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Cohomology of Local systems

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§1. Introduction

This survey is intended to provide a background for the authors paper [23]. The latter was the subject of the talk given by the second author at the Arrangement Workshop. The central theme of this survey is the cohomology of local systems on quasi-projective varieties, especially on the complements to algebraic curves and arrangements of lines in $\mathbf{P}^2$. A few of the results of [23] are discussed in section 4 while the first part of this paper contains some of highlights of Deligne's theory [7] and several examples from the theory of Alexander invariants developed mostly by the first author in the series of papers [17] - [22]. We also included several problems indicating possible further development. The second author uses the opportunity to thank M. Oka and H. Terao for the hard labor of organizing the Arrangement Workshop.

§2. Background on cohomology of local systems

Local systems. A local system of rank n on a topological space X is a homomorphism $\pi_1(X) \to GL(n, \mathbf{C})$. Such a homomorphism defines a vector bundle on X with discrete structure group or a locally constant bundle (cf. [7], I.1). Indeed, if $\tilde{X}_u$ is the universal cover of X then $\tilde{X}_u \times_{\pi_1(X)} \mathbf{C}^n$ is such a bundle (this product is the quotient of $\tilde{X}_u \times \mathbf{C}^n$ by the equivalence relation $(x, v) \sim (x', v')$ if and only if there is $g \in \pi_1(X)$ such that $x' = gx, v' = gv$; this quotient has the projection onto $\tilde{X}_u/\pi_1(X) = X$ with the fiber $\mathbf{C}^n$). Vice versa, any locally constant bundle defines a representation of the fundamental group of the base.

If X is a *complex* manifold, then there is a one-to-one correspondence between the local systems and pairs consisting of a *holomorphic* vector bundle on X and an *integrable* connection on the latter (cf. [7] I.2, Theorem 2.17). If V is a vector bundle then a connection can be viewed as a $\mathbf{C}$-linear map defined for each open set U of X and acting as follows:

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$\nabla : V(U) \to \Omega^1(X)(U) \otimes_{\mathcal{O}(U)} V(U))$. Here $V(U)$ (resp. $\mathcal{O}(U)$, resp. $\Omega(X)(U))$ is the space of sections of V (resp. the space of functions and the space of 1-forms) holomorphic on U. It is required from ∇ to satisfy the Leibniz rule $\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$. The integrability requirement is that if one extends ∇ to the maps $\nabla_1 : \Omega^1(X)(U) \otimes_{\mathcal{O}(U)} V(U) \to \Omega^2(X)(U) \otimes_{\mathcal{O}(U)} V(U)$ using the rule $\nabla_1(\omega \otimes v) = d\omega \otimes v - \omega \wedge \nabla v$ then $\nabla_1 \circ \nabla = 0$.

The above correspondence can be described as follows. If V is a locally constant bundle, then on the holomorphic bundle $V \otimes_{\mathbf{C}} \mathcal{O}$, where $\mathcal{O}$ is the trivial bundle, one can define the connection by $\nabla(f \otimes v) = df \otimes v$ where f (resp. v) is a holomorphic function (resp. a section of a locally constant bundle V) on U. The sections v of $V \otimes_{\mathbf{C}} \mathcal{O}$ which are flat with respect to this connection, i.e., such that $\nabla(v) = 0$, coincide with the sections of V. Vice versa, the sections of any holomorphic bundle with integrable connection form a locally constant bundle, i.e., a local system.

Cohomology. The homology of a local system can be defined as the homology of chain complex:

$$\dots \to C_i(\tilde{X}_u) \otimes_{\pi_1(X)} \mathbf{C}^n \to \dots$$

Here the chain complex for $\tilde{X}_u$ can be the complex of singular chains, or chains corresponding to a triangulation, or chains with a support, etc.

It is well known that the cohomology of X with constant coefficients can be calculated using the de Rham complex $\mathcal{A}^*(X)$ of C^∞-differential forms(the de Rham theorem). In the case where X is a non singular algebraic variety which is the complement to the union Y of smooth divisors on a projective variety $\bar{X}$ one can define a subcomplex $\mathcal{A}\langle Y \rangle$ of de Rham complex $\mathcal{A}^*(X)$ called log-complex. It consists of C^∞ forms ω on X with the property that near a point of $\bar{X}$ at which Y has local equation $Q = 0$ both $Q\omega$ and $Qd\omega$ admit extension to $\bar{X}$. If the components of Y intersect transversally then the cohomology of the complex $\mathcal{A}\langle Y \rangle$ is also isomorphic to $H^*(X)$. Otherwise it is not valid in general, though under some conditions on the singularities of Y (e.g. if Y is *free*) it is still true (see [4, 34]).

