* (of height h^0_i) for a piecewise constant function u_0. We first consider the case that $h^0_{i-1} \neq h^0_i$ for $i = 1, \ldots, m-1$.

3.2.1. *Evolution before merging of plateaus*

Theorem on persistency. *Assume that a and b are positive and continuous on $\overline{\Omega} = [z_0, z_1]$. Assume that $a(x_1) \leq a(x)$ for all $x \in [x_0, x_1]$ and $a(x_{m-1}) \leq a(x)$ for all $x \in [x_{m-1}, x_m]$ and that*

(C_i)

$$\{a(x_{i+1}) - a(x_i)\} \Big/ \int_{x_i}^{x_{i+1}} b(\tau)\, d\tau \leq \{a(x) - a(x_i)\} \Big/ \int_{x_i}^{x} b(\tau)\, d\tau \qquad \text{for all } x \in (x_i,\ x_{i+1}]$$

holds for all $i = 1, \ldots, m-2$. Let u be the solution of (3.1) with initial data u_0 given by (3.7) with Dirichlet boundary condition $u|_{\partial\Omega} = u_0|_{\partial\Omega}$. Assume that $h^0_i \neq h^0_{i-1}$ for $i = 1, \ldots, m-1$. Then for each $i = 0, 1, \ldots, m-1$ the speed $u_t(x,t)$ is independent of $x \in (x_i, x_{i+1})$ and $t \in (0, t_0)$, where t_0 is the first time that some plateau of $u(\,\cdot\,,t)$ merges to another one. Moreover $u_t(x,t) \equiv 0$ in (x_0, x_1) and (x_{m-1}, x_m) for $0 < t < t_0$. (Thus the function $u(\,\cdot\,,t)$ is piecewise constant and it jumps at $x_1, \ldots, x_{m-1}$ for $t \in (0, t_0)$.)

Remark 1. If we use the length with respect to the metric $b\,dx$, the condition (C_i) is rewritten as

$$\{a_b(y_{i+1}) - a_b(y_i)\}/(y_{i+1} - y_i) \leq \{a_b(y) - a_b(y_i)\}/(y - y_i) \qquad \text{for all } y \in (y_i, y_{i+1}],$$

where $a_b(y) = a(x(y))$, $y_i = \int_{x_1}^{x_i} b(\tau)\, d\tau$ and $x(y)$ is the inverse function of $y(x) = \int_{x_1}^{x} b(\tau)\, d\tau$. In other words the convex hull of a_b on $[y_i, y_{i+1}]$ is affine.

Remark 2. It turns out that the condition (C_i) is necessary so that u_t is constant on (x_i, x_{i+1}) for any initial data u_0 given by (3.7) with $h_j^0 \neq h_{j-1}^0$ $(j = 1, \dots, m-1)$. Also $a(x) \geq a(x_1)$ on $[x_0, x_1]$ and $a(x) \geq a(x_{m-1})$ on $[x_{m-1}, x_m]$ are necessary so that u_t is zero on (x_0, x_1) and (x_{m-1}, x_m), respectively. We shall explain the reasons in the proof of Lemma 2.

Remark 3. It is clear that (C_i) is fulfilled if a is concave when b is constant on $[x_i, x_{i+1}]$. It is more difficult to see that (C_i) is fulfilled if a is concave on $[x_i, x_{i+1}]$ when b is proportional to a. We shall prove this statement at the end of this subsection §3.2.1.

Since u_0 is discontinuous, our Lemma on subdifferentials does not apply. We seek a nice element of subdifferential at a piecewise constant function.

Lemma 1. *Assume that a and b are positive and continuous on $[z_0, z_1]$. Assume that u_0 is given by* (3.7). *Let $f \in L^2(\Omega)$ be of form*

$$f(x) = -\frac{1}{b(x)}\{a(x)\xi(x)\}_x \quad |\xi(x)| \leq 1,\ x \in \Omega = (z_0, z_1) \tag{3.8}$$

for some continuous ξ in Ω that satisfies

$$\xi(x_i) = \begin{cases} 1, & \text{if } h_{i-1}^0 < h_i^0, \\ -1, & \text{if } h_{i-1}^0 > h_i^0, \end{cases} \tag{3.9}$$

for $i = 1, 2, \dots, m-1$. (If $h_{i-1}^0 = h_i^0$, no condition on $\xi(x_i)$ is imposed.) Then $f \in \partial\varphi(u_0)$, where the boundary data g is taken as the boundary value of u_0 and φ is given in (2.5). *Conversely, if $f \in \partial\varphi(u_0)$ then f is of form* (3.8) *satisfying* (3.9).

Proof of Lemma 1. We shall check (2.1) or

$$\langle v - u_0, f\rangle_b \leq \varphi(v) - \varphi(u_0)$$

for all $v \in D(\varphi)$. By definition

$$\langle v - u_0, f\rangle_b = -\int_\Omega (v - u_0)(a\xi)_x\, dx. \tag{3.10}$$

Since $|\xi| \leq 1$, integrating by parts we see

$$-\int_\Omega v(a\xi)_x\, dx = \int_\Omega v_x a\xi\, dx - u_0 a\xi|_{z_0}^{z_1} \leq \varphi(v) - u_0 a\xi|_{z_0}^{z_1}; \tag{3.11}$$

here v_x is regarded as a measure and $\varphi(v) = \int |v_x| a(x)\, dx$ is defined for a Radon measure v_x, for example

$$\varphi(u_0) = \sum_{i=1}^{m-1} a(x_i)|h_i - h_{i-1}|.$$

Since $\xi(x_i) = \pm 1$ depending on the sign of $h_i - h_{i-1}$, we see

$$\begin{aligned}(3.12) \qquad \int_\Omega u_0 (a\xi)_x \, dx &= u_0 a\xi|_{z_0}^{z_1} - \sum_{i=1}^{m-1} a(x_i)|h_i - h_{i-1}| \\ &= u_0 a\xi|_{z_0}^{z_1} - \varphi(u_0).\end{aligned}$$

The formula (3.10)–(3.12) now yields

$$\begin{aligned}\langle v - u_0, f\rangle_b &\le \varphi(v) - u_0 a\xi|_{z_0}^{z_1} + u_0 a\xi|_{z_0}^{z_1} - \varphi(u_0) \\ &= \varphi(v) - \varphi(u_0),\end{aligned}$$

which implies $f \in \partial\varphi(u_0)$.

Conversely, assume that $f \in \partial\varphi(u_0)$. Let ζ by a primitive of $-bf$. Since $f \in L^2(\Omega)$, ζ must be absolutely continuous in Ω. The condition $f \in \partial\varphi(u_0)$ is equivalent to

$$(*) \qquad -\int_\Omega (v - u_0)\zeta_x \, dx \le \varphi(v) - \varphi(u_0).$$

