Advanced Studies in Pure Mathematics 28, 2000
Combinatorial Methods in Representation Theory
pp. 81–101

Plane partitions II: $5\frac{1}{2}$ symmetry classes

Mihai Ciucu and Christian Krattenthaler

Abstract.

We present new, simple proofs for the enumeration of five of the ten symmetry classes of plane partitions contained in a given box. Four of them are derived from a simple determinant evaluation, using combinatorial arguments. The previous proofs of these four cases were quite complicated. For one more symmetry class we give an elementary proof in the case when two of the sides of the box are equal. Our results include simple evaluations of the determinants $\det\left(\delta_{ij} + \binom{x+i+j}{i}\right)_{0\le i,j\le n-1}$ and $\det\left(\binom{x+i+j}{2j-i}\right)_{0\le i,j\le n-1}$, notorious in plane partition enumeration, whose previous evaluations were quite intricate.

1. Introduction

A plane partition is an array of nonnegative integers with the property that all rows and columns are weakly decreasing. By a well-known bijection (see [**9**] or [**18**]), plane partitions contained in an $a \times b$ rectangle and with entries at most c can be identified with lozenge tilings of a hexagon $H(a, b, c)$ with side-lengths a, b, c, a, b, c (in cyclic order) and angles of 120° (a lozenge tiling of a region on the triangular lattice is a tiling by unit rhombi with angles of 60° and 120°).

In [**19**] Stanley describes ten natural symmetry classes of plane partitions. Strikingly, the number of elements in each symmetry class is

Received March 15, 1999.
2000 *Mathematics Subjects Classification.* Primary 05A15, 05B45, 05A17. Secondary 52C20, 11P81.
Key words and phrases. Plane partitions, symmetry classes, determinant evaluations, lozenge tilings, non-intersecting lattice paths, tiling enumeration, perfect matchings.
The first author was supported by a Membership at the Institute for Advanced Study and NSF grant DMS-9802390.

given by a simple product formula. The available proofs, however, are in many cases quite intricate (see [**19**], [**3**], [**13**] and [**21**]). In this paper we present simple proofs for five symmetry classes, and for one more we give an elementary proof in the case when two of the numbers a, b and c are equal.

Our proofs employ Kuperberg's observation [**13**] that the bijection mentioned in the first paragraph maps symmetry classes of plane partitions to symmetry classes of tilings of $H(a,b,c)$. The three basic symmetries, in the context of tilings T, are:

(1) the reflection $t : T \mapsto T^t$ (called *transposition*) in the diagonal joining the two vertices of $H(a,b,c)$ where sides of lengths a and b meet (this assumes $a = b$),

(2) the *rotation* $r : T \mapsto T^r$ by 120° around the center of $H(a,b,c)$ (assuming $a = b = c$), and

(3) the rotation $k : T \mapsto T^k$ by 180° (called *complementation*) around the center of $H(a,b,c)$.

If a tiling is invariant under one of these symmetries, it is called symmetric, cyclically-symmetric or self-complementary, respectively.

We employ simple combinatorial arguments to deduce four difficult symmetry classes from a determinant evaluation due to Andrews and Burge [**4**], which was later generalized by Krattenthaler [**12**], and then proved in a very simple way by Amdeberhan [**1**]. The main tool used in our proofs is the Factorization Theorem for perfect matchings presented in [**6**].

The first of this group of four symmetry classes is the case of cyclically symmetric plane partitions (i.e., $T^r = T$), first proved by Andrews [**2**] (for another proof and a q-version, see [**15**]). In fact, Andrews' result [**2**, Theorem 8] is a generalization of this case, and it gives a simple product formula for

$$\det\left(\delta_{ij} + \binom{x+i+j}{i}\right)_{0\le i,j\le n-1}. \tag{1.1}$$

Our proof also addresses this more general result, and answers thus the problem suggested by Mills, Robbins and Rumsey [**16**] of finding a simple solution for the evaluation of (1.1).

The next case we treat is that of cyclically symmetric transposed-complementary plane partitions (i.e., $T^r = T$ and $T^t = T^k$), first proved by Mills, Robbins and Rumsey [**16**]. Again, we solve the more general problem of evaluating the determinant

$$\det\left(\binom{x+i+j}{2j-i}\right)_{0\le i,j\le n-1}. \tag{1.2}$$

(It is in fact this more general result that is obtained in [**16**].)

The last two cases in this group of four are those of cyclically symmetric self-complementary (i.e., $T^r = T^k = T$) and totally symmetric self-complementary (i.e., $T^t = T^r = T^k = T$) plane partitions, which were first proved by Kuperberg [**13**] and Andrews [**3**], respectively.

The fifth case we deal with is that of transposed-complementary plane partitions (i.e., $T^t = T^k$), which was first proved by Proctor [**17**] using arguments from representation theory. We deduce it as a simple consequence of results in [**8**] on the tiling generating function of certain regions on the triangular lattice. (We note here that three of the five cases mentioned so far — those of invariance under the groups $\langle r\rangle$, $\langle tk\rangle$, and $\langle r, tk\rangle$ — are among the four that were given uniform solutions by Kuperberg [**14**] using representation theory; the fourth case covered in [**14**] is the base case.)

Finally, the "half" case — which we provide a simple proof for, based on the aforementioned results of [**8**], in case two of the numbers a, b and c are equal — is that of self-complementary plane partitions (i.e., $T^k = T$), which was first proved by Stanley [**19**] using the theory of symmetric functions.

In fact, one more "half-case" could be added to the ones mentioned above: if two of the numbers a, b and c are equal, the base case (i.e., no symmetry requirements) follows directly by specializing $k = 0$ in [**8**, Theorem 1.1(a)].

2. A determinant with two tiling interpretations

The determinant evaluation mentioned in the Introduction from which we will derive the first four symmetry classes is the following.