The cohomology of local systems also can be described using differential forms. Before stating this result let us recall that, though the *holomorphic* log-de Rham complex is too small to give full cohomology groups, there is, nevertheless, a way to reconstruct cohomology of X using holomorphic log-forms. Namely, one can consider a double complex $\mathcal{F}^{i,j}$ of sheaves such that all $\mathcal{F}^{i,j}$ are acyclic and $\mathcal{F}^{i,*}$ form a resolution of the sheaf of holomorphic log-forms $\Omega^i(X)\langle Y \rangle$. The cohomology of the double complex $\Gamma(\mathcal{F}^{i,j})$, i.e., the cohomology of the complex $\oplus_{i+j=k}\Gamma(\mathcal{F}^{i,j})$, is called the hypercohomolgy of $\Omega^*(X)\langle Y\rangle$. This construction of hypercohomology, applied verbatim to any complex of sheaves F^* instead of log-complex, yields hypercohomology groups $\mathbf{H}^*(F^*)$. A theorem of Deligne states that the hypercohomolgy $\mathbf{H}^i(\Omega^*(X)\langle Y\rangle)$ is isomorphic to $H^i(X,\mathbf{C})$ (cf. [8]). On the other hand, if X is affine, e.g. a complement to a hypersurface in projective space (cf. section 4.), then the cohomology $H^i(X,\mathbf{C})$ can be found using the complex of *rational* forms (the algebraic de Rham theorem, cf. [12]).

Hypercohomology also yields the cohomology of local systems in terms of differential forms, i.e., give a version of the de Rham theorem for local systems. The (holomorphic) de Rham complex in this case is formed by the sheaves of holomorphic forms with values in the holomorphic bundle $\mathcal{V}$ corresponding to the local system V, i.e., $\Omega^p(\mathcal{V})=\Omega^p\otimes_{\mathcal{O}}\mathcal{V}$ with the differential given by $\nabla_p(\omega\otimes v)=d\omega\otimes v+(-1)^{\deg(v)}\omega\wedge\nabla v$ (∇_1 above is a special case of the differential in this de Rham complex). Note that integrability $\nabla_1\circ\nabla=0$ yields that $\nabla_{p+1}\circ\nabla_p=0$, i.e., $\Omega^p(\mathcal{V})$ form indeed a complex of sheaves. This de Rham complex is a resolution of the holomorphic bundle $\mathcal{V}$ and it yields "de Rham theorem" $\mathbf{H}^p(\Omega(X)(\mathcal{V}))=H^p(X,V)$ (if V is a trivial local system one obtains the standard de Rham theorem). Moreover, if X is affine, the de Rham theorem with twisted coefficients still holds, i.e., the cohomology of the complex of rational forms with values in $\mathcal{V}$ is isomorphic to $H^i(X,V)$ (cf. [7], II, cor.6.3)

Calculation of cohomology of local systems using logarithmic complex is more subtle (even in the case of normal crossing), i.e., hypercohomology of log-complex yield the cohomology of the local system only if certain conditions are met. Deligne describes such sufficient conditions. The conditions are stated in the case where the connection ∇ has logarithmic poles along Y in the following sense. One assumes that the bundle $\mathcal{V}$ on X is a restriction of a holomorphic bundle $\bar{\mathcal{V}}$ on compactification $\bar{X}$ of X where $Y=\bar{X}-X$ is a divisor with normal crossings. The logarithmic property of ∇ means that in a sufficiently small neighborhood U_p of any point $p\in Y$, such that there exists a choice of sections $e_i\in\Gamma(U_p,\bar{\mathcal{V}})$ forming a basis of any fiber of $\bar{V}$ in U_p, the matrix of ∇ consists of 1-forms having logarithmic poles along Y. The entries of this matrix are $a_{i,j}\in\Omega^1(U_p\cap X\otimes\mathcal{V})$ such that $\nabla(e_i)=\Sigma a_{i,j}\otimes e_j$. The matrix of ∇ can be described in invariant terms as an element of $\Omega^1(X)(\operatorname{End}\mathcal{V})$. On the other hand, near $p\in Y$ where Y is given by $z_1\cdots z_k=0$, a log-1-form ω on X can be written as $\Sigma\alpha_i dz_i/z_i$ where α_i are holomorphic in U_p and hence defines a holomorphic section $\operatorname{Res}_Y(\omega)=\Sigma_i\alpha_i|Y$ on Y called the residue of ω. If ∇ is a connection with logarithmic poles then one can define $\operatorname{Res}_Y(\nabla)$ as a matrix formed by the residues of log-1-forms $a_{i,j}$. This matrix can be viewed as an element of $\mathrm{End}(\bar{\mathcal{V}}|_Y)$. Deligne's fundamental theorem ([7], 3.15) states that if none of the eigenvalues of matrices $\mathrm{Res}_Y(\nabla)(p)$ $(p \in Y)$ is a positive integer then one has the isomorphism of hypercohomology:

