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Hardy Spaces, Carleson Measures and a Gradient Estimate for Harmonic Functions on Negatively Curved Manifolds

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Abstract.

In this paper we study Hardy spaces, BMO, Carleson measures, Green potential and Bloch functions on a Cartan-Hadamard manifold M of pinched negative curvature. Further, using our results on Carleson measure and BMO, we give a gradient estimate for harmonic functions on M. It is different from Yau's gradient estimates, and is applied to the existence problem of harmonic Bloch functions described in §10. We deal also with boundary behavior of harmonic Bloch functions on M.

§1. Introduction

In their paper [22], C. Fefferman and E. Stein developed the theory of Hardy spaces of harmonic functions on the upper-half spaces endowed with the Euclidean metric. As is well known, their theory have played crucial roles in the classical harmonic analysis. In 1981 and 1982, D. Jerison, E. Fabes, C. Kenig and U. Neri extended some important parts of the theory of Hardy spaces on the upper-half spaces to more general Euclidean domains with non-smooth boundaries ([21], [20] and [26]).

On the other hand, after the work of A. Korányi on the boundary behavior of harmonic functions on symmetric spaces ([28]), Hardy spaces have been investigated also for symmetric spaces. In particular, D. Geller ([23]) and A. Debiard ([16], [17]) studied Hardy spaces on Siegel upper-half spaces of type II, and in somewhat later, P. Cifuentes extended the classical theorems on the probabilistic characterization and area integral characterization of Hardy spaces to rank one symmetric spaces of noncompact type ([12] and [13]).

Now in this paper we study Hardy spaces, BMO and Carleson measures on a complete, simply connected n-dimensional Riemannian manifold (M, g) such that the sectional curvatures K_M of M satisfy $-\kappa_2^2 \leq$

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$K_M \leq -\kappa_1^2$, for some constants κ_1 and κ_2 with $0 < \kappa_1 \leq \kappa_2 < +\infty$. A typical example of such a manifold is a rank one symmetric space of noncompact type, but also many other examples are known.

For the manifold M, the boundary behavior of harmonic functions has been studied by several authors: For instance, the solvability of the Dirichlet problem for the manifold M was proved by Anderson [2] and D. Sullivan [46], and moreover in Anderson and Schoen [3] it was proved that the Eberlein-O'Neill compactification of M is homeomorphic to the Martin compactification (see also Kifer [27], Ancona [1]). Then Anderson and Schoen [3], Ancona [1], Arai [5], Mouton [38], and Cifuentes and Korányi [14] studied boundary behavior of harmonic functions on M.

This paper consists of two parts. First part is concerned with the Hardy spaces of harmonic functions on M, and the second part with Carleson measures. Then we will give an application to Bloch functions on M.

We will begin in §2 with a quick review of some preliminaries about harmonic functions on M. In §3 we will define analogues to the manifold M of the classical Stoltz domain and of the classical Hardy spaces of harmonic functions. Section 4 contains a review real analysis on the sphere at infinity. In §5 we prove some elementary properties of Hardy spaces H^p, and then in §6 we prove that the Hardy space H^1, atomic Hardy space H^1_{atom} and probabilistic Hardy space H^1_{prob} are mutually equivalent. Some results in this section were announced already in our paper [5], but in the announcement we assumed an additional geometric condition in order to show that every $(1,\infty)$-atom is in H^1. In the present paper we call it the condition (β). As pointed out in [5], there are some examples of manifolds having the condition (β). However, recently Cifuentes and Korányi [14] proved that M possesses always the condition. Therefore combining theorems announced in [5] with their result, we gain the equivalence of the three different definitions of Hardy spaces.

In the second part of this paper we study Carleson measure and its application to Bloch functions on M. In §7 we are concerned with relationship between Carleson measures and L^p boundedness of the Martin integral, and in §8 we give a characterization of Carleson measure in terms of a certain Green potential. Using it, in §9 we prove Carleson measure characterization of BMO functions. In the classical Euclidean case, this characterization was found by C. Fefferman and E. Stein ([22]), and in the case of the complex unit ball endowed with the Bergman metric, the characterization was proved by Jevtic [25]. However, his proof is based on the nature of the ball. Our proof is different from it. In §10 we will study harmonic Bloch functions defined on M. From our

Carleson measure characterization of BMO functions, we give a gradient estimate for harmonic functions on M (see (45)), and prove existence of unbounded harmonic Bloch function on M. Moreover in §11 we study boundary behavior of unbounded harmonic Bloch functions, and prove a generalization of Lyons' theorem on the law of iterated logarithm.

Notation: In this paper we fix a point o in M as a reference point. The constants depending only on g, n, κ_1, κ_2 and o will usually be denoted by C, C' or C_j $(j = 1, 2, \ldots)$. But C and C' may change in value from one occurrence to the next, while constants C_j $(j = 1, 2, \ldots)$ retain a fixed value. For two nonnegative functions f and g defined on a set U, the notation $f \lessapprox g$ indicate that $f(x) \leq Cg(x)$ for all $x \in U$, and $f \approx g$ means that $f \lessapprox g$ and $g \lessapprox f$.

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§2. Preliminaries

In this section, we review Harnack type inequalities for positive harmonic functions and some facts about the Martin compactification. Both are very important for us.

A C^2 function f in an open set U of M is called harmonic in U if $\Delta_M f = 0$ in U, where Δ_M is the Laplace-Beltrami operator of (M, g). For $x \in M$ and $r > 0$, let $B(x, r) := \{y \in M : d(x, y) < r\}$, where $d(\ ,\ )$ is the distance function with respect to the Riemannian metric g. Then Moser's Harnack inequality implies

Theorem H (Interior Harnack inequality). *Let $R > 0$. Then for every positive harmonic function u on a ball $B(x, 2R)$,*

$$C_{1,R}^{-1} u(y) \leq u(x) \leq C_{1,R} u(y), \tag{1}$$

for all $y \in B(x, R)$, where $C_{1,R}$ is a positive constant depending only on M and R.

In this paper we will use the so-called boundary Harnack inequalities. They were proved firstly by Anderson and Schoen [3] and then also by Ancona [1]. To describe them we need some notation. Denote by $S(\infty)$ the sphere at infinity of M and by $\overline{M}$ the Eberlein and O'Neill compactification $M \cup S(\infty)$ of M (see [19] for definitions). For $x \in M$ and $y \in \overline{M}$ $(x \neq y)$, let γ_{xy} be the unit speed geodesic such that

$\gamma_{xy}(0) = x$ and $\gamma_{xy}(t) = y$ for some $t \in (0, +\infty]$. Since such a number t is unique, we denote it by t_{xy}.

For $p \in M$, $v \in T_pM$ and $\delta > 0$, let $C(p, v, \delta)$ be the cone about the tangent vector v of angle δ defined by

$$C(p, v, \delta) := \{x \in \overline{M} \setminus \{p\} : \angle_p(v, \dot{\gamma}_{px}(0)) < \delta\},$$

where $\angle_p$ denotes the angle in T_pM and $\dot{\gamma}_{px}(t)$ is its tangent vector at t.

The following is called boundary Harnack inequality:

Theorem BH1 (Anderson and Schoen [3]: see also [1], [27]). *Let $p \in M$ and $v \in T_pM$ with $g_p(v, v) = 1$. Denote $C = C(p, v, \pi/4)$ and $T = C(p, v, \pi/8) \setminus B(p, 1)$. Let u and h be positive harmonic functions on $C \cap M$, continuous up to the closure $\overline{C}$ of C in $\overline{M}$ and vanishing on $\overline{C} \cap S(\infty)$. Then*

$$C_1 \exp\{-C_2 d(p, x)\} \leq \frac{u(x)}{u(p_0)} \leq C_3 \exp\{-C_4 d(p, x)\}, \tag{2}$$

$$C_5^{-1} \frac{u(p_0)}{h(p_0)} \leq \frac{u(x)}{h(x)} \leq C_6 \frac{u(p_0)}{h(p_0)}, \tag{3}$$

for $x \in T$, where $p_0 = \exp_p(v)$, and $C_1, \ldots, C_6$ are constants depending only on M.

For $z \in M \setminus \{o\}$ and $t \in \mathbf{R}$, we denote

$$C(z, t) = C(\gamma_{oz}(t_{oz} + t), \dot{\gamma}_{oz}(t_{oz} + t), \pi/4), \text{ and } z(t) = \gamma_{oz}(t_{oz} + t).$$

In this paper we will use the following variation of Theorem BH1:

Theorem BH2 (Ancona [1]). (1) *Let u and h be positive harmonic functions on a cone $C(z, t) \cap M$ and vanishing continuously on $C(z, t) \cap S(\infty)$. Then*

$$C_7^{-1} \frac{u(z(t+1))}{h(z(t+1))} \leq \frac{u(x)}{h(x)} \leq C_7 \frac{u(z(t+1))}{h(z(t+1))}, \quad \text{for all } x \in C(z, t+1) \cap M,$$

where C_7 is a positive constant depending only on M.

(2) *Let u and h be positive harmonic functions on $M \setminus C(z, t+1)$, and vanishing continuously on $(\overline{M} \setminus C(z, t+1)) \cap S(\infty)$. Then*

$$C_8^{-1} \frac{u(z(t))}{h(z(t))} \leq \frac{u(x)}{h(x)} \leq C_8 \frac{u(z(t))}{h(z(t))}, \quad \text{for all } x \in M \setminus C(z, t),$$

where C_8 is a positive constant depending only on M.

The second statement in Theorem BH2 seems to be a little different from (1), but it is actually a special case of Theorem 5′ in [1]. For reader's convenience, we give a proof of Theorem BH2 (2) in Appendix 1.

The following theorem is an important consequence of boundary Harnack inequalities:

Theorem AS (Anderson and Schoen [3]; [1], [27]). (1) *The Martin compactification of M with respect to the Laplacian Δ_M is homeomorphic to Eberlein and O'Neill's compactification $\overline{M}$, and the Martin boundary consists only of minimal points.*

(2) *For every $z \in M$, there exists a unique function $K_z(x, Q)$ ($Q \in S(\infty)$, $x \in \overline{M} \setminus \{Q\}$) such that for every $Q \in S(\infty)$,*

(4) $K_z(\,\cdot\,, Q)$ is positive harmonic on M,

(5) $K_z(\,\cdot\,, Q)$ is continuous on $\overline{M} \setminus \{Q\}$,

(6) $K_z(Q', Q) = 0$ for all $Q' \in S(\infty) \setminus \{Q\}$, and

(7) $K_z(z, Q) = 1$.

(*This function is called the Poisson kernel normalized at z.*)

(3) *For every $z \in M$ and for every positive harmonic function u on M, there exists a unique Borel measure m_u^z on $S(\infty)$ such that*

$$u(x) = \int_{S(\infty)} K_z(x, Q)\, dm_u^z(Q), \quad x \in M. \tag{8}$$

(*The measure m_u^z is called a Martin representing measure relative to u and z.*)

Throughout this paper, we denote $K(x, Q) = K_o(x, Q)$, and write simply ω^x the Martin representing measure relative to the constant function 1 and $x \in M$. It is called the harmonic measure relative to x. In particular, let $\omega = \omega^o$. Note that $\omega^x(S(\infty)) = 1$ and $d\omega^x(Q) = K(x, Q)\, d\omega(Q)$, for all $x \in M$.

Theorem H yields that for any compact sets $E \subset M$, there exists a positive constant c_E satisfying

$$c_E^{-1} \omega^x(A) \le \omega^y(A) \le c_E \omega^x(A) \tag{9}$$

for all $x, y \in E$ and for all Borel sets $A \subset S(\infty)$.

§3. Approach regions and Hardy spaces

In order to study Hardy space H^1 and BMO on M, we begin with recalling two analogues for M of the classical Stoltz region. First is the following:

Definition 3.1 (Anderson and Schoen [3]). For $Q \in S(\infty)$ and $d > 0$, let

$$T_d(Q) = \bigcup_{t>0} B(\gamma_{oQ}(t), d). \tag{10}$$

Following [3], we call such a set the nontangential region at $Q \in S(\infty)$.

In this paper we will be mainly concerned with another analogue of the Stoltz region in some technical reasons: For $x \in M$ and $t \in \mathbf{R}$, let

$$\Delta(x,t) = C(x,t) \cap S(\infty). \tag{11}$$

Definition 3.2 ([5]). For $Q \in S(\infty)$ and $\alpha \in \mathbf{R}$, let

$$\Gamma_\alpha(Q) = \{z \in M : Q \in \Delta(z,\alpha)\}, \tag{12}$$

and we call this set an admissible region at Q.

This definition is motivated by the following consideration about classical Stoltz domains in the upper half-plane $\mathbf{R}^2_+$: Recall that the Stoltz domain $S_\alpha(x)$ at $x \in \mathbf{R}$ with angle $\alpha > 0$ is the set $\{(y,t) \in \mathbf{R} \times (0,+\infty) : |y-x| < \alpha t\}$. For $z\,(= (z_0,t)) \in \mathbf{R}^2_+$, let $\mathcal{D}(z,\alpha) := \{y \in \mathbf{R} :$ the angle at z between the segment $\overline{zy}$ and $\overline{zz_0}$ is less than $\alpha\}$, where $\overline{pq}$ is the segment joining p and q for $p,q \in \overline{\mathbf{R}}^2_+$. Denote $S'_\alpha(x) = \{z \in \mathbf{R}^2_+ : x \in \mathcal{D}(z,\alpha)\}$. Then $S_\alpha(x) = S'_\alpha(x)$. Now we can easily see that our set $\Delta(z,\alpha)$ corresponds to the domain $\mathcal{D}(z,\alpha)$, and $\Gamma_\alpha(x)$ to the set $S'_\alpha(x)$.

We note that if we define Hardy spaces by using our admissible domains, we may apply "tent" method to our case as we will show later. By this reason, in this paper, we use the admissible regions instead of nontangential regions in the sense of [3]. However, it should be noted that both are closely related to each other:

Theorem CK1 (Cifuentes and Korányi [14]). *Two families of approach regions $\{T_d(Q)\}$ and $\{\Gamma_\alpha(Q)\}$ are equivalent in the sense of* [14], *that is, for all $\alpha \in \mathbf{R}$, there exists $d > 0$ and $R > 0$ such that for all $Q \in S(\infty)$,*

$$\Gamma_\alpha(Q) \cap (M \setminus B(o,R)) \subset T_d(Q) \cap (M \setminus B(o,R)), \tag{13}$$

and vice versa.

For the notational convenience, for every $f \in L^1(\omega)$ $(= L^1(S(\infty), \omega))$, let

$$\tilde{f}(x) = \begin{cases} \int_{S(\infty)} K(x,Q) f(Q)\, d\omega(Q), & x \in M, \\ f(x), & x \in S(\infty). \end{cases}$$

Then $\tilde{f}$ is harmonic on M. Now Hardy spaces $H^p(\omega, \alpha)$ $(1 \le p \le \infty$, $\alpha \in \mathbf{R})$ are defined as follows: For a function u on M, let

$$N_\alpha(u)(Q) := \sup_{x \in \Gamma_\alpha(Q)} |u(x)|, \quad Q \in S(\infty),$$

and let

$$H^p(\omega, \alpha) = \{ f \in L^p(\omega) : N_\alpha(\tilde{f}) \in L^p(\omega) \}.$$

Denote $\|f\|_{H^1(\omega,\alpha)} := \|N_\alpha(\tilde{f})\|_{L^p(\omega)}$.

Here we should note that when u is a continuous function on M, then $N_\alpha(u)$ is lower semicontinuous on $S(\infty)$. Indeed for every $\lambda > 0$, the set $E := \{Q \in S(\infty) : N_\alpha(u) > \lambda\}$ is open. For if $Q \in E$, then there exist $\varepsilon > 0$ and $z \in \Gamma_\alpha(Q)$ such that $|u(z)| > \lambda + \varepsilon$. By the definition of $\Gamma_\alpha(Q)$, we have $Q \in \Delta(z, \alpha)$. We can take an open subset U of $S(\infty)$ such that $Q \in U \subset \Delta(z, \alpha)$. Since $z \in \Gamma_\alpha(Q')$ for all $Q' \in U$, we have $U \subset E$. This implies that E is open.