Theorem 2.1 (Krattenthaler [12]). *Let x, y and n be nonnegative integers with $x+y>0$, and set*

$$K_n(x,y) = \left(\frac{(x+y+i+j-1)!}{(x+2i-j)!\,(y+2j-i)!}\right)_{0\le i,j\le n-1}. \tag{2.1}$$

Then we have

(2.2)

$$\det(K_n(x,y)) = \prod_{i=0}^{n-1} \frac{i!\,(x+y+i-1)!\,(2x+y+2i)_i\,(x+2y+2i)_i}{(x+2i)!\,(y+2i)!},$$

where $(a)_k := a(a+1)\cdots(a+k-1)$ *is the shifted factorial.*

Proof (AMDEBERHAN [**1**]). We use the fact that for any matrix $A = (a_{ij})_{0\le i,j\le n-1}$ we have

$$\det A = \frac{(\det A_0^0)(\det A_{n-1}^{n-1}) - (\det A_0^{n-1})(\det A_{n-1}^0)}{\det A_{0,n-1}^{0,n-1}}, \tag{2.3}$$

where $A_{i_1,\dots,i_k}^{j_1,\dots,j_k}$ is the submatrix of A obtained by removing rows indexed by $i_1,\dots,i_k$ and columns indexed by $j_1,\dots,j_k$ (see e.g. [**11**]).

Take $A = K_n(x,y)$ in (2.3). It is readily seen that the five determinants on the right hand side can be written in the form $\det K_m(x',y')$, with $m < n$ and suitable x' and y'. More precisely, we obtain

$$\det K_n(x,y) = (\det K_{n-1}(x+1,y+1)\det K_{n-1}(x,y) - \det K_{n-1}(x+2,y-1)\det K_{n-1}(x-1,y+2))/\det K_{n-2}(x+1,y+1).$$

It is easy to check that the expression on the right hand side of (2.2) also satisfies the above recurrence. Thus, (2.2) follows by induction on n. □

We define a *region* to be any subset of the plane that can be obtained as the union of finitely many unit triangles of the regular triangular lattice. In a lozenge tiling of a region, we allow the tile positions to be weighted. The weight of a tiling is the product of the weights of the positions occupied by lozenges. The tiling generating function $L(R)$ of a region R is the sum of the weights of all its tilings.

We now introduce two types of regions whose tiling generating functions turn out to be very closely related to the determinant in the statement of Theorem 2.1.

Let n and x be nonnegative integers. Consider the pentagonal region illustrated by Figure 2.1, where the top side has length x, the southeastern side has length n, and the western and northeastern sides follow zig-zag paths of length $2n$. Weight the n tile positions fitting in the indentations of the northeastern boundary by $1/2$ (we indicate weightings by $1/2$ in our figures by placing shaded ovals in the corresponding

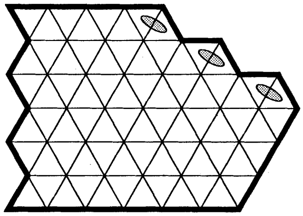

FIGURE 2.1. $A_{3,4}$.

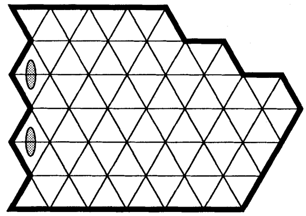

FIGURE 2.2. $B_{3,4}$.

tile positions; see Figure 2.1); weight all the others by 1. Denote the resulting region by $A_{n,x}$.

Let $B_{n,x}$ be the region with the same boundary as $A_{n,x}$, but having the $n-1$ tile positions fitting in the indentations of the western boundary weighted by 1/2, and all other tile positions weighted by 1 (see Figure 2.2).

The close connection between these regions and the determinant of the matrix (2.1) is expressed by the following result.

Proposition 2.2.

$$L(A_{n,x}) = \frac{1}{2^n}\det(K_n(x,0))\prod_{i=0}^{n-1}(x+3i) \tag{2.4}$$

$$L(B_{n,x}) = \frac{1}{2^n}\det(K_n(x,0))\prod_{i=0}^{n-1}(2x+3i). \tag{2.5}$$

Proof. We use the well-known procedure of encoding tilings of a region as families of non-intersecting lattice paths (see e.g. [**8**, §4]). By this, every tiling T of $A_{n,x}$ is identified with an n-tuple of paths of rhombi of T, each going from the western boundary of $A_{n,x}$ to its northeastern boundary. It follows that $L(A_{n,x})$ is equal to the generating function of n-tuples of non-intersecting lattice paths on the square lattice taking steps north and east, starting at the points $u_i = (i, 2n-2i-1)$ and ending at $v_i = (x+2i, 2n-i-1)$, $i = 0, \dots, n-1$, where paths with the last step horizontal are weighted by 1/2 (the weight of a family of paths is the product of the weights of its elements).

It is immediate to check that the u_i's and v_j's satisfy the requirements in the hypothesis of the basic theorem of Gessel and Viennot on non-intersecting lattice paths (see e.g. [**20**, Theorem 1.2] or [**10**]). We obtain that the above generating function of non-intersecting lattice paths equals

$$\det\left((a_{ij})_{0\le i,j\le n-1}\right), \tag{2.6}$$

where a_{ij} is the generating function of lattice paths from u_i to v_j. A straightforward calculation yields

$$\begin{aligned} a_{ij} &= \frac{1}{2}\binom{x+i+j-1}{2i-j} + \binom{x+i+j-1}{2i-j-1} \\ &= \frac{x+3i}{2}\frac{(x+i+j-1)!}{(x-i+2j)!\,(2i-j)!}. \end{aligned} \tag{2.7}$$

Therefore, by factoring out $(x+3i)/2$ along row i of the matrix in (2.6), we obtain (2.4).

To prove (2.5) we proceed similarly. Encoding tilings as lattice paths, we obtain that $L(B_{n,x})$ is equal to the generating function of n-tuples of non-intersecting lattice paths starting and ending at the same points as above, but now with paths having the *first* step *vertical* weighted by 1/2. It is easy to see that in this case we have

$$\begin{aligned} a_{ij} &= \frac{1}{2}\binom{x+i+j-1}{2i-j-1} + \binom{x+i+j-1}{2i-j} \\ &= \frac{2x+3j}{2}\frac{(x+i+j-1)!}{(x-i+2j)!\,(2i-j)!}. \end{aligned} \tag{2.8}$$

By factoring out $(2x+3j)/2$ along column j we obtain (2.5). □

3. Cyclically symmetric plane partitions

By (2) of the Introduction, this case amounts to enumerating r-invariant tilings of $H(n,n,n)$, where n is a positive integer.

We generalize this problem as follows. Consider the hexagonal region having sides of lengths n, $n+x$, n, $n+x$, n, $n+x$ (in cyclic order), where $n \ge 1$ and $x \ge 0$ are integers. Let $H_{n,x}$ be the region obtained from this hexagon by removing a triangular region of side x from its center, so that its vertices point towards the shorter edges of the hexagon (this is illustrated in Figure 3.1 for $n=4$, $x=2$). Denote by $CS(n,x)$ the number of lozenge tilings of $H_{n,x}$ that are invariant under rotation by 120°. Clearly, $CS(n,0)$ is the number of r-invariant tilings of $H(n,n,n)$.

The result below was inspired by Stembridge's proof of the special case $x=0$ (see [**21**, Lemma 2.4]), first proved by Andrews [**2**, Theorem 4].

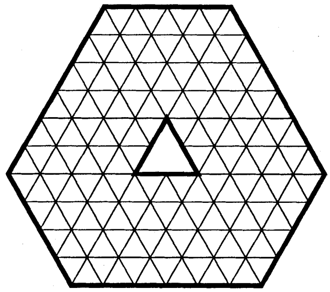

FIGURE 3.1.

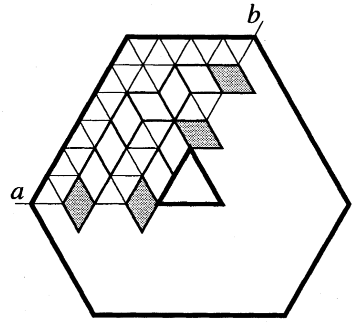

FIGURE 3.2.

Lemma 3.1.

$$CS(n,x) = \det\left(\delta_{ij} + \binom{x+i+j}{i}\right)_{0\le i,j\le n-1}.$$

Proof. Let a and b be two lattice rays originating at two vertices of the removed triangle so that they determine a fundamental region F for the action of r on $H_{n,x}$ (see Figure 3.2). Both a and b dissect n tile positions in $H_{n,x}$. Label these positions, starting with the ones closest to the removed triangle, by $0, 1, \dots, n-1$.

Clearly, for any r-invariant tiling of $H_{n,x}$, the sets of tiles crossed by a and b have the same labels. We claim that the number of tilings for which this set of labels is $0 \le i_1 < \cdots < i_k \le n-1$, $1 \le k \le n$ is equal to the principal minor of the matrix $B = \left(\binom{x+i+j}{i}\right)_{0\le i,j\le n-1}$ corresponding to these labels.

Indeed, r-invariant tilings are determined by their intersection with the fundamental region F, so such tilings with corresponding labels $i_1, \dots, i_k$ can be identified with tilings of the region $F(i_1, \dots, i_k)$ obtained from F by removing unit triangles along a and b in positions $i_1 \dots, i_k$.

In turn, using the standard encoding of lozenge tilings as families of non-intersecting lattice paths, the tilings of $F(i_1, \dots, i_k)$ are easily seen to be in bijection with k-tuples of non-intersecting lattice paths on the square lattice, taking steps north and east, starting at $u_\mu = (n-i_\mu-1, 0)$ and ending at $v_\mu = (n-1, x+i_\mu)$, $\mu = 1, \dots, k$. Apply the Gessel-Viennot theorem [**20**, Theorem 1.2]. The determinant corresponding to (2.6) is easily seen to be in this case precisely the principal minor of B corresponding to row and column indices $i_1, \dots, i_k$, thus proving our claim.

We obtain that $CS(n,x)$ is equal to the sum of all principal minors of B, i.e., to $\det(I+B)$. □

Theorem 3.2. *For $n, x \geq 1$ we have*

(3.1)
$$CS(2n,2x+1) = \frac{n!\,(x-1)!}{(2n)!} \prod_{i=0}^{n} \frac{(x+2i)_{i+1}}{(x+n+i)!} \prod_{i=0}^{n-1} \frac{[i!]^2\,[(2x+2i+2)_{i+1}]^2\,(x+i)!\,(x+2i+1)_i}{[(2i)!]^2}$$

(3.2)
$$CS(2n-1,2x+1) = \frac{(x-1)!\,(2x+2n)_n}{(x+n-1)!} \prod_{i=0}^{n-1} \frac{[i!]^2\,[(2x+2i)_i]^2\,(x+i)!\,(x+2i)_{i+1}\,(x+2i+1)_i}{[(2i)!]^2\,(x+n+i)!}.$$

Proof. Lozenge tilings of $H_{2n,2x+1}$ can naturally be identified with perfect matchings of the "dual" graph G, i.e., the graph whose vertices are the unit triangles of $H_{2n,2x+1}$ and whose edges connect precisely those unit triangles that share an edge (a perfect matching of a graph is a collection of vertex-disjoint edges collectively incident to all vertices of the graph; we will often refer to a perfect matching simply as a *matching*). Therefore, $CS(2n,2x+1)$ is the number of matchings of G invariant under the rotation r by 120° around the center of G.

Consider the action of the group generated by r on G, and let $\tilde{G}$ be the orbit graph. It follows easily that the r-invariant matchings of G can be identified with the matchings of $\tilde{G}$.

As illustrated in Figure 3.3 (for $n=3$, $x=2$), the graph $\tilde{G}$ can be embedded in the plane so that it admits a symmetry axis ℓ. Moreover, it can be readily checked that the Factorization Theorem [**6**, Theorem 1.2] for perfect matchings can be applied to $\tilde{G}$. We obtain that

$$M(\tilde{G}) = 2^{2n} M(\tilde{G}^+) M(\tilde{G}^-), \tag{3.3}$$

where $M(G)$ denotes the matching generating function of G, and $\tilde{G}^+$ and $\tilde{G}^-$ (illustrated in Figure 3.4) are the connected components of the

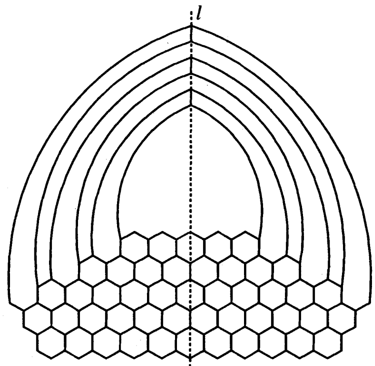

FIGURE 3.3.

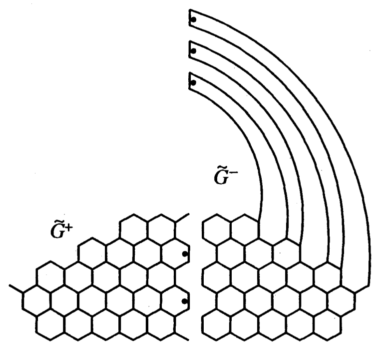

FIGURE 3.4.

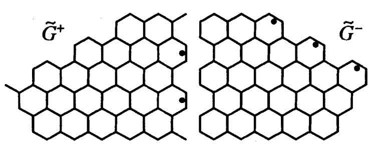

FIGURE 3.5.

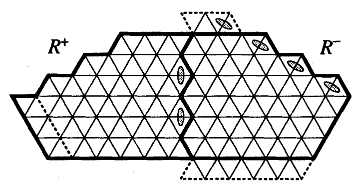

FIGURE 3.6.

subgraph obtained from $\tilde{G}$ by deleting the top $2n$ edges immediately to the left of ℓ, the bottom $2n$ edges immediately to the right of ℓ, and changing the weight of the $2n-1$ edges along ℓ to $1/2$ (the matching generating function of a graph is the sum of the weights of all its perfect matchings, the weight of a matching being the product of weights of its edges).

Clearly, the graphs $\tilde{G}^+$ and $\tilde{G}^-$ can be redrawn as shown in Figure 3.5. Using again the duality between matchings and tilings, we arrive at two regions R^+ and R^- whose tilings can be identified, preserving weights, with the matchings of $\tilde{G}^+$ and $\tilde{G}^-$ (see Figure 3.6; the boundaries of R^+ and R^- are shown in solid lines). However, because of forced tiles, it is readily seen that $L(R^+) = L(B_{n,x+1})$ and $L(R^-) = 2L(A_{n+1,x})$ (compare Figure 3.6 to Figures 2.1 and 2.2; the places where the boundaries of $B_{n,x+1}$ and $A_{n+1,x}$ differ from those of R^+ and R^- are indicated by dashed lines in Figure 3.6). Therefore, (3.3) can be rewritten as

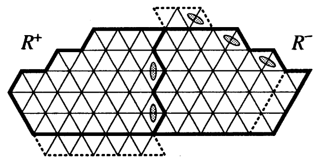

Figure 3.7.

$$CS(2n, 2x+1) = 2^{2n+1} L(A_{n+1,x}) L(B_{n,x+1}). \tag{3.4}$$

By Proposition 2.2 and Theorem 2.1, the above equality yields an explicit product formula for $CS(2n, 2x+1)$. After some manipulation one arrives at (3.1).

To prove (3.2) we proceed similarly. Take G to be the graph dual to $H_{2n-1,2x+1}$, construct the orbit graph $\tilde{G}$ as above and apply the Factorization Theorem to $\tilde{G}$. One obtains

$$M(\tilde{G}) = 2^{2n-1} M(\tilde{G}^+) M(\tilde{G}^-) \tag{3.5}$$

(the change in the exponent of 2 is due the fact that the "width" of $\tilde{G}$ — cf. [**6**], half the number of vertices on ℓ — is now $2n-1$).

The regions R^+ and R^- dual to $\tilde{G}^+$ and $\tilde{G}^-$ satisfy this time $L(R^+) = L(B_{n,x+1})$ and $L(R^-) = 2L(A_{n,x})$ (Figure 3.7 illustrates the case $n = 3$, $x = 2$). Therefore, (3.5) implies

$$CS(2n-1, 2x+1) = 2^{2n} L(A_{n,x}) L(B_{n,x+1}). \tag{3.6}$$

This provides, by Proposition 2.2 and Theorem 2.1, a product formula for $CS(2n-1, 2x+1)$, which one easily brings to the form (3.2). □

By Lemma 3.1, for fixed n, the expressions on the right hand side in (3.1) and (3.2) are polynomials in x. Define $P_{2n}(x)$ and $P_{2n-1}(x)$ to be the polynomials on the right hand side in (3.1) and (3.2), respectively.

Corollary 3.3. *With the above definition of the polynomials P_n, for all $n \geq 1$ we have*

$$CS(n, x) = P_n\left(\frac{x-1}{2}\right) \tag{3.7}$$

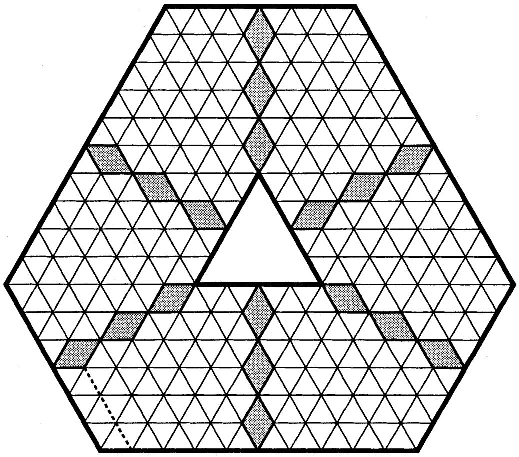

Figure 4.1.

as polynomials in x.

Proof. By Theorem 3.2, (3.7) holds if x is odd and $x \geq 3$. Since the two sides of (3.7) are polynomials (the left hand side by Lemma 3.1), they must be equal. □

REMARK 3.4. By Lemma 3.1 and Corollary 3.3 we obtain an expression for

$$\det\left(\delta_{ij} + \binom{x+i+j}{i}\right)_{0\leq i,j\leq n-1}$$

as a product of linear factors in x. This is equivalent to Theorem 8 of [**2**].

4. Cyclically symmetric transposed-complementary plane partitions

By (1), (2) and (3) of the Introduction, this case is equivalent to counting tilings of $H(n,n,n)$ that are invariant under the rotation r and the reflection t' across a symmetry axis of $H(n,n,n)$ not containing any of its vertices. More generally, we determine the number $CSTC(n,x)$ of r,t'-invariant tilings of the regions $H_{n,x}$ (defined at the beginning of Section 3). It is easy to see that $H_{n,x}$ has no such tilings unless n and x are both even.

Define the region $C_{n,x}$ to be the region having the same boundary as $A_{n,x}$ (see Figure 2.1), but with all tile positions weighted by 1.

Lemma 4.1. $CSTC(2n, 2x) = L(C_{n,x})$.

Proof. Suppose T is an r, t'-invariant tiling of $H_{2n,2x}$. It follows that T is invariant under reflection in the three symmetry axes of $H_{2n,2x}$. This implies that in T the $6n$ tile positions along these symmetry axes are occupied by lozenges (see Figure 4.1). The set of these $6n$ lozenges disconnects $H_{2n,2x}$ in six congruent pieces. Removing n forced lozenges from one of these pieces one obtains a region congruent to $C_{n,x}$ (this is indicated by the dotted line in Figure 4.1). The group generated by r and t' acts transitively on the set of these pieces. Therefore, the restriction of T to one of the pieces gives a bijection between r, t'-invariant tilings of $H_{2n,2x}$ and tilings of $C_{n,x}$. □

Theorem 4.2.

$$2\frac{CSTC(2n+2, 2x)}{CSTC(2n, 2x)} = \frac{CS(2n+1, 2x)}{CS(2n, 2x)}. \tag{4.1}$$

Proof. We deduce (4.1) by working out the analogs of (3.4) and (3.6) for the case when the second argument on the left hand side is even.

We proceed along the same lines as in the proof of Theorem 3.2. Let G be the graph dual to $H_{2n,2x}$, and let $\tilde{G}$ be the orbit graph of the action of $\langle r \rangle$ on G. As in the proof of Theorem 3.2, we can embed $\tilde{G}$ in the plane so that it admits a symmetry axis, and we can apply the Factorization Theorem of [**6**]. This expresses the number of perfect matchings of $\tilde{G}$ as a product involving the matching generating functions of two subgraphs. These two subgraphs can be redrawn in the plane such that they are the dual graphs of two regions R_1^+ and R_1^- on the triangular lattice. For $n = 2$, $x = 1$, these regions are illustrated in Figure 4.2 (their boundaries are shown in solid lines).

Therefore, since $M(\tilde{G}) = CS(2n, 2x)$, we can phrase the result of applying the Factorization Theorem to $\tilde{G}$ as

$$CS(2n, 2x) = 2^{2n} L(R_1^+) L(R_1^-). \tag{4.2}$$

However, by removing the n forced lozenges along the left boundary of R_1^+, we are left with a region congruent to $C_{n,x}$ (this is indicated by the dotted line in Figure 4.2). Thus, (4.2) implies

$$CS(2n, 2x) = 2^{2n} L(C_{n,x}) L(R_1^-). \tag{4.3}$$

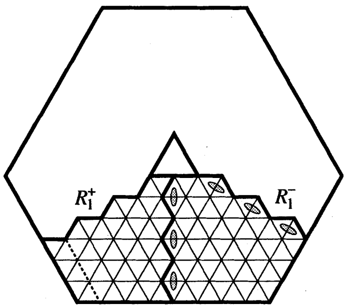

FIGURE 4.2.

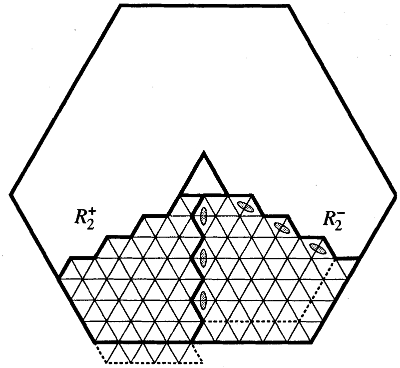

FIGURE 4.3.

Similarly, starting from the graph dual to $H_{2n+1,2x}$, considering its orbit graph under the action of $\langle r \rangle$ and applying the Factorization Theorem to it, we obtain after rephrasing everything in terms of tilings that

$$(4.4.) \qquad CS(2n+1,2x) = 2^{2n+1} L(R_2^+) L(R_2^-),$$

for two regions R_2^+ and R_2^- which are illustrated in Figure 4.3 for $n = 2$, $x = 1$ (their boundaries are shown in solid lines). However, R_2^+ is congruent to the region obtained from $C_{n+1,x}$ after removing the $n+x$ forced lozenges along its base. Moreover, the region obtained from R_2^- by removing all forced lozenges is isomorphic to R_1^-. Therefore, (4.4) becomes

$$(4.5.) \qquad CS(2n+1,2x) = 2^{2n+1} L(C_{n+1,x}) L(R_1^-).$$

Dividing (4.3) and (4.5) side by side and using Lemma 4.1 we obtain (4.1). □

Corollary 4.3.

$$(4.6) \qquad CSTC(2n,2x) = \frac{1}{2^n} \prod_{k=0}^{n-1} \frac{CS(2k+1,2x)}{CS(2k,2x)}.$$

(By Corollary 3.3, this provides an explicit formula for $CSTC(2n,2x)$.)

Proof. Take the side by side product of (4.1) for $n = 0, 1, \ldots, n-1$.
□

REMARK 4.4. Using the standard encoding of lozenge tilings as families of non-intersecting lattice paths, and then employing the Gessel-Viennot theorem [20, Theorem 1.2], it is easy to see that

$$L(C_{n,x}) = \det\left(\binom{x+i+j}{2j-i}\right)_{0\le i,j\le n-1}. \tag{4.7}$$

Therefore, by (4.7), Lemma 4.1 and Corollary 4.3 we obtain an expression for

$$\det\left(\binom{x+i+j}{2j-i}\right)_{0\le i,j\le n-1}$$

as a product of linear factors in x. Such a formula was first proved by Mills, Robbins and Rumsey in [16, Theorem 7].

REMARK 4.5. Following the notation of [5], let

$$Z_n(x) = \det\left(\delta_{ij} + \binom{x+i+j}{i}\right)_{0\le i,j\le n-1},$$

$$T_n(x) = \det\left(\binom{x+i+j}{2j-i}\right)_{0\le i,j\le n-1},$$

$$R_n(x) = \det\left(\binom{x+i+j}{2i-j} + 2\binom{x+i+j+2}{2i-j+1}\right)_{0\le i,j\le n-1}.$$

Encode the tilings of the region R_1^- in (4.3) as n-tuples of non-intersecting paths of rhombi going from the western boundary to the northeastern boundary of R_1^- (see Figure 4.2). Identify, as usual, these paths of rhombi with lattice paths on $\mathbf{Z}^2$. Apply the Gessel-Viennot theorem on non-intersecting lattice paths [20, Theorem 1.2]. It is easy to see that the (i,j)-entry of the Gessel-Viennot matrix M is in this case

$$M_{ij} = \binom{x+i+j}{2i-j} + \frac{1}{2}\binom{x+i+j}{2i-j-1} + \frac{1}{2}\binom{x+i+j}{2i-j+1} + \frac{1}{4}\binom{x+i+j}{2i-j},$$

for $i,j = 0,\ldots,n-1$. A simple calculation shows that

$$M_{ij} = \frac{1}{4}\left(R_n(x)\right)_{ij}.$$

Therefore, (4.3) and (4.5) can be written as

$$Z_{2n}(2x) = T_n(x)R_n(x),$$
$$Z_{2n+1}(2x) = 2T_{n+1}(x)R_n(x).$$

These are precisely relations (2.5) and (2.6) of [**5**], which were deduced there from [**16**, Theorem 5].

REMARK 4.6. The case $x = 0$ of Theorem 4.2 is the object of Theorem 6.2 of [**6**].

5. Cyclically symmetric self-complementary and totally symmetric self-complementary plane partitions

It is easy to see that in order for the hexagon $H(a, b, c)$ to have tilings in any of the two symmetry classes mentioned in the title of this section one needs to have $a = b = c = 2n$, with n a positive integer. Denote by $CSSC(2n)$ and $TSSC(2n)$ the number of tilings of $H(2n, 2n, 2n)$ in the two symmetry classes, respectively.

The following result was first proved by Kuperberg [**13**].

Theorem 5.1.

$$CSSC(2n) = \left(\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}\right)^2. \tag{5.1}$$

Proof. In [**7**] it is shown (see [**7**, (2.3)]) that a simple consequence of the Factorization Theorem for perfect matchings [**6**, Theorem 1.2] is that

$$CSSC(2n) = 2^n L(A_{n,1}). \tag{5.2}$$

(The derivation of this result follows along the lines of the proofs of (3.4), (3.6), (4.3) and (4.5).) Using Proposition 2.2 and Theorem 2.1 we obtain a product formula for $CSSC(2n)$, which is easily seen to be equivalent to (5.1). □

REMARK 5.2. Following the notation of [**5**], let

$$W_n(x) = \left(\binom{x+i+j+1}{2i-j+1} + \binom{x+i+j}{2i-j}\right)_{0 \le i,j \le n-1}.$$

Based on the fact that the related determinants $Z_n(x)$ and $T_n(x)$ defined in Remark 4.5 have close connections with plane partition enumeration problems, Andrews and Burge suggest in [**5**] that the same might be true for $\det W_n(x)$. Relation (5.2) allows us to give what appears to be the first such connection.

Indeed, let

$$w_n(x) = \left(\binom{x+i+j+1}{2i-j} + \binom{x+i+j}{2i-j-1}\right)_{0\le i,j\le n-1}.$$

It is readily checked that the matrix obtained from $w_n(x)$ by removing the first row and column is precisely $W_{n-1}(x+2)$. Since the top left entry of $w_n(x)$ is 1, we obtain that

$$\det W_{n-1}(x+2) = \det w_n(x). \tag{5.3}$$

A straightforward calculation reveals that $w_n(x)_{ij} = (x+3i+1)K_n(x+1,0)_{ij}$. Therefore, by (2.4), we deduce that

$$\det w_n(x) = 2^n L(A_{n,x+1}). \tag{5.4}$$

From (5.3) and (5.4), it follows that $\det W_{n-1}(x+2) = 2^n L(A_{n,x+1})$. Therefore, by (5.2), we obtain that $\det W_{n-1}(2) = CSSC(2n)$.

In [**7**] there is presented a direct proof of the fact that

$$CSSC(2n) = TSSC(2n)^2.$$

(In outline, by combinatorial arguments, an expression is derived for $CSSC(2n)$ as the determinant of a certain matrix, which is then transformed by elementary row and column operations to an antisymmetric matrix whose Pfaffian was previously known to give $TSSC(2n)$).

Therefore, we obtain by Theorem 5.1 the following result, first proved by Andrews [**3**].

Corollary 5.3.

$$TSSC(2n) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

Remark 5.4. By (3.4) and (3.6) we have

$$\frac{CS(2n, 2x+1)}{CS(2n-1, 2x+1)} = 2\frac{L(A_{n+1,x})}{L(A_{n,x})}. \tag{5.5}$$

On the other hand, from (5.2) we deduce

$$\frac{CSSC(2n+2)}{CSSC(2n)} = 2\frac{L(A_{n+1,1})}{L(A_{n,1})}.$$

This relation and (5.5) specialized to $x = 1$ imply

$$\frac{CSSC(2n+2)}{CSSC(2n)} = \frac{CS(2n, 3)}{CS(2n-1, 3)}. \tag{5.6}$$

One may regard (5.6) as giving a proof of the cyclically symmetric, self-complementary case based on the solution of the cyclically symmetric case, which was solved fifteen years earlier (see [**2**] and [**13**]).

6. Transposed-complementary plane partitions

By (1) and (3) of the Introduction, this case amounts to finding the number $TC(a, a, 2b)$ of tilings of the hexagon $H(a, a, 2b)$ that are symmetric with respect to its symmetry axis ℓ perpendicular to the sides of length $2b$ (see Figure 6.1; it is easy to see that the indicated form of the arguments represents the general case).

The following result was first proved (in an equivalent form) by Proctor [**17**].

Theorem 6.1.

$$TC(a, a, 2b) = \prod_{i=1}^{\lceil \frac{a}{2} \rceil} \frac{(b+i)_{a-2i+1}}{(i)_{a-2i+1}} \prod_{i=1}^{\lceil \frac{a-1}{2} \rceil} \frac{(2b+2i+1)_{a-2i;\,2}}{(2i+1)_{a-2i;\,2}},$$

where $(a)_{k;\,s} := a(a+s)(a+2s)\cdots(a+(k-1)s)$ is the shifted factorial of step s.

Proof. In any tiling T of $H(a, a, 2b)$ symmetric with respect to ℓ, the a tile positions along ℓ are occupied by lozenges. This set of lozenges divides our hexagon in two congruent pieces, and T is determined by its restriction to the left piece S, say (see Figure 6.1). Therefore, $TC(a, a, 2b)$ is just the number of tilings of S.

However, the region obtained from S by removing the forced lozenges (see Figure 6.1) is readily recognized as being a member of the family of

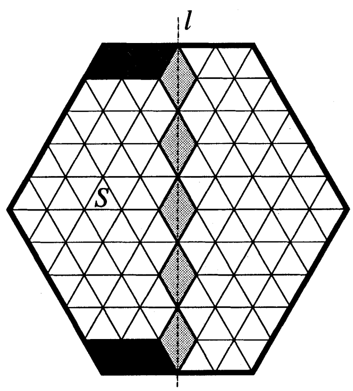

Figure 6.1.

regions $\bar{R}_{\mathbf{l},\mathbf{q}}(x)$ defined in [**8**, §2] (here **l** and **q** are lists of strictly increasing positive integers, and x is integer). More precisely, S is congruent to the region $\bar{R}_{[a-1],\emptyset}(b)$, where $[n]$ denotes the list $(1,\dots,n)$.

Therefore, Proposition 2.1 of [**8**] and formulas (1.6), (1.2) and (1.4) of [**8**] provide an expression for $L(S)$ (hence, for $TC(a,a,2b)$) as a product of linear polynomials in b. After some manipulation, this expression becomes the right hand side of the equality in the statement of the Theorem. □

7. Self-complementary plane partitions

This case amounts to enumerating tilings of $H(a,b,c)$ that are invariant under the rotation k by 180°, and it was first proved by Stanley [**19**]. In this section we give a simple proof in the case when two of the numbers a, b and c are equal.

Let $SC(a,a,b)$ be the number of k-invariant tilings of $H(a,a,b)$. It is easy to see that this number is 0 unless a or b is even. Let $PP(a,b,c)$ be the total number of tilings of $H(a,b,c)$ (for an explicit product formula, due to Macmahon, see e.g.[8, p.2]).

Theorem 7.1.

$$SC(2x,2x,2y) = PP(x,x,y)^2 \tag{7.1}$$

$$SC(2x,2x,2y+1) = PP(x,x,y)PP(x,x,y+1) \tag{7.2}$$

$$SC(2x+1,2x+1,2y) = PP(x,x+1,y)^2. \tag{7.3}$$

Proof. Following the same reasoning as in proving (3.4), (3.6), (4.3), (4.5) and (5.1), one sees that the Factorization Theorem of [**6**] can be

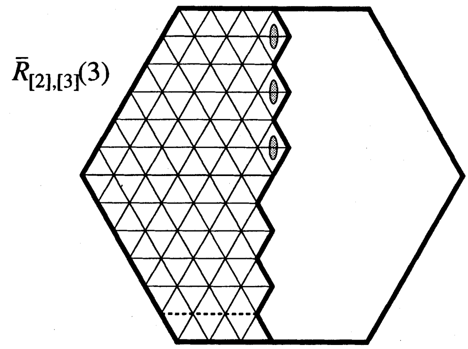

Figure 7.1. $a = 6$, $b = 6$.

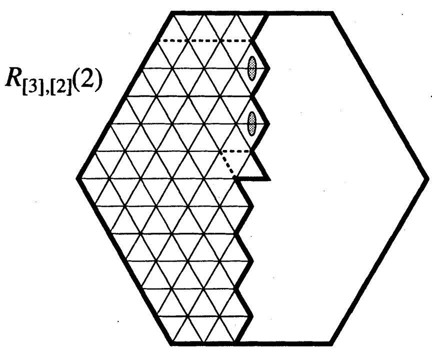

FIGURE 7.2. $a = 6$, $b = 5$.

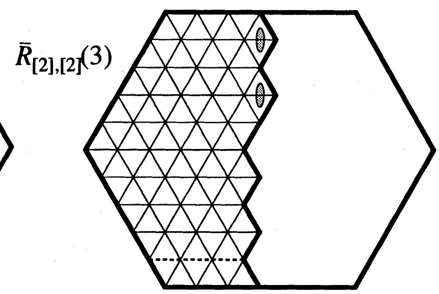

FIGURE 7.3. $a = 5$, $b = 6$.

used to express the number of k-invariant tilings of $H(a, a, b)$ as a power of 2 times the tiling generating function of a certain subregion with some tile positions weighted by $1/2$ (the precise shape of this region depends on the parities of a and b). Furthermore, after removing the forced lozenges from this region, the leftover piece is readily recognized to belong to one of the families $R_{\mathbf{l},\mathbf{q}}(x)$ or $\bar{R}_{\mathbf{l},\mathbf{q}}(x)$ defined in [**8**, §2].

More precisely, for $a = 2x$, $b = 2y$, we obtain that

$$L(H(2x, 2x, 2y)) = 2^x L(\bar{R}_{[x-1],[x]}(y)) \tag{7.4}$$

(see Figure 7.1; as usual, the dotted lines indicate removal of forced lozenges). Similarly, we deduce

$$L(H(2x,2x,2y+1)) = 2^x L(R_{[x],[x-1]}(y)) \tag{7.5}$$

$$L(H(2x+1,2x+1,2y)) = 2^x L(\bar{R}_{[x],[x]}(y)) \tag{7.6}$$

(see Figures 7.2 and 7.3).

By Proposition 2.1 of [**8**], (7.4)–(7.6) provide product formulas for $SC(a,a,b)$, and these are easily seen to agree with (7.1)–(7.3). □

Acknowledgments. The first author would like to thank John Stembridge, David Robbins and Dennis Stanton for stimulating conversations and for their interest in this work.

References

[**1**] T. Amdeberhan, Lewis strikes again!, unpublished manuscript dated 1997.

[**2**] G. E. Andrews, Plane partitions, III: The weak Macdonald conjecture, *Invent. Math.* **53** (1979), 193–225.

[**3**] G. E. Andrews, Plane Partitions, V: The T.S.S.C.P.P. conjecture, *J. Combin. Theory Ser. A* **66** (1994), 28–39.

[**4**] G. E. Andrews and W. H. Burge, Determinant identities, *Pacific J. Math.* **158** (1993), 1–14.

[**5**] G. E. Andrews and D. W. Stanton, Determinants in plane partitions enumeration, preprint (available at http://www.math.umn.edu/~stanton/PAPERS/det.ps).

[**6**] M. Ciucu, Enumeration of perfect matchings in graphs with reflective symmetry, *J. Combin. Theory Ser. A* **77** (1997), 67–97.

[**7**] M. Ciucu, The equivalence between enumerating cyclically symmetric, self-complementary and totally symmetric, self-complementary plane partitions, *J. Combin. Theory Ser. A* **86** (1999), 382–389.

[**8**] M. Ciucu, Plane partitions I: A generalization of MacMahon's formula, e-print math.CO/9808017, Los Alamos preprint server, http://xxx.lanl.gov/ps/math.CO/9808017.

[**9**] G. David and C. Tomei, The problem of the calissons, *Amer. Math. Monthly* **96** (1989), 429–431.

[**10**] I. M. Gessel and X. Viennot, Binomial determinants, paths, and hook length formulae, *Adv. in Math.* **58** (1985), 300–321.

[**11**] C. G. J. Jacobi, De formatione et proprietatibus determinantium, *J. Reine Angew. Math.* **22** (1841), 285–318.

[**12**] C. Krattenthaler, Determinant identities and a generalization of the number of totally symmetric self-complementary plane partitions, *Elect. J. Combin.* **4** No. 1 (1997), R27.

[**13**] G. Kuperberg, Symmetries of plane partitions and the permanent-determinant method, *J. Combin. Theory Ser. A* **68** (1994), 115–151.

[**14**] G. Kuperberg, Four symmetry classes of plane partitions under one roof, *J. Combin. Theory Ser. A* **75** (1996), 295–315.

[**15**] W. H. Mills, D. P. Robbins and H. Rumsey Jr., Proof of the Macdonald conjecture, *Invent. Math.* **66** (1982), 73–87.

[**16**] W. H. Mills, D. P. Robbins and H. Rumsey Jr., Enumeration of a symmetry class of plane partitions, *Discrete Math.* **67** (1987), 43–55.

[**17**] R. A. Proctor, Odd symplectic groups, *Invent. Math.* **92** (1988), 307–332.

[**18**] D. P. Robbins, The story of 1, 2, 7, 42, 429, 7436, ..., *Math. Intelligencer* **13** (1991), No. 2, 12–19.

[**19**] R. P. Stanley, Symmetries of plane partitions, *J. Combin. Theory Ser. A* **43** (1986), 103–113.

[**20**] J. R. Stembridge, Nonintersecting paths, Pfaffians and plane partitions, *Adv. in Math.* **83** (1990), 96–131.

[**21**] J. R. Stembridge, The enumeration of totally symmetric plane partitions, *Adv. in Math.* **111** (1995), 227–243.

Mihai Ciucu
Georgia Institute of Technology
School of Mathematics
686 Cherry Street, Skiles Building, Atlanta, GA 30332-0160, USA

Christian Krattenthaler
Institut für Mathematik der Universität Wien
Strudlhofgasse 4, A-1090 Wien, Austria