$$\mathbf{H}(X, \Omega^*_{\bar{X}}\langle Y\rangle(\mathcal{V})) = \mathbf{H}(X, \Omega^*_X(\mathcal{V}))$$

Rank one local systems. Rank one local systems on X are just the characters of fundamental group or equivalently of $H_1(X,\mathbb{Z})$. We will assume for simplicity that the latter group is torsion free. In this case the "moduli space" of local systems of rank one is just the torus $\mathrm{Char}(X) = \mathbf{C}^{*b_1}$ where $b_1 = \dim H_1(X,\mathbf{R})$ (presence of torsion in $H_1(X,\mathbf{Z})$ will yield $\mathrm{Char}(X)$ with several connected components, each being a translation of a torus). For higher rank the construction of moduli spaces in considerably more complicated (cf. [31]).

The torus $\mathrm{Char}(X)$ contains subvarieties Σ^k_i that consist of local systems V such that $rkH^k(X,V) \geq i$. Σ^k_i are important invariants of X. They play a crucial role in several problems.

First, these subvarieties of $\mathrm{Char}(X)$ are closely related to the structure of the fundamental group of X or more precisely to the Alexander invariants of the latter. Those can be defined as follows (cf. [22]). Let $\tilde{X}_A$ be an abelian cover of X corresponding to the kernel of a surjection $\phi_A : \pi_1(X) \to A$. The group A is an abelian group of deck transformations. Though $\tilde{X}_A$ of course depends on ϕ_A we shall not specify this dependence since in this paper this wouldn't cause a confusion. The group $H_1(\tilde{X}_A, \mathbf{C})$ can be considered as a module over the group ring of the group of deck transformations of $\tilde{X}_A$, i.e., over $R = \mathbf{C}[A]$. The latter, after a choice of independent generators in A, can be identified with the ring of Laurent polynomials of $rk(A)$ variables. This module is the Alexander invariant of X corresponding to A and is denoted below by $\mathcal{A}(X,A)$. A particularly important case is where $A = H_1(X,\mathbb{Z})$, i.e., the case of universal abelian cover, since in this case $\mathcal{A}(X,A)$ is a homotopy invariant of X.

Definition 2.1. *Let $R^m \to R^n \to \mathcal{A}(X,A) \to 0$ be a presentation of the Alexander invariant. The i-th characteristic variety is the set of zeros in $(\mathbf{C}^*)^{rk(A)}$ of the polynomials in the ideal in R generated by minors of order $n-i+1$ (i-th Fitting ideal of $\mathcal{A}(X,A)$).*

If $A = H_1(X,\mathbb{Z})$ then $H_1(\tilde{X}_A) = \pi_1'(X)/\pi_1''(X) \otimes \mathbf{C}$, i.e., depends only on the fundamental group of X. For any group G, the Alexander invariant of X such that $\pi_1(X) = G$ provides an invariant of a pair

(G, A) where A is a (free) abelian quotient of G. For any A the module $\mathcal{A}(X, A)$ can be computed directly using Fox calculus.

It turns out (cf. [14], [22]) that the i-th characteristic variety coincides with Σ_i^1 (considered as subvariety of $\mathrm{Char}(A)$ of the space of rank one local systems which factor through A). In the case where $\pi_1(X)$ is abelian and $\pi_j(X) = 0$ for $j < k$ one can similarly interpret Σ_i^k as the set of zeros of polynomials in the i-th Fitting ideal of the module $H_k(\tilde{X}_{H_1(X,\mathbb{Z})})$ (cf. [20] where typical case of such situation, i.e., when X is a complement to a hypersurface in $\mathbf{C}^{k+1}$, is considered).

Second, the characteristic varieties Σ_i^1 determine the one dimensional cohomology of branched and unbranched covers of X (cf. [18], [29] and the next section)

Third, the varieties Σ_i^1 allow one to detect dominant maps of X on curves. These results are going back to classical works of deFranchis on the existence of irrational pencils on algebraic surfaces and to more recent work of Green-Lazarsfeld, Beauville, Simpson, Deligne (when X is projective) and D. Arapura (when X is quasiprojective).

Theorem 2.2 ((D.Arapura) [2]). *Let X be a quasiprojective variety then any irreducible component of characteristic variety is a coset of a subgroup of* $\mathrm{Char}\,\pi_1(X)$*. Moreover each component having a positive dimension corresponds to a holomorphic map $f : X \to C$ on a curve C such that local systems in this component have the form $E \otimes f^*(L)$ where L runs through the local systems on C.*

§3. Local systems on complements to algebraic curves

Now we will restrict our attention to the case where X is a complement to an algebraic curve $\mathcal{C}$ in affine plane $\mathbf{C}^2 = \mathbf{P}^2 - L$. The case where L is in general position relative to the projective closure $\bar{\mathcal{C}}$ of $\mathcal{C}$ is of particular interest since in this case $\pi_1(\mathbf{P}^2 - \bar{\mathcal{C}})$ is just a quotient of $\pi_1(\mathbf{C}^2 - \mathcal{C})$ by an element of its center (cf. [19]) which we will assume here. A closely related case of local systems on complements to hypersurfaces with isolated singularities is considered in [21].

With curve $\mathcal{C}$ and surjection $H_1(\mathbf{C}^2 - \mathcal{C}, \mathbb{Z}) \to A$ on a group A (cf. section 2) one can associate unbranched cover $(\widetilde{\mathbf{C}^2 - \mathcal{C}})_A$ of $\mathbf{C}^2 - \mathcal{C}$ and branched cover of $\mathbf{P}^2$ branched over the projective closure of $\mathcal{C}$ with A as the group of the cover. Since the first Betti number of an algebraic surface is a birational invariant, the first Betti number of a resolution of singularities of the latter cover depends only on $\mathcal{C}$ and the group A. We shall denote a resolution of singularities of a cover of $\mathbf{P}^2$ branched along $\mathcal{C}$ by $Z_{\mathcal{C},A}$ (though it depends, of course, on the surjection on

$H_1(\mathbf{C}^2 - \mathcal{C}) \to A)$. Moreover, the first Betti numbers of $(\widetilde{\mathbf{C}^2 - \mathcal{C}})_A$ and $Z_{\mathcal{C},A}$ depend only on the characteristic varieties of $\mathbf{C}^2 - \mathcal{C}$. More precisely we have (cf. [18])

$$rkH_1((\widetilde{\mathbf{C}^2 - \mathcal{C}})_A) = \Sigma_{\chi \in \mathrm{Char}(A)} \max\{i | \chi \in \Sigma_i^1(\mathbf{C}^2 - \mathcal{C})\}.$$

For the branched case, for $\chi \in \mathrm{Char}\,\pi_1(\mathbf{C}^2 - \mathcal{C})$ denote by $\mathcal{C}_\chi$ the curve formed by the components C_i of $\mathcal{C}$ such that if γ is the boundary of a small 2-disk transversal to C_i then $\chi(\gamma_i) \neq 1$. Then (cf. [29])

$$rkH_1(Z_{\mathcal{C},A}) = \Sigma_{\chi \in \mathrm{Char}(A)} \max\{i | \chi \in \Sigma_i^1(\mathbf{C}^2 - \mathcal{C}_\chi)\}.$$

If (G, A) is a pair as in section 2 where $A = \mathbb{Z}$ then all ideals in $R = \mathbf{C}[A]$ are principal. A generator of the i-th Fitting ideal for the module $\mathcal{A}(X, A)$, (defined up to a unit of R) where X is a space with $\pi_1(X) = G$ is a polynomial called i-th Alexander polynomial $\Delta_i(\mathcal{C})$ of $\mathcal{C}$. Its set of zeros is Σ_i^1. In the case when $rkA > 1$, Fitting ideals for $\mathrm{Ker}(G \to A)/\mathrm{Ker}(G \to A)' \otimes \mathbf{C}$ are not principal in general, though in special case where G is the fundamental group of a complement to a link in a 3-sphere the first Fitting ideal *is* a product of a power of the maximal ideal of the identity of $\mathrm{Char}\,G$ and a principal ideal (whose generator is the multivariable Alexander polynomial). A special feature of the case where $G = \pi_1(\mathbf{C}^2 - \mathcal{C})$ is that one can determine the characteristic varieties in terms of local type of singularities of $\mathcal{C}$ and the geometry of the set of singular points of $\mathcal{C}$ in the projective plane containing $\bar{\mathcal{C}}$. In fact, in the cyclic case, one obtains an expression for the whole Alexander polynomial (cf. [33]). In the rest of this section we will describe this calculation and give some examples only briefly indicating how this can be generalized to the abelian case (i.e., when $rk(A) > 1$) referring the reader to [22] for complete details.

Let us first describe the local data which comes into the description of the Alexander polynomials of algebraic curves (cf. [17]). We want to associate with each germ of a plane curve singularity, say $f(x, y) = 0$ at the origin, a sequence of rational numbers $\kappa_1, ..., \kappa_l$ and corresponding ideals $\mathcal{A}_{\kappa_1}, ..\mathcal{A}_{\kappa_l}$ in the local ring $\mathcal{O}_{(0,0)}$.

Recall that the adjoint ideal of an isolated singularity of a hypersurface V at the origin near which V is given by the equation $g(x_1, ..., x_r) = 0$ consists of germs ϕ in the local ring of the origin such that $\phi \cdot (dx_1 \wedge ... \wedge \hat{dx_i} \wedge ... dx_r)/g_{x_i}$ admits a holomorphic extension over the exceptional set of some resolution of the singularity of V. The adjoint ideal will be denoted as $Adj(g = 0)$. In the case where $g(x_1, ..., x_r)$ is generic for its Newton polytope, a monomial $x_1^{i_1} \cdots x_r^{i_r}$ belongs to the adjoint ideal if and only if $(i_1+1, ..., i_r+1)$ is strictly above the Newton polytope of g (cf. [25]).

In order to define the constants $\kappa_1, ..., \kappa_l$ (constants of quasiadjunction of the singularity of a germ of plane curve $f(x,y)$) let us consider for each element ϕ in the local ring of the origin, the function

$$\Psi_\phi(p) = \min\{k | z^k \cdot \phi \in Adj(z^p = f(x,y))\}.$$

One can show that this function can be written for an appropriate rational number κ_ϕ as $\Psi_\phi(p) = [\kappa_\phi \cdot p]$ where $[\cdot]$ denotes the integer part (this is immediate, in the case where $f(x,y)$ is generic for its Newton polytope, from the description of the adjoint ideal mentioned in the previous paragraph, since in this case the germ of $z^p = f(x,y)$ is generic for its Newton polytope). Moreover, the set of rational numbers $\kappa_\phi, \phi \in \mathcal{O}_{0,0}$, is finite. In fact the set of numbers $-\kappa_\phi$ forms a subset of Arnold-Steenbrink spectrum of $f(x,y) = 0$ belonging to the interval $(0,1)$ (cf. [24]). In particular $\exp(2\pi i\kappa)$ is a root of the local Alexander polynomial of the link of the singularity $f(x,y) = 0$.

It follows from the definition that for each κ the germs ϕ such that $\kappa_\phi < \kappa$ form an ideal called *an ideal of quasiadjunction* and denoted $\mathcal{A}_\kappa$. Now we are ready to describe the Alexander polynomial of $\mathcal{C} \subset \mathbf{C}^2$ (cf. [17]). For each rational κ let us consider the ideal sheaf $\mathcal{I}_\kappa \subset \mathcal{O}_{\mathbf{P}^2}$ such that $\mathcal{O}_{\mathbf{P}^2}/\mathcal{I}_\kappa$ is supported at the singular locus of $\bar{\mathcal{C}}$ and such that the stalk of $\mathcal{I}_\kappa$ at a singular point p of $\mathcal{C}$ consists of germs belonging to the ideal $\mathcal{A}_\kappa$ of the singularity of $\mathcal{C}$ at p.

Theorem 3.1. *The Alexander polynomial of $\mathcal{C}$ having degree d is $\Pi_\kappa[(t-\exp(2\pi i\kappa))(t-\exp(-2\pi i\kappa)]^{\dim H^1(\mathcal{I}_\kappa(d-3-d\cdot\kappa))}$ where the product is over all constants of quasiadjunction of all singular points of $\mathcal{C}$ such that $d\cdot\kappa \in \mathbb{Z}$.*

Examples 1. Let $\bar{\mathcal{C}}$ be given by the equation $f_{3k}^2 + f_{2k}^3 = 0$ where f_l is a generic form of degree l. Then $\bar{\mathcal{C}}$ is a curve of degree $6k$ having $6k^2$ ordinary cusps (i.e., locally given by $x^2+y^3=0$) located at the set of solutions of $f_{2k} = f_{3k} = 0$. Ordinary cusp has only one constant of quasiadjunction $\kappa = 1/6$ and the corresponding ideal of quasiadjunction is just the maximal ideal (this follows directly from the above since the ordinary cusp is weighted homogeneous and hence generic for its Newton polytope). The corresponding sheaf $\mathcal{I}_{1/6}$ admits Koszul resolution $0 \to \mathcal{O}_{\mathbf{P}^2}(-5k) \to \mathcal{O}_{\mathbf{P}^2}(-3k) \oplus \mathcal{O}_{\mathbf{P}^2}(-2k) \to \mathcal{I}_{1/6} \to 0$ which yields $\dim H^1(\mathbf{P}^2, \mathcal{I}_{1/6}(6k-3-6k/6)) = 1$. Therefore the Alexander polynomial is equal to t^2-t+1. This, of course, provides complete description of the cohomology of local systems on the complement to this curve.

2. Let $\bar{\mathcal{C}}$ be a sextic with 3 cusps and one singularity of type $x^4 = y^5$ (cf. [26]). We start by describing the constants of quasiadjunction of singularities of this curve which may contribute to the Alexander polynomial. First, the constant of quasiadjunction of singularity $x^4 = y^5$ corresponding to $\phi = x^i \cdot y^j$ is equal to $\kappa_\phi = \max\{(11 - 5i - 4j)/20, 0\}$ as follows from the description of adjoint ideals for polynomials generic for their Newton polytopes mentioned earlier. We noted already that $x^2 = y^3$ has only one constant of quasiadjunction, i.e., $1/6$. Second, since the degree of the curve is 6, the contributing into Alexander polynomial constants of quasiadjunction κ should satisfy $6 \cdot \kappa \in \mathbb{Z}$. Third, it follows again from the description of adjoint ideals that the monomial $x^i y^j$ belongs to the ideal of quasiadjunction corresponding to the constant of quasiadjunction $1/6$ in the local ring of the singularity $x^4 = y^5$ if and only if $x^i y^j z^{[p/6]}$ belongs to the adjoint ideal of $z^p = x^4 - y^5$. This happens if an only if $5p(i+1) + 4p(j+1) + 20((p/6)+1) \geq 20p$ for any positive p. This is equivalent to $5i + 4j > 7(2/3)$, i.e., either $i \geq 2$ or $j \geq 2$ or both $i, j \geq 1$. Hence ϕ, which is a combination of $x^i y^j$, is in the ideal of quasiadjunction of $1/6$ of singularity $x^4 = y^5$ if and only if it is in the square of the maximal ideal. Therefore the intersection index of the $\phi = 0$ with $x^4 = y^5$ is at least 8. The ideal of quasiadjunction corresponding to the constant $1/6$ for the ordinary cusp is the maximal ideal. It follows from the Bezout theorem that $H^0(\mathcal{I}_{1/6}(2)) = 0$. Now $\chi(I_{1/6}(2)) = 0$, because the sum of dimensions of stalks of $\mathcal{O}_{\mathbf{P}^2}(2)/\mathcal{I}_{1/6} = \dim H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2)) = 6$, whence $H^1(\mathcal{I}_{1/6}(2)) = 0$ and the Alexander polynomial of this curve is 1.

§4. Local systems on the complements to arrangements of hyperplanes

An interesting class of examples where cohomology of local systems and characteristic varieties can be often explicitly computed is formed by complements to hyperplane arrangements. Tools for computations are given by combinatorial invariants of arrangements: the intersection lattice and its Orlik-Solomon algebra.

Let $\mathcal{B}$ be an arrangement $\{H_1, \ldots, H_n\}$ of hyperplanes in a complex projective space $\mathbf{P}$ and L its intersection lattice (i.e., the set of all intersections of the hyperplanes ordered opposite to inclusion and augmented by the maximum element 1). Fix some homogeneous linear forms $\alpha_1, \ldots, \alpha_n$ such that the zero locus of α_i is H_i . Recall that the Orlik-Solomon algebra S of $\mathcal{B}$ (or of L) is the factor of the exterior algebra over $\mathbb{C}$ on generators $e_1, \ldots, e_n$ by the ideal generated by

$$\sum_{j=1}^{p} (-1)^j e_{i_1} \cdots \hat{e}_{i_j} \cdots e_{i_p}$$

for all linearly dependent sets $\{\alpha_{i_1}, \ldots, \alpha_{i_p}\}$. Algebra S is graded and generated in degree one. Denote by $\bar{S}$ the subalgebra of S generated by the elements $\sum_{i=1}^{n} a_i e_i$ ($a_i \in \mathbb{C}$) with $\sum_{i=1}^{n} a_i = 0$. According to the projective version of the Brieskorn-Orlik-Solomon theorem ([3, 27]), $\bar{S}$ is isomorphic to the algebra $H^*(M, \mathbb{C})$ where M is the complement of the divisor $D = \bigcup_i H_i$. The isomorphism is given by sending each e_i to the closed 1-form $\omega_i = d\alpha_i/\alpha_i$ and taking the cohomology class of the latter.

The forms ω_i can be used to produce matrix-valued logarithmic forms and local systems of higher rank on M. For a positive integer r, let P_i ($i = 1, \ldots, n$) be $r \times r$-matrices over $\mathbb{C}$ such that $\sum_i P_i = 0$ and $\omega \in H^0(\mathbf{P}, \Omega^1\langle\mathcal{B}\rangle \otimes \mathcal{O}^r)$ be defined as

$$\omega = \sum_{i=1}^{n} \omega_i \otimes P_i.$$

Via the construction mentioned in section 2, the form ω defines a connection on $\mathcal{O}_M^r$ which is integrable if an only if $\omega \wedge \omega = 0$. This connection defines the local system of rank r on M and since M is affine the cohomology of this system is the cohomology of the complex of rational forms:

$$\Gamma = \Gamma(M, \Omega_M^* \otimes \mathbb{C}^r)$$

with differential $d + \omega\wedge$ (cf. section 2). The correspondence $e_i \mapsto \omega_i$ defines also an embedding

$$\phi : \bar{S}^* \otimes \mathbb{C}^r \subset \Gamma$$

where $\bar{S}^* \otimes \mathbb{C}^r$ is the complex on $\bar{S} \otimes \mathbb{C}^r$ whose differential is the (left) multiplication by the element $a \in \bar{S}_1$ corresponding to the form ω. In the rank one case which is of the main interest in this note we denote the cohomology of that complex by $H^*(\bar{S}, a)$.

For the arrangement of hyperplanes in general position or for a general position a the embedding ϕ is a quasi-isomorphism. More precise sufficient conditions were obtained in [9, 30] by blowing up at non-normal crossings and applying Deligne's theorem (see section 2). To state the stronger version from [30] note that each $X \in L$ defines the subarrangement $\mathcal{B}_X = \{H \in \mathcal{B} | X \subset H\}$ of $\mathcal{B}$. We put $P_X = \sum_{H_i \in \mathcal{B}_X} P_i$ and call the subspace X *dense* if $\mathcal{B}_X$ is not the product of two non-empty arrangements.

Theorem 4.1 (Schechtman-Terao-Varchenko). *If for any dense subspace $X \in L$ none of the eigenvalues of P_X is a positive integer then ϕ is a quasi-isomorphism.*

Theorem 4.1 brings up the combinatorial problem of computing $H^*(\bar{S}, a)$ for various $a \in \bar{S}_1$. In particular an important question for applications (e.g. to hypergeometric functions, see [1]) is when this cohomology vanishes in all but the maximum dimensions. This question was studied in [35] (cf. also [16]) using sheaves on posets. In particular it was proved there that a sufficient condition for the vanishing is

$$\sum_{H_i \in \mathcal{B}_X} a_i \neq 0$$

for every dense $X \in L$. This work was continued in [11] where a basis of cohomology of the maximum dimension was found that is independent of ω.

Another interesting problem is to investigate connections between the two types of cohomology (of rank 1 local system and of the complex $\bar{S}^*$) when the conditions of Theorem 4.1 cease to hold (so called *resonance* case). There are at least two related but different ways to do that. One way is to relate the characteristic varieties of an arrangement with the respective subvarieties of $\bar{S}_1$. Let us define the latter. The first relevant definition was given by Falk [10] who studied invariants of S.

For an arrangement $\mathcal{B}$ define the *resonance variety*

$$R_k^\ell = R_k^\ell(\mathcal{B}) = \{a \in \bar{S}_1 \,|\, \dim H^\ell(\bar{S}, a) \geq k\}.$$

Clearly each R_k^ℓ is an algebraic subvariety of the linear space $\bar{S}_1$ and the easiest one to study is R_k^1. The studies of these varieties were started in [10]. Their relations with the characteristic varieties were first investigated in [22] and then in [5].

Since we focus on the cohomology of dimension 1 it suffices to consider arrangements of lines in the projective plane since by twisted version of Lefschetz theorem [32] the fundamental group of the complement to an arrangement is the same as the one for the intersection of this arrangement with a generic plane. For this case, the irreducible components of R_k^1 are linear and there is a bijection $\mathcal{W} \mapsto W$ between the set of these components and the set of all the positive dimension components of Σ_k^1 passing through 1. The exponentiation defines the universal covering of W by $\mathcal{W}$ [22, 23] and R_k^1 is the tangent cone of Σ_k^1 at the point 1 [5]. Characteristic varieties also yield a different sufficient condition for the conclusion of Theorem 4.1 [22] to be true. Namely ϕ

is a quasi-isomorphism if

$$(\exp(2\pi i a_1), \dots, \exp(2\pi i a_n)) \notin \mathrm{Char}^1.$$

The other way to relate the two kinds of cohomology is for individual elements $a \in \bar{S}_1$. This is the main theme of [23]. It starts with the following inequality

$$\dim H^p(M, \mathcal{V}(a)) \geq \dim H^p(\bar{S}, a + N) \qquad (*)$$

for every p and $a \in \bar{S}_1$ where $\mathcal{V}(a)$ is the rank 1 local system defined by the 1-form corresponding to a and N is an arbitrary element of $\bar{S}_1$ with integer coordinates in the canonical basis $(e_1, \dots, e_n)$. This inequality follows immediately from the two observations. First, by multiplying a by $1 + \epsilon$ with $|\epsilon|$ small one makes it satisfy the conditions of Theorem 4.1 and does not change $H^p(\bar{S}, a)$. Now the upper semicontinuity of the dimension of cohomology gives $(*)$ with $N = 0$. Second, adding N to the right hand side of $(*)$ does not change its left hand side since the local system $\mathcal{V}(a)$ is defined by the character of S_1 given by $e_k \mapsto \exp(2\pi i a_k)$. (Note that the differentials $d + \omega_a \wedge$ and $d + \omega_{a+N} \wedge$ are different though isomorphic via multiplication by a rational function.)

The main result of [23] is the following theorem.

Theorem 4.2. *The left-hand side of* $(*)$ *is the supremum of its right-hand side while* N *is running through* $\mathbb{Z}^n$ *for all but finitely many cosets mod* $\mathbb{Z}^n$ *of elements of* R^1.

The proof of this theorem required certain further resutls about irreducible components of both the characteristic and resonance varieties. With every $a \in R^1_k$ one can associate the set $\mathcal{X}(a)$ of multiple points of intersection of lines such that the vector $(a_i | H_i \supset X)$ is not zero but $\sum_{H_i \supset X} a_i = 0$. The set $\mathcal{X}(a)$ defines the collection of subsets of $\mathcal{B}$ of lines passing through a point $X \in \mathcal{X}$. The incidence matrix $J(a)$ of the collection defines the symmetric matrix $Q(a) = J^t J - E$, where E is the matrix whose every entry is 1, that satisfies the conditions of a theorem of Vinberg's ([15], p.48) except that it is decomposable in general. An application of this theorem to the indecomposable components of Q shows that they should be either affine or finite with at least three affine ones. In particular this implies that if $\mathcal{W}$ is an irreducible component of R^1_k and a is an arbitrary nonzero vector form $\mathcal{W}$ then $\mathcal{W}$ is the $k+1$-dimensional subspace of $\bar{S}_1$ given by the linear system $\sum_{H_i \supset X} x_i = 0$ for all $X \in \mathcal{X}(a)$. In particular $\mathcal{W}$ is defined by $\mathcal{X}(a)$. In the ring $\bar{S}$, $\mathcal{W}$ is the annihilator of a in degree one whence any two irreducible components intersect at 0.

On the other hand, let W be a positive dimension component of Σ^1_k containing 1 with the universal cover $\mathcal{W}$. Suppose W is essential, i.e., it is not the image of a component for a proper subarrangement $\mathcal{B}'$ of $\mathcal{B}$ under the inclusion map $H^1(M', \mathbb{C}^*) \to H^1(M, \mathbb{C}^*)$. Then one can associate with W a pencil of curves whose fibers do not have a common component. Some degenerate fibers are the unions of lines from $\mathcal{B}$ passing through a point from $\mathcal{X}(a)$ where $a \in \mathcal{W}$. If P' is the blow up of the projective plane at $\mathcal{X}(a)$ then matrix Q can be recovered as the minus intersection form on P'. Using the pencil of curves, the Euler characteristic of P' can be computed in two different ways that gives strong restrictions on the size and amount of indecomposable blocks of Q. Combining this with the condition on the blocks of being affine one obtains strong restrictions on arrangements of lines with characteristic varieties of positive dimension. For instance, if each line has precisely three multiple points then the arrangement can be embedded into the Hesse arrangement consