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Robinson-Schensted Correspondence and Left Cells

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§1. Introduction

This is based on [A]. In [A], I explained several theorems which focused on a famous theorem of [KL] that two elements of the symmetric group belong to a same left cell if and only if they share a common Q-symbol. The first half of [A] was about a direct proof of this theorem (Theorem A), and the second half was about relation between primitive ideals and left cells, and I explained another proof of this theorem.

The reason why I gave a direct proof which was different from the proof in [KL] was that the proof in [KL] was hard to read: It relied on [V1, 6], which in turn relied on [Jo1], the full paper of which is not yet available even today. Note also that the theorem itself is not stated in [KL]. But the beginning part of the proof of [KL, Theorem 1.4] gives some explanation on the relation between left cells in the sense of Kazhdan and Lusztig and Vogan's generalized τ-invariants in the theory of primitive ideals. In this picture, Theorem A is derived from Joseph's theorem.

Lack of a clear proof in the literature lead Garsia and McLarnan to the publication of [GM]. [1] The proof given in [GM] is close to [A], but the line of the proof in [GM] is interrupted with combinatorics of tableaux, which is not necessary. In fact, after we read to the fourth section of [KL], which is the section for some preliminaries to the proof of [KL, Theorem 1.4], we can give a short and elementary proof of Theorem A in a direct way, as I will show below.

I rush to say that my proof was not so original: It copied argument in [Ja1, Satz 5.25] for Joseph's theorem. This was the reason why I

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[1]It is worth mentioning that Garsia and McLarnan wrote in [GM] that they were benefitted by A.Björner's lecture notes and R.King's lecture notes. Both of these notes are still not available, and it seems that preliminary version of them were circulated in a very restricted group of people around the time.

did not publish it in English. But after a decade has passed, we still have no suitable literature which includes the direct proof. Further, we have new development in the last decade. For example, we have better understanding of this theorem in *jeu de taquin* context (see [H],[BSS]); the study of solvable lattice models in Kyoto school lead to the theory of crystal bases and canonical bases, by which we can understand Theorem A in the crystal base theory context.

I have therefore decided to add this short note to this volume in order to explain this proof and new development. I also prepare enough papers in the references for reader's convenience. I note that there is a sketch of proof of Theorem A in [BV, p.172]. It involves the notion of wave front sets. I do not recommend [BV] for knowing the proof of Theorem A. One reason is that we do not need wave front sets for the proof of Theorem A itself.

The non-direct proof I explained in the second half of [A] is the proof which is indicated in [KL], and the proof was taken from Jantzen's lecture notes [Ja1]. Thus I do not reproduce it, and I only give statements of several theorems (Theorem B,C,D) which concludes Theorem A.

I give some bibliographical comments on the second proof. It is obtained by combining two theorems; one known as Joseph's theorem, which states that two primitive ideals with a same integral regular central character coincide if and only if their Q-symbol coincide (Theorem B), and another theorem due to Joseph and Vogan, which relates the inclusion relation of primitive ideals to non-vanishing condition of certain multiplicities (Theorem C), and thus to order relation of left cells of the symmetric group (Theorem D).

In a survey [Bo], it is stated that Theorem A was proved in [Jo1], and simple proof could be found in [V1] and [Ja1]. But as I stated, [Jo1] is not published, and [V1] is based on [Jo1]. Thus to read [V1], one has to reproduce arguments by oneself. Nevertheless, it is Joseph's theorem, and his idea came from the explicit form of Goldie rank representations in type A case [Jo2, Proposition 8.4], which makes the number of primitive ideals with a common regular central character equal to the number of involutions. That is, Duflo's map is bijective, which proves Theorem B. The proof of Theorem B is given in [Jo3, Corollary 5.3]. We do not follow his line and I refer to [Ja1, Satz 5.25].

Theorem C is proved in [V2, Theorem 3.2]. One implication is due to [Jo5, Theorem 5.3], which is reformulated in [V2, Proposition 3.1]. It is not difficult to derive Theorem D: That Theorem C implies Theorem D (see also [Jo5, Conjecture C]) is stated in the introduction of [V2]. The proof here is based on [Ja1, Corollar 7.13] and [Ja1, Lemma 14.9]. Since the Kazhdan-Lusztig conjecture proved by Brylinski-Kashiwara and Beilinson-Berstein is very well known, I take it for granted when I explain Theorem D. But it is of cource a very deep result.

Joseph's Goldie rank representation was related to Springer representation and the theory evolved into a beautiful geometric representation theory. Since this part is not at all combinatorial, I do not mention it. I only refer to [BB3] for this development. There is also new direction for the generalization of the Robinson-Schensted correspondence related to the primitive ideal theory. I refer [Ga] and [Tr]. [2]

§2. Preliminaries

2.1. P-symbols and Q-symbols

Let S_n be the symmetric group of degree n. Namely, its underlying set is the set of bijective maps from $\{1,\dots,n\}$ to itself, and the group structure is given by composition of maps. Let w be an element in S_n, and denote the image of $i \in \{1,\dots,n\}$ under the map w by w_i. Throughout the paper, we identify w with the sequence $w_1 \cdots w_n$, which is a permutation of $1,\dots,n$.

Let $\mathbb{N} = \{1,2,\dots\}$ be the set of natural numbers. A **Young diagram** λ is a finite subset of $\mathbb{N}\times\mathbb{N}$ which satisfies the condition that if $(x,y)\in\lambda$, then $\{(x-1,y),(x,y-1)\}\cap\mathbb{N}\times\mathbb{N}\in\lambda$. $(x,y)\in\lambda$ is called a **node** of λ. x is called the row number of the node, and y is called the column number of the node.

A **tableau** T of shape λ is a map from λ to $\mathbb{N}$. The image of a node (x,y) of λ under the map is called the entry of the node, and is denoted by $T(x,y)$. We only consider the tableaux satisfying

$$T(x,y) \le T(x',y') \quad (x\le x', y\le y').$$

If it also satisfies $T(x,y) < T(x',y)$ $(x < x')$ (resp. $T(x,y) < T(x,y')$ $(y < y')$), T is called a **column strict** (resp. **row strict**) **semi-standard tableau**. If the entries of T are precisely $\{1,\dots,n\}$, we call T a **standard tableau.**

Let T be a column strict semi-standard tableau, k be a natural number. We denote the set of nodes in the i th row by $R_i(T)$, and denote the maximal column number of the nodes in $R_i(T)$ by c_i. Assume that we are given a natural number k_i. If k_i is equal or greater than all entries of $R_i(T)$, we add the node (i, c_i+1) to $R_i(T)$ and make its entry be k_i.