We take various v in this inequality to derive properties of ζ. If $u_{0x} \not\equiv 0$, there is an index $i \in \{1, \dots, m-1\}$ such that $h_i^0 \neq h_{i-1}^0$. We may assume that $h_{i-1}^0 < h_i^0$. For $\hat{x} \in (z_0, z_1) \setminus \{x_j\}_{j=1}^{m-1}$ we take

$$v(x) = u_0(x) + \lambda \int_{z_0}^{x} (\delta(\tau - \hat{x}) - \delta(\tau - x_i)), \quad \lambda < h_i^0 - h_{i-1}^0$$

in $(*)$ and integrate by parts to get $\lambda(\zeta(\hat{x}) - \zeta(x_i)) \le |\lambda| a(\hat{x}) - \lambda a(x_i)$. If we set $\zeta(x_i) = a(x_i)$, then this inequality yields $|\zeta(\hat{x})| \le a(\hat{x})$ for all $\hat{x} \in (z_0, z_1) \backslash \{x_j\}_{j=1}^{m-1}$ by taking λ positive or negative. By continuity this implies $|\zeta(x)| \le a(x)$ for all $x \in (z_0, z_1)$. If $h_{i+1}^0 > h_i^0$ we take $\hat{x} = x_{i+1}$ and plug above v in $(*)$ to get $\lambda(\zeta(x_{i+1}) - \zeta(x_i)) \le \lambda(a(x_{i+1}) - a(x_i))$ for $\lambda \in \mathbf{R}$ with small $|\lambda|$. This yields $\zeta(x_{i+1}) = a(x_{i+1})$. Similarly, we have $-\zeta(x_{i+1}) = a(x_{i+1})$ if $h_{i+1}^0 < h_i^0$. Repeating this argument for both sides of x_i we conclude that f is of form (3.8) satisfying (3.9) when $u_{0x} \not\equiv 0$. (If u_0 is a constant, we normalize ζ so that $\max(\zeta - a) = 0$ to get $\zeta(x_*) = a(x_*)$ for some $x_* \in [z_0, z_1]$. We set v as above in $(*)$ with $\hat{x} = x_*$, $x_1 = x$ and $\lambda = 1$ to get $-\zeta(x) \le a(x)$ which yields $|\zeta(x)| \le a(x)$.) Q.E.D.

Lemma 2. *Assume that u_0 is given by (3.7) with $h_i^0 \neq h_{i-1}^0$ for $i = 1, \dots, m-1$. Assume the same hypothesis of theorem on persistency concerning a and b. Then there is continuous ξ on Ω satisfying (3.8) and (3.9) such that f given by (3.8) is constant $-\nu^i$ on each interval (x_i, x_{i+1}) $(i = 0, 1, \dots, m-1)$ and that $\nu^0 = \nu^{m-1} = 0$. The constant ν^i is of form*

$$\nu^i = \frac{(a\xi)(x_{i+1}) - (a\xi)(x_i)}{\int_{x_i}^{x_{i+1}} b\,dx}, \quad i = 0, 1, \dots, m-1,$$

which depends on u_0 only through the order of h_i, h_{i-1}, h_{i+1}.

Proof of Theorem on persistency. Since $h_i^0 \neq h_{i-1}^0$ $(i = 1, \dots, m-1)$, by Lemmas 1 and 2 (and the unique existence theorem) we see

$$u(x,t) = h_i(t) \quad \text{on } (x_i, x_{i+1}) \text{ with } \frac{dh_i}{dt}(t) = \nu^i$$

for $i = 0, 1, \dots, m-1$ is the unique solution of (3.1) with initial data u_0 given in (3.7), until the first time t_0 when $h_i(t_0) = h_{i+1}(t_0)$ for some $i = 0, 1, \dots, m-1$. Actually we have used a version of uniqueness of a local solution for (2.2) on $[0, T)$ which is proved in the same way as the unique existence theorem in §2. (The time t_0 may be infinite. Indeed, if $a \equiv b \equiv 1$, and $h_i^0 < h_{i+1}^0$ for $i = 0, 1, \dots, m-2$, then $u_0(x)$ itself is the solution of (3.1) with (3.7) and no plateau merges for all $t > 0$.) Q.E.D.

Remark 4. We did not use the fact that f in Lemma 2 is the canonical restriction of $\partial\varphi(u_0)$. By the Theorem on characterization of the speed we see that $f \in \partial\varphi(u_0)$ in Lemma 2 is actually equals $\partial\varphi^0(u_0)$ a posteriori.

In the rest of this subsection we shall prove Lemma 2.

Lemma 3 (Constant velocity profile). *Assume that a and b are positive and continuous on a nontrivial interval $[\alpha, \beta]$. The following two conditions are equivalent.*

(A) *$\{a(\beta) - a(\alpha)\}/\int_\alpha^\beta b(\tau)\,d\tau \leq \{a(x) - a(\alpha)\}/\int_\alpha^x b(\tau)\,d\tau$ for all $x \in (\alpha, \beta]$.*

(B) *For any* $\delta_1, \delta_2 \in \{-1, 1\}$ *there is a unique function* ξ *on* $[\alpha, \beta]$ *fulfilling the properties*:

(3.13) $\quad a\xi \quad$ is C^1 on $[\alpha, \beta]$;

(3.14) $\quad |\xi(x)| \leq 1 \quad$ for all $x \in [\alpha, \beta]$;

(3.15) $\quad \xi(\alpha) = \delta_1, \quad \xi(\beta) = \delta_2$;

(3.16) $$\frac{1}{b}\frac{d}{dx}(a\xi)(x) = \frac{(a\xi)(\beta) - (a\xi)(\alpha)}{\int_\alpha^\beta b(\tau)\,d\tau} \quad \text{for all } x \in [\alpha, \beta].$$

Proof. We first prove that (A) implies (B). We may assume that $\xi(\beta) = 1$ since the argument for $\xi(\beta) = -1$ is symmetric. We denote the constant $\{(a\xi)(\beta) - (a\xi)(\alpha)\}/\int_\alpha^\beta b$ by V. Integrating the equation $\partial_x(a\xi) = bV$, we have a representation formula for ξ

$$(a\xi)(x) - (a\xi)(\alpha) = V\int_\alpha^x b(\tau)\,d\tau.$$

The regularity of $a\xi$ is clear. It remains to check the constraint $|\xi(x)| \leq 1$, which is equivalent to

(3.17) $$-a(x) \leq (a\xi)(\alpha) + V\int_\alpha^x b(\tau)\,d\tau \leq a(x) \quad \text{for all } x \in [\alpha, \beta].$$

If $\xi(\alpha) = 1$, the condition (A) is equivalent to the rightest inequality of (3.17). Since $\int_\alpha^x b \leq \int_\alpha^\beta b$ so that $\{-a(x) - a(\alpha)\}/\int_\alpha^x b \leq -a(\alpha)/\int_\alpha^\beta b \leq \{a(\beta) - \alpha(\alpha)\}/\int_\alpha^\beta b$, the leftest inequality of (3.17) has been proved. If $\xi(\alpha) = -1$, the inequalities (3.17) read

(3.18) $$\frac{-a(x) + a(\alpha)}{\int_\alpha^x b(\tau)\,d\tau} \leq \frac{a(\beta) + a(\alpha)}{\int_\alpha^\beta b(\tau)\,d\tau} \leq \frac{a(x) + a(\alpha)}{\int_\alpha^x b(\tau)\,d\tau}.$$

Since $\int_\alpha^x b \leq \int_\alpha^\beta b$, the condition (A) implies

$$\frac{a(\beta) + a(\alpha)}{\int_\alpha^\beta b(\tau)\,d\tau} \leq \frac{a(\beta) - a(\alpha)}{\int_\alpha^\beta b(\tau)\,d\tau} + \frac{2a(\alpha)}{\int_\alpha^\beta b(\tau)\,d\tau} \leq \frac{a(x) - a(\alpha)}{\int_\alpha^x b(\tau)\,d\tau} + \frac{2a(\alpha)}{\int_\alpha^x b(\tau)\,d\tau}.$$

Thus we get the rightest inequality of (3.18). By the condition (A) we have

$$\frac{-a(x) + a(\alpha)}{\int_\alpha^x b(\tau)\,d\tau} \leq \frac{a(\alpha) - a(\beta)}{\int_\alpha^\beta b(\tau)\,d\tau} \leq \frac{a(\beta) + a(\alpha)}{\int_\alpha^\beta b(\tau)\,d\tau}$$

which implies the leftest hand of (3.18). We thus obtained (3.17).

The converse follows from (3.17) by taking $\xi(\alpha) = \xi(\beta) = 1$.

Q.E.D.

Lemma 4 (Constant velocity profile – boundary version). *Assume that a and b are positive and continuous on a nontrivial interval $[\alpha,\beta]$. Let p be a boundary point of $[\alpha,\beta]$, i.e., $p=\alpha$ or $p=\beta$. The following two conditions are equivalent.*

(i) *$a(p)\le a(x)$ for all $x\in[\alpha,\beta]$.*

(ii) *For any $\delta\in\{-1,1\}$, there is a unique function ξ on $[\alpha,\beta]$ fulfilling* (3.13), (3.14) *and $\xi(p)=\delta$, $\frac{d}{dx}(a\xi)=0$ on $[\alpha,\beta]$.*

Proof. There is a unique solution $\xi(x)=a(p)\delta/a(x)$ of $\xi(p)=\delta$, $d(a\xi)/dx=0$ on $[\alpha,\beta]$ (satisfying (3.13)). The condition (i) is equivalent to (3.14) so (i) and (ii) are equivalent. Q.E.D.

Proof of Lemma 2. We apply Lemma 1 and Lemma 3 on each interval $[x_i,x_{i+1}]$ $(i=1,2,\dots,m-2)$ (and Lemma 4 on $[x_0,x_1]$ and $[x_{m-1},x_m]$) to get Lemma 2.

The necessity of (C_i) in Remark 2 follows from Lemma 3 and Lemma 1. Similarly the necessity of $a(x)\ge a(x_1)$ on $[x_0,x_1]$, $a(x)\ge a(x_{m-1})$ on $[x_{m-1},x_m]$ in Remark 2 follows from Lemma 4. Q.E.D.

We conclude this subsection by proving that the concavity of a on $[x_i,x_{xi+1}]$ implies (C_i) when a is propotional to b as stated in Remark 3. It follows from the next lemma.

Lemma 5. *Assume that a is concave and positive on a nontrivial interval $[\alpha,\beta]$. Assume that a is proportional to b. Then*

$$\{a(\beta)-a(\alpha)\}\Big/\int_\alpha^\beta b(\tau)\,d\tau\le\{a(x)-a(\alpha)\}\Big/\int_\alpha^x b(\tau)\,d\tau \tag{3.19}$$

for all $x\in(\alpha,\beta]$.

Proof. We may assume that $a\equiv b$ on $[\alpha,\beta]$. We first discuss the case $a(\alpha)\le a(\beta)$. Let I be the interval of form

$$I=\{x\in[\alpha,\beta]\,;\,a(x)\ge a(\beta)\}$$

which may be a singleton (cf. Figure 2). Since $a(x)-a(\alpha)\ge 0$ and $\int_\alpha^x a\le\int_\alpha^\beta a$ for $x\in I$, (3.19) holds for all $x\in I$.

By concavity of a we have

$$\{a(\beta)-a(\alpha)\}/(\beta-\alpha)\le\{a(x)-a(\alpha)\}/(x-\alpha) \tag{3.20}$$

for all $x\in(\alpha,\beta]$. Moreover, if $x\notin I$, then

$$\frac{1}{x-\alpha}\int_\alpha^x a(\tau)\,d\tau\le\frac{1}{\beta-\alpha}\int_\alpha^\beta a(\tau)\,d\tau. \tag{3.21}$$

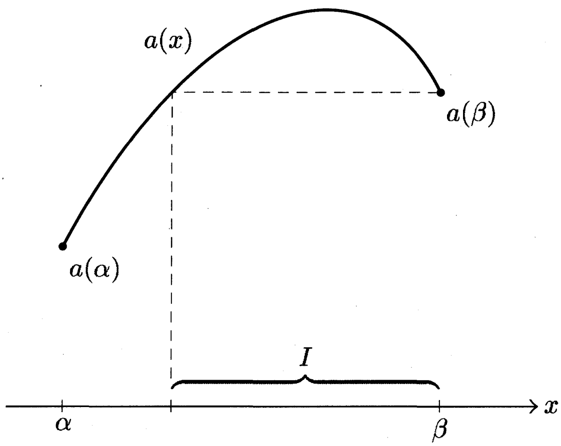

Fig. 2. Interval I defined in the proof of Lemma 5.

From (3.20) and (3.21) it follows that

$$\frac{a(\beta)-a(\alpha)}{\int_\alpha^\beta a(\tau)\,d\tau} = \frac{a(\beta)-a(\alpha)}{\beta-\alpha}\cdot\frac{\beta-\alpha}{\int_\alpha^\beta a(\tau)\,d\tau} \le \frac{a(x)-a(\alpha)}{x-\alpha}\cdot\frac{x-\alpha}{\int_\alpha^x a(\tau)\,d\tau}$$

for $x \notin I$. Thus (3.19) holds for all $x \in (\alpha, \beta]$.

If $a(\alpha) \geq a(\beta)$ a symmetric argument yields

$$\{a(\beta)-a(\alpha)\} \bigg/ \int_\alpha^\beta b(\tau)\,d\tau \geq \{a(\beta)-a(x)\} \bigg/ \int_x^\beta b(\tau)\,d\tau$$

or

$$\{a_b(\tilde\beta)-a_b(0)\}/\tilde\beta \geq \{a_b(\tilde\beta)-a_b(y)\}/(\tilde\beta-y), \quad 0 \leq y < \tilde\beta$$

where $a_b(y) = a(x(y))$, $\tilde\beta = \int_\alpha^\beta b(\tau)\,d\tau$ and $x(y)$ is the inverse function of $y(x) = \int_\alpha^x b(\tau)\,d\tau$; see Remark 1. This evidently implies

$$\{a_b(\tilde\beta)-a_b(0)\}/\tilde\beta \leq \{a_b(y)-a_b(0)\}/y, \quad 0 < y \leq \tilde\beta$$

which is the same as (3.19). Q.E.D.

3.2.2. *Evolution at the time of plateau merging*

We consider the solution u of (3.1) with initial data u_0 of form (3.7) when $h_i^0 \neq h_{i-1}^0$ $(i = 1, 2, \ldots, m-1)$. Under the assumptions of Theorem on persistency concerning a and b it follows from Lemma 1 and Lemma 2 that

$$u(t,x) = \nu^i t + h_i^0 \,(= h_i(t)) \quad \text{on } (x_i, x_{i+1}) \ (i = 0, 1, \ldots, m-1)$$

until the first time t_0 at which some plateau merges, i.e., t_0 is the first time t that $h_i(t) = h_{i+1}(t)$ for some i, $0 \le i \le m-1$.

At time t_0 some consecutive plateau merges. We shall discuss whether or not these merged plateaus are forced to split for $t > t_0$ close to t_0. The answer depends on a and b. For this purpose we shall extend Lemma 3 to calculate velocity profile on merged plateaus.

Lemma 6. *Assume that a and b are positive and continuous on $[\alpha, \beta]$ with $\alpha = r_0 < r_1 < r_2 < \cdots < r_k = \beta$, where $k \ge 1$. Assume that (C_i) in Theorem on persistency with x_i replaced by r_i for all $i = 0, 1, \ldots, k-1$.*

(i) (*Monotone velocity profile*) *There is a unique function ξ on $[\alpha, \beta]$ fulfilling* (3.14), (3.15) *with $\delta_1 = \delta_2 \in \{-1, 1\}$ such that the following properties* (a)–(c) *hold.*

(a) *$a\xi$ is continuous on $[\alpha, \beta]$ and C^1 as a function on each (r_i, r_{i+1}) $(i = 0, 1, \ldots, k-1)$.*

(b) *$v(x) = \frac{1}{b(x)} \frac{d}{dx}(a\xi)(x)$ is constant on each interval (r_i, r_{i+1}) $(i = 0, 1, \ldots, k-1)$ and $v(x)$ is nondecreasing for $\delta_1 = 1$ (nonincreasing for $\delta_1 = -1$) as a function of x in (α, β) outside the set of jump discontinuities $\Sigma \subset \{r_1, \ldots, r_{k-1}\}$ of v.*

(c) *$\xi(r_i) = \delta_1$ for $r_i \in \Sigma$ and $i = 1, \ldots, k-1$.*

(ii) (*One peak velocity profile*) *There is a unique function ξ on $[\alpha, \beta]$ fulfilling* (3.14), (3.15) *with $\delta_1 = -\delta_2 \in \{-1, 1\}$ and* (a) *such that the following properties* (d), (e) *hold.*

(d) *$v(x) = \frac{1}{b(x)} \frac{d}{dx}(a\xi)(x)$ is constant on each interval (r_i, r_{i+1}) $(i = 0, 1, \ldots, m-1)$ and $v(x)$ is nondecreasing for $\delta_1 = 1$ (nonincreasing for $\delta_1 = -1$) on $(\alpha, (r_j + r_{j+1})/2) \backslash \Sigma'$ and nonincreasing for $\delta_1 = 1$ (nondecreasing for $\delta_1 = -1$) on $((r_j + r_{j+1})/2, \beta) \backslash \Sigma'$ for some $j \in \{0, 1, \ldots, k-1\}$, where Σ' is the set of jump discontinuities of v.*

(e) *$\xi(r_i) = \delta_1$ for $i = 0, 1, \ldots, j$ and $\xi(r_i) = \delta_2$ for $i = j+1, \ldots, k$ provided that $r_i \in \Sigma'$.*

Remark 5. If ξ is constructed for $\delta_1 = \delta_2 = 1$ (resp. $\delta_1 = -\delta_2 = 1$) then $-\xi$ is the desired ξ for $\delta_1 = \delta_2 = -1$ (resp. $\delta_1 = -\delta_2 = -1$).

Remark 6. To prove Lemma 6 (as well as Lemmas 3 and 4) we may assume that $b \equiv 1$ by changing coordinate and replacing a by a_b as defined in Remark 1.

Remark 7. To prove Lemma 6 it is helpful to give an elementary interpretation of the condition $|\xi| \leq 1$. Assume that $(a\xi)_x =: p$ is a constant on $(\alpha_1, \beta_1) \subset [\alpha, \beta]$. Then $|\xi| \leq 1$ on $[\alpha_1, \beta_1]$ if and only if $|\eta| \leq a$ on $[\alpha_1, \beta_1]$, where η is affine with slope p and $\eta(a_1) = (a\xi)(\alpha)$, $\eta(\beta_1) = (a\xi)(\beta_1)$, i.e., $\eta(x) = p(x - \alpha_1) + (a\xi)(\alpha_1)$.

Proof. (i) By Remark 5 we may assume $\delta_1 = \delta_2 = 1$. By Remark 6 we may assume that $b \equiv 1$. Let ζ be the convex hull of a in $[\alpha, \beta]$. By (C_i) $(i = 0, 1, \ldots, k-1)$, ζ is a piecewise linear convex function whose jumps of derivatives are contained in $\{r_i\}_{i=1}^{k-1}$. If we take $\xi = \zeta/a$, then ξ satisfies all desired properties. Indeed, $v = d\zeta/dx$ is piecewise constant and nondecreasing since ζ is convex. Also $|\xi| \leq 1$ is fulfilled (by Remark 7), since $0 \leq \zeta \leq a$. The set Σ should be the set of jump discontinuities of $d\zeta/dx$. For $x \in \Sigma$ and $x = \alpha$, $x = \beta$ we see $\zeta(x) = a(x)$ so that $\xi(x) = 1$.

To see uniqueness for given ξ satisfying (3.14), (3.15) and (a)–(c) we set $\tilde{\zeta} = a\xi$. By (b), $\tilde{\zeta}$ must be convex and piecewise linear. For $x \in \Sigma$ and $x = \alpha, \beta$ we see $\tilde{\zeta}(x) = a(x)$, by (c) and (3.15). By (3.14) $\tilde{\zeta} \leq a$ on $[\alpha, \beta]$. Thus $\tilde{\zeta}$ must be the convex hull of a so the uniqueness of ξ has been proved.

(ii) We may assume that $\delta_1 = -\delta_2 = 1$ and $b \equiv 1$. Let ζ^+ be the convex hull of a in $[\alpha, \beta]$ and set $\zeta^- = -\zeta^+$. We would like to construct a new piecewise linear function ζ such that

1° $\zeta^- \leq \zeta \leq \zeta^+$ on $[\alpha, \beta]$,

2° $\zeta = \zeta^+$ on $[\alpha, r_\ell]$ and $\zeta = \zeta^-$ on $[r_\sigma, \beta]$ for some ℓ, σ satisfying $0 \leq \ell < \sigma \leq k$,

3° ζ is linear (affine) in $[r_\ell, r_\sigma]$ and its slope $V = d\zeta/dx$ fulfills $V \geq v_+(x)$ on $[\alpha, r_\ell]$ and $V \geq v_-(x)$ on $[r_\sigma, \beta]$ where $v_\pm = d\zeta^\pm/dx$.

Such a function ζ is easy to construct. Indeed, we set

$$v_*(x) = \min(v_+(x), v_-(x)) \quad \text{and} \quad \lambda = \min_{[\alpha,\beta]} \zeta^+ \, (> 0)$$

and find that there is a unique negative number V fulfilling

$$\int_\alpha^\beta \max\{v_*(x) - V, 0\}\, dx = 2\lambda.$$

If we set

$$\zeta(x) = \int_{\alpha}^{x} \min(v_*(z), V)\, dz + \zeta^+(\alpha),$$

then ζ satisfies 1°, 2°, 3° by setting

$$r_\ell = \max\{r_i\ ;\ v_+(r_i - 0) \le V\}, \quad r_\sigma = \min\{r_i\ ;\ v_-(r_i + 0) \le V\}.$$

For notational convenience we set $v_+(r_0 - 0) = -\infty$ and $v_-(r_k + 0) = -\infty$. (The choice of V is important so that $\zeta(r_\sigma) = \zeta^-(r_\sigma)$.) We do not present the proof of 1°, 2°, 3° since it is elementary. Instead, we present an example of the graph of v_* and the value V in Figure 3 (a) and the graph of $\zeta^\pm$ and ζ when $\max_x(\min(v_+(x), 0)) > 0 > \max_x(\min(v_-(x), 0))$ in Figure 3 (b).

We set $\xi = \zeta/a$ and observe that ξ satisfies all desired properties. For example, by Remark 7 $|\xi| \le 1$ is fulfilled since $\zeta^- \le \zeta \le \zeta^+$ implies $|\zeta| \le a$.

For given ξ satisfying (3.14), (3.15) and (d), (e) it is easy to see that $a\xi = \zeta$ fulfills 1°, 2°, 3°. Since ζ satisfying 1°, 2°, 3° is unique, so is ξ. Q.E.D.

There is a boundary version corresponding to Lemma 6 (i) but we do not state it explicitly. By Lemma 1 and Lemma 6 together its boundary version one is able to find an element $f \in \partial\varphi(u_0)$, (which is piecewise constant) such that $u(x,t) = -f(x)t + u_0(x)$ is a solution of (3.7) with $u|_{\partial\Omega} = u_0|_{\partial\Omega}$ for small $t > 0$. By the uniqueness of a solution Theorem on persistency is generalized as follows.

General theorem on persistency. *Assume the same hypotheses of Theorem on persistency concerning a and b. Let u be the solution of (3.1) with initial data u_0 given by (3.7) with $u|_{\partial\Omega} = u_0|_{\partial\Omega}$. Then for each $i = 0, 1, \ldots, m-1$ the speed $u_t(x,t)$ is independent of $x \in (x_i, x_{i+1})$ and $t \in (0, t_0)$, where t_0 is the first time that some plateau of $u(\,\cdot\,, t)$ merges to another one. Moreover, $u_t(x,t) \equiv 0$ in (x_0, x_1) and (x_{m-1}, x_m) for $0 < t < t_0$.*

Note that some plateau $(x_i, x_{i+k}) \times \{h_i^0\}$, $k \ge 2$ at $t = 0$ may break instantaneously or stay as a plateau. By the above theorem we are able to calculate solution u explicitly by calculating the speed based on Lemma 6 until the time t_0 when some plateau merges. We calculate new speed for $u(x, t_0)$ and use above theorem to find explicit value u until the time t_1 when another plateau merges. We repeat this procedure to calculate u globally in time. Because of monotone nature of velocity profile in Lemma 6 we see that the number of peaks of u does not

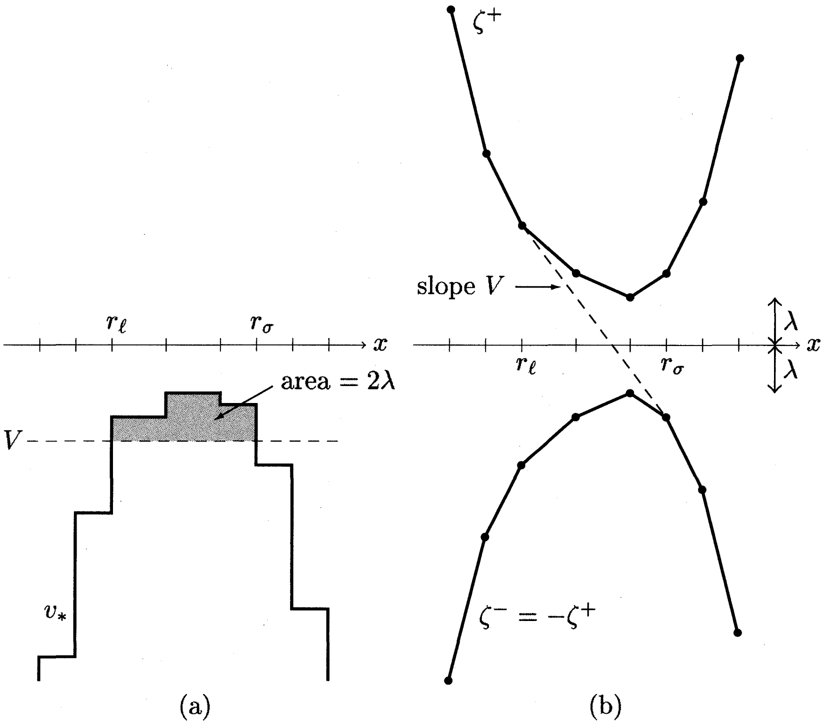

Fig. 3. (a) The graph of v_* and the value V. (b) The graph of $\zeta^\pm$. The dotted line indicates a part of the graph of ζ.

increase. For a piecewise constant function u_0 given by (3.7) with $h_i^0 \neq h_{i-1}^0$ $(i = 1, \ldots, m-1)$ let $M(u_0)$ (resp. $\mu(u_0)$) denote number of local maximum (resp. minimum) of function $i \mapsto h_i^0$ defined in $\{0, \ldots, m-1\}$. (For such a function $i_0 \in \{1, \ldots, m-2\}$ is called a local maximum (resp. minimum) if $h_{i_0 \pm 1}^0 < h_{i_0}^0$ (resp. $h_{i_0 \pm 1} > h_{i_0}$).) If $m = 2$, we set $M(u_0) = \mu(u_0) = 0$. If $h_i^0 = h_{i-1}^0$ for some i we renumber x_i's to identify u_0 of form (3.7) with $h_i^0 \neq h_{i-1}^0$ $(i = 1, \ldots, m'-1)$ for some $m' < m$. For such identification we define $M(u_0)$ and $\mu(u_0)$.

Theorem on nonincrease of peaks. *Under the same hypotheses and notations of General Theorem on persistency we have*

$$M(u(\,\cdot\,, t)) \text{ and } \mu(u(\,\cdot\,, t)) \text{ are constant on } (0, t_0).$$

Moreover, for all $t \geq 0$ *we have*

$$M(u(\cdot, t)) \leq M(u_0), \quad \mu(u(\cdot, t)) \leq \mu(u_0).$$

Remark 8. If $b \equiv 1$, it is enough to consider convexification of a on $[x_i, x_{i+k}]$ (instead of original a) to see whether a plateau $(x_i, x_{i+k}) \times \{h_i^0\}$ $(k \geq 2)$ is forced to break instanteneously when $h_{i-1}^0 < h_i^0 < h_{i+k+1}^0$ or $h_{i-1}^0 > h_i^0 > h_{i+k+1}^0$. However, if $b \not\equiv 1$ a simple convexification does not give a right answer. We have to convexify a_b instead of a (cf. Remark 1).

In the following, several numerical simulations will be demonstrated which support our theoretical results. For these calculations, we approximate (3.1) by $u_t = b^{-1}(a\chi_\gamma(u_x)u_x)_x$, $\chi_\gamma(p) = (\tanh \gamma p)/p$ for large γ and adopt the numerical scheme introduced in [KG]. This approximation is justified by the stability theorem. Note that our numerical scheme does not assume any persistency properties of plateaus. Let us give the graph of $a(x)$ by connecting $(0,1)$, $(1/6, a_1)$, $(2/6, 1)$, $(3/6, a_2)$, $(4/6, 1)$, $(5/6, a_3)$ and $(1,1)$ in this order as shown in Figure 4 (a). Here a_1, a_2 and a_3 are assumed to satisfy the relation $0 < a_1 < a_2 < a_3 < 1$. Related to the profile of $a(x)$, the sequence $\{x_i\}$ is given as follows; $x_0 = 0$, $x_1 = 1/6$, $x_2 = 3/6$, $x_3 = 5/6$ and $x_4 = 1$. We take the initial data with $h_1(0) = h_2(0)$ as shown in Figure 4 (b), and the global minimizer of φ defined by (2.5) is indicated in Figure 4 (c). The problem is whether the plateau $(x_1, x_3) \times \{h_1(0)\}$ is broken or not during the transition from the initial state to the final.

We present two simulations in Figure 5 by taking $b(x) \equiv 1$. If a_i's are selected so that the point (x_2, a_2) locates above the line segment connecting (x_1, a_1) and (x_3, a_3), the plateau is kept unbroken as indicated in Figure 5 (c). If it locates below, the plateau splits into the two plateaus as shown in Figure 5 (d). These results assures the validity of the convexity check stated in Remark 8.

However, the convexity check does not always give the right answer if $b(x)$ is not constant. For example, define $a(x)$ by connecting $(0,1)$, $(1/6, 0.2)$, $(2/6, 1)$, $(3/6, 0.401)$, $(4/6, 0.6)$, $(5/6, 0.6)$ and $(1,1)$ (Figure 6 (a) and (b)) and take the same initial data as the one in the previous simulation. According to the convexity check, the plateau should not split as long as $b(x) \equiv 1$ is adopted as is shown in Figure 6 (c). On the other hand, the condition (C_i) is violated on the interval (x_1, x_3) for $x = x_2$ if $b(x) \equiv a(x)$ is assumed. Therefore the plateau must be broken, which is also confirmed in Figure 6 (d).

We present one more example with $b(x) \equiv 1$ and $a(x)$ given in Figure 7 (a), and see what will happen to the big plateau (consists of

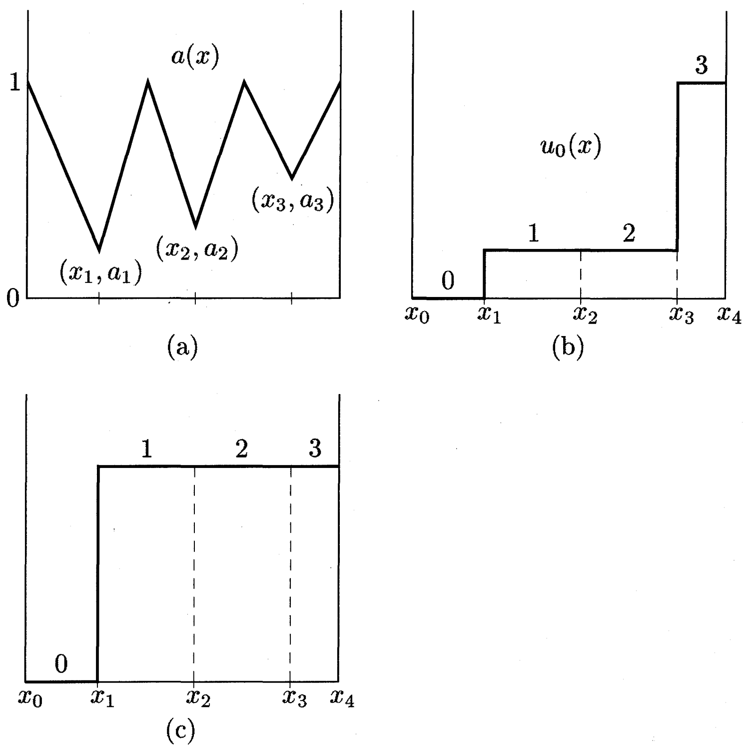

Fig. 4. (a) Piecewise linear profile of $a(x)$ with the three local minima. (b) Initial data $u_0(x)$ given by $h_0(0) = 0.0$, $h_1(0) = h_2(0) = 0.2$ and $h_3(0) = 1.0$. (c) The final state (global minimizer). Small numbers on each segment indicate the index i of $h_i(t)$.

seven segments) of the initial data (Figure 7 (b)). Graphical checking tells us that the plateau will be broken into four pieces, and it is actually observed in the simulation as shown in Figure 7 (c) and (d).

§4. Formation of jumps

We consider (3.1) with $b \equiv 1$, i.e.,

$$u_t = \partial_x(a(x)u_x/|u_x|) \quad \text{on } \Omega \times (0, \infty) \tag{4.1}$$

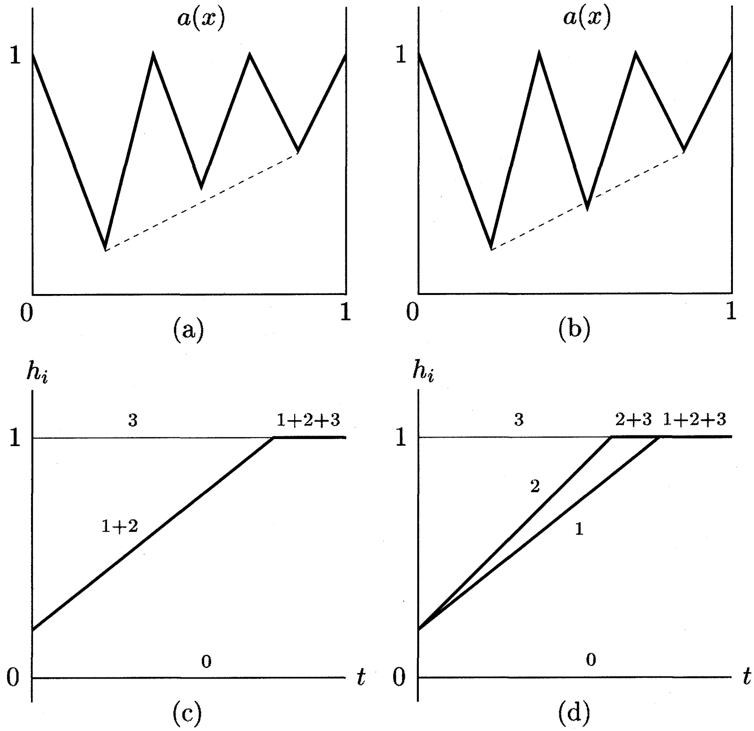

Fig. 5. (a) The function $a(x)$ is given by $a_1 = 0.2$, $a_2 = 0.42$ and $a_3 = 0.6$, and $b(x) \equiv 1$. (b) The function $a(x)$ is given by $a_1 = 0.2$, $a_2 = 0.38$ and $a_3 = 0.6$, and $b(x) \equiv 1$. (c) Time evolution of all the segments for $a(x)$ given in (a) and $b(x) \equiv 1$. (d) Time evolution of all the segments for $a(x)$ given in (b) and $b(x) \equiv 1$.

with initial data u_0 and the boundary condition $u|_{\partial\Omega} = u_0|_{\partial\Omega}$, where Ω is a bounded open interval. The regularity property of solutions of (4.1) is different from that for $u_t = \partial_x(a(x)u_x)$. For example, for the latter equation if $a(x)$ is Hölder continuous on $\overline{\Omega}$ (and $a > 0$ on $\overline{\Omega}$) then solution u is C^2 in x and C^1 in time. For the problem (4.1) a jump discontinuity of solutions may be formed instantaneously. We give such an example and discuss other properties of solutions.

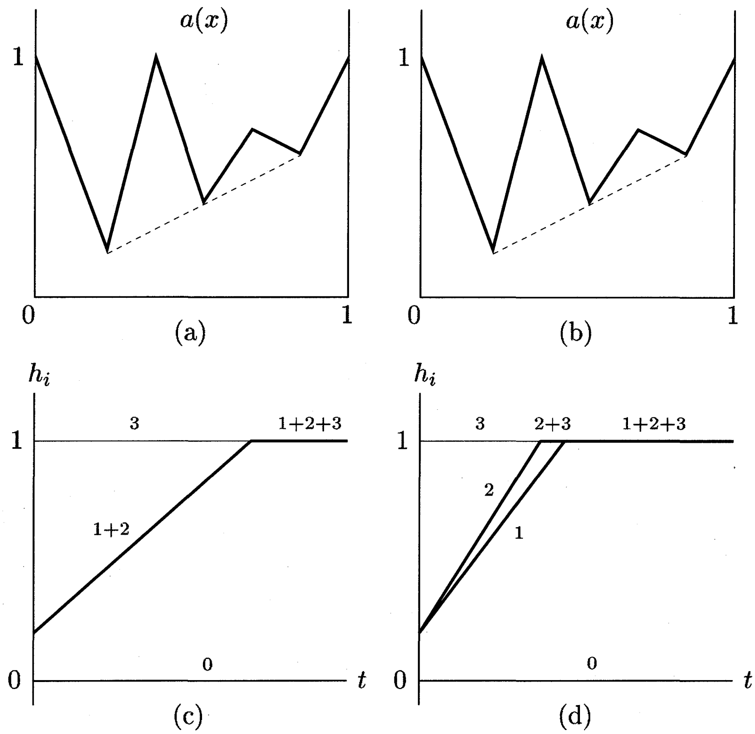

Fig. 6. (a) The point (x_2, a_2) is slightly above the line segment connecting (x_1, a_1) and (x_3, a_3) although it is hard to judge by the figure. (b) Exactly the same graph with (a). (c) Time evolution of all the segments for $a(x)$ given in (a) and $b(x) \equiv 1$. (d) Time evolution of all the segments for $a(x)$ given in (b) and $b(x) \equiv a(x)$.

Example 4.1. Assume that $a(x) = a_0 + |x|$, $x \in \Omega = (-1, 1)$, $a_0 > 0$. If $u_0(x) = x$, then the solution u of the initial-boundary value problem for (4.1) is of form

$$u(x,t) = \begin{cases} \min\{1, x+t\}, & x > 0, \\ \max\{-1, x-t\}, & x < 0. \end{cases}$$

Evidently, $u(x,t)$ has a jump at $x = 0$ for $t > 0$. Moreover, $u(x,t)$ becomes the global minimizer sgn x of energy φ at $t = 1$ and $u(x,t) =$

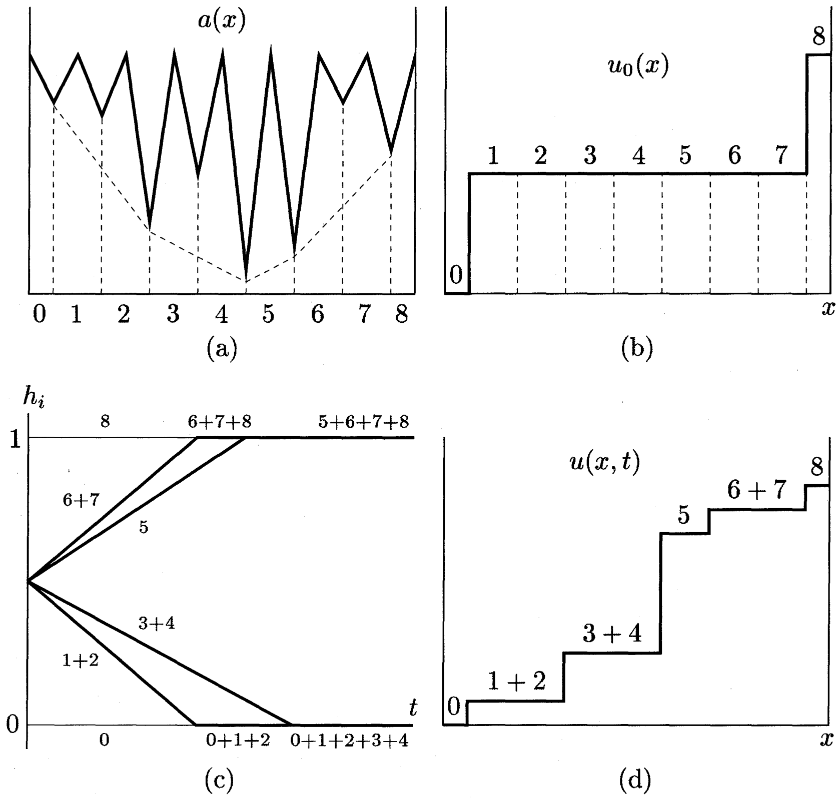

Fig. 7. (a) The sequence $\{x_i\}$ is given by $x_0 = 0$, $x_i = (i - 1/2)/8$ $(i = 1, \dots, 8)$, $x_9 = 1$ and a_i's are appropriately chosen. (b) Initial data given by $h_0(0) = 0$, $h_1(0) = \cdots = h_7(0) = 1/2$, $h_8(0) = 1$. (c) Time evolution of all the segments for $a(x)$ given in (a) and $b(x) \equiv 1$. (d) Snapshot of $u(x,t)$ for $t = 0.21$.

$\operatorname{sgn} x$ for $t \geq 1$. For general initial data u_0 the convergence property to the global minimizer for given initial data holds if $u_0(-1) \neq u_0(+1)$ as stated below.

Theorem. *Assume that* $a(x) = a_0 + |x|$, $x \in \Omega = (-1,1)$, $a_0 > 0$. *Assume that* $u_0 \in BV(\Omega)$ *with* $u_0(-1) \neq u_0(+1)$. *Then the solution* u *of the initial-boundary value problem for* (4.1) *with* $u|_{t=0} = u_0$ *and*

$u|_{\partial\Omega} = u_0|_{\partial\Omega}$ *converges to*

$$U(x) = \begin{cases} u_0(+1), & x > 0, \\ u_0(-1), & x < 0 \end{cases}$$

in a finite time.

Proof. By symmetry we may assume that $u_0(-1) < u_0(+1)$. We may assume that $u_0(-1) = -1$, $u_0(+1) = 1$ so that $U(x) = \operatorname{sgn} x$ by adding and multiplying a constant to u. By assumption u_0 is bounded, i.e., there is a positive constant $M > 1$ such that $|u_0(x)| \le M$ for all $x \in \Omega$. We note that

$$v(x,t) = \begin{cases} \max\{M - (1+a_0)t, 1\}, & 0 < x < 1, \\ \max\{M - (1+a_0)t + a_0\big(t - \frac{M-1}{1+a_0}\big)_+, -1\}, & -1 < x < 0 \end{cases}$$

is the solution of the initial-boundary value problem for (3.1) with the boundary condition $u(\pm 1, t) = \pm 1$ and initial condition $u|_{t=0} = M$. This can be proved as in §3.2 if we pay attention that v has a jump at $x = \pm 1$. (The solution v is a typical example that the initial data is incompatible with the boundary condition.) Thus we see $u(x,t) \le U(x)$ in a finite time by the comparison principle. Here $p_+ = \max(p, 0)$. A symmetric argument implies $u(x,t) \ge U(x)$ in a finite time. Q.E.D.

Example 4.2. Assume that $a(x) = a(-x) > 0$, $x \in (-1,1)$ and that a is Lipschitz continuous. Assume that a is C^1 and nondecreasing in x for $x > 0$. If $u_0(x) = x$, then the solution u of the initial-boundary value problem for (4.1) is of the form

$$u(x,t) = \begin{cases} \min\{1, x + \big(\frac{da}{dx}(x)\big)t\}, & x > 0, \\ \max\{-1, x + \big(\frac{da}{dx}(x)\big)t\}, & x < 0 \end{cases}$$

which generalizes Example 4.1. If a is C^1 at the origin, $u(x,t)$ stays continuous for $t \ge 0$ since $da/dx \to 0$ as $x \to 0$. It tends to $U(x)$ in the preceeding theorem as $t \to \infty$, but $u(x,t) \not\equiv U(x)$ for any finite t. It is possible to prove a convergence result to $U(x)$ as $t \to \infty$ for a general initial data with $u_0(-1) \ne u_0(+1)$ under suitable assumptions on a, however, we do not state it here. Instead, we present several numerical calculations.

The simulation of Example 4.1 is indicated in Figure 8, and Figure 9 corresponds to Example 4.2 with $a(x) = a_0 + x^2$. In both cases, the numerical solutions approximate the exact solutions given above quite well.

Note added in the proof. In a recent preprint "The Dirichlet problem for the total variation flow" by F. Andreu, C. Ballester, V. Caselles and J. M. Mazón L^1 framework for (2.7) with $a \equiv b \equiv 1$ is established instead of L^2 framework given in this paper.

The authors are grateful to Professor Naoyuki Ishimura for valuable comments.

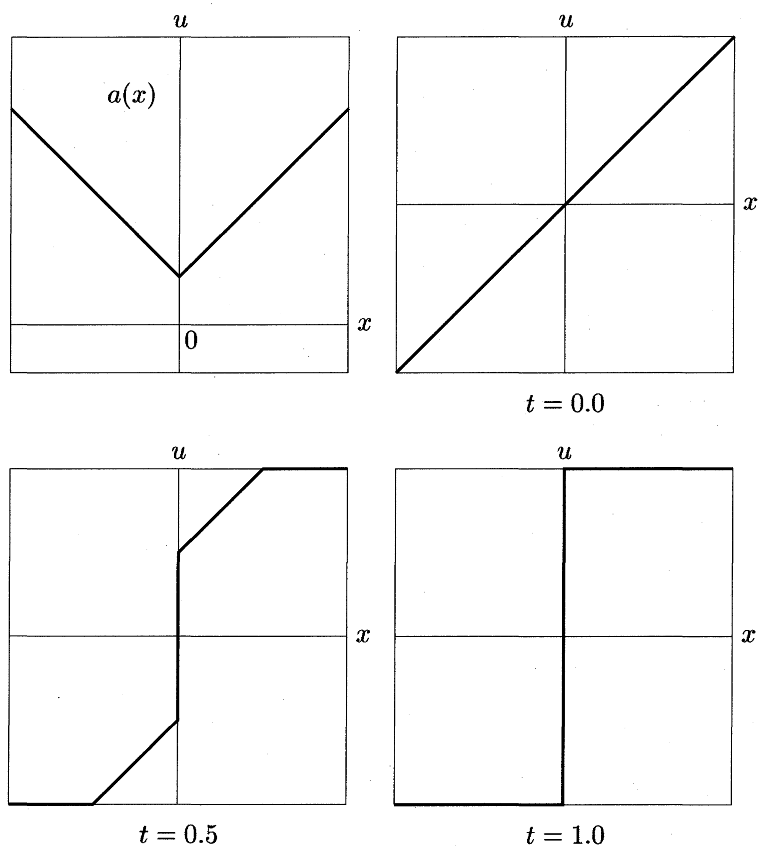

Fig. 8. Simulation of Example 4.1 with $a(x) = 0.2 + |x|$, $b(x) \equiv 1$ and $u_0(x) = x$. Discontinuity appears instantaneously and the solution reaches to the final state in a finite time.

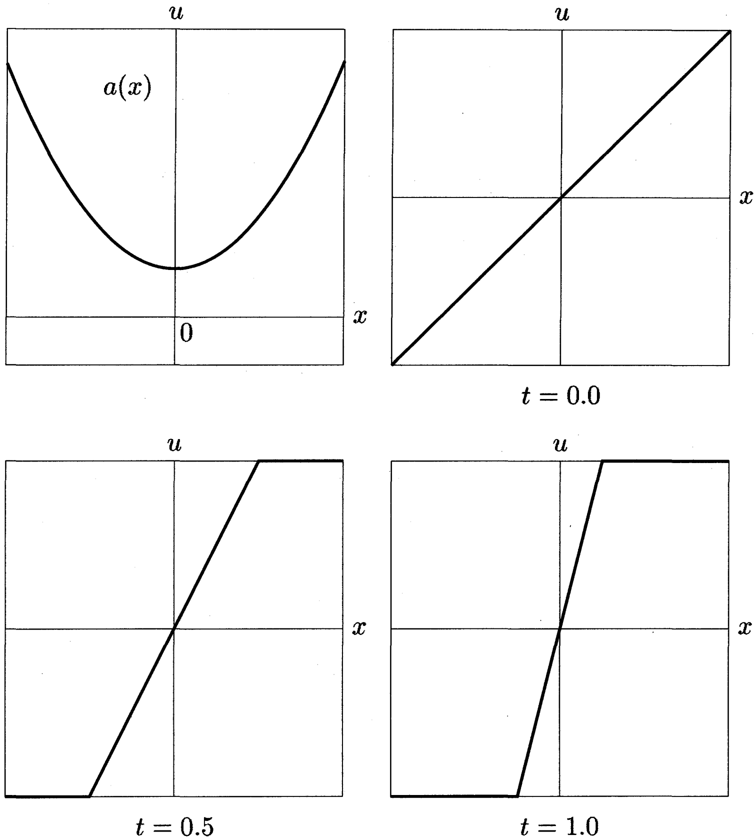

Fig. 9. Simulation of Example 4.2 with $a(x) = 0.2 + x^2$, $b(x) \equiv 1$ and $u_0(x) = x$. Discontinuity never appears and the solution converges to the final state while it doesn't reach the final state in a finite time.

References

[A] S. B. Angenent, The zeroset of a solution of a parabolic equation, J. Reine Angew. Math., **390** (1988), 79–96.

[AG] S. B. Angenent and M. E. Gurtin, Multiphase thermomechanics with interfacial structure 2. Evolution of an isothermal interface, Arch. Rational Mech. Anal., **108** (1989), 323–391.

[AD] H. Attouch and A. Damlamian, Application des methods de convexité et monotonie a l'etude de certaines equations quasi lineares, Proc. Roy. Soc. Edinburgh, **79A** (1977), 107–129.

[Ba] V. Barbu, "Nonlinear Semigroups and Differential Equations in Banach Space", Noordhoff, Groningen, 1976.

[BP] H. Brezis and A. Pazy, Convergence and approximation of semigroups of nonlinear operators in Banach spaces, J. Funct. Anal., **9** (1972), 63–74.

[ESG] C. M. Elliott, A. R. Gardiner and R. Schätzle, Crystalline curvature flow of a graph in a variational setting, Adv. in Math. Sci, Appl., **8** (1998), 425–460.

[CGG] Y.-G. Chen, Y. Giga and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, J. Diff. Geom., **33** (1991), 749–786.

[D] A. Damlamian, "Non Linear Evolution Equations with Variable Norms", Thesis, Harvard University, 1974.

[ES] L. C. Evans and J. Spruck, Motion of level sets by mean curvature I, J. Diff. Geom., **33** (1991), 635–681.

[FG] T. Fukui and Y. Giga, Motion of a graph by nonsmooth weighted curvature, in "World Congress of Nonlinear Analysis '92", (V. Lakshmikantham, ed.), de Gruyter, Berlin, vol I, 1996, pp. 47–56.

[G] Y. Giga, A level set method for surface evolution equations, Sugaku Expositions, **10** (1997), 217–241.

[GG1] M.-H. Giga and Y. Giga, Evolving graphs by singular weighted curvature, Arch. Rational Mech. Anal., **141** (1998), 117–198.

[GG2] M.-H. Giga and Y. Giga, A subdifferential interpretation of crystalline motion under nonuniform driving force, Proc. of the International Conference on Dynamical Systems and Differential Equations, Springfield, Missouri (1996); in "Dynamical Systems and Differential Equations" (W.-X. Chen and S.-C. Hu, eds.), Southwest Missouri State Univ. vol. **1**, 1998, pp. 276–287.

[GG3] M.-H. Giga and Y. Giga, Stability for evolving graphs by nonlocal weighted curvature, Commun. in Partial Differential Equations, **24** (1999), 109–184.

[GG4] M.-H. Giga and Y. Giga, Motion driven by nonlocal curvature in the plane, in "Asymptotics in Nonlinear Diffusive Systems" (Nishiura et al, eds.), Tohoku Math. Publ., **8** (1998), pp. 75–83.

[Giu] E. Giusti, "Minimal Surfaces and Functions of Bounded Variations", Birkhauser, Boston, 1984.

[HZ] R. Hardt and X. Zhou, An evolution problem for linear growth functionals, Commun. in Partial Differential Equations, **19** (1994), 1879–1907.

[KG] R. Kobayashi and Y. Giga, Equations with singular diffusivity, J. Stat. Phys., **95** (1999), 1187–1220.

[KWC] R. Kobayashi, A. Warren and Carter, Modeling grain boundaries using a phase field technique, Hokkaido Univ. Preprint Series in Math., **422** (1998).

[Ko] Y. Kōmura, Nonlinear semi-groups in Hilbert space, J. Math. Soc. Japan, **19** (1967), 493–507.

[M] H. Matano, Nonincrease of the lapnumber for a solution of a one-dimensional semilinear parabolic equation, J. Fac. Sci. Univ. Tokyo, Sect. IA **29** (1982), 401–441.

[T] J. Taylor, Constructions and conjectures in crystalline curvature, including triple junctions and boundary points, in "Differential Geometry: Partial Differential Equations on Manifolds", (R. Greene and S. T. Yau, eds.), Proc. Symp. Pure Math., **54**, Part I, 1993, pp. 417–438. Amer. Math. Society, Providence.

[Te] R. Temam, Solutions généralisées de certain équations du type hypersurfaces minimals, Arch. Rational Mech. Anal., **44** (1991), 121–156.

[V] A. Visintin, "Models of Phase Transitions", Birkhäuser, 1996.

[W] J. Watanabe, Approximation of nonlinear problems of a certain type in "Numerical Analysis of Evolution Equations", (H. Fujita and M. Yamaguti, eds.), Lecture Notes in Num. Appl., **1**, Kinokuniya, Tokyo, 1979, pp. 147–163.

Mi-Ho Giga and Yoshikazu Giga
Department of Mathematics
Hokkaido University
Sapporo, 060-0810
Japan

Ryo Kobayashi
Research Institute for Electronic Science
Hokkaido University
Sapporo, 060-0810
Japan