As we will see in §5, these spaces $H^p(\omega, \alpha)$ and $H^p(\omega, \beta)$ are equivalent for every $\alpha, \beta \in \mathbf{R}$. We denote

$$H^p(\omega) = H^p(\omega, 0), \quad \text{and} \quad \|f\|_{H^p(\omega)} = \|f\|_{H^p(\omega,0)}.$$

Remark 1. As in the classical case, another Hardy spaces of harmonic functions on M are defined by

$$H^p(M) := \{u : u \text{ is harmonic on } M \text{ and } N_0(u) \in L^p(\omega)\}, \quad 1 \le p \le \infty.$$

See Appendix 3 for these Hardy spaces.

§4. Real analysis at infinity – Quick Review –

Before going to the main body of this paper, we set down the basic facts about real analysis on the sphere at infinity $S(\infty)$ of M. All theorems stated in this section follow immediately from results in [3] and, in particular, from the abstract theory of real analysis due to Coifman and Weiss [15]. For any $Q \in S(\infty)$ we define $\Delta_t(Q)$ to be the "ball" in $S(\infty)$ centered at Q,

$$\Delta_t(Q) := \Delta(\gamma_{oQ}(t), 0) \; (= C(\gamma_{oQ}(t), \dot{\gamma}_{oQ}(t), \pi/4) \cap S(\infty)),$$

when $t \geq 0$, and let $\Delta_t(Q) = S(\infty)$ when t is negative. Then we can see that the family of the sets $\{\Delta_t(Q)\}$ defines a quasi-distance ρ on $S(\infty)$ which makes the triple $(S(\infty), \rho, \omega)$ is a space of homogeneous type in the sense of Coifman and Weiss [15] as follows: By [3] the family of "balls" $\{\Delta_t(Q)\}$ satisfies the following properties:

(H1) For all $s > 0$ and $r > 0$

$$S(\infty) = \lim_{t \to -\infty} \Delta_t(Q) \supset \Delta_r(Q) \supset\supset \Delta_{r+s}(Q) \supset \lim_{t \to \infty} \Delta_t(Q) = \{Q\},$$

where $A \supset\supset B$ means that A contains the closure of B. Furthermore, $\{\Delta_r(Q) : r \in \mathbf{R}\}$ is a fundamental system of neighborhoods of Q.

(H2) Let $Q_1, Q_2 \in S(\infty)$ and $r \in \mathbf{R}$. If $\Delta_r(Q_1) \cap \Delta_r(Q_2) \neq \emptyset$, then $\Delta_{r-k}(Q_1) \supset \Delta_r(Q_2)$, where k is a positive integer depending only on the curvature bounds κ_1 and κ_2.

(H3) $0 < \omega(\Delta_r(Q)) \leq 1$ for every $Q \in S(\infty)$ and $r \in \mathbf{R}$.

(H4) For every $\Delta_r(Q)$ and $l > 0$, $\omega(\Delta_{r-l}(Q)) \leq C(l)\omega(\Delta_r(Q))$, where $C(l)$ is a positive constant depending only on M, o and l.

Without loss of generality, we may assume $k \geq 2$. Note that the function

$$\rho_0(Q, Q') := \inf\{e^{-t} : Q' \in \Delta_t(Q)\}, \quad Q, Q' \in S(\infty)$$

satisfies that

(D1) $\rho_0(Q, Q') = 0$ implies $Q = Q'$,

(D2) $\rho_0(Q, Q') \leq e^k \rho_0(Q', Q)$, where k is the constant in (H2),

(D3) $\rho_0(Q, Q'') \leq e^{2k}(\rho_0(Q, Q') + \rho(Q', Q''))$,

(D4) $\{Q' : \rho_0(Q, Q') < r\} = \Delta_{\log(1/r)}(Q)$ $(r > 0)$ and

$$\omega(\{Q' : \rho_0(Q, Q') < 2r\}) \leq C\omega(\{Q' : \rho_0(Q, Q') < r\}). \tag{14}$$

Consequently, the symmetrization

$$\rho(Q, Q') = \frac{\rho_0(Q, Q') + \rho_0(Q', Q)}{2}$$

is a quasi-distance in the sense of [15] such that $(S(\infty), \rho, \omega)$ is a space of homogeneous type, because there exists positive constants k_1 and k_2 depending only on M such that

$$\Delta_{\log(1/r)+k_1}(Q) \subset \{Q' : \rho(Q, Q') < r\} \subset \Delta_{\log(1/r)-k_2}(Q). \tag{15}$$

Therefore the abstract theory in [15] can be transplanted to our case. For instance, some covering lemmas, theorems on atomic Hardy spaces and BMO on spaces of homogeneous type hold true for $(S(\infty), \omega, \rho)$. We will sketch statements of some of them. As first, we deal with a covering lemma of Vitali type. Since the family of balls defined by the quasi-distance ρ and the family of sets $\{\Delta_t(Q)\}$ are equivalent (see (15)), we can state Vitali's covering lemma in terms of $\{\Delta_t(Q)\}$:

Lemma V (Vitali type covering lemma: see [3], [15]). *Let $E \subset S(\infty)$. Suppose $\{\Delta_{r(Q)}(Q) : Q \in E\}$ is a covering of E. Then there exist $Q_1, Q_2, \ldots$ in E such that*
(16) $\Delta_{r(Q_i)}(Q_i) \cap \Delta_{r(Q_j)}(Q_j) = \emptyset$, *$i \neq j$, and*
(17) *for every $Q \in S(\infty)$, there exists i with $\Delta_{r(Q)}(Q) \subset \Delta_{r(Q_i)-k'}(Q_i)$, where k' is a positive constant depending only on M and o.*

As known, from this lemma it follows the Hardy-Littlewood maximal theorem. To mention the theorem, we need the uncentered Hardy-Littlewood maximal function of $f \in L^1(\omega)$ defined as

$$(18)\quad \mathfrak{M}(f)(Q) := \sup_{\Delta_t(Q')\,:\,Q\in\Delta_t(Q')} \frac{1}{\omega(\Delta_t(Q'))}\int_{\Delta_t(Q')} |f|\,d\omega,\quad Q \in S(\infty).$$

Then we have

Theorem HL (see [15]). (1) *There exists a positive constant C_9 such that*

$$\omega(\{Q \in S(\infty) : \mathfrak{M}(f)(Q) > \lambda\}) \leq C_9\lambda^{-1}\|f\|_{L^1(\omega)}$$

for all $f \in L^1(\omega)$ and for all $\lambda > 0$.

(2) *For every $1 < p \leq \infty$, there exists a positive constant $C_{10,p}$ such that*

$$\|\mathfrak{M}(f)\|_{L^p(\omega)} \leq C_{10,p}\|f\|_{L^p(\omega)}$$

for every $f \in L^p(\omega)$.

Now let us mention the definition of atomic Hardy spaces on $S(\infty)$. In [15], atomic Hardy spaces and BMO on a space of homogeneous type are defined in terms of its quasi-distance. However in our case, as we have seen, the family of balls defined by ρ is equivalent to $\{\Delta_t(Q)\}$. For this reason, one can define atomic Hardy spaces and BMO in terms of $\{\Delta_t(Q)\}$ which are equivalent to those defined by the quasi-distace ρ: Let $0 < p < q$ and $p \leq 1 \leq q \leq \infty$. A function a on $S(\infty)$ is called (p,q)-atom if the support of a is contained in a "ball" $\Delta_r(Q)$, $\int_{S(\infty)} a\,d\omega = 0$, and $\|a\|_{L^q(\omega)} \leq \omega(\Delta_r(Q))^{1/q-1/p}$. Since $\omega(S(\infty)) = 1$, we regard also the constant function 1 as a (p,q)-atom.

For a continuous function f on $S(\infty)$, let

$$|f|_\alpha = \sup\left\{\frac{|f(Q)-f(Q')|}{\omega(\Delta_r(Q''))^\alpha} : r \in \mathbf{R},\ Q'' \in S(\infty) \text{ with } Q, Q' \in \Delta_r(Q'')\right\}.$$

Let Λ^α be the set of all continuous functions f with $|f|_\alpha < \infty$. The atomic Hardy spaces $H^{p,q}(\omega)$ $(= H^{p,q}(S(\infty), \omega))$ are defined as follows:

(i) If $0 < p < 1 \leq q \leq \infty$, then $H^{p,q}(\omega)$ is the subspace of the dual of $\Lambda^{1/p-1}$ consisting of those linear functionals admitting an atomic decomposition

$$h = \sum_{j=1}^{\infty} \lambda_j a_j, \tag{19}$$

where $\lambda_j \in \mathbf{R}$, and a_j's are (p,q)-atoms and $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$.

(ii) If $p = 1 < q \leq \infty$, then $H^{1,q}(\omega)$ is defined as the set of all functions h in $L^1(S(\infty), \omega)$ such that h has an atomic decomposition (19), where a_j's are $(1,q)$-atom and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$.

In any case we set

$$|h|_{p,q}^p = \inf\left\{\sum_{j=1}^{\infty} |\lambda_j|^p : h = \sum_{j=1}^{\infty} \lambda_j a_j,\ a_j\text{'s are } (p,q)\text{-atoms}\right\}$$

for $h \in H^{p,q}(\omega)$. Then the function $\phi(h,f) = |h-f|_{p,q}$ is defines a complete metric on $H^{p,q}(\omega)$.

By Coifman and Weiss [15], we obtain that $H^{p,q}(\omega)$ is isomorphic to $H^{p,\infty}(\omega)$, for $1 < q < \infty$.

Let $\mathrm{BMO}(\omega)$ be the set of all functions $f \in L^1(S(\infty), \omega)$ such that

$$|f|_{\mathrm{BMO}(\omega)} = \sup_{Q \in S(\infty),\, r \in \mathbf{R}} \frac{1}{\omega(\Delta_r(Q))} \int_{\Delta_r(Q)} |f - m_{\Delta_r(Q)} f|\, d\omega + \|f\|_{L^1(\omega)} < \infty,$$

where

$$m_{\Delta_r(Q)} f = \frac{1}{\omega(\Delta_r(Q))} \int_{\Delta_r(Q)} f\, d\omega.$$

Moreover, since the definitions of $H^{p,q}(\omega)$ and $\mathrm{BMO}(\omega)$ are equivalent to those by [15], we have the following:

Theorem CW ([15]). *(1) $H^{p,q}(\omega) = H^{p,\infty}(\omega)$, and moreover the (quasi-) norms $|\cdot|_{p,q}$ and $|\cdot|_{p,\infty}$ are equivalent $(0 < p \leq 1,\ 1 < q < \infty)$.*

(2) If $p < 1$, $\alpha = 1/p - 1$ and $1 < q \leq \infty$, then Λ^α is isomorphic to the dual of $H^{p,q}(\omega)$.

(3) For every $1 < q \leq \infty$, the dual of $H^{1,q}(\omega)$ is regarded as the space $\mathrm{BMO}(\omega)$ in the following sense: If $h = \sum \lambda_j a_j \in H^{1,q}(\omega)$, then for each $\ell \in \mathrm{BMO}(\omega)$

$$\langle h, \ell \rangle := \lim_{m \to \infty} \lambda_j \int_X \ell a_j \, d\omega$$

is a well defined continuous linear functional and its norm is equivalent to $|\ell|_{\mathrm{BMO}}$. Moreover, every linear continuous functional on $H^{1,q}(\omega)$ has this form.

In this paper we write

$$H^1_{\mathrm{atom}}(\omega) = H^{1,\infty}(\omega), \quad \text{and} \quad \|\cdot\|_{1,\mathrm{atom}} = |\cdot|_{1,\infty}.$$

§5. Some basic properties of Hardy spaces

This section is concerned with an elementary properties of Hardy spaces. As first we prove the equivalence of $H^p(\omega, \alpha)$ $(\alpha \in \mathbf{R})$. From now on, for $z \in M \setminus \{o\}$, we denote $z_\infty = \gamma_{oz}(+\infty)$ and $t(z) = t_{oz}$, where t_{oz} is a unique positive number such that $\gamma_{oz}(t_{oz}) = z$.

Proposition 5.1. *Suppose $-\infty < \alpha < \beta < \infty$. For every $1 \leq p \leq \infty$, $H^p(\omega, \alpha) = H^p(\omega, \beta)$. Moreover, the norms $\|f\|_{H^p(\omega,\beta)}$ and $\|f\|_{H^p(\omega,\alpha)}$ are equivalent.*

Proof. This proposition is a direct consequence of the following lemma:

Lemma 5.2. *For every continuous function u on M,*

$$\begin{aligned} &\omega(\{Q \in S(\infty) : N_\beta(u)(Q) > \lambda\}) \\ &\quad \leq \omega(\{Q \in S(\infty) : N_\alpha(u)(Q) > \lambda\}) \\ &\quad \leq C_{\alpha,\beta}\omega(\{Q \in S(\infty) : N_\beta(u)(Q) > \lambda\}), \end{aligned}$$

for all $\lambda > 0$, where $C_{\alpha,\beta}$ is a positive constant depending only on α, β, o and M.

Proof of Lemma 5.2. We adapt a standard argument (cf. [10]) to our case. Since by definition, $\Gamma_\beta(Q) \subset \Gamma_\alpha(Q)$, it is sufficient to prove the second inequality. Let f be the characteristic function of the set $\{N_\beta(u) > \lambda\}$. For every $Q \in \{N_\alpha(u) > \lambda\}$, there is a point $z \in M$ such that $|u(z)| > \lambda$ and $Q \in \Delta(z,\alpha)$. Then since $Q \in \Delta(z,\alpha) = \Delta_{t(z)+\alpha}(z_\infty)$, we have

$$\mathfrak{M}(f)(Q) \geq \frac{1}{\omega(\Delta(z,\alpha))} \int_{\Delta(z,\alpha)} |f|\, d\omega. \tag{20}$$

On the other hand, (H2) implies that

$$\begin{aligned}\Delta(z,\beta) \subset \Delta(z,\alpha) &= \Delta_{t(z)+\alpha}(z_\infty) \subset \Delta_{t(z)+\alpha-k}(Q) \\ &\subset \Delta_{t(z)+\alpha-2k}(z_\infty) = \Delta_{t(z)+\beta-(\beta+2k-\alpha)}(z_\infty).\end{aligned}$$

Hence by (20),

$$\begin{aligned}\mathfrak{M}(f)(Q) &\geq \frac{1}{\omega(\Delta_{t(z)+\beta-(\beta+2k-\alpha)}(z_\infty))} \int_{\Delta(z,\beta)} |f|\, d\omega \\ &\geq C(\beta+2k-\alpha)^{-1} \frac{1}{\omega(\Delta_{t(z)+\beta}(z_\infty))} \int_{\Delta(z,\beta)} |f|\, d\omega \quad (:= \text{(I), say}).\end{aligned}$$

Further, since for $Q' \in \Delta(z,\beta)$, $N_\beta(u)(Q') > \lambda$, so we have that $|f(Q')| \geq 1$, and therefore (I) $\geq C(\beta+2k-\alpha)^{-1} > 0$. Consequently, $\{N_\alpha(u) > \lambda\} \subset \{\mathfrak{M}(f) > C(\beta+2k-\alpha)^{-1}\}$. Accordingly by Theorem HL (1) we obtain that

$$\begin{aligned}\omega(\{N_\alpha(u) > \lambda\}) &\leq C_9 C(\beta+2k-\alpha)\|f\|_{L^1(\omega)} \\ &= C_9 C(\beta+2k-\alpha)\omega(\{N_\beta(u) > \lambda\}).\end{aligned}$$

Q.E.D.

Next we prove some estimates for the Poisson kernel which will be used in this paper:

Lemma 5.3. (i) *Suppose $r > 0$. Then there exists a positive constant $C_{0,r}$ such that*

$$C_{0,r}^{-1} \leq K(x,Q) \leq C_{0,r}, \quad \text{for all } x \in B(o,r) \text{ and } Q \in S(\infty).$$

(ii) *There exist positive constants C_{11}, C_{12} and C_{13} satisfying the following* (a) *and* (b):

(a) *For every $x \in M \setminus B(o,3)$ and for every positive integer j with $d(o,x) > j+1$,*

$$\sup\{K(x,Q) : Q \in \Delta(x,-j-1) \setminus \Delta(x,-j)\} \leq C_{11} \frac{\exp(-jC_4)}{\omega(\Delta(x,-j-1))}.$$

Let N be the largest positive integer N with $d(o,x) > N+1$. Then

$$\sup\{K(x,Q) : Q \in S(\infty) \setminus \Delta(x,-N-1)\} \leq C_{12} \exp(-C_4 d(o,x)).$$

(b) *For every $x \in M \setminus \{o\}$,*

$$\sup\{K(x,Q) : Q \in \Delta(x,0)\} \leq C_{13} \frac{1}{\omega(\Delta(x,0))}.$$

(iii) *Suppose $r > 0$ and $Q \in S(\infty)$. There exists a positive constants C_{14} depending only on M such that for all $t > r$ and $Q' \in S(\infty) \setminus \Delta_r(Q)$,*

$$K(\gamma_{oQ}(t), Q') \leq C_{14} C_{0,r} \exp(-C_4(t-r)).$$

Proof. (i) This is proved easily by Harnack inequality: Let $x \in B(o,r)$. Then Theorem H implies that for all $y \in M \setminus B(o,r+1)$, $G(x,y) \leq C_r G(o,y)$, where C_r is a positive constant depending only on M and r. Hence the construction of the Martin kernel (see [3]),

$$K(x,Q) \leq \sup_{y \in M \setminus B(o,r+1)} \frac{G(x,y)}{G(o,y)} \leq C_r,$$

and $1 = K(o,Q) \lessapprox K(x,Q)$. Consequently we have (i).

(ii) Suppose $x \in M \setminus B(o,3)$, and let j be a positive integer such that $d(o,x) > j+1$. For simplicity, let $x(-j) = \gamma_{ox}(t(x)-j)$. Let F be an arbitrary Borel subset of $\Delta(x,-j-1) \setminus \Delta(x,-j)$. Then by Theorem BH2, we have that

$$\frac{\omega^x(F)}{G(x,o)} \approx \frac{\omega^{x(-j+1)}(F)}{G(x(-j+1),o)} \approx \frac{\omega^{x(-j)}(F)}{G(x(-j),o)}.$$

On the other hand, Theorems BH1 and BH2 imply that

$$\begin{aligned} \frac{G(o,x)}{G(o,x(-j))} &\approx \frac{G(x(-j-1),x)}{G(x(-j-1),x(-j))} \approx G(x(-j-1),x) \\ &\lessapprox \exp(-C_4 d(x(-j-1),x)). \end{aligned}$$

Combining these inequalities we have

$$\omega^x(F) \lessapprox \exp(-C_4 j)\omega^{x(-j-1)}(F) \approx \exp(-C_4 j)\omega^{x(-j-1)}(F).$$

Since there exists a positive constant c such that for every $z \in M \setminus \{o\}$,

(21) $\qquad \omega^z(\Delta(z,0)) \geq c, \quad$ (see the proof of [3, Lemma 7.4]),

we have by Theorem BH2 (2) that

$$\omega^{x(-j-1)}(F) \lessapprox \frac{\omega^{x(-j-1)}(F)}{\omega^{x(-j-1)}(\Delta(x,-j-1))} \approx \frac{\omega(F)}{\omega(\Delta(x,-j-1))}.$$

Hence

$$\frac{\omega^x(F)}{\omega(F)} \lessapprox \exp(-C_4 j)\omega(\Delta(-j-1))^{-1}.$$

Therefore for $Q \in \Delta(x,-j-1) \setminus \Delta(x,-j)$, we have the first inequality in (a):

$$K(x,Q) = \lim_{F \to \{Q\}} \frac{\omega^x(F)}{\omega(F)} \leq C \exp(-C_4 j)\omega(\Delta(-j-1))^{-1}.$$

The second inequality in (a) is a direct consequence of Theorems BH2 and BH1: From (i) and Theorem BH2 it follows that for $Q \in S(\infty) \setminus \Delta(x,-N-1)$,

$$\frac{K(x,Q)}{G(x,o)} \approx \frac{K(x(-N),Q)}{G(x(-N),o)} \approx 1,$$

and consequently $K(x,Q) \approx G(x,o) \lessapprox \exp(-C_4 d(o,x))$.

To prove (b) we use (21). For $Q \in \Delta(x,0)$,

$$K(x,Q) \approx \frac{K(x,Q)}{\omega^x(\Delta(x,0))} \approx \frac{K(o,Q)}{\omega(\Delta(x,0))} = \frac{1}{\omega(\Delta(x,0))}.$$

This yields (b).

We prove (iii). Let $t > r+1$. Then by Theorem BH2 we have

$$\begin{aligned}\frac{K(\gamma_{oQ}(t),Q')}{G(\gamma_{oQ}(t),\gamma_{oQ}(r))} &\approx \frac{K(\gamma_{oQ}(r+1),Q')}{G(\gamma_{oQ}(r+1),\gamma_{oQ}(r))} \approx K(\gamma_{oQ}(r+1),Q') \\ &\approx K(\gamma_{oQ}(r),Q') \leq C_{0,r}.\end{aligned}$$

Hence $K(\gamma_{oQ}(t),Q') \leq C_{0,r} G(\gamma_{oQ}(t),\gamma_{oQ}(r))$, and this implies (iii).

Q.E.D.

Using Lemma 5.3, we have

Lemma 5.4. *There exists a positive constant C_{15} such that for every $f \in L^1(\omega)$,*

$$N_0(\tilde{f})(Q) \leq C_{15}\mathfrak{M}(f)(Q), \quad Q \in S(\infty).$$

We can prove this lemma by combining [3, Theorem 7.3] with Theorem CK1. However here we give a direct proof using Lemma 5.3:

Proof. Let x be an arbitrary point in $\Gamma_0(Q) \setminus B(o,3)$. Then $Q \in \Delta(x,0)$. Let N be the largest positive integer with $d(o,x) > N+1$.

$$
\begin{aligned}
|\tilde{f}(x)| \le & \int_{S(\infty)\setminus\Delta(x,-N-1)} + \sum_{j=0}^{N} \int_{\Delta(x,-j-1)\setminus\Delta(x,-j)} \\
& + \int_{\Delta(x,0)} K(x,Q)|f(Q)|\, d\omega(Q) \\
& (= \text{(I)} + \text{(II)} + \text{(III)}, \quad \text{say}).
\end{aligned}
$$

Then by Lemma 5.3,

$$
\text{(I)} \le C_{12} \int_{S(\infty)} |f|\, d\omega \le C_{12}\mathfrak{M}(f)(Q), \quad \text{and}
$$

$$
\begin{aligned}
\text{(II)} &\le C_{11} \sum_{j=0}^{N} \frac{\exp(-jC_4)}{\omega(\Delta(x,-j-1))} \int_{\Delta(x,-j-1)} |f|\, d\omega \\
&\le C_{11} \sum_{j=0}^{N} \exp(-jC_4)\mathfrak{M}(f)(Q) \lessapprox \mathfrak{M}(f)(Q).
\end{aligned}
$$

Further,

$$
\text{(III)} \le \frac{C_{13}}{\omega(\Delta(x,0))} \int_{\Delta(x,0)} |f|\, d\omega \le C_{13}\mathfrak{M}(f)(Q).
$$

Consequently, we have $|\tilde{f}(x)| \lessapprox \mathfrak{M}(f)(Q)$.

Now we consider the case of $x \in B(o,3)$. By Lemma 5.3 (i),

$$
\begin{aligned}
|\tilde{f}(x)| &\le \int_{S(\infty)} K(x,\zeta)|f(\zeta)|\, d\omega(\zeta) \\
&\le C_{0,3} \int_{S(\infty)} |f(\zeta)|\, d\omega(\zeta) \le C_{0,3}\mathfrak{M}(f)(Q).
\end{aligned}
$$

Thus we obtain the desired inequality. Q.E.D.

Now we have the following theorem as known in the classical case.

Theorem 5.5. (i) *For* $1 \le p \le \infty$, $\|f\|_{L^p(\omega)} \le \|f\|_{H^p(\omega)}$, *for all* $f \in H^1(\omega)$.

(ii) *Suppose* $1 < p \leq \infty$. *Then* $H^p(\omega) = L^p(\omega)$, *and there exists a positive constant* $C_{16}(p)$ *such that* $\|f\|_{H^p(\omega)} \leq C_{16}(p)\|f\|_{L^p(\omega)}$ *for all* $f \in H^p(\omega)$.

Proof. Note that for a given function $f \in L^1(\omega)$ and for a Lebesgue point Q of f,

$$\lim_{t\to\infty} \tilde{f}(\gamma_{oQ}(t)) = f(Q).$$

Indeed by Lemma 5.4 and Lemma 5.3 (iii), we can prove this assertion by a similar way as the classical case (see [44, p.244]). Accordingly (i) is obvious.

Theorem HL and Lemma 5.4 guarantee (ii). Q.E.D.

We close this section with making some remarks on truncated maximal functions. For $r > 0$, let

$$N_{0,r}(f)(Q) := \sup\{|\tilde{f}(z)| : z \in \Gamma_0(Q) \cap (M \setminus B(o,r))\}.$$

By the same way as in the case of $N_0(f)$, we have that the function $N_{0,r}(f)$ is lower semicontinuous on $S(\infty)$. Using Lemma 5.3 (ii) we obtain that

$$N_{0,r}(f)(Q) \leq N_0(f)(Q) \leq C_r\|f\|_{L^1(\omega)} + N_{0,r}(f)(Q), \quad Q \in S(\infty). \tag{22}$$

Therefore for every $r > 0$,

$$\|f\|_{H^p} \approx \|N_{0,r}(f)\|_{L^p(\omega)}. \tag{23}$$

§6. Hardy spaces, atoms and Brownian motion

In this section we prove the equivalence of the spaces $H^1(\omega)$, H^1_{atom} and probabilistic analogues of Hardy spaces which will be mentioned later. To describe the analogues we recall some facts and notions in probability theory:

Let W be the set of all continuous maps from $[0,\infty)$ to M, and let $Z_t(w) = w(t)$, $w \in W$. Since by Yau [50] the life time of Brownian motion on M is equal to $+\infty$, so there exists a system of probability measures $\{P_x\}_{x\in M}$ on W such that (P_x, Z_t) is a Brownian motion starting at x. From Sullivan [46] or Kifer [27] it follows the following (A) and (B):

(A) There exists a limit $Z_\infty(w) := \lim_{t\to\infty} Z_t(w)$ for almost sure $w \in W$ with respect to P_x, $x \in M$. Moreover, $Z_\infty(w) \in S(\infty)$ for P_x-a.s. $w \in W$.

(B) For every $x \in M$ and for every Borel subset F of $S(\infty)$,

$$\omega^x(F) = P_x(\{w \in W : Z_\infty(w) \in F\}).$$

Since for every $f \in L^1(\omega)$, we have $f \in L^1(\omega^x)$ for all $x \in M$ by (9). Therefore for every $f \in L^1(\omega)$, $\tilde{f}(x) = E_x[f(Z_\infty)]$ for all $x \in M$ and $\lim_{t\to\infty} \tilde{f}(Z_t) = f(Z_\infty)$ P_x-a.s., where $E_x[\ ]$ denotes the expectation with respect to P_x $(x \in M)$.

We denote $P = P_o$ and $E[\ ] = E_o[\ ]$. First we describe a probabilistic analogue of Hardy spaces:

$$H^p_{\mathrm{prob}} := \left\{ f \in L^p(\omega) : \|f\|_{H^p_{\mathrm{prob}}} = E\left[\sup_{0\le t<\infty} \left|\tilde{f}(Z_t)\right|^p\right]^{1/p} < \infty \right\},$$
$$1 \le p < \infty.$$

Next we will deal with another probabilistic analogue of Hardy spaces. To define it, we recall some facts on Markov properties of $\{P_x\}_{x\in M}$: Let $\mathcal{B}$ (resp. $\mathcal{B}_t$) be the smallest σ-field for which all random variables Z_s, $s \ge 0$ (resp. Z_s, $0 \le s \le t$) are measurable. For a probability Borel measure μ on M, let $P_\mu(A) = \int_{S(\infty)} P_x(A)\, d\mu(x)$, $A \subset W$. We denote by $(W, \mathcal{F}^\mu, \mathcal{F}^\mu_t, P_\mu)$ the usual P_μ augmentation of $(W, \mathcal{B}, \mathcal{B}_t, P_\mu)$ in the sense of [43, III 9]. In particular, $(W, \mathcal{F}^x, \mathcal{F}^x_t, P_x)$ denotes the P_x-augmentation of $(W, \mathcal{B}, \mathcal{B}_t, P_\mu)$. Put $\tilde{\mathcal{F}} := \bigcap \mathcal{F}^\mu$ and $\tilde{F}_t := \bigcap \mathcal{F}^\mu_t$, where the intersection is taken over all probability Borel measures μ on M. Then $(Z_t, W, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, P_x : x \in M)$ is a strong Markov process. If fact, considering that M is diffeomorphic to $\mathbf{R}^n$, it is a honest FD diffusion in the sense of [43, III 3, III 13].

It is known that the usual P_x-augmentation $(W, \mathcal{F}^x, \mathcal{F}^x_t, P_x)$ satisfies the so-called usual condition (see [43, III 9]). Moreover, for every harmonic function u on M, the process $u(Z_t)$ is a continuous local $(P_x, \mathcal{F}^x_t)$-martingale. Denote by $(W, \mathcal{F}, \mathcal{F}_t, P)$ the usual P_o-augmentation $(W, \mathcal{F}^o, \mathcal{F}^o_t, P_o)$. As usual, Hardy spaces of martingales are defined as follows:

$$\mathcal{M}^p := \left\{ X \in L^1(W, \mathcal{F}, P) : \|X\|_{\mathcal{M}^p} := E\left[\sup_{0\le t<\infty} |E[X|\mathcal{F}_t]|^p\right]^{1/p} < \infty \right\},$$

$(1 \le p < \infty)$, where and always $E[\,\cdot\,|\mathcal{C}]$ denotes the conditional expectation with respect to P and a sub σ-field $\mathcal{C}$ of $\mathcal{F}$. Note that Meyer's previsibility theorem ([43, VI 15, Theorem 15.4]) implies that for every $X \in L^1(W, P)$, the process $(E[X/\mathcal{F}_t])_{t\ge0}$ is an $(\mathcal{F}_t)$-continuous martingale.

For $X \in L^1(W, \mathcal{F}, P)$, let $\mathcal{N}'(X) := E[X|\sigma(Z_\infty)]$, where $\sigma(Z_\infty)$ is the sub σ-field of $\mathcal{F}$ generated by the random variable Z_∞. Then by (A) there exists a unique element $f \in L^1(\omega)$ such that $\mathcal{N}'(X) = f(Z_\infty)$, P-a.s. Denote the function f by $\mathcal{N}X$. Now we can mention the second probabilistic analogue of Hardy spaces:

$$H^p_{\mathrm{mart}} := \{\mathcal{N}(X) : X \in \mathcal{M}^p\}, \quad 1 \le p < \infty,$$

and $\|\mathcal{N}(X)\|_{H^p_{\mathrm{mart}}} := \inf\{\|Y\|_{\mathcal{M}^p} : \mathcal{N}(Y) = \mathcal{N}(X),\, Y \in \mathcal{M}^p\}$.

To describe our results, we use the following notation: For two normed spaces $(A, \|\ \|_A)$ and $(B, \|\ \|_B)$, we denote by $A \preceq B$ that $A \subset B$ and $\|x\|_B \le C\|x\|_A$ for every $x \in A$, where C is a constant independent of x. Further we set $A \simeq B$ if $A \preceq B$ and $B \preceq A$.

Theorem 6.1.

$$H^1(\omega) \simeq H^1_{\mathrm{atom}}(\omega) \simeq H^1_{\mathrm{prob}} \simeq H^1_{\mathrm{mart}}.$$

Before proving this theorem, we would like to refer to both a work of Cifuentes and Korányi ([14]) and our previous announcement [5]. As pointed out briefly in Introduction, we announced in [5] the following two theorems:

Theorem 6.2.

$$H^1(\omega) \preceq H^1_{\mathrm{prob}} \preceq H^1_{\mathrm{mart}} \preceq H^1_{\mathrm{atom}}(\omega).$$

Theorem 6.3. *Consider the following geometric condition:*
(β) *For every* $Q \in S(\infty)$, $t > 0$ *and* $z \in C(\gamma_{oQ}(t), 0)$,

$$\Delta_t(\gamma_{oz}(+\infty)) \cap \Delta_t(Q) \ne \emptyset.$$

If our manifold M *satisfies the condition* (β), *we have* $H^1_{\mathrm{atom}}(\omega) \preceq H^1(\omega)$.

It is easy to see that when M is rotationally symmetric at o or the dimension of M is two, the condition (β) is satisfied. However recently, Cifuentes and Korányi proved the following

Theorem CK2 ([14]). *The manifold* M *satisfies always the condition* (β).

Therefore combining our Theorems 6.2 and 6.3 with Theorem CK2, we gain finally Theorem 6.1. For this reason, in order to get Theorem 6.1, we prove in this section Theorems 6.2 and 6.3.

First we prove the following

Proposition 6.4. *For every continuous function u on M, and for every $\lambda > 0$,*

$$P\left(\left\{w \in W : \sup_{0\le t<\infty} |u(Z_t)| > \lambda\right\}\right) \lessapprox \omega(N_0(u) > \lambda).$$

In particular, we have $H^1(\omega) \preceq H^1_{\mathrm{prob}}$.

Proof. We adapt to our case of an idea of Burkholder, Gundy and Silverstein [10]. Let $F = \{N_0(u) > \lambda\}$. By the definition of admissible regions, when $|u(z)| > \lambda$, then $\Delta(z,0) \subset F$. Hence by (21) $\omega^z(F) \ge \omega^z(\Delta(z,0)) \ge c > 0$. Denote by χ_F the characteristic function of F. From Doob's maximal theorem it follows that

$$\begin{aligned} P(\{\sup_t |u(Z_t)| > \lambda\}) &\le P(\{\sup_t \omega^{Z_t}(F) > c\}) = P(\{\sup_t E[\chi_F/\mathcal{F}_t] > c\}) \\ &\le C\|\sup_t E[\chi_F/\mathcal{F}_t]\|^2_{L^2(W,P)} \le CE[\chi_F(Z_\infty)] = C\omega(F). \end{aligned}$$

Q.E.D.

For $f \in H^1_{\mathrm{prob}}$, we have that $\mathcal{N}f(Z_\infty) = f$ and $E[f(Z_\infty)/\mathcal{F}_t] = \tilde{f}(Z_t)$. Accordingly $H^1_{\mathrm{prob}} \preceq H^1_{\mathrm{mart}}$.

Next we prove $H^1_{\mathrm{mart}} \preceq H^1_{\mathrm{atom}}$. For this aim, we need to recall a probabilistic version of BMO: For $f \in L^1(\omega)$, let

$$\|f\|_{\mathrm{BMO_{prob}}} := \sup_{0\le t<\infty} \left\| E\left[|\tilde{f}(Z_\infty) - \tilde{f}(Z_t)|/\mathcal{F}_t\right]\right\|_{L^\infty(W,P)} + \|f\|_{L^1(\omega)},$$

and let $\mathrm{BMO_{prob}} := \{f \in L^1(\omega) : \|f\|_{\mathrm{BMO_{prob}}} < \infty\}$.

As in the classical case, one can consider the following version of BMO norm called "Garsia norm":

$$\|f\|_G := \sup_{x\in M} \int_M |f(Q) - \tilde{f}(x)|\, d\omega^x(Q) + \|f\|_{L^1(\omega)}\ (\le \infty),$$

for $f \in L^1(\omega)$.

Before proving $H^1_{\mathrm{mart}} \preceq H^1_{\mathrm{atom}}(\omega)$, we show the following relation among these variants of BMO norms by using ideas in [48]:

Proposition 6.5. *Let $f \in L^1(\omega)$. Then*

$$\|f\|_{\mathrm{BMO_{prob}}} \lessapprox \|f\|_G \lessapprox \|f\|_{\mathrm{BMO}}. \tag{24}$$

Proof. Let $F_s(w) := |\tilde{f}(w(s)) - \tilde{f}(w(0))|$, $w \in W$, and let θ_t be the shift operator, i.e., $\theta_t(w)(s) := w(s+t)$. Then the Markov property of Brownian motion on M, we have that $E[F_s \circ \theta_t / \mathcal{F}_t] = E_{X_t}[F_s]$. Hence $E[|f(X_{t+s}) - f(X_t)| / \mathcal{F}_t] = E_{X_t}[|f(X_s) - f(X_0)|]$ P-a.s. Letting $s \to \infty$, we have

$$\begin{aligned} E[|f(X_\infty) - f(X_t)| / \mathcal{F}_t] &= E_{X_t}[|f(X_\infty) - f(X_0)|] \\ &= \int_M |f(Q) - f(X_t)| \, d\omega^{X_t}(Q), \end{aligned}$$

P-a.s. Consequently, we obtain the first inequality of (24).

For $x \in M \setminus B(o,3)$, we set $\Delta(x) = \Delta(x,0)$. Then

$$\begin{aligned} &\int_{S(\infty)} |f(Q) - \tilde{f}(x)| \, d\omega^x(Q) \\ &\le \int_{S(\infty)} |f(Q) - m_{\Delta(x)} f| \, d\omega^x(Q) + \int_{S(\infty)} |m_{\Delta(x)} f - \tilde{f}(x)| \, d\omega^x \\ &= \int_{S(\infty)} |f(Q) - m_{\Delta(x)} f| \, d\omega^x(Q) + |\tilde{f}(x) - m_{\Delta(x)} f| \\ &\le 2 \int_{S(\infty)} |f - m_{\Delta(x)} f| \, d\omega^x \\ &\le 2 \int_{S(\infty) \setminus \Delta(x)} |f - m_{\Delta(x)} f| \, d\omega^x + 2 \int_{\Delta(x)} |f - m_{\Delta(x)} f| \, d\omega^x \\ &(:= (\mathrm{I}) + (\mathrm{II})) \end{aligned}$$

By Lemma 5.3 (ii) (b), we have

$$(\mathrm{II}) \le C \frac{1}{\omega(\Delta(x))} \int_{\Delta(x)} |f - m_{\Delta(x)} f| \, d\omega \le C \|f\|_{\mathrm{BMO}}.$$

To estimate (I), we use Lemma 5.3 (ii) (a): Let N be the largest positive integer with $d(o,x) > N+1$, and let $\Delta(j) = \Delta(x,-j)$. Then

$$\begin{aligned} (\mathrm{I}) &= \left(\int_{S(\infty) \setminus \Delta(N+1)} + \sum_{j=1}^{N} \int_{\Delta(j+1) \setminus \Delta(j)} \right) |f - m_{\Delta(x)}| \, d\omega^x \\ &=: (\mathrm{III}) + (\mathrm{IV}). \end{aligned}$$

From Lemma 5.3 it follows that

$$\begin{aligned}(\mathrm{IV}) &\lessapprox \sum_{j=1}^{N} \frac{\exp(-jC_4)}{\omega(\Delta(j+1))} \int_{\Delta(j+1)} |f - m_{\Delta(x)} f| \, d\omega \\ &\lessapprox \sum_{j=1}^{N} \frac{\exp(-jC_4)}{\omega(\Delta(j+1))} \left\{ \int_{\Delta(j+1)} |f - m_{\Delta(j+1)} f| \, d\omega \right. \\ &\quad + \int_{\Delta(j+1)} |m_{\Delta(j+1)} f - m_{\Delta(j)} f| \, d\omega \\ &\quad \left. + \cdots + \int_{\Delta(j+1)} |m_{\Delta(1)} f - m_{\Delta(0)} f| \, d\omega \right\}.\end{aligned}$$

Since (H4) implies $|m_{\Delta(j+1)} f - m_{\Delta(j)} f| \lessapprox \|f\|_{\mathrm{BMO}}$, we have

$$(\mathrm{IV}) \lessapprox \sum_{j=1}^{N} \exp(-jC_4) \left\{ \|f\|_{\mathrm{BMO}} + j \|f\|_{\mathrm{BMO}} \right\} \lessapprox \|f\|_{\mathrm{BMO}}.$$

Moreover, by Lemma 5.3 (ii) (a),

$$\begin{aligned}(\mathrm{III}) &\lessapprox \exp(-C_4 d(o,x)) \int_{S(\infty) \setminus \Delta(N+1)} |f - m_{\Delta(x)}| \, d\omega \\ &\leq 2 \exp(-C_4 d(o,x)) \|f\|_{L^1(\omega)}.\end{aligned}$$

When $x \in B(o,3)$, applying Theorem H we have

$$\int_M |f - \tilde{f}| \, d\omega^x \lessapprox \int_M |f - \tilde{f}| \, d\omega \leq 2 \|f\|_{L^1(\omega)}.$$

Consequently we gain $\|f\|_G \lessapprox \|f\|_{\mathrm{BMO}}$. Q.E.D.

As a consequence of this proposition and of a probabilistic version of Fefferman's inequality ([24]), we have

Proposition 6.6. *For $X \in \mathcal{M}^1$,*

$$\|\mathcal{N} X\|_{1,\mathrm{atom}} \leq C \|X\|_{\mathcal{M}^1}.$$

Therefore $H^1_{\mathrm{mart}} \preceq H^1_{\mathrm{atom}}$.

Proof. Suppose $X \in L^2(W,P)$. Then from H^1_{atom}-BMO duality theorem in [15] it follows that

$$\begin{aligned}
\|\mathcal{N}X\|_{1,\mathrm{atom}} &= \sup\left\{|\Psi(\mathcal{N}X)| : \Psi \in (H^1_{\mathrm{atom}})^*, \|\Psi\|_{(H^1_{\mathrm{atom}})^*} \leq 1\right\} \\
&\lessapprox \sup\left\{\left|\int_{S(\infty)} \psi \mathcal{N}X\, d\omega\right| : \psi \in \mathrm{BMO}(\omega),\ \|\psi\|_{\mathrm{BMO}} \leq 1\right\} \\
&=: (\mathrm{I}).
\end{aligned}$$

Now using a martingale version of Fefferman's inequality ([24]), we have that

$$\begin{aligned}
(\mathrm{I}) &= \sup\left\{|E[\psi(Z_\infty)\mathcal{N}X(Z_\infty)]| : \psi \in \mathrm{BMO}(\omega), \|\psi\|_{\mathrm{BMO}} \leq 1\right\} \\
&= \sup\left\{|E[\psi(Z_\infty)X]| : \psi \in \mathrm{BMO}(\omega), \|\psi\|_{\mathrm{BMO}} \leq 1\right\} \\
&\lessapprox \|X\|_{\mathcal{M}^1} \sup\left\{\|\psi\|_{\mathrm{BMO}_{\mathrm{prob}}} : \psi \in \mathrm{BMO}(\omega), \|\psi\|_{\mathrm{BMO}} \leq 1\right\} \\
&\lessapprox \|X\|_{\mathcal{M}^1},
\end{aligned}$$

where the last inequality is proved by Proposition 6.5.

Now suppose $X \in \mathcal{M}^1$. It is well known that for $X \in \mathcal{M}^1$, there exist $X^k \in \mathcal{M}^2$ $(k = 1, 2, \ldots)$ such that $\|X^k - X\|_{\mathcal{M}^1} \to 0$ as $k \to \infty$. From what we have proved it follows that $\|\mathcal{N}X^k - \mathcal{N}X^l\|_{H^1_{\mathrm{atom}}} \lessapprox \|X^k - X^l\|_{\mathcal{M}^1} \to 0$ $(k, l \to \infty)$. Since H^1_{atom} is complete ([15]), there exists $h \in H^1_{\mathrm{atom}}$ such that $\|h - \mathcal{N}X^k\|_{1,\mathrm{atom}} \to 0$ as $k \to \infty$. Therefore

$$\begin{aligned}
\|\mathcal{N}X - h\|_{L^1(\omega)} &\leq \|\mathcal{N}X - \mathcal{N}X^k\|_{L^1(\omega)} + \|\mathcal{N}X^k - h\|_{L^1(\omega)} \\
&\leq \|X - X^k\|_{L^1(W,P)} + \|\mathcal{N}X^k - h\|_{1,\mathrm{atom}} \\
&\leq \|X - X^k\|_{\mathcal{M}^1} + \|\mathcal{N}X^k - h\|_{1,\mathrm{atom}} \to 0, \quad (k \to \infty).
\end{aligned}$$

Hence $\mathcal{N}X = h$ ω-a.e. Therefore

$$\begin{aligned}
\|\mathcal{N}X\|_{1,\mathrm{atom}} = \|h\|_{1,\mathrm{atom}} &= \lim_{k\to\infty} \|\mathcal{N}X^k\|_{1,\mathrm{atom}} \lessapprox \lim_{k\to\infty} \|X^k\|_{\mathcal{M}^1} \\
&= \|X\|_{\mathcal{M}^1}.
\end{aligned}$$

Q.E.D.

When M is the open unit disc, this proposition for BMO was proved in [48]. See also [51] for balls.

What we have proved implies Theorem 6.2.

In order to prove Theorem 6.3, we need the following estimate:

Lemma 6.7. *Suppose $Q_0 \in S(\infty)$, $r > 3k$ and $Q \in \Delta_r(Q_0)$. Let N be the biggest positive integer such that $r > (N+2)k$. Let $C(j) = C(\gamma_{oQ_0}(r - jk), 0)$ $(j = 0, 1, \ldots, N)$. Then for every $j \in \{0, 1, \ldots, N\}$ and for every $x \in M \setminus C(j)$,*

$$|K(x,Q) - K(x,Q_0)| \lessapprox c^j K(x,Q_0),$$

where c is a positive constant such that $c < 1$ and is depending only on M.

Proof. The following proof is based on an idea in Anderson and Schoen [3, p.449]. Let $D(j) = M \setminus C(j)$, and let

$$\overline{\varphi}_j = \sup_{z \in D(j)} \frac{K(z,Q)}{K(z,Q_0)}, \quad \underline{\varphi}_j = \inf_{z \in D(j)} \frac{K(z,Q)}{K(z,Q_0)}.$$

Let $u_j(z) = K(z,Q) - \underline{\varphi}_{j-1} K(z,Q_0)$. Then u_j is harmonic on M and positive on $D(j-1)$. Hence by Theorem BH2 (2),

$$\sup_{z \in D(j)} \frac{u_j(z)}{K(z,Q_0)} \le C_8 \inf_{z \in D(j)} \frac{u_j(z)}{K(z,Q_0)}. \tag{25}$$

(Note that $C_8 > 1$.) By (25), we have

$$\overline{\varphi}_j - \underline{\varphi}_{j-1} \le C_8 \big(\underline{\varphi}_j - \underline{\varphi}_{j-1}\big). \tag{26}$$

On the other hand, if we consider the function $v_i(z) = \overline{\varphi}_{j-1} K(z,Q_0) - K(z,Q)$ in stead of u_j, we obtain the following estimate:

$$\overline{\varphi}_{j-1} - \underline{\varphi}_j \le C_8 \big(\overline{\varphi}_{j-1} - \overline{\varphi}_j\big). \tag{27}$$

Let $\mathrm{osc}(j) = \overline{\varphi}_j - \underline{\varphi}_j$. Then by (26) and (27), we have

$$\mathrm{osc}(j) \le \frac{C_8 - 1}{C_8 + 1} \mathrm{osc}(j-1).$$

Hence when $x \in M \setminus C(j)$, then

$$\begin{aligned} \left| \frac{K(x,Q)}{K(x,Q_0)} - 1 \right| = \left| \frac{K(x,Q)}{K(x,Q_0)} - \frac{K(o,Q)}{K(o,Q_0)} \right| &\le \mathrm{osc}(j) \\ &\le \left(\frac{C_8 - 1}{C_8 + 1} \right)^{j-1} \mathrm{osc}(1). \end{aligned}$$

Moreover, since

$$\frac{K(z,Q)}{K(z,Q_0)} \approx \frac{K(o,Q)}{K(o,Q_0)} = 1,$$

for all $z \in M \setminus C(j-1)$ by Theorem BH2 (2), we have that osc(1) is bounded by a positive constant depending only on M. Now what we have obtained implies the desired inequality. Q.E.D.

To prove the following lemma, we need the condition (β) stated in Theorem 6.3:

Lemma 6.8. *For every $Q \in S(\infty)$, $t > k$ and $Q' \in S(\infty) \setminus \Delta_{t-k}(Q)$, we have $\Gamma_0(Q') \cap C(\gamma_{oQ}(t), 0) = \emptyset$.*

Proof. Suppose that there exists a point $z \in \Gamma_0(Q') \cap C(\gamma_{oQ}(t), 0)$. Then $Q' \in \Delta(z, 0)$ and $d(o, z) > t$. Further by the condition (β), we have $\Delta_t(\gamma_{oz}(+\infty)) \cap \Delta_t(Q) \neq \emptyset$. Therefore from (H3) it follows that

$$\Delta_{t-k}(Q) \supset \Delta_t(\gamma_{oz}(+\infty)) \supset \Delta_{d(o,z)}(\gamma_{oz}(+\infty)) = \Delta(z,0) \ni Q'.$$

This contradicts to that $Q' \in S(\infty) \setminus \Delta_{t-k}(Q)$. Q.E.D.

Now we are ready to prove Theorem 6.3:

Proof of Theorem 6.3. It is sufficient to prove that for every $(1,\infty)$-atom a, $\|a\|_{H^1} \leq C$, where C is a positive constant depending only on M. Let a be a $(1,\infty)$-atom, that is, there exists a ball $\Delta_r(Q_0)$ such that

$$\operatorname{supp} a \subset \Delta_r(Q_0), \qquad \int_M a\, d\omega = 0, \qquad \|a\|_{L^\infty} \leq 1/\omega(\Delta_r(Q_0)).$$

Suppose $r > 3k$. For simplicity, let $\Delta = \Delta_r(Q_0)$, $\Delta(j) = C(\gamma_{oQ_0}(r - jk), 0) \cap S(\infty)\ (= \Delta_{r-jk}(Q_0))$, and $A(j) = \gamma_{oQ_0}(r - jk)$. Then

$$\|N_0(a)\|_{L^1(\omega)} = \int_{S(\infty)\setminus\Delta(N+2)} N_0(a)\, d\omega + \sum_{j=1}^{N} \int_{\Delta(j+2)\setminus\Delta(j+1)} N_0(a)\, d\omega$$
$$+ \int_{\Delta(2)} N_0(a)\, d\omega\ (:= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})),$$

where N is the biggest integer with $r > (N+2)k$. First we estimate (II). Let $p \in \Delta(j+2) \setminus \Delta(j+1)$. Then by Lemma 6.8, $\Gamma_0(p) \cap C(\gamma_{oQ_0}(r - jk), 0) = \emptyset$. For $x \in M \setminus C(\gamma_{oQ_0}(r-jk), 0)$,

$$K(x, Q_0) \leq \frac{K(x,Q_0)}{\omega^x(\Delta(j))} \approx \frac{K(o,Q_0)}{\omega^o(\Delta(j))} = \frac{1}{\omega(\Delta(j))}.$$

Hence by Lemma 6.7, for every $x \in \Gamma_0(p)$,

$$\begin{aligned}|\tilde{a}(x)| &= \left|\int_\Delta a(Q)\{K(x,Q) - K(x,Q_0)\}\, d\omega\right| \\ &\le \|a\|_{L^\infty} \int_\Delta |K(x,Q) - K(x,Q_0)|\, d\omega \\ &\lessapprox c^j K(x,Q_0) \lessapprox c^j \frac{1}{\omega(\Delta(j))}.\end{aligned}$$

Using this estimate, we have that

$$\int_{\Delta(j+2)\setminus\Delta(j+1)} N_0(a)\, d\omega \lessapprox c^j \int_{\Delta(j+2)} \omega(\Delta(j))^{-1}\, d\omega \lessapprox c^j.$$

Accordingly (II) $\le C$. To estimate (I), let $p \in S(\infty) \setminus \Delta(N+2)$. Then $\Gamma_0(p) \cap C(\gamma_{oQ_0}(r - Nk), 0) = \emptyset$. Note that $\operatorname{dist}\big(\partial C(\gamma_{oQ_0}(r - Nk), 0), \partial C(\gamma_{oQ_0}(r - (N-1)k), 0)\big) \ge c$ for some positive constant depending only on the curvature bounds κ_1 and κ_2 (see [6, pp.310–311] for a related estimate). Therefore for every $x \in \Gamma_0(p)$ and $Q \in \Delta_r(Q_0)$,

$$\begin{aligned}K(x,Q) &\le C\frac{K(x,Q)}{G(x, A(N-1))} \approx \frac{K(A(N),Q)}{G(A(N), A(N-1))} \\ &\approx K(A(N),Q) \le C,\end{aligned}$$

where the last inequality is proved by Lemma 5.3 (ii). Hence $|\tilde{a}(x)| \le C$, and therefore $N_0(a)(p) \le C$. Consequently we have (I) $\le C$.

Next we estimate (III). For $x \in M$,

$$\begin{aligned}|\tilde{a}(x)| &\le \int_{\Delta_r(Q_0)} K(x,Q)|a(Q)|\, d\omega(Q) \\ &\le \|a\|_{L^\infty} \int_{S(\infty)} d\omega^x \le \frac{1}{\omega(\Delta_r(Q_0))}.\end{aligned} \tag{28}$$

This implies that (III) $\le C$.

Lastly we consider the case $r < 3k$. By Theorem H and (21),

$$\omega(\Delta_r(Q_0)) \ge \omega(\Delta_{3k}(Q_0)) \approx \omega^{A(3)}(\Delta_{3k}(Q_0)) \ge c$$

for some positive constant c depending only on M. Therefore (28) yields that $N_0(a) \le c^{-1}$, and that $\|a\|_{H^1(\omega)} \le c^{-1}$. Q.E.D.

Now we have proved Theorems 6.2 and 6.3. Therefore by combining these theorems with Theorem CK2, we gain Theorem 6.1.

As an immediate consequence of Theorem 6.1, we have the equivalence of BMO norm, Garsia norm and probabilistic BMO norm.

Corollary 6.9. *Let $f \in L^1(\omega)$. Then*

$$\|f\|_{\mathrm{BMO}} \approx \|f\|_G \approx \|f\|_{\mathrm{BMO}_{\mathrm{prob}}}.$$

Proof. By Proposition 6.5, it is sufficient to prove $\|f\|_{\mathrm{BMO}} \lessapprox \|f\|_{\mathrm{BMO}_{\mathrm{prob}}}$. Note that if $f \in \mathrm{BMO}_{\mathrm{prob}}$, then $f \in L^2(\omega)$. Moreover, $L^2(\omega)$ is dense in H^1_{atom}. Therefore by H^1_{atom}-BMO duality theorem (cf. [15]) and Theorem 6.1, we have that

$$\begin{aligned}\|f\|_{\mathrm{BMO}} &\approx \sup\left\{\left|\int_{S(\infty)} fh\,d\omega\right| : h \in L^2(\omega),\ \|h\|_{1,\mathrm{atom}} = 1\right\}\\ &\approx \sup\left\{|E[f(Z_\infty)h(Z_\infty)]| : h \in L^2(\omega),\ \|h\|_{1,\mathrm{prob}} = 1\right\}\\ &\lessapprox \|f\|_{\mathrm{BMO}_{\mathrm{prob}}},\end{aligned}$$

where for the last inequality, a probabilistic version of Fefferman's inequality is used. Q.E.D.

Remark 2. For $f \in L^1(\omega)$, let

$$\|f\|_{\mathrm{BMO},p} := \sup_{t\in\mathbf{R},\, Q\in S(\infty)} m_{\Delta_t(Q)}\left(|f - m_{\Delta_t(Q)}(f)|^p\right)^{1/p} + \|f\|_{L^p(\omega)},$$

$$\|f\|_{G,p} := \sup_{z\in M}\left(\int_M |f - \tilde{f}(z)|^p\,d\omega^z\right)^{1/p} + \|f\|_{L^p(\omega)},$$

$$\|f\|_{\mathrm{BMO}_{\mathrm{prob}},p} := \sup_{0\le t<\infty}\left\|E\left[|\tilde{f}(Z_\infty) - \tilde{f}(Z_t)|^p\right]^{1/p}\right\|_{L^\infty(W,P)} + \|f\|_{L^p(\omega)}.$$

It is known that if $f \in \mathrm{BMO}(\omega)$, then $f \in L^p(\omega)$ and $\|f\|_{\mathrm{BMO},p} \approx \|f\|_{\mathrm{BMO}}$ (cf. [15]). The same is true for probabilistic BMO, that is, if $f \in \mathrm{BMO}_{\mathrm{prob}}$, then $\tilde{f}(Z_\infty) \in L^p(W,P)$ and $\|f\|_{\mathrm{BMO}_{\mathrm{prob}},p} \approx \|f\|_{\mathrm{BMO}_{\mathrm{prob}}}$ (cf. [36]). Now by a similar way as the proof of Proposition 6.5 we can prove that $\|f\|_{\mathrm{BMO}_{\mathrm{prob}},p} \le C\|f\|_{G,p} \le C\|f\|_{\mathrm{BMO},p}$, for every $f \in L^p(\omega)$. Therefore what we have noted guarantees that

$$\|f\|_{G,p} \approx \|f\|_G. \tag{29}$$

§7. Carleson measures and Martin integrals

In this section we study a condition on a measure μ on M in order that the Martin integral operator,

$$K[f](z) = \int_{S(\infty)} K(z,Q)f(Q)\,d\omega(Q)\,(= \tilde{f}(z)), \quad z \in M,$$

is bounded from $L^p(\omega)$ to $L^p(M,\mu)$. This problem was studied by L. Carleson in the classical Euclidean case, and he found a necessary and sufficient condition called now "Carleson condition". We will study a version to M of "Carleson condition":

Definition 7.1. For a set $A \subset S(\infty)$ and $r > 0$, let

$$S_r[A] := \{z \in M \setminus B(o,r) : \Delta(z,0) \subset A\}.$$

A given complex Borel measure μ on M is said to be a Carleson measure on M if for every $r > 0$,

$$\|\mu\|_{c,r} := \sup_{Q\in S(\infty),\, t>1} \frac{|\mu|(S_r[\Delta_t(Q)])}{\omega(\Delta_t(Q))} + |\mu|(M) < \infty,$$

where $|\mu|$ is the total variation of μ. We wirte $\|\mu\|_c = \|\mu\|_{c,1}$.

As an analogue of the classical Carleson-Hörmander's theorem, we will prove the following by using Stein's idea:

Theorem 7.1. *Let μ be a complex Borel measure on M. Then the following are equivalent*:

(i) *μ is a Carleson measure on M.*

(ii) *$\|\mu\|_{c,r} < \infty$ for some $r > 0$.*

(iii) *For every $1 \le p < \infty$, the Martin integral operator K is bounded from $H^p(\omega)$ to $L^p(M,|\mu|)$.*

(iv) *For every $1 < p < \infty$, the operator K is bounded from $L^p(\omega)$ to $L^p(M,|\mu|)$.*

(v) *For some $1 < p < \infty$, the operator K is bounded from $L^p(\omega)$ to $L^p(M,|\mu|)$.*

Furthermore, for every $r > 0$, there is a constant C'_r depending only on M, o and r such that

$$C_r'^{-1}\|\mu\|_{c,r} \le \|\mu\|_c \le C'_r\|\mu\|_{c,r}.$$

Proof. We begin with proving "(ii) $\Rightarrow$ (iii)". We may assume that $r > k' + 1$, where k' is the positive constant in Lemma V. Suppose

$f \in H^p(\omega)$ and $\lambda > 0$. Let $E := \{Q \in S(\infty) : N_{0,r}(f) > \lambda\}$, and $G := \{z \in M \setminus B(o,r) : |\tilde{f}(z)| > \lambda\}$. Since $\{\Delta(z,0) : z \in S_r(E)\}$ is a covering of the bounded open set E, Vitali type covering lemma (Lemma V) guarantees that there exist $z_1, z_2, \ldots \in S_r(E)$ satisfying that $\Delta(z_i,0) \cap \Delta(z_j,0) = \emptyset$ $(i \neq j)$, and that for every $z \in S_r(E)$, $\Delta(z,0) \subset \Delta(z_i,-k')$ for some i.

Now let $z \in G$, and $Q \in \Delta(z,0)$. Then $z \in \Gamma_0(Q) \cap (M \setminus B(o,r))$, and therefore $N_{0,r}(f)(Q) > \lambda$. Hence $\Delta(z,0) \subset E$. Hence there exists i such that $\Delta(z,0) \subset \Delta(z_i,-k')$. This implies that $z \in S_r[\Delta(z,0)] \subset S_r[\Delta(z_i,-k')]$. Consequently, $G \subset \bigcup_j S_r[\Delta(z_j,-k')]$. Using this we have

$$\begin{aligned}|\mu|(G) &\leq \sum_j |\mu|(S_r[\Delta(z_j,-k')]) \leq \|\mu\|_{c,r} \sum_j \omega(\Delta(z_j,-k')) \\ &\leq C(k')\|\mu\|_{c,r} \sum_j \omega(\Delta(z_j,0)) \leq C(k')\|\mu\|_{c,r}\omega(E).\end{aligned}$$

Accordingly by these estimates, Lemma 5.3 (1) and (23) we have that

$$\begin{aligned}\int_M |\tilde{f}|^p \, d|\mu| &= \int_{B(o,r)} |\tilde{f}|^p \, d|\mu| + \int_{M\setminus B(o,r)} |\tilde{f}|^p \, d|\mu| \\ &\lessapprox \sup_{z\in B(o,r)} |\tilde{f}(z)|^p |\mu|(M) + \|\mu\|_{c,r} \int_{S(\infty)} N_{0,r}(f)^p \, d\omega \\ &\leq C_r |\mu|(M) \|f\|_{L^1(\omega)} + \|\mu\|_{c,r} \|N_{0,r}(f)\|_{L^p(\omega)} \\ &\leq (C_r+1)\|\mu\|_{c,r}\|f\|_{H^p(\omega)}.\end{aligned}$$

The part "(iii) $\Rightarrow$ (iv)" and "(iv) $\Rightarrow$ (v)" are obvious. We prove "(v) $\Rightarrow$ (i)". Let f be the characteristic function of $\Delta_t(Q)$ $(t > 1, Q \in S(\infty))$. Suppose $r > 0$. If $z \in S_r[\Delta_t(Q)]$, then $\Delta(z,0) \subset \Delta_t(Q)$. Hence for every $z \in S_r[\Delta_t(Q)]$, $\tilde{f}(z) \geq \omega^z(\Delta(z,0)) > c/2$, where c is the positive constant in (21). Denote by C_p the operator norm of K from $L^p(\omega)$ to $L^p(|\mu|)$. Therefore,

$$\begin{aligned}|\mu|(S_r[\Delta_t(Q)]) &\leq |\mu|(\{z \in M : \tilde{f}(z) > c/2\}) \leq (c/2)^{-p} \int_M \tilde{f}^p \, d\mu \\ &\leq (c/2)^{-p} C_p^p \|f\|_{L^p(\omega)}^p = (c/2)^{-p} C_p^p \omega(\Delta_t(Q)).\end{aligned}$$

Moreover, $|\mu|(M) \leq C_p^p \|1\|_{L^p(\omega)} \leq C_p^p$. Thus we have $\|\mu\|_{c,r} \leq ((c/2)^{-p}+1)C_p^p$. Q.E.D.

§8. Carleson measures and Green potentials

For a Borel measure μ on M, the function

$$G[\mu](x) = \int_M G(x,y)\,d\mu(y), \quad x \in M$$

is called the Green potential of μ. In this section we study boundary behavior of the Green potentials of the following weighted measures: for a nonnegative Borel measure μ on M, let

$$\mu_0(A) = \int_A \frac{1}{G(o,w)}\,d\mu(w), \quad A \subset M.$$

A nonnegative function f on M is said to be asymptotically bounded if there exists a positive constant $R > 0$ such that $\sup_{x \in M \setminus B(o,R)} f(x) < \infty$. Then we have the following

Theorem 8.1. *Let μ be a nonnegative Borel measure on M. Suppose that $\mu(H) < \infty$ for every compact set H in M. Then the following are equivalent:*

(i) *$G[\mu_0]$ is asymptotically bounded on M.*

(ii) *μ is a Carleson measure and satisfies the following condition* (F):

(F) *There exist positive constants r and C such that*

$$\int_{B(z,1)} G(z,w)\,d\mu(w) \le CG(o,z) \quad \text{for every } z \in M \setminus B(o,r). \tag{30}$$

We denote by $C_{r,\mu}$ the infimum of constants C in the condition (F).

This theorem is used in order to prove a Carleson measure characterization of BMO stated in the next section.

Proof. First we prove "(ii) $\Rightarrow$ (i)". In order to prove this we need the following lemma:

Lemma 8.2. *For $Q \in S(\infty)$ and $t > 0$, let $C(Q,t) = C(\gamma_{oQ}(t), 0)$. Then for $0 < r' < t + k$,*

$$C(Q, t+k) \subset S_{r'}[\Delta_t(Q)], \quad (Q \in S(\infty)).$$

Proof of Lemma 8.2. Let $w \in C(Q, t+k)$. By the condition (β) in Theorem 6.3 (see also Theorem CK2), we have that $\Delta_{t+k}(w(+\infty)) \cap$

$\Delta_{t+k}(Q) \neq \emptyset$. Hence (H3) implies that $\Delta_{t+k}(w(+\infty)) \subset \Delta_t(Q)$. Since $\Delta(w,0) \subset \Delta_{t+k}(w(+\infty))$, we have $w \in S_{r'}[\Delta_t(Q)]$.

End of the proof of Lemma 8.2.

We proceed to prove Theorem 8.1 "(ii) $\Rightarrow$ (i)". By Lemma 8.2, we have

$$\frac{\mu(C(Q,t+k))}{\omega(\Delta_{t+k}(Q))} \lessapprox \frac{\mu(C(Q,t+k))}{\omega(\Delta_t(Q))} \leq \frac{\mu(S_{r'}[\Delta_t(Q)])}{\omega(\Delta_t(Q))} \lessapprox \|\mu\|_c, \tag{31}$$

where r' is a positive number with $r' < t+k$.

We may assume that the number r in the condition (F) is greater than $100k$. Now let $z \in M \setminus B(o,r+2)$. Denote $t(z) = d(o,z)$ and $z(t) = \gamma_{oz}(t(z)+t)$. Let N be the biggest positive integer with $t(z) > N+2$. Let

$$\begin{aligned} E_0 &= C(z(+\infty),t(z)) \quad \text{and} \\ E_j &= C(z(+\infty),t(z)-j) \setminus C(z(+\infty),t(z)-j+1), \end{aligned}$$

$(j = 1,2,\ldots,N)$. Then $\mu(E_j) \lessapprox \|\mu\|_c \omega(\Delta_{t(z)-j}(z(+\infty)))$.

We estimate each integral $\int_{E_j} G(z,w)\,d\mu(w)$. Let $w \in E_j$. Note that when $w \in M \setminus B(z(-j),3)$, then $G(z(-j+2),w) \approx G(z(-j-1),w)$. Suppose $N \geq j \geq 4$. By [1, Theorem 1] we have that

$$\begin{aligned} G(w,z) &\lessapprox G(z(-j+1),z)G(w,z(-j+2)) \\ &\approx G(z(-j+1),z)G(w,z(-j-1)). \end{aligned} \tag{32}$$

Hence for $w \in E_j \setminus B(z(-j),3)$

$$\begin{aligned} \frac{\omega(\Delta_{t(z)-j}(z(+\infty)))}{G(o,w)} &\approx \frac{\omega^{z(-j-1)}(\Delta_{t(z)-j}(z(+\infty)))}{G(z(-j-1),w)} \\ &\lessapprox \frac{\omega^{z(-j-1)}(\Delta_{t(z)-j}(z(+\infty)))G(z(-j+1),z)}{G(w,z)} \approx \frac{G(z,z(-j+1))}{G(w,z)} \\ &\approx \frac{G(z,z(-j+2))}{G(w,z)} \lessapprox \exp(-C_4 d(z,z(-j+2)))\frac{1}{G(w,z)} \\ &\lessapprox \exp(-C_4 j)\frac{1}{G(w,z)}. \end{aligned}$$

Accordingly, for $w \in E_j \setminus B(z(-j),3)$,

$$\frac{G(z,w)}{G(o,w)} \leq C\exp(-C_4 j)\frac{1}{\omega(\Delta_{t(z)-j}(z(+\infty)))}. \tag{33}$$

Suppose $w \in E_j \cap B(z(-j),3)$. In this case, we have $G(z,w) \approx G(z,z(-j))$ and $G(o,w) \approx G(o,z(-j))$. Moreover,

$$\begin{aligned}\frac{\omega(\Delta_{t(z)-j}(z(+\infty)))}{G(o,w)} &\approx \frac{\omega(\Delta_{t(z)-j}(z(+\infty)))}{G(o,z(-j))} \\ &\approx \frac{\omega^{z(-j-1)}(\Delta_{t(z)-1}(z(+\infty)))}{G(z(-j-1),z(-j))} \approx 1.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{G(z,w)}{G(o,w)} &\approx \frac{G(z,z(-j))}{G(o,w)} \approx \frac{G(z,z(-j))}{G(o,w)} \frac{G(o,w)}{\omega(\Delta_{t(z)-j}(z(+\infty)))} \\ &\approx \exp(-C_4 j) \frac{1}{\omega(\Delta_{t(z)-j}(z(+\infty)))}.\end{aligned}$$

Consequently for $4 \le j \le N$ and for $w \in E_j$ we have

$$\frac{G(z,w)}{G(o,w)} \le C \exp(-C_4 j) \frac{1}{\omega(\Delta_{t(z)-j}(z(+\infty)))}. \tag{34}$$

Using (34),

$$\begin{aligned}\int_{E_j} G(z,w)\,d\mu_0(w) &\lessapprox \exp(-C_4 j) \frac{\mu(E_j)}{\omega(\Delta_{t(z)-j}(z(+\infty)))} \\ &\lessapprox \exp(-C_4 j) \|\mu\|_c.\end{aligned} \tag{35}$$

Next we consider the case of $w \in E_0' = E_0 \cup \cdots \cup E_3$. Suppose $w \in E_0' \setminus B(z,r)$, where r is a constant in the condition (F). Then by Theorem BH2 and (21),

$$\frac{\omega(E_0')}{G(o,w)} \approx \frac{\omega^{z(-1)}(E_0')}{G(z(-1),w)} \approx \frac{1}{G(z,w)}.$$

Hence for $0 < t < t(z) - 3$

$$\begin{aligned}\int_{E_0' \setminus B(z,r)} G(z,w)\,d\mu_0(w) &\lessapprox \frac{\mu(C(z(+\infty),t(z)-3))}{\omega(\Delta_{t(z)-3}(z(+\infty)))} \\ &\lessapprox \frac{\mu(S_t[\Delta_{t(z)-3-k}(z(+\infty))])}{\omega(\Delta_{t(z)-3-k}(z(+\infty)))} \le C\|\mu\|_c.\end{aligned}$$

From (F) it follows that

$$\int_{B(z,r)} G(z,w)\,d\mu_0(w) \lessapprox \frac{C_r}{G(o,z)} \int_{B(z,r)} G(z,w)\,d\mu(w) \lessapprox C_r C_{r,\mu}.$$

We consider the case of $w \in M \setminus C(z(+\infty), t(z) - N)$. By Theorem BH2 we have that

$$\frac{G(w,z)}{G(o,z)} \approx \frac{G(w,z(-N+1))}{G(o,z(-N+1))} \approx G(w,z(-N+1)) \leq C.$$

Therefore

$$\int_{M\setminus C(z(+\infty),t(z)-N)} G(z,w)\,d\mu_0(w) \leq C\int_M d\mu \leq C\|\mu\|_c.$$

Summing up, we have that

$$(36) \qquad G[\mu_0](z) \leq C(\|\mu\|_c + C_{r,\mu}), \quad \text{whenever } z \in M \setminus B(o, r+2).$$

Next we prove (i) $\Rightarrow$ (ii). Suppose $H := \sup_{z\in M\setminus B(o,R)} G[\mu_0](z) < \infty$ for a positive number R. It is easy to see that $S_r[\Delta_t(Q)] \subset C(Q,t)$ for all $r > 0$. Suppose $t > R+2$. For $w \in S_R[\Delta_t(Q)]$, $z = \gamma_{oQ}(t-1/4)$, and $z' = \gamma_{oQ}(t-1/2)$,

$$\frac{G(z,w)}{G(o,w)} \approx \frac{G(z',w)}{G(o,w)} \approx \frac{G(z',z)}{G(o,z)} \approx \frac{1}{G(o,z)}.$$

On the other hand, for every $x \in M \setminus B(o,1/2)$,

$$(37) \qquad G(o,x) \approx \omega(\Delta_{d(o,x)}(x(+\infty))),$$

because for $x' = \gamma_{ox}(d(o,x)-1/2)$, Theorem BH2 implies that

$$\frac{G(o,x)}{\omega^o(\Delta_{d(o,x)}(x(+\infty)))} \approx \frac{G(x',x)}{\omega^{x'}(\Delta_{d(o,x)}(x(+\infty)))} \approx 1.$$

Therefore $G(o,z) \approx \omega(\Delta_t(Q))$. Accordingly,

$$\frac{G(z,w)}{G(o,w)} \approx \frac{1}{\omega(\Delta_t(Q))}.$$

From this estimates it follows that

$$H \geq \int_{S_R[\Delta_t(Q)]} \frac{G(z,w)}{G(o,w)}\,d\mu(w) \approx \frac{\mu(S_R[\Delta_t(Q)])}{\omega(\Delta_t(Q))}.$$

To prove $\mu(M) < \infty$, let z be a point in M with $d(o,z) = R+1/2$. Then

$$H \geq \int_{M\setminus B(o,R+1)} \frac{G(z,w)}{G(o,w)}\,d\mu(w) + \int_{B(o,R+1)\setminus B(o,R)} \frac{G(z,w)}{G(o,w)}\,d\mu(w).$$

Note that

$$\frac{G(z,w)}{G(o,w)} \geq c_1, \quad w \in M \setminus B(o,R+1),$$

$$\frac{G(z,w)}{G(o,w)} \geq c_2 G(z,w) \geq c_3, \quad w \in B(o,R+1) \setminus B(o,R),$$

where c_i $(i = 1,2,3)$ are positive constants depending only on M and R. Using this we have that $\mu(M \setminus B(o,R)) \lessapprox H$, and consequently by the assumption, $\mu(M) \lessapprox H + \mu(B(o,R)) < \infty$. Therefore $\|\mu\|_{c,R} < \infty$.

For $z \in M \setminus B(o,R+2)$,

$$\begin{aligned} H &\geq \int_M \frac{G(z,w)}{G(o,w)}\, d\mu(w) \geq \int_{B(z,1)} \frac{G(z,w)}{G(o,w)}\, d\mu(w) \\ &\approx \frac{1}{G(o,z)} \int_{B(z,1)} G(z,w)\, d\mu(w). \end{aligned}$$

Thus μ satisfies the condition (F). Q.E.D.

§9. Littlewood-Paley measures and BMO

In this section we prove a Carleson measure characterization of BMO functions. To state our theorem, we consider Littlewood-Paly type measure on M: for $f \in L^1(\omega)$, let

$$d\mu_f(w) = G(o,w)|\nabla \tilde{f}(w)|^2\, dV(w),$$

where dV is the volume measure with respect to the metric g, and $|\nabla \tilde{f}(w)|$ is the norm of the gradient of $\tilde{f}$ with respect to g, that is, in a local coordinate neighborhood,

$$|\nabla \tilde{f}(w)|^2 = \sum_{ij} g^{ij}(w) \frac{\partial f(w)}{\partial x_i} \frac{\partial f(w)}{\partial x_j},$$

where $(g^{ij}(w))$ is the inverse matrix of the metric $(g_{ij}(w))$. This is an analogue to M of the classical Littlewood-Paley measure.

First we prove the following theorem

Theorem 9.1. *Let $f \in L^2(\omega)$. Then*

$$\mu_f(M) = \int_{S(\infty)} |f(Q) - \tilde{f}(o)|^2\, d\omega(Q) < \infty.$$

Proof. This is an immediate consequence of Dynkin's formula: By (9), $f \in L^2(\omega^z)$. Let $h = f - \tilde{f}(z)$. Then $M_t^h := \tilde{h}(Z_t)$ is an L^2 bounded martingale with respect to $(W, \mathcal{F}^z, \mathcal{F}_t^z, P^z)$. Note that

$$G(z,w) = \int_0^\infty p(t,z,w)\,dt,$$

where $p(t,z,w)$ is the minimal fundamental solution of the equation $\partial/\partial t - \Delta_g$. Hence by Ito's formula we have that

(38)
$$\begin{aligned}\int_{S(\infty)} |h(Q)|^2\,d\omega^z(Q) &= E^z[|h(Z_\infty)|^2] = E^z\left[\int_0^\infty d\langle M^h, M^h\rangle\right] \\ &= E^z\left[\int_0^\infty |\nabla\tilde{h}(Z_t)|^2\,dt\right] = \int_0^\infty E^z[|\nabla\tilde{h}(Z_t)|^2]\,dt \\ &= \int_0^\infty \int_M p(t,z,w)|\nabla\tilde{h}(w)|^2\,dV(w)\,dt \\ &= \int_M G(z,w)|\nabla h(w)|^2\,dV(x).\end{aligned}$$

Taking $z = o$, the theorem was proved. Q.E.D.

We can characterize BMO functions in terms of Carleson measures and Green potentials:

Theorem 9.2. *Let $f \in L^2(\omega)$. Then the following are equivalent*:
(i) $f \in \mathrm{BMO}(\omega)$.
(ii) μ_f *is a Carleson measure on* M.
(iii) *The Green potential*

$$G_f(x) := \int_M G(x,w)|\nabla\tilde{f}(w)|^2\,dV(w)$$

is asymptotically bounded.
(iv) *The potential G_f defined in* (iii) *is bounded on M.*

Proof. First we prove "(i) $\Rightarrow$ (ii)". Let $f \in \mathrm{BMO}(\omega)$. Then by Corollary 6.9, $f \in \mathrm{BMO}_{\mathrm{prob}}$. Therefore $M_t := \tilde{f}(Z_t) - \tilde{f}(Z_0)$ is a BMO-martingale with respect to $(W, \mathcal{F}, \mathcal{F}_t, P)$. Hence by [36, p.333], we have that for every $(\mathcal{F}_t)$-stopping time T,

$$E\left[\int_T^\infty d\langle M, M\rangle\right] \le \|f\|_{\mathrm{BMO}_{\mathrm{prob}}} P(T < \infty). \tag{39}$$

Using this inequality we will prove the desired part. For this aim we need the following variant of (38):

Lemma 9.3. *Let $h \in L^1(\omega)$ with $\tilde{h}(0) = 0$, and let $M_t^h = \tilde{h}(Z_t)$. Then for every nonnegative Borel function F on M,*

$$E\left[\int_0^\infty F(Z_t)\, d\langle M^h, M^h\rangle\right] = \int_M F(x)G(o,x)|\nabla\tilde{h}(x)|^2\, dV(x).$$

Since this is proved by the same way as (38), we omit the proof of this lemma. We proceed to prove Theorem 9.2. We will prove that μ_f satisfies the condition (v) in Theorem 7.1 with $p = 2$. Let $\psi \in L^2(\omega)$. For $\lambda > 0$, let $T_\lambda = \inf\{t > 0 : |\tilde{\psi}(Z_t)| > \lambda\}$. Since $\{w \in W : |\tilde{\psi}(Z_s)| > \lambda\} \subset \{w \in W : T_\lambda(w) \leq s\}$, we have

$$L := \{(s,w) \in [0,\infty) \times W : |\tilde{\psi}(Z_s)| > \lambda\} \subset [T_\lambda, \infty[,$$

where $[T_\lambda, \infty[$ is the usual stochastic interval, i.e., $\{(s,w) : T_\lambda(w) \leq t < \infty\}$. Denote $d\nu(s,w) = d\langle M, M\rangle(dw)$. By (39) we have the following inequalities:

$$\begin{aligned}\nu(L) &\leq \nu([T_\lambda, \infty[) \leq \|f\|_{\mathrm{BMO_{prob}}} P(T_\lambda < \infty)\\ &\leq \|f\|_{\mathrm{BMO_{prob}}} P(\{\sup_t |\tilde{\psi}(Z_t)| > \lambda\}).\end{aligned}$$

Therefore we have by Lemma 9.3 and Doob's maximal inequality that

$$\begin{aligned}\int_M |\tilde{\psi}(w)|^2\, d\mu_f(w) = E\left[\int_0^\infty |\tilde{\psi}(Z_t)|^2\, dt\right] &\leq \|f\|_{\mathrm{BMO_{prob}}} E[\sup_t |\tilde{\psi}(Z_t)|^2]\\ &\leq 4\|f\|_{\mathrm{BMO_{prob}}} E[|\psi(Z_\infty)|^2] = 4\|f\|_{\mathrm{BMO_{prob}}} \|\psi\|^2_{L^2(\omega)}.\end{aligned}$$

By the proof of Theorem 7.1 (v) $\Rightarrow$ (i), we have

$$\|\mu\|_c \lessapprox \|f\|_{\mathrm{BMO}}. \tag{40}$$

Next we prove "(ii) $\Rightarrow$ (iii)". By Theorem 8.1, we need to prove only that μ_f satisfies the condition (F). To prove (F), we use the following inequality due to Mouton (see [38, p.502 and p.501]): there exists a positive constant C such that for every harmonic function u on M, and for every $z \in M$,

$$|\nabla u(z)|^2 \leq C \int_{B(z,1)} |\nabla u(w)|^2\, dV(w). \tag{41}$$

Suppose $z \in M \backslash B(o,r)$ for a sufficient large number r to be chosen later. Then $G(o,z) \approx G(o,y)$ $(y \in B(z,1))$. By (37), $G(o,z) \approx \omega(\Delta(z,0))$. Hence

$$\begin{aligned}|\nabla \tilde{f}(z)|^2 &\le C \int_{B(z,1)} \frac{G(o,y)}{G(o,z)} |\nabla \tilde{f}(y)|^2 \, dV(y) \\ &\le \frac{C}{\omega(\Delta(z,0))} \int_{B(z,1)} G(o,y) |\nabla \tilde{f}(y)|^2 \, dV(y) \\ &\le \frac{C}{\omega(\Delta(z,0))} \int_{C(z(+\infty), d(o,z)-c)} G(o,y) |\nabla \tilde{f}(y)|^2 \, dV(y),\end{aligned}$$

where c is a positive constant depending only on κ_1 and κ_2 such that $B(z,1) \subset C(z(\infty), d(o,z)-c)$. We suppose $r > c+1$. Then that μ_f is a Carleson measure implies that

$$\frac{1}{\omega(\Delta(z,0))} \int_{C(z(+\infty), d(o,z)-c)} G(o,y) |\nabla \tilde{f}(y)|^2 \, dV(y) \le C \|\mu_f\|_c,$$

that is,

$$\sup_{z \in M \backslash B(o,r)} |\nabla \tilde{f}(z)|^2 \le C \|\mu_f\|_c. \tag{42}$$

Therefore for $z \in M \setminus B(o, r+1)$,

$$\begin{aligned}\int_{B(z,1)} G(z,w) \, d\mu_f(w) &= \int_{B(z,1)} G(z,w) G(o,w) |\nabla \tilde{f}(w)|^2 \, dV(w) \\ &\le C \|\mu_f\|_c G(o,z) \int_{B(z,1)} G(z,w) \, dV(w) \le C' \|\mu_f\|_c G(o,z).\end{aligned}$$

This implies that μ_f satisfies the condition (F) with $r+1$.

Now we prove "(iii) $\Rightarrow$ (i)". By (38) we have

$$\int_{S(\infty)} |f - \tilde{f}(z)|^2 \, d\omega^z = \int_M G(z,w) |\nabla \tilde{f}(w)|^2 \, dV(w). \tag{43}$$

Hence the Hölder inequality and the condition (iii) yield that for some $r > 0$,

$$\sup_{z \in M \backslash B(o,r)} \int_{S(\infty)} |f - \tilde{f}(z)| \, d\omega^z < \infty.$$

However, (9) implies that for every $z \in B(o,r)$,

$$\begin{aligned}\int_{S(\infty)} |f - \tilde{f}(z)| \, d\omega^z &\le C_r \int_{S(\infty)} |f - \tilde{f}(z)| \, d\omega \\ &\le C_r \|f\|_{L^1(\omega)} + C_r^2 \|f\|_{L^1(\omega)}.\end{aligned}$$

Consequently $\|f\|_G < \infty$, and by Corollary 6.9 we have $f \in \mathrm{BMO}(\omega)$.

Lastly, we prove (i) $\Rightarrow$ (iv). By (43), Corollary 6.9 and Remark 2 we have

$$\|f\|_{G,2}^2 \geq \sup_{x \in M} G_f(x).$$

The part of "(iv) $\Rightarrow$(iii)" is obvious. Thus the theorem was proved.

Q.E.D.

Remark 3. (1) In the classical Euclidean case, the part "(i) $\Leftrightarrow$ (ii)" was proved by C. Fefferman and E. Stein ([22]), and in the case of the complex unit ball endowed with the Bergman metric, the similar reseult to Theorem 9.2 was proved by Jevtic [25]. However, his proof is based on the nature of the ball, and our proof is different from it. See also [26] and [31] for related results.

(2) The proof of the part "(i) $\Rightarrow$ (ii)" is based on an idea in Arai [4]. However this part can be proved also by using Theorem 8.1. Indeed, we have (i) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (ii) by Theorem 8.1.

§10. Bloch functions on manifolds and a gradient estimate for harmonic functions

In this section we study Bloch functions on M and give an application of Theorem 9.2.

Bloch functions were defined originally on the open unit disc D in $\mathbf{C}$: a holomorphic function f on D is said to be a Bloch function on D if

$$\sup_{z \in D}(1 - |z|)|f'(z)| < \infty. \tag{44}$$

In other word, f is a Bloch function if and only if the norm of gradient $|\nabla f|$ with respect to the Poincáre metric is bounded. Taking this fact into account, Bloch functions on an n-dimensional Riemannian manifold $(\mathcal{R}, h)$ are defined as follows:

Definition 10.1. Let f be a harmonic function on $\mathcal{R}$. Then f is said to be a harmonic Bloch function on $\mathcal{M}$ if

$$\|f\|_B := \sup_{x \in \mathcal{R}} |\nabla f(x)| < \infty,$$

where $|\nabla f|$ is the norm of gradient of f with respect to the metric h, i.e., $|\nabla f(x)|^2 = \sum_{i,j} h^{ij}(x)(\partial f(x)/\partial x_i)(\partial f(x)/\partial x_j)$, where $(h^{ij}(x))$ is the inverse matrix of the Riemannian metric $(h_{ij}(x))$. Denote by $\mathcal{B}(\mathcal{R}, h)$ the linear space consisting of all harmonic Bloch functions on $\mathcal{R}$.

In particular, if $(\mathcal{R}, h)$ is a Kähler manifold, then a function u is said to be a holomorphic Bloch function on M if u is a harmonic Bloch function and holomorphic on $\mathcal{R}$.

The first question we have to ask is whether there exists a nonconstant harmonic Bloch function. As known, existence problem for nonconstant bounded harmonic functions is a crucial theme in geometric analysis. Indeed, this problem has been a motivation of analysis on negatively curved manifold, and M. T. Anderson ([2]) and D. Sullivan ([46]) proved existence of a lot of nonconstant bounded harmonic functions on M. Fortunately, their result implies also existence of nonconstant Bloch functions on M, because a gradient estimate of harmonic functions due to S.-T. Yau ([45, Corollary 3.1], [49]) tells us the following

Proposition Y. *Suppose $(\mathcal{R}, h)$ is a complete Riemannian manifold such that its Ricci curvature is bounded below by a constant. Then a bounded harmonic function on M is a harmonic Bloch function on M.*

Therefore if the manifold $\mathcal{R}$ has a nonconstant bounded harmonic function, then it possesses a nonconstant harmonic Bloch function. However, the converse is not true because of the following easy fact:

Proposition 10.1. *Suppose that $\mathcal{R} = \mathbf{R}^n$ and h is the Euclidean metric. Then $\mathcal{B}(\mathcal{R}, h)$ is an $(n+1)$-dimensional linear space.*

Proof. Since h is the Euclidean metric, we have for $u \in C^1(\mathbf{R}^n)$, $\|u\|_B^2 = \sup_{x\in\mathbf{R}^n} \sum_{j=1}^n |(\partial u(x)/\partial x_j)|^2$. Therefore the coordinate functions $u_j(x) = x_j$ and the constant function $u_0(x) = 1$ are harmonic Bloch functions on $\mathbf{R}^n$. Now let u be a harmonic Bloch function $\mathbf{R}^n$. Then $\partial u/\partial x_j$ is also harmonic on $\mathbf{R}^n$ $(j = 1, \ldots, n)$, and by definition it is bounded on $\mathbf{R}^n$. By Liouville's theorem $\partial u/\partial x_j$ must be a constant. Consequently u must be an affine function on $\mathbf{R}^n$. Q.E.D.

Since by Liouville's theorem there is no nonconstant bounded harmonic functions on $\mathbf{R}^n$, it is interested to find a geometric condition in order that a unbounded harmonic or holomorphic Bloch function exists, but it is beyond the scope of this paper to study the problem. (See Remark 4 (1) and Li and Tam [32].)

However it might be worthwhile to point out that in the case of our manifold M, Theorem 9.2 guarantees that harmonic extensions of unbounded BMO functions are unbounded harmonic Bloch functions:

Theorem 10.2. *Suppose $f \in \mathrm{BMO}(\omega)$. Then $\tilde{f}$ is a harmonic Bloch function on M. Indeed*

$$\sup_{x \in M} |\nabla \tilde{f}(x)| \leq C \|f\|_{\mathrm{BMO}}, \tag{45}$$

where C is a positive constant depending only on M and o.

In particular, there exists a unbounded BMO *function b, and $\tilde{b}$ is a unbounded harmonic Bloch function on M.*

Proof. We begin with proving the first assertion: By Theorem 9.2 we have that the Littlewood-Paley measure μ_f is a Carleson measure on M. Hence from (42) it follows that

$$\|\tilde{f}\|_B^2 \lessapprox \sup_{z \in B(o,r)} |\nabla f(z)|^2 + \|\mu_f\|_c \lessapprox \sup_{z \in B(0,r)} |\nabla \tilde{f}(z)|^2 + \|f\|_{\mathrm{BMO}}^2 < \infty,$$

where r is a positive constant in (42). In addition, by [45, Corollary 3.2] and Lemma 5.3 (i) we have

$$\begin{aligned}\sup_{z \in B(o,r)} |\nabla \tilde{f}(z)| &\lessapprox \sup_{z \in B(o,2r)} |\tilde{f}(z)| \leq \sup_{z \in B(o,2r)} \int_{S(\infty)} K(z,Q)|f(Q)|\, d\omega(Q) \\ &\lessapprox \|f\|_{L^1(\omega)} \leq \|f\|_{\mathrm{BMO}}.\end{aligned}$$

Hence $\|\tilde{f}\|_B \lessapprox \|f\|_{\mathrm{BMO}}$.

The second assertion is an immediate consequence of the first one. For if u is a bounded harmonic function on M, then by Fatou's theorem for M ([3]), we have that there exists $f \in L^\infty(\omega)$ satisfying $u = \tilde{f}$ on M. However, $L^\infty(\omega) \subsetneq \mathrm{BMO}(\omega)$ (see Appendix 2). Therefore, a unbounded harmonic Bloch function exists. Q.E.D.

Suppose u is a bounded harmonic function on M. Then there exists $f \in L^\infty(S(\infty), \omega)$ such that $\tilde{f} = u$. From Yau [45, Corollary 3.1] it follows that

$$\sup_{x \in M} |\nabla u(x)| \lessapprox \|f\|_{L^\infty(\omega)}. \tag{46}$$

On the other hand our inequality (45) implies that

$$\sup_{x \in M} |\nabla u(x)| \lessapprox \|f\|_{\mathrm{BMO}} \, \left(\leq 3\|f\|_{L^\infty(\omega)}\right),$$

which refines (46).

Remark 4. (1) Suppose $\mathcal{R}$ is a complete manifold with nonnegative Ricci curvature and u is a harmonic function on $\mathcal{R}$. Then it is known that by Yau's estimate u is a linear growth harmonic function if and only if $|\nabla u|$ is bounded (see [32]).

(2) Let $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$, and BMOA($\mathbf{T}$) the set of all functions f in BMO($\mathbf{T}$) such that the Poisson integral of f is holomorphic in D. Then it is known that if $f \in$ BMOA($\mathbf{T}$), then its Poisson integral is a holomorphic Bloch function on $D = \{z \in \mathbf{C} : |z| < 1\}$ (cf. [41]). This was extended to bounded, strongly pseudoconvex domain with smooth boundary by Krantz and Ma [30]. Our proof of Theorem 10.2 is different from their proofs. We note that the inequality (45) is an analogue to M of Jerison and Kenig [26, Lemma 9.9].

§11. Boundary behavior of harmonic Bloch functions

In the classical case of the unit disc in $\mathbf{C}$, a lot of unbounded holomorphic Bloch functions are known. For instance, $u(z) = \sum_{k=m}^{\infty} z^{15^k}$ ($z \in D$) is a holomorphic Bloch function, and it is known that for large m,

$$\limsup_{r\to 1} \frac{|u(re^{i\theta})|}{\sqrt{\log(1-r)^{-1}\log\log\log(1-r)^{-1}}} > 0.685\|u\|_B \text{ a.e. } \theta \in [0, 2\pi)$$

(see [41, p.194]). This means that u is not only unbounded, but also it has no boundary limits at almost every boundary point. On the other hand, Makarov proved the following

Theorem M (Makarov [34]; see also Pommerenke [41, p.186]). *Let u be a holomorphic Bloch function on D. Then for almost every $\theta \in [0, 2\pi)$,*

$$\limsup_{r\to 1} \frac{|u(re^{i\theta})|}{\sqrt{\log(1-r)^{-1}\log\log\log(1-r)^{-1}}} \le \|u\|_B.$$

Somewhat later a probabilistic version of Theorem M was proved in [33]:

Theorem L (Lyons [33]). *Let u be a holomorphic Bloch function on D. Let X_t be hyperbolic Brownian motion on D. Then*

$$\limsup_{t\to\infty} \frac{|u(X_t)|}{\sqrt{\log(1-|X_t|)^{-1}\log\log\log(1-|X_t|)^{-1}}} \le \|u\|_B.$$

In the higher dimensional case, little is known about unbounded Bloch functions. Recently, D. Ullrich constructed a holomorphic Bloch function on the open unit ball in $\mathbf{C}^n$ which has no finite radial limits ([47]).

In the rest of this section, we study boundary behavior of harmonic Bloch functions on M, and generalize Theorem L to the manifold M. As first we characterize Bloch functions in terms of Brownian motion:

Theorem 11.1. *For a harmonic function u on M, the following* (i) *and* (ii) *are equivalent:*

(i) *u is a harmonic Bloch function on M.*

(ii) *The stochastic process $\{u(Z_t)\}_t$ satisfies that*

$$\|u\|_{B,\mathrm{prob}}^2 := \sup_{x\in M}\left\{\frac{E_x[|u(Z_T)-u(Z_0)|^2]}{E_x[T]} : T\in\mathcal{T}_x,\ E_x[T]>0\right\}<\infty,$$

where $\mathcal{T}_x$ is the set of all $(\mathcal{F}_t^x)$-stopping times. Furthermore, $\|u\|_B \leq \|u\|_{B,\mathrm{prob}} \leq \sqrt{2}\,\|u\|_B$.

In the case of the open unit disc in $\mathbf{C}$, a martingale characterization of holomorphic Bloch functions was given in Muramoto [39]. We will prove Theorem 11.1 by simplifying and exploiting the method in [39] by combining an idea in Lyons [33].

Proof. (i) $\Rightarrow$ (ii). Let u be a harmonic Bloch function on M. By Ito's formula we have that

$$E_x[|u(Z_T)-u(Z_0)|^2] = 2E_x\left[\int_0^T |\nabla u(Z_s)|^2\,ds\right] \leq 2\|u\|_B^2 E_x[T],$$

for every $x\in M$ and $T\in\mathcal{T}_x$. Therefore $\|u\|_{B,\mathrm{prob}} \leq \sqrt{2}\,\|u\|_B$.

(ii) $\Rightarrow$ (i). Suppose $\|u\|_{B,\mathrm{prob}}<\infty$. Let α be an arbitrary number with $0<\alpha<\|u\|_B$. Then there exists a geodesic ball $B(z,\varepsilon)$ such that $\alpha \leq |\nabla u(x)|$ for all $x\in B(z,\varepsilon)$. Let $x\in B(z,\varepsilon)$ and $T=\inf\{t>0 : Z_t\notin B(z,\varepsilon)\}$. By the definition of $\|u\|_{B,\mathrm{prob}}$ we have

$$\begin{aligned}\alpha^2 E_x[T] &= E_x\left[\int_0^T \alpha^2\,ds\right] \leq E_x\left[\int_0^T |\nabla u(Z_t)|^2\right]\\ &= E_x[|u(Z_T)-u(Z_0)|^2] \leq \|f\|_{B,\mathrm{prob}}^2 E_x[T].\end{aligned}$$

Therefore $\alpha \leq \|u\|_{B,\mathrm{prob}}$. Thus $\|u\|_B \leq \|f\|_{B,\mathrm{prob}}$. Q.E.D.

Now we discuss on boundary behavior along Brownian paths of Bloch functions u on M. Since $M^u := \{u(Z_t) - u(Z_0)\}_t$ is a continuous local $(\mathcal{F}_t, P)$-martingale, it is known that the sets $\{\langle M^u, M^u\rangle_\infty < \infty\}$ and $\{\lim_{t\to\infty} M_t^u \text{ exists}\}$ are almost surely equal. Therefore we are interested to the behavior of M^u on the set $\{\langle M^u, M^u\rangle_\infty = \infty\}$:

Theorem 11.2. *Let u be a harmonic Bloch functions on M. Then*

$$\limsup_{t\to\infty} \frac{|u(Z_t)|}{\sqrt{d(o, Z_t)\log\log d(o, Z_t)}} \le C\|u\|_B \quad P\text{-a.s.}$$

Proof. By virtue of Theorem 11.1 we can apply an idea in [33] to our setting: Let $M_t = u(Z_t) - u(Z_0)$ and $T(t) := \inf\{s : \langle M, M\rangle_s > t\}$. By [42, Theorem 1.7, p.182], there exists an enlargement $(\tilde{W}, \tilde{\mathcal{F}}_t, \tilde{P})$ and a Brownian motion $\tilde{\beta}$ on $\tilde{W}$ independent of M^u such that the process

$$B_t := \begin{cases} M_{T(t)}, & t < \langle M, M\rangle_\infty, \\ M_\infty + \tilde{\beta}_{t-\langle M,M\rangle_\infty}, & t \ge \langle M, M\rangle_\infty, \end{cases}$$

is a standard linear Brownian motion. Therefore by the classical law of the iterated logarithm we have

$$\limsup_{t\to\infty} \frac{|B_t|}{\sqrt{t\log\log t}} = 1 \quad \tilde{P}\text{-a.s.} \tag{47}$$

Now on the set $A = \{\langle M, M\rangle_\infty = \infty\}$, we have $M_t = B_{\langle M,M\rangle_t}$, and

$$\langle M, M\rangle_s = 2\int_0^s |\nabla u(Z_r)|^2\, dr \le 2\|u\|_B^2 s.$$

Consequently, we have

$$\limsup_{t\to\infty} \frac{|u(Z_t) - u(o)|}{\sqrt{t\log\log t}} \le C\|u\|_B$$

P-a.s. on $\{\langle M, M\rangle_\infty = \infty\}$. On the other hand $t \approx d(o, Z_t)$ as $t \to \infty$ (see [35, (3.2), p.254]). Therefore we have the desired inequality P-a.s. on $\{\langle M, M\rangle_\infty = \infty\}$. Thus the theorem was proved. Q.E.D.

As an immediate consequence of Theorem 11.2 we have the following

Corollary 11.3. *Let $M = \{x \in \mathbf{R}^n : |x| < 1\}$ and let g be the hyperbolic metric on M. Then for a harmonic Bloch function u on (M, g),*

$$\limsup_{t\to\infty} \frac{|u(Z_t) - u(o)|}{\sqrt{\log(1 - |Z_t|)^{-1} \log\log\log(1 - |Z_t|)^{-1}}} \leq C\|u\|_B \quad a.s.\ P^o.$$

§12. Appendix 1 (A proof of Theorem BH2 (2))

Proof of Theorem BH2 (2). Let $\mathcal{C}_0$ (resp. $\mathcal{C}_0'$) be the cone with vertex $z(t)$ (resp. $z(t+1)$), direction tangent to $\partial C(z,t)$ (resp. $\partial C(z,t+1)$) of sufficiently small angle θ defined as in [1, p.518]. Consider sequences of cones $\mathcal{C}_1, \ldots, \mathcal{C}_k$ and $\mathcal{C}_1', \ldots, \mathcal{C}_l'$ obtained by iterated 1-shift of $\mathcal{C}_0$ and $\mathcal{C}_0'$ respectively. Then by our curvature condition, there exists a positive number ε depending only on κ_1 and κ_2 such that for an angle $\theta > \varepsilon$ the sequence of cones $\mathcal{C}_k^c, \ldots, \mathcal{C}_0^c$, $C(z, t+1/2)$, $\mathcal{C}_0', \ldots, \mathcal{C}_l'$ together with their vertices is a Φ-chain through $z(t+1/2)$ in the sense of Ancona [1], where Φ is depending only on κ_1 and κ_2. Moreover, by the remark after Proposition 15 in [1], we have that the sequence $(\mathcal{C}_l')^c, \ldots, (\mathcal{C}_0')^c$, $C(z, t+1/2)^c$, $\mathcal{C}_0, \ldots, \mathcal{C}_k$ together with their vertices is a Ψ-chian through $z(t+1/2)$ where Ψ depends only on Φ. Therefore from this observation it follows that the sets $M \cap C(z, t+1)^c$, $M \cap C(z, t+1/2)^c$ and $M \cap C(z,t)^c$ satisfy the assumption of [1, p.519, Theorem 5′]. Q.E.D.

Remark 5. It is easy to see that also Theorem BH1 for cones with more general aperture implies Theorem BH2.

§13. Appendix 2 (Unbounded BMO functions)

As is well known, the function $\log|x-1|$ on the unit sphere is unbounded but belongs to the classical BMO space. However, it seems to be difficult to construct a unbounded BMO function on the sphere at infinity. In this section we give a nonconstructive proof of existence of unbounded BMO function:

Proposition 13.1. *Suppose that (X, ρ, μ) is a space of homogeneous type in the sense of Coifman and Weiss [15], and that $\mu(X) = 1$. For $x \in X$ and $r > 0$, let $B(x,r) = \{y \in X : \rho(x,y) < r\}$. Assume that $\mu(B(x,r)) > 0$ for all $x \in X$ and $r > 0$, and that $\lim_{r\to 0} \mu(B(x,r)) = 0$ for every $x \in X$. Then $L^\infty(X) \neq \mathrm{BMO}(X)$.*

Proof. Suppose $L^\infty(X) = \mathrm{BMO}(X)$. Then it is easy to see that the norms $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{\mathrm{BMO}}$ must be equivalent. Therefore by H^1_{atom}-BMO duality and L^1-L^∞ duality, we have that the norms $\|\cdot\|_{L^1(\omega)}$ and $\|\cdot\|_{H^1_{\mathrm{atom}}}$ are equivalent on L^2. Since L^2 is dense in L^1 and H^1_{atom}, we have $H^1_{\mathrm{atom}} = L^1$. By [15], H^1_{atom} is isomorphic to the dual space of the Banach space VMO. Therefore L^1 also is isomorphic to the dual of VMO. However this is not the case as we will prove below (see Remark 6 after the proof). Using this isomorphism we can define on the space L^1 the topology induced from the weak $*$ topology of the dual of VMO. Moreover, with this topology, L^1 is a locally convex topological linear space. From Banach-Alaoglu theorem it follows that the closed unit ball $B(L^1)$ of L^1 is compact convex set in the induced topology. Therefore by Krein-Milman's theorem $B(L^1)$ must have at least one extreme point. However L^1 has no extreme points. For if $f \in B(L^1)$, then we can consider the following two cases: (Case 1) There exists a Borel set $E \subset X$ such that $\int_E |f|\,d\mu \in (0,1)$. (Case 2) For every Borel set $E \subset X$, $\int_E |f|\,d\mu = 1$ or $= 0$. In the first case, f is not an extreme point of $B(L^1)$, because $f = \alpha f_1 + (1-\alpha) f_2$, where $\alpha = \int_E |f|\,d\mu \in (0,1)$, $f_1 = \alpha^{-1} f\chi_E$, and $f_2 = (1-\alpha)^{-1} f\chi_{X\setminus E}$ (χ_A is the characteristic function of A). The second case does not happen: Let $\|f\|_{L^1} > 0$. Then we have $\mu(\{0 < |f| < \infty\}) > 0$. By Lebesgue's differential theorem, for almost every $x \in \{0 < |f| < \infty\}$,

$$\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f|\,d\mu = |f(x)| \in (0,\infty). \tag{48}$$

However, in the second case,

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f|\,d\mu = \frac{1}{\mu(B(x,r))} \quad \text{or} \quad 0,$$

and $1/\mu(B(x,r)) \to \infty$ as $r \to 0$. This contradicts (48). Q.E.D.

Remark 6. A. Pełczyński proved that for a σ-finite and non purely atomic measure space (Σ, μ), the space $L^1(\Sigma, \mu)$ is not isomorphic to any conjugate Banach spaces ([40]).

§14. Appendix 3 (Local Fatou-Doob theorem revisited)

In this section we study local version of Doob-Fatou type theorem on boundary behavior of harmonic functions, and yet another definition of Hardy spaces.

For $t > 0$, let $T^t_d(Q) = T_d(Q) \cap (M \setminus B(o,t))$ $(d > 0)$ and $\Gamma^t_\alpha(Q) = \Gamma_\alpha(Q) \cap (M \setminus B(o,t))$ $(\alpha \in \mathbf{R})$. We say a function f converges admissibly (resp. nontangentially) to l at $Q \in S(\infty)$ or has an admissible limit (resp. a nontangential limit) l if $\lim_{k\to\infty} f(x_k) = l$ for every $\alpha \in \mathbf{R}$ (resp. $d > 0$) and for every sequence $\{x_k\}_k \subset \Gamma_\alpha(Q)$ (resp. $T_d(Q)$) with $x_k \to Q$ as $k \to \infty$. We wirte the admissible limit (resp. nontangential limit) by ad-$\lim_{x\to Q} f(x)$ (resp. K-$\lim_{x\to Q} f(Q)$).

In 1989 we proved a local version of Fatou-Doob theorem under a technical assumption on admissible regions (see [5, Theorem 5.1]). However, by virtue of results in Cifuentes and Korányi [14] we can remove the assumption, and obtain the following theorem by the same way as in [5, Theorem 5.1]:

Theorem 14.1. *Let h be a positive harmonic function on M. Let E be a Borel subset of $S(\infty)$, and u a harmonic function on M. Assume that for each $Q \in E$, there exist $t > 0$ and $\alpha \in \mathbf{R}$ such that u/h is bounded below on $\Gamma^t_\alpha(Q)$. Then there exist $F_1, F_2 \subset S(\infty)$ such that $\omega(F_1) = \mu_h(F_2) = 0$, where μ_h is the Martin representing measure for h, and that u/h converges admissibly at every point in $E \setminus (F_1 \cup F_2)$.*

Recently, F. Mouton [38] proved local versions of Fatou theorem and of Calderón-Stein type theorem. We note that the former follows also from Theorem 14.1 with $h = 1$.

As an application of Theorem 14.1 we prove the equivalence of $H^p(\omega)$ and $H^p(M)$ which is defined in §3:

Theorem 14.2. *Suppose $1 \le p \le \infty$. Let $u \in H^p(M)$. Then there exists $f \in H^p(\omega)$ such that $\tilde{f} = u$ on M.*

Proof. Here we prove only the case of $p = 1$. We can apply the following proof to other $p \in (1,\infty)$. Because of Theorem 14.1 (or [38, Theorem 5.1] and Theorem CK1), for ω-a.e. $Q \in S(\infty)$, the admissible limit $f(Q) =$ ad-$\lim_{x\to Q} u(x)$ exists. Furthermore, $u_t(Q) = u(\gamma_{oQ}(t))$ is continuous on $S(\infty)$ and $f(Q) = \lim_{t\to\infty} u_t(Q)$ ω-a.e. $Q \in S(\infty)$ and $|u_t(Q)| \le N_0(u)(Q)$. Therefore we have that the function $f : Q \mapsto f(Q)$ is measurable and $f \in L^1(\omega)$.

It remains to prove that $u = \tilde{f}$. Let $z \in M$. By (9), $N_0(f) \in L^1(\omega^z)$. Since Theorem 6.2 holds true also for ω^z, we have that $\sup_{0\le t<\infty} |u(Z_t)| \in L^1(W, P_z)$. Therefore by the martingale convergence theorem there exists $F_z \in L^1(W, P_z)$ such that $\lim_{t\to\infty} u(Z_t) = F_z$ and $u(Z_t) = E_z[F_z/\mathcal{F}^z_t]$ (P_z-a.s.). For $Q \in S(\infty)$, let (P^Q_z, Z_t) be the conditioned Brownian motion to exit M at Q (see [38, 3.3] for the definition and basic properties). Then

$$\int_{S(\infty)} P_z^Q(\{\lim_{t\to\infty} u(Z_t) = F_z\})\, d\omega^z(Q) = P_z(\{\lim_{t\to\infty} u(Z_t) = F_z\}) = 1.$$

Hence for ω^z-a.e. $Q \in S(\infty)$, $P_z^Q(\{\lim_{t\to\infty} u(Z_t) = F_z\}) = 1$. From this we prove the following assertion:

Assertion. *There exists a constant $G_{Q,z}$ such that $\lim_{t\to\infty} u(Z_t) = G_{Q,z}$ P_z^Q-a.e.*

Proof of Assertion. Let $I_j = [j, j+1)$, and $L_j = \{\lim_{t\to\infty} u(Z_t)$ exists in $I_j\}$ $(j \in \mathbf{Z})$. Then L_j is asymptotic in the sense of [38, p.482], and therefore the 0-1 law implies that $P_z^Q(L_j) = 1$ or 0. Therefore there exists a unique number $j \in \mathbf{Z}$ such that $P_z^Q(L_j) = 1$ and $P_z^Q(L_i) = 0$ for $i \neq j$. We write the interval I_j by H_1. Consider the dyadic decomposition of H_1, namely $H_{11} = [j, j+1/2)$ and $H_{12} = [j+1/2, j+1)$. Let $L_{1i} = \{\lim_{t\to\infty} u(Z_t)$ exists in $H_{1i}\}$ $(i = 1, 2)$. Then again by 0-1 law we have either $P_z^Q(L_{11}) = 1$ or $P_z^Q(L_{12}) = 1$. We denote by H_2 the interval H_{1i} with $P_z^Q(L_{1i}) = 1$. Continuing this procedure we get a decreasing sequence of intervals $\{H_k\}_{k=1,2,\ldots}$ such that $P_z^Q(\{\lim_{t\to\infty} u(Z_t)$ exists in $H_k\}) = 1$. Since there exists a point $G_{Q,z} \in \mathbf{R}$ such that $\{G_{Q,z}\} = \bigcap_k H_k$, the assertion is proved.

End of the proof of Assertion.

Because of (9) and Theorem CK1, K-$\lim_{x\to Q} u(x) = f(Q)$ for ω^z-a.e. $Q \in S(\infty)$. Hence by [38, Corollary 4.4] we have that $f(Q) = G_{Q,z}$ for ω^z-a.e. $Q \in S(\infty)$.

Since $Z_\infty = \lim_{t\to\infty} Z_t = Q$ a.s. P_z^Q for every $Q \in S(\infty)$, we have that for ω^z-a.e. $Q \in S(\infty)$,

$$\begin{aligned} 1 &= P_z^Q(\{\lim_{t\to\infty} u(Z_t) = G_{Q,z}\}) = P_z^Q(\{\lim_{t\to\infty} u(Z_t) = f(Q)\}) \\ &= P_z^Q(\{\lim_{t\to\infty} u(Z_t) = f(Z_\infty)\}). \end{aligned}$$

Therefore $P_z(\{\lim_{t\to\infty} u(Z_t) = f(Z_\infty)\}) = 1$. This implies that $F_z = f(Z_\infty)$ P_z-a.s. Thus

$$u(z) = E_z[u(Z_0)] = E_z[E_z[F_z/\mathcal{F}_0^z]] = E_z[f(Z_\infty)] = \int_{S(\infty)} f\, d\omega^z.$$

Q.E.D.

From this theorem it follows that $H^p(M)$ is naturally identified with $H^p(\omega)$.

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