[2] There is one more dirction: generalization of the Steinberg's theorem [St] is given by M. van Leeuwen. This is the direction to the geometry of flag varieties.

If it is not the case, we consider the nodes of $R_i(T)$ whose entries are greater than k_i, and pick up the node of minimal column number among them. We then change its entry to k_i, and we make k_{i+1} be the original entry of the node. This latter procedure is called **bumping procedure**. We set $k_1 = k$, and continue the bumping procedure until no bumping occurs. This is called a **row insertion algorithm**, and it results in a new column strict semi-standard tableau, which we denote by $T \leftarrow k$. In the similar way, we can define a column insertion algorithm $k \rightarrow T$.

Definition 2.1. *Let $w = w_1 \cdots w_n$ be a permutation. Two standard tableaux $P(w)$ and $Q(w)$ defined by*

$$\begin{aligned} P(w) &= \emptyset \leftarrow w_1 \leftarrow \cdots \leftarrow w_n, \\ Q(w) &= P(w^{-1}) \end{aligned}$$

are called the **P-symbol** *of w and the* **Q-symbol** *of w respectively. The correspondence between w and the pair $(P(w), Q(w))$ is called the* **Robinson-Schensted correspondence**. *We often write $P(w) = \emptyset \leftarrow w_1 \cdots w_n$ for short.*

It is known that $P(w) = w_1 \rightarrow \cdots \rightarrow w_k \rightarrow \emptyset \leftarrow w_{k+1} \leftarrow \cdots \leftarrow w_n$ holds for any k.

Example 2.2. *If $w = 31524$, then we have*

$$P(w) = \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & \end{array} \qquad Q(w) = \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & \end{array}.$$

More familiar definition of the Q-symbol is by the "recording" tableau, which records the node added by each insertion procedure. It is a well known theorem that it coincides with $P(w^{-1})$.

Remark We have a two dimensional pictorial algorithm to compute P-symbols and Q-symbols due to S.V.Fomin [Fo3, 4.2.4]. In his picture, we know at a glance that $Q(w)$ equals $P(w^{-1})$.

Remark If two elements in S_n have a common P-symbol, we say that these belong to a same left Knuth class. Similarly, if these have a common Q-symbol, we say that these belong to a same right Knuth class.

Although we do not go into the combinatorial structures of the Robinson-Schensted correspondence, it is worth referring to the relation of the Robinson-Schensted correspondence to the *jeu de taquin* sliding algorithm. Namely, the insertion algorithm may be viewed as *jeu de taquin* moves, and *jeu de taquin* equivalence classes are the same as left

Knuth classes. On the other hand, right Knuth classes are the same as Haiman's dual equivalence classes.

To be more precise, let λ_n be the staircase Young diagram of size $n(n-1)/2$. Then λ_{n+1}/λ_n consists of n one-node components. The tableaux of shape λ_{n+1}/λ_n are called **permutation tableaux**. By reading entries from left to right, we identify permutation tableaux with permutations of $1, \ldots, n$. This rule extends to more general tableaux. We read entries of such a tableau row by row from left to right starting with the node on the south-west end and ending up with the node on the north-east end. Then the corresponding permutation tableau is *jeu de taquin* equivalent to the original tableau.

We take a tableau T of shape λ_n, and compute **switching** of a permutation tableau w and T [BSS]. Since all tableaux of a same non skew shape are dual equivalent [BSS, Proposition 4.2], the non skew tableau produced by the switching is independent of the choice of T [BSS, Theorem 4.3], and it is *jeu de taquin* equivalent to the permutation tableau w [BSS, Theorem 3.1]. This is the P-symbol of w. Further, two permutation tableaux are in a same dual equivalence class if and only if they have a common Q-symbol [H, Theorem 2.12]. These give the Robinson-Schensted corespondence in the *jeu de taquin* context. In fact, this view point also appears in the crystal base theory [BKK].

2.2. KL polynomials

Let q be a variable, and let $\mathcal{H}_n$ be the Hecke algebra of the symmetric group S_n. Namely, $\mathcal{H}_n$ is the algebra over $\mathbb{Q}(q)$ defined by generators $T_1, \ldots, T_{n-1}$ and relations

$$(T_i - q)(T_i + 1) = 0, \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad T_iT_j = T_jT_i \ (j \geq i+2).$$

Let $s_i = (i, i+1)$ be the transposition of i and $i+1$. We set $T_{s_i} = T_i$. For general $w \in S_n$, we find a reduced expression $w = s_{i_1} \cdots s_{i_r}$ and set $T_w = T_{i_1} \cdots T_{i_r}$. The length r of reduced expressions is constant on the set of reduced expressions of w, which is denoted by $l(w)$. It is also well known that T_w does not depend on the choice of the reduced expression, and $\{T_w | w \in S_n\}$ is a basis of $\mathcal{H}_n$. If y is obtained by the product of a subword of a reduced expression of w, we write $y \leq w$. This order is called **Bruhat order**.

Definition 2.3. *The following two conditions uniquely define the polynomials $P_{y,w}(q) \in \mathbb{Z}[q]$ $(y \leq w)$, which are called* **Kazhdan-Lusztig polynomials***:*

$$\begin{aligned} C_w &= \textstyle\sum_{y \leq w} (-1)^{l(w)-l(y)} q^{\frac{l(w)}{2} - l(y)} P_{y,w}(q^{-1}) T_y \\ &= \textstyle\sum_{y \leq w} (-1)^{l(w)-l(y)} q^{-\frac{l(w)}{2} + l(y)} P_{y,w}(q) T_{y^{-1}}^{-1}, \end{aligned}$$

and $P_{w,w}(q) = 1, \quad \deg P_{y,w}(q) \leq \frac{l(w)-l(y)-1}{2} \quad (y < w).$

We call the first property the bar invariance property, and the second property the degree property. For the definition, we can use the following element instead of C_w.

$$C'_w = q^{-\frac{l(w)}{2}} \sum_{y \leq w} P_{y,w}(q) T_y$$

If $\frac{l(w)-l(y)-1}{2}$ is an integer, we denote the coefficient of $q^{\frac{l(w)-l(y)-1}{2}}$ in $P_{y,w}(q)$ by $\mu(y,w)$. If $\mu(y,w) \neq 0$ for $y < w$ or $\mu(w,y) \neq 0$ for $y > w$ occurs, we write $\mu(y|w) \neq 0$. Note that if we write $\mu(y|w) \neq 0$, it particularly implies that $y > w$ or $y < w$ holds.

The welldefinedness of $P_{y,w}(q)$ is non trivial, and in fact is one of the main theorems [KL, Theorem 1.1]. The uniqueness of $P_{y,w}(q)$ is easy to prove, but for the existence of these polynomials, we need to construct C_w $(w \in S_n)$. In [KL, 2.2], these C_w are inductively constructed by setting $C_e = 1$ and

$$C_w = C_{s_i} C_{s_i w} - \sum_{\substack{z < w, s_i z < z \\ \mu(z,w) \neq 0}} \mu(z, s_i w) C_z$$

for $s_i \in \mathcal{L}(w) = \{s_j | \ s_j w < w\}$. Since $C_{s_i} = q^{-\frac{1}{2}} T_i - q^{\frac{1}{2}}$, we can give an inductive definition of Kazhdan-Lusztig polynoimals as follows.

Definition 2.4.

$$\begin{aligned} P_{y,w}(q) &= q^{1-c} P_{s_i y, s_i w}(q) + q^c P_{y, s_i w}(q) \\ &\quad - \sum_{\substack{y \leq z \leq s_i w, s_i z < z \\ \mu(z, s_i w) \neq 0}} \mu(z, s_i w) q^{\frac{l(w)-l(z)}{2}} P_{y,z}(q) \end{aligned}$$

where $c = 1$ *if* $s_i y < y$ *and* $c = 0$ *if* $s_i y > y$.

That the right hand side does not depend on the choice of s_i comes from the welldefinedness result. We can also construct C_w by $C_e = 1$ and

$$C_w = C_{w s_i} C_{s_i} - \sum_{\substack{z < w, z s_i < z \\ \mu(z,w) \neq 0}} \mu(z, s_i w) C_z$$

for $s_i \in \mathcal{R}(w) = \{s_j | \ w s_j < w\}$, which leads to a similar inductive definition of Kazhdan-Lusztig polynomials. Note that $\mathcal{R}(w) = \mathcal{L}(w^{-1})$.

Lemma 2.5. *(1)* $P_{y,w}(0) = 1$.
(2) $y < w, s_i y > y, s_i w < w$ *imply* $P_{y,w}(q) = P_{s_i y,w}(q)$.
(3) $P_{y^{-1},w^{-1}}(q) = P_{y,w}(q)$.

(1) follows from the inductive definition 2.4. (2) is proved by induction on $l(y)$, which also uses 2.4. (3) follows from the first definition of Kazhdan-Lusztig polynomials: If we replace $P_{y,w}(q)$ by $P_{y^{-1},w^{-1}}(q)$ in the definition of C_w and apply the anti-involution defined by $T_i \mapsto T_i$, we have $C_{w^{-1}}$. Thus we have the bar invariance property. The degree property is obvious. Hence $P_{y,w}(q)$ and $P_{y^{-1},w^{-1}}(q)$ must coincide.

Another corollary of this construction of C_w is the **Kazhdan-Lusztig representation** of the regular representation, which is the matrix representation with respect to the basis $\{C_w\}$. For the left regular representation, we have

$$T_i C_w = \begin{cases} -C_w & (\text{if } s_i w < w) \\ qC_w + q^{\frac{1}{2}} C_{s_i w} + \sum\limits_{\substack{z<w, s_i z < z \\ \mu(z,w) \neq 0}} \mu(z,w) q^{\frac{1}{2}} C_z & (\text{if } s_i w > w). \end{cases}$$

We have the same formula for the right regular representation. We can now introduce the notion of left cells and right cells.

Definition 2.6. *If there exists a sequence* $y = x_1, x_2, \ldots, x_r = w$ *such that* $\mathcal{L}(x_i) \not\subset \mathcal{L}(x_{i+1})$, $\mu(x_i|x_{i+1}) \neq 0$ *for* $1 \leq i < r$, *we write* $y \underset{L}{\leq} w$.

If there exists a sequence $y = x_1, x_2, \ldots, x_r = w$ *such that* $\mathcal{R}(x_i) \not\subset \mathcal{R}(x_{i+1})$, $\mu(x_i|x_{i+1}) \neq 0$ *for* $1 \leq i < r$, *we write* $y \underset{R}{\leq} w$.

Note that $y \underset{R}{\leq} w$ *if and only if* $y^{-1} \underset{L}{\leq} w^{-1}$. *If both* $y \underset{L}{\leq} w$ *and* $w \underset{L}{\leq} y$ *hold, we write* $y \underset{L}{\sim} w$. *Similarly, if both* $y \underset{R}{\leq} w$ *and* $w \underset{R}{\leq} y$ *hold, we write* $y \underset{R}{\sim} w$.

These relations partition S_n *into equivalence classes, which are called* **left cells** *and* **right cells** *respectively.*

At a first look, the definition of the relation $y \underset{L}{\leq} w$ seems to be very artificial. To understand it in a more natural way, we set $q = 1$ and denote $C_w|_{q=1}$ by $a(w)$. (The specialization to $q = 1$ is only for simplifying the situation to more familiar setting of the symmetric group, and is not at all essential.) Then we have the following lemma by using Lemma 2.5 and the Kazhdan-Lusztig representation specialized to $q = 1$.

Lemma 2.7. *Let $y \neq w$ be two elements of S_n. Then we have (1)$\Leftrightarrow$(2) where*

(1) $s_i \in \mathcal{L}(y) \setminus \mathcal{L}(w)$ and $\mu(y|w) \neq 0$.

(2) $a(y)$ appears in $s_i a(w)$.

Hence, the left regular representation of the symmetric group with the specific basis $\{a(w)\}$ gives a natural meaning of the relation $y \underset{L}{\leq} w$ as follows.

Let $\overline{V_w}^L$ be the left ideal uniquely defined by the following three conditions.

(1) $a(w) \in \overline{V_w}^L$.

(2) $\overline{V_w}^L$ is spanned by a subset of $\{a(x)\}$.

(3) If a left ideal satisfies (1) and (2), it contains $\overline{V_w}^L$.

Then we have $y \underset{L}{\leq} w \Leftrightarrow \overline{V_y}^L \subset \overline{V_w}^L$. Similar formula exists for $y \underset{R}{\leq} w$.

§3. The RS correspondence and the left cell

3.1. The Kazhdan-Lusztig theorem

The following theorem is the theorem of Kazhdan and Lusztig which we are going to prove.

Theorem A For $y, w \in S_n$, we have $y \underset{L}{\sim} w \Leftrightarrow Q(y) = Q(w)$.

Example 3.1 (The S_3 case). *Left cells are* $\{123\}$, $\{213, 312\}$, $\{132, 231\}$ *and* $\{321\}$. *Their Q-symbols are*

$$\begin{matrix} 1 & 2 & 3 \end{matrix}\ , \quad \begin{matrix} 1 & 3 \\ 2 & \end{matrix}\ , \quad \begin{matrix} 1 & 2 \\ 3 & \end{matrix}\ , \quad \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\ .$$

For the S_4 case, see [Shi, p.20].

3.2. A theorem of Knuth

We write $y \equiv w$ if $P(y) = P(w)$. To describe this equivalence relation, we introduce Knuth relations.

Definition 3.2. *Let $y_1 \cdots y_n$ be a permutation of $1, \ldots, n$. We set w as follows.*

If $y_{i+1} < y_i < y_{i+2}$, we set $w = y_1 \cdots y_i y_{i+2} y_{i+1} \cdots y_n$.

If $y_{i+1} < y_{i+2} < y_i$, we set $w = y_1 \cdots y_{i+1} y_i y_{i+2} \cdots y_n$.

We have $y \equiv w$, and we say that y and w are in Knuth relation.

The following theorem is due to Knuth [K, Theorem 6].

Theorem 3.3. *Let $y, w \in S_n$. Then $y \equiv w$ if and only if these permutations are connected by a chain of Knuth relations.*

Let $D_{ij} := \{ w \mid ws_i < w, ws_j > w \}$ where $j = i \pm 1$. If $y \in D_{ij}$, we consider the right coset $y < s_i, s_j >$ and take the distinguished coset representative y^0. Then we have either $y = y^0 s_i$ or $y^0 s_j s_i$. We set $K_{ij}(y) = y^0 s_i s_j$ in the former case, and $K_{ij}(y) = y^0 s_j$ in the latter case. Note that K_{ij} is a bijective map from D_{ij} to D_{ji}. If $j = i+1$, this is the rule to obtain w from y in the Knuth relation, and if $j = i-1$, this is the rule to obtain y from w in the Knuth relation. This description of Knuth relations is convenient for our purpose. The following lemma shows that two elements in Knuth relation are in a right cell.

Lemma 3.4. *If $w \in D_{ij}$, we have $K_{ij}(w) \underset{R}{\sim} w$.*

(Proof) Since $w \in D_{ij}$, we have $w = w^0 s_i$ or $w = w^0 s_j s_i$ where w^0 is the distinguished coset representative of $w\langle s_i, s_j\rangle$. By the same proof as in Lemma 2.7 we have $\mu(w^0 s_i, w^0 s_i s_j) = 1$, and $\mu(w^0 s_j, w^0 s_j s_i) = 1$. In either cases, we have $\mu(w|K_{ij}(w)) \neq 0$. Since $s_i \in \mathcal{R}(w) \setminus \mathcal{R}(K_{ij}(w))$ and $s_j \in \mathcal{R}(K_{ij}(w)) \setminus \mathcal{R}(w)$, we have $\mathcal{R}(w) \not\subset \mathcal{R}(K_{ij}(w))$ and $\mathcal{R}(w) \not\supset \mathcal{R}(K_{ij}(w))$. We have the result. Q.E.D

Remark We have another way to describe the Knuth relation as follows.

If $w < s_i w$ and $\mathcal{L}(w) \not\subset \mathcal{L}(s_i w)$, then we have $w^{-1} \equiv w^{-1} s_i$.

In fact, if we take $s_j \in \mathcal{L}(w) \setminus \mathcal{L}(s_i w)$, the choice of s_i, s_j leads to $s_j w < w$ and $s_j s_i w > s_i w$. These s_i and s_j can not be commutable elements. Thus $w^{-1} \in D_{ji}$ and we are in the latter case in the definition of K_{ij}. If we consider the case that $y < s_j y$ and $\mathcal{L}(y) \not\subset \mathcal{L}(s_j y)$, where we take $s_i \in \mathcal{L}(y) \setminus \mathcal{L}(s_j y)$ such that $y^{-1} \in D_{ij}$, we meet the former case in the definition of K_{ij}, and we have $y^{-1} \equiv y^{-1} s_j$. But this statement is the same as the previous case.

3.3. Preparatory results for the proof of Theorem A

The following three propositions are proved in [KL]. I avoid repetition as long as the readability of the proof is guaranteed.

Proposition 3.5 ([KL, Proposition 2.4]). *If $y \underset{L}{\leq} w$, then we have $\mathcal{R}(y) \supset \mathcal{R}(w)$.*

(Proof) It is enough to prove it for the case that $\mathcal{L}(y) \not\subset \mathcal{L}(w)$ and $\mu(y|w) \neq 0$. By Lemma 2.7, we have $y = s_i w > w$ for some i or $y < w$ and $\mu(y, w) \neq 0$. In the former case, we consider the double coset $\langle s_i \rangle w \langle s_j \rangle$ for each $s_j \in \mathcal{R}(w)$. Then we can easily conclude that $s_j \in \mathcal{R}(s_i w)$. Thus we have $\mathcal{R}(y) \supset \mathcal{R}(w)$. In the latter case, we assume to the contrary that there is $s_j \in \mathcal{R}(w) \setminus \mathcal{R}(y)$. By Lemma 2.5(3), our assumption $\mu(y|w) \neq 0$ is equal to $\mu(y^{-1}|w^{-1}) \neq 0$. We also have $s_j \in \mathcal{L}(w^{-1}) \setminus \mathcal{L}(y^{-1})$. By Lemma 2.7, we know that $a(w^{-1})$ appears in $s_j a(y^{-1})$. Since $y < w$, we have $w^{-1} > y^{-1}$ and thus we have $w^{-1} = s_j y^{-1}$. By the same argument in the former case, $w = y s_j > y$ implies $\mathcal{L}(y) \subset \mathcal{L}(w)$. It contradicts our assumption that $\mathcal{L}(y) \not\subset \mathcal{L}(w)$. Q.E.D

Proposition 3.6 ([KL, Theorem 4.2]). *If $y \neq w \in D_{ij}$ and $\mu(y|w) \neq 0$, then we have $\mu(K_{ij}(y)|K_{ij}(w)) \neq 0$.*

Remark By the definition of D_{ij}, there are two possibilities for y and w respectively. Namely,

$$\begin{aligned} ys_i < y &= K_{ij}(y)s_j < ys_j = K_{ij}(y) < K_{ij}(y)s_i, \\ ys_j > y &= K_{ij}(y)s_i > ys_i = K_{ij}(y) > K_{ij}(y)s_j, \end{aligned}$$

$$\begin{aligned} ws_i < w &= K_{ij}(w)s_j < ws_j = K_{ij}(w) < K_{ij}(w)s_i, \\ ws_j > w &= K_{ij}(w)s_i > ws_i = K_{ij}(w) > K_{ij}(w)s_j. \end{aligned}$$

Let y_i, w_i, $(i = 1, 2)$ and s, t be as follows.
(a) If both y and w are in the former case, we set

$$y_1 = K_{ij}(y),\ y_2 = y,\ s = s_j,\ t = s_i,\ w_1 = K_{ij}(w),\ w_2 = w.$$

(b) If both y and w are in the latter case, we set

$$y_1 = y,\ y_2 = K_{ij}(y),\ s = s_i,\ t = s_j,\ w_1 = w,\ w_2 = K_{ij}(w).$$

(c) If y is in the latter case and w is in the former case, we set

$$w_1 = K_{ij}(y),\ w_2 = y,\ s = s_j,\ t = s_i,\ y_1 = K_{ij}(w),\ y_2 = w.$$

Then we are reduced to the following two cases.

$$\begin{array}{ll}(1)\ y_2t < y_2 = y_1s < y_1 < y_1t, & w_2t < w_2 = w_1s < w_1 < w_1t,\\ (2)\ y_2t < y_2 = y_1s < y_1 < y_1t, & w_1s < w_1 < w_1t = w_2 < w_2s.\end{array}$$

We have to show $\mu(y_1|w_1) = \mu(y_2|w_2)$ for these two cases. Then we have come to the beginning of the proof in [KL, Theorem 4.2(ii)].

Proposition 3.7 ([KL, Corollary 4.3]). *Let $y, w \in D_{ij}$. Then $y \underset{L}{\sim} w$ implies $K_{ij}(y) \underset{L}{\sim} K_{ij}(w)$.*

(Proof) We can assume that $\mathcal{L}(y) \not\subset \mathcal{L}(w)$ ($\mathcal{L}(y) \not\supset \mathcal{L}(w)$), and $\mu(y|w) \neq 0$. By Lemma 3.4 and Proposition 3.5, we have $\mathcal{L}(K_{ij}(y)) = \mathcal{L}(y)$, $\mathcal{L}(K_{ij}(w)) = \mathcal{L}(w)$. Hence we have $\mathcal{L}(K_{ij}(y)) \not\subset \mathcal{L}(K_{ij}(w))$ $((\mathcal{L}(K_{ij}(y)) \not\supset \mathcal{L}(K_{ij}(w)))$. We also have $\mu(K_{ij}(y)|K_{ij}(w)) \neq 0$ by Proposition 3.6. We are through. Q.E.D

3.4. Proof of Theorem A

One implication is easy.

Proposition 3.8. *If $Q(y) = Q(w)$, then $y \underset{L}{\sim} w$.*

(Proof) Since $Q(y) = P(y^{-1})$ and $Q(w) = P(w^{-1})$, y^{-1} is connected to w^{-1} by a chain of Knuth relations. Thus it is enough to prove that $w^{-1} = K_{ij}(y^{-1})$ $(y^{-1} \in D_{ij})$ implies $y \underset{L}{\sim} w$. But Lemma 3.4 shows that $y^{-1} \underset{R}{\sim} w^{-1}$, which is $y \underset{L}{\sim} w$. Q.E.D

It remains to prove that $y \underset{L}{\sim} w$ implies $Q(y) = Q(w)$. For each partition π, we define a standard tableau P_π by setting the entries of the i th column of P_π to be $\sum_{j=1}^{i-1} l_j + 1, \dots, \sum_{j=1}^{i} l_j$ from top to bottom, where $l_1, l_2, \dots$ are column lengths of π. We denote the shapes of $Q(y)$ and $Q(w)$ by π_1 and π_2 respectively. We define $\hat{y}$, $\hat{w}$ by $(P(\hat{y}), Q(\hat{y})) = (P_{\pi_1}, Q(y))$ and $(P(\hat{w}), Q(\hat{w})) = (P_{\pi_2}, Q(w))$. By Proposition 3.8, we have $y \underset{L}{\sim} \hat{y}$ and $w \underset{L}{\sim} \hat{w}$. Thus we have $\hat{y} \underset{L}{\sim} \hat{w}$. To prove that $Q(\hat{y}) = Q(\hat{w})$, we define y' and $w"$ by

$$(P(y'), Q(y')) = (P_{\pi_1}, P_{\pi_1}) \qquad (P(w"), Q(w")) = (P_{\pi_2}, P_{\pi_2}).$$

By the theorem of Knuth, we can write

$$\begin{aligned} y' &= K_{i_1j_1} \circ \cdots \circ K_{i_rj_r}(\hat{y}) \\ w" &= K_{i'_1j'_1} \circ \cdots \circ K_{i'_sj'_s}(\hat{w}) \end{aligned}$$

We shall define w' and y" by

$$\begin{aligned} w' &= K_{i_1j_1}\circ\cdots\circ K_{i_rj_r}(\hat{w}) \\ y" &= K_{i'_1j'_1}\circ\cdots\circ K_{i'_sj'_s}(\hat{y}) \end{aligned}$$

Recall that Proposition 3.5 tells that $\mathcal{R}(\hat{y}) = \mathcal{R}(\hat{w})$. Hence $\hat{y} \in D_{i_rj_r}$ implies $\hat{w} \in D_{i_rj_r}$. We then have welldefined $K_{i_rj_r}(\hat{w})$, which satisfies $K_{i_rj_r}(\hat{y}) \underset{L}{\sim} K_{i_rj_r}(\hat{w})$ by Proposition 3.7. We continue the argument and conclude that these y" and w' are welldefined. y' and w' satisfy $\mathcal{R}(y') = \mathcal{R}(w')$, $P(y') = Q(y') = P_{\pi_1}$ and $P(w') = P_{\pi_2}$. Similarly, y" and w" satisfy $\mathcal{R}(y") = \mathcal{R}(w")$, $P(y") = P_{\pi_1}$ and $P(w") = Q(w") = P_{\pi_2}$. Note that y' is the permutation

$$l_1, l_1-1, \cdots, 1, l_1+l_2, \cdots, l_1+1, \cdots.$$

Similarly, w" is the permutation

$$l'_1, l'_1-1, \cdots, 1, l'_1+l'_2, \cdots, l'_1+1, \cdots.$$

where we denote column lengths of π_2 by $l'_1, l'_2, \ldots$. Since $\mathcal{R}(y') = \mathcal{R}(w')$, the first l_1 letters of w' are in the decreasing order, the next l_2 letters are in the decreasing order, etc. Similarly, the first l'_1 letters of y" are in the decreasing order, the next l'_2 letters are in the decreasing order, etc.

By inserting the first l_1 letters of w' to $\emptyset$, we know that the first column of π_2 must have the length equal or greater than l_1. By using y", we have the opposite inequality. We have $l_1 = l'_1$. It also implies that the next l_2 decreasing letters of w' do not produce bumping, since if otherwise we have $l'_1 > l_1$. Thus we have that $l'_2 \geq l_2$. We use y" to have the opposite inequality. Continuing the same argument, we conclude that $\pi_1 = \pi_2$ and $Q(w') = P_{\pi_2}$. (We also have $Q(y") = P_{\pi_1}$.) Therefore, we have $y' = w'$, which implies $\hat{y} = \hat{w}$. We have proved $Q(y) = Q(w)$. Q.E.D

3.5. Theorem A in the crystal base theory context

An occurence of the Robinson-Schensted algorithm in the tensor product representation of the vector representation of $U_q(\mathfrak{gl}_n)$ was first observed in [DJM]. The tensor product representation itself can be viewed as an example of Demazure modules [KMOTU, Theorem 3.1], and we may consider generalization into this direction, but we restrict ourselves to the original case. Then the crystal base is induced by the canonical base and we now have a good understanding of the base (see [SV]) and of the Robinson-Schensted algorithm in the crystal base theory context.

Let U_q be the quantum algebra of $\mathfrak{gl}_r$, and Δ be Lusztig's coproduct:

$$\begin{aligned}\Delta(e_i) = &\quad e_i \otimes 1 + q^{\epsilon_i-\epsilon_{i+1}} \otimes e_i \quad (i = 1, \dots, r\text{–}1),\\ \Delta(f_i) = &\quad 1 \otimes f_i + f_i \otimes q^{-\epsilon_i+\epsilon_{i+1}} \quad (i = 1, \dots, r\text{–}1),\\ \Delta(q^h) = &\quad q^h \otimes q^h \quad (h \in \mathbb{Z}\epsilon_1 + \cdots + \mathbb{Z}\epsilon_n).\end{aligned}$$

Let $V = \mathbb{Q}(q)^r$ be its vector representation given by

$$e_i = E_{i,i+1}, \quad f_i = E_{i+1,i}, \quad q^{\epsilon_i} = qE_{ii} + \sum_{j\neq i} E_{jj}$$

where E_{ij} are matrix units. Natural base elements $v_1 = (1, 0, \dots, 0)^{\mathrm{T}}$, $v_2 = (0, 1, \dots, 0)^{\mathrm{T}}, \dots$ induce a base at $q = \infty$ in the sense of Kashiwara-Lusztig. In the following, we exclusively work with bases at infinity, and call them crystal bases instead of bases at $q = \infty$. We set $L = \oplus_{i=1}^{r}\mathbb{Q}[q^{-1}]_{(q^{-1})}v_i$, $B = \{ v_i \bmod q^{-1}L\} \subset L/q^{-1}L$. Then (L, B) is the crystal base of V stated above, and $V^{\otimes n}$ has $(L^{\otimes n}, B^{\otimes n})$ as its crystal base. To describe the tensor structure on $B^{\otimes n}$, we introduce $\varphi_i(b), \epsilon_i(b)$ by

$$\varphi_i(b) = \max\{ k|\ \tilde{f}_i^k(b) \neq 0\}, \quad \epsilon_i(b) = \max\{ k|\ \tilde{e}_i^k(b) \neq 0\}$$

where $\tilde{e}_i$ and $\tilde{f}_i$ are Kashiwara operators. Then

$$\begin{aligned}\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{e}_i(b_2) & (\epsilon_i(b_1) \leq \varphi_i(b_2)),\\ \tilde{e}_i(b_1) \otimes b_2 & (\epsilon_i(b_1) > \varphi_i(b_2)),\end{cases}\\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{f}_i(b_2) & (\epsilon_i(b_1) < \varphi_i(b_2)),\\ \tilde{f}_i(b_1) \otimes b_2 & (\epsilon_i(b_1) \geq \varphi_i(b_2)).\end{cases}\end{aligned}$$

Let $V_q(\lambda)$ be the irreducible highest weight module of U_q associated with $\lambda = \sum \lambda_i\epsilon_i$. We identify λ with the corresponding Young diagram. Then it is well known that $V_q(\lambda) \otimes V$ is multiplicity free. Hence, we can uniquely define the submodule of $V^{\otimes n}$ for each increasing sequence of Young diagrams. We identify the increasing sequence with a standard tableau Q, which we call the recording tableau. We denote the submodule by $V_q(Q)$. If the shape of Q is λ, we have $V_q(Q) \simeq V_q(\lambda)$.

Proposition 3.9. *(1) Let Q be a standard tableau of size n–1, and $\mathcal{T}_Q$ be the set of tableaux obtained from Q by adding $\boxed{n}$. Let $(L(Q), B(Q))$ be a crystal base of $V_q(Q)$. We set $L(T) = (L(Q) \otimes L) \cap V_q(T)$. Then we have*

$$L(Q) \otimes L = \bigoplus_{T\in\mathcal{T}_Q} L(T).$$

We nextly set $B(T) = (B(Q) \otimes B) \cap (L(T)/q^{-1}L(T))$. Then we have

$$(L(Q) \otimes L, B(Q) \otimes B) = \bigoplus_{T \in \mathcal{T}_Q} (L(T), B(T)).$$

(2) Let $L(Q) = L^{\otimes n} \cap V_q(Q)$. Then we have $L^{\otimes n} = \oplus L(Q)$. If we further set $B(Q) = B^{\otimes n} \cap (L(Q)/q^{-1}L(Q))$, we have $(L^{\otimes n}, B^{\otimes n}) = \oplus(L(Q), B(Q))$.

(Proof) (1) Let v_T be the highest weight vector which generates the highest weight space of $L(T)$. Since $L(T)$ and the lattice generated by $\tilde{f}_{i_1} \cdots \tilde{f}_{i_*} v_T$ are crystal lattices of $V_q(T)$, the uniqueness theorem of crystal bases concludes that they coincide. The uniqueness theorem also guarantees that there exists an automorphism of $V_q(Q) \otimes V$ such that it maps $L(Q) \otimes L$ to $\oplus L(T)$. Since $V_q(Q) \otimes V$ is multiplicity free, the automorphism is scalar multiplication on each $V_q(T)$. Thus by looking at highest weight spaces, we have that the automorphism is the identity. By descending induction on weights, we can prove $B(Q) \otimes B = \sqcup B(T)$.
(2) We prove it by induction on n. Assume that it holds for n. Then we have $(L(Q) \otimes L, B(Q) \otimes B) = \oplus(L(T), B(T))$ by (1) where

$$\begin{aligned} L(T) &= (L(Q) \otimes L) \cap V_q(T) = ((L^{\otimes n} \cap V_q(Q)) \otimes L) \cap V_q(T) \\ &= \left(L^{\otimes n+1} \cap V_q(Q) \otimes L\right) \cap V_q(T) \\ &= \left(L^{\otimes n+1} \cap V_q(Q) \otimes V\right) \cap V_q(T) \\ &= L^{\otimes n+1} \cap V_q(T). \end{aligned}$$

Q.E.D

The crystal graph of $V_q(\lambda)$ has description in terms of semistandard tableaux as follows [KN].

We write $\boxed{i}$ for $v_i \bmod q^{-1} \in B$. Let $B(\lambda)$ be the set of column strict semi-standard tableaux of shape λ. For each $T \in B(\lambda)$, we read its entries row by row, starting from the bottom row. This reading gives an injection from $B(\lambda)$ to $B^{\otimes n}$. For example, we have

$$\begin{matrix} 1 & 1 & 2 & 4 \\ 2 & 3 & & \\ 4 & & & \end{matrix} \qquad \Longrightarrow \boxed{4} \otimes \boxed{2} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{4}.$$

We induce the crystal structure on $B(\lambda)$ through this inclusion: the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ act on these monomial tensors by changing the leftmost $\boxed{i+1}$ or the rightmost $\boxed{i}$ of the sequence which is obtained by removing consecutive $\boxed{i+1} \otimes \boxed{i}$ as many as possible. Thus if $\tilde{e}_i T \neq$

$0(\tilde{f}_i T \neq 0)$, then $\tilde{e}_i T \in B(\lambda)(\tilde{f}_i T \in B(\lambda))$. This embedding is in fact the embedding of $B(\lambda)$ into the set of permutation tableaux by *jeu de taquin* moves, and the inverse is given by taking P-symbols, namely by the insertion algorithm. See [Fo3] for example.

Let Q be a standard tableau of shape λ. We identify $B(Q)$ with $B(\lambda)$. Note that there exists a unique isomorphism of the crystals $(L(Q), B(Q))$ and $(L(\lambda), B(\lambda))$. The following is the modern version of the Date-Jimbo-Miwa theorem. We refer [BKK] for its generalization to superalgebras.

Theorem 3.10. *We identify $B(Q)$ with $B(\lambda)$ as above. Then the following hold.*

(1) If $b = \boxed{i_1} \otimes \cdots \otimes \boxed{i_n} \in B(Q)$, then the Q-symbol of $\emptyset \leftarrow i_1 i_2 \cdots i_n$ is Q.
(2) Let $P(b)$ be the P-symbol of $\emptyset \leftarrow i_1 i_2 \cdots i_n$. Then the identification of $B(Q)$ with $B(\lambda)$ is given by the map $b \mapsto P(b)$.

(Proof) We first recall that the bumping procedure $(T, i) \mapsto (T \leftarrow i)$ gives the isomorphism of crystals between $B(\lambda) \otimes B$ and $\sqcup_{|\mu/\lambda|=1} B(\mu)$. (As I have explained, we can think of the insertion via *jeu de taquin* moves. Hence it is enough to establish the isomorphism for a *jeu de taquin* move, which is easy.)

This isomorphism leads to a crystal automorphism on $B(Q) \otimes B$ as follows.

$$B(Q) \otimes B \xrightarrow{\sim} B(\lambda) \otimes B \xrightarrow{\sim} \bigsqcup_{|\mu/\lambda|=1} B(\mu) \xleftarrow{\sim} \bigsqcup_{T \in \mathcal{T}_Q} B(T) = B(Q) \otimes B$$

where the second isomorphism is given by the insertion algorithm. Since $V_q(Q) \otimes V$ is multiplicity free, the automorphism must be the identity. Hence the isomorphism $B(T) \xrightarrow{\sim} B(\mu)$ for $T \in \mathcal{T}_Q$ of shape μ is given by restricting the following isomorphism to $B(T)$. Note again that the second isomorphism is given by the insertion algorithm.

$$B(Q) \otimes B \xrightarrow{\sim} B(\lambda) \otimes B \xrightarrow{\sim} \bigsqcup_{|\mu/\lambda|=1} B(\mu)$$

Thus if the Robinson-Schensted algorithm gives the isomorphism $B(Q) \xrightarrow{\sim} B(\lambda)$ such that the Q-symbols of the elements in its image are constant Q, then the Robinson-Schensted algorithm gives the isomorphism $B(T) \xrightarrow{\sim} B(\mu)$, and the Q-symbols of the elements in its image are constant T. Therefore the induction proceeds. Q.E.D

We now turn to the q^2-Schur algebra. We refer [Du] for the details. We consider the Hecke algebra whose deformation parameter q is replaced by q^2. We also denote it by $\mathcal{H}_n$ by abuse of notion. $V^{\otimes n}$ has $\mathcal{H}_n$ action given by

$$v_{i_1}\otimes\cdots\otimes v_{i_n}T_k=\begin{cases} qv_{i_1}\otimes\cdots\otimes v_{i_{k+1}}\otimes v_{i_k}\otimes\cdots\otimes v_{i_n} & (i_k>i_{k+1})\\ q^2v_{i_1}\otimes\cdots\otimes v_{i_n} & (i_k=i_{k+1})\\ qv_{i_1}\otimes\cdots\otimes v_{i_{k+1}}\otimes v_{i_k}\otimes\cdots\otimes v_{i_n} & \\ \qquad +(q^2-1)v_{i_1}\otimes\cdots\otimes v_{i_n} & (i_k<i_{k+1})\end{cases}$$

It commutes with U_q action. The endomorphism ring $\mathrm{End}_{\mathcal{H}_n}(V^{\otimes n})$ is called the **q-Schur algebra**, which is denoted by $\mathcal{S}_{r,n}$. It is well known that it is a quotient algebra of U_q. If we denote the μ-weight space of $V^{\otimes n}$ by V_μ, then we obviously have $\mathcal{S}_{r,n}=\oplus_{\mu,\nu}\mathrm{End}_{\mathcal{H}_n}(V_\nu,V_\mu)$.

We now assume $r=n$ and set $\omega=\epsilon_1+\cdots+\epsilon_n$. Then $\mathcal{H}_n\simeq\mathrm{End}_{\mathcal{H}_n}(V_\omega,V_\omega)$, and we can identify $\mathcal{H}_n$ with the subalgebra of $\mathcal{S}_{n,n}$.

On the other hand, if we set $x_\mu=\sum_{w\in S_\mu}T_w$ where S_μ is the Young subgroup associated with μ, the weight space V_μ is isomorphic to $x_\mu\mathcal{H}_n$. Hence we can also identify V_ω with $\mathcal{H}_n$. This identification is given by

$$v_{w_1}\otimes\cdots\otimes v_{w_n}\mapsto (q^2)^{-l(ww_0)/2}T_{ww_0}.$$

In particular, the Kazhdan-Lusztig basis element C'_w is identified with

$$\sum_{y\le w}P_{y,w}(q^2)q^{l(y)-l(w)}v_{y_n}\otimes\cdots\otimes v_{y_1}.$$

We have $P_{y,w}(q^2)q^{l(y)-l(w)}\in\mathbb{Z}[q^{-1}]$, and $C'_w\equiv v_{w_n}\otimes\cdots\otimes v_{w_1} \bmod q^{-1}$.

The tensor space and the q^2-Schur algebra have bar operations, which satisfy $\bar{x}\bar{v}=\overline{xv}$ $(x\in\mathcal{S}_{n,n}, v\in V^{\otimes n})$, and $\overline{v_n\otimes\cdots\otimes v_1}=v_n\otimes\cdots\otimes v_1$. The bar operation on the tensor space coincides with the bar operation introduced in 2.2 if restricted to $\mathcal{H}=V_\omega\subset V^{\otimes n}$.

By these reasons, we conclude that these are canonical basis elements arising from the crystal base we have considered above. We also remark that the canonical basis of the q^2-Schur algebra is the image of the canonical basis of the modified quantized enveloping algebra by the work [SV]. In fact, because of $(V^{\otimes n})_\omega=\oplus S_q(Q)$ where $S_q(Q)=V_q(Q)_\omega$, these Kazhdan-Lusztig basis elements are partitioned into the disjoint union $\sqcup B(Q)_\omega$ at $q=\infty$.

Recall that these C'_w are obtained from C_w by applying a $\mathbb{Q}$-algebra automorphism of $\mathcal{H}_n$. Thus the vector spaces $S_{\underset{L}{\le} w}$, $S_{\underset{L}{<} w}$ generated by

$\{C'_y | y \underset{L}{\leq} w\}$, $\{C'_y | y \underset{L}{<} w\}$ respectively are $\mathcal{H}_n$-modules. It is known that the factor module $S_{\underset{L}{\leq} w}/S_{\underset{L}{<} w}$ is irreducible. We now take the U_q-submodules $V_{\underset{L}{\leq} w}$, $V_{\underset{L}{<} w}$ of $V^{\otimes n}$ generated by $S_{\underset{L}{\leq} w}$, $S_{\underset{L}{<} w}$ respectively. By applying compositions of $\tilde{e}_i, \tilde{f}_i$ to $\{C'_y | y \underset{L}{\leq} w\}$, $\{C'_y | y \underset{L}{<} w\}$, we also have crystal bases of $V_{\underset{L}{\leq} w}$ and $V_{\underset{L}{<} w}$, which we denote by $(L_{\underset{L}{\leq} w}, B_{\underset{L}{\leq} w})$, $(L_{\underset{L}{<} w}, B_{\underset{L}{<} w})$. $B_{\underset{L}{<} w}$ is a union of connected components of $B_{\underset{L}{\leq} w}$. Since $V_{\underset{L}{\leq} w}/V_{\underset{L}{<} w}$ is irreducible, $B_{\underset{L}{\leq} w} \setminus B_{\underset{L}{<} w}$ coincides with one of $B(Q)_\omega$. Therefore, we have Theorem A again.

3.6. Theorem A derived from the primitive ideal theory

Definition 3.11. *The annihilator ideal of $L(\lambda)$ in $U(\mathfrak{g})$ is denoted by $I(\lambda) := Ann(L(\lambda))$, and is called a* **primitive ideal**.

The following is a theorem of Joseph.

Theorem B $Q(y) = Q(w) \Leftrightarrow I(y \cdot 0) = I(w \cdot 0)$.

By the translation principle, 0 can be replaced by any dominant integral weight.

The proof of this theorem depends on the following proposition.

Proposition 3.12. **(1)** *Let $y, w \in D_{ij}$ and assume $I(y \cdot 0) \subset I(w \cdot 0)$. Then we have $I(K_{ij}(y) \cdot 0) \subset I(K_{ij}(w) \cdot 0)$.*
(2) *If $Q(y) = Q(w)$, we have $I(y \cdot 0) = I(w \cdot 0)$.*

(1) is proved in [Ja1, Satz 5.9]. (2) is proved in [Ja1, Satz 5.18]. Once this proposition is established, the proof of Theorem B goes precisely the same as the proof of Theorem A.

By [Ja1, Corollar 6.26], [Ja1, Satz 7.9], [Ja1, Satz 7.12], we have the following theorem of Vogan. We state it in weaker form since it is enough for our purpose.

Theorem C Let λ, μ_1, μ_2 be dominant integral weights. Then we have that $I(y \cdot \lambda) \subset I(w \cdot \lambda)$ holds if and only if there exists a finite dimensional module E such that

$$\left[L(y^{-1} \cdot \mu_1) \otimes E : L(w^{-1} \cdot \mu_2)\right] \neq 0.$$

This theorem leads to Theorem D below. Recall that Kazhdan-Lusztig conjecture states that if we define $a(y,w)$ by

$$L(y\cdot 0)=\sum_{y\le w} a(y,w)M(w\cdot 0),$$

we have $a(y,w)=(-1)^{l(w)-l(y)}P_{w_0w,w_0y}(1)$. This is proved by Brylinski and Kashiwara, Beilinson and Bernstein. Thus, there is a linear isomorphism between $K_0(\mathcal{O}_0)$ and $\mathbb{Z}W$ which sends $M(w_0w^{-1}\cdot 0)$ to w and $L(w_0w^{-1}\cdot 0)$ to $a(w)$. By introducing W-action on $K_0(\mathcal{O}_0)$ by $\tau M(w_0w^{-1}\cdot 0)=M(w_0w^{-1}\tau^{-1}\cdot 0)$, we can make it into a W-module isomorphism. Hence it is possible to translate statements for $K_0(\mathcal{O}_0)$ to those for the Weyl group. The following theorem is due to Joseph and Vogan. The formulation is due to Joseph [Jo5], and Vogan gives the proof in proving Theorem C. See [Ja1, Lemma 14.9] for the proof.

Theorem D $I(yw_0\cdot 0)\subset I(ww_0\cdot 0)\Leftrightarrow a(w)\in \overline{V_y}^L$.

This theorem shows that $y \underset{L}{\sim} w \Leftrightarrow I(yw_0\cdot 0)=I(ww_0\cdot 0)$. We then use Theorem B to conclude that $y \underset{L}{\sim} w \Leftrightarrow Q(yw_0)=Q(ww_0)$. Schützenberger's theorem [Sch1] tells that if we apply evacuation procedure to $Q(w)$, we obtain the transpose of $Q(ww_0)$ [S, Theorem 3.114]. Thus we have established Theorem A again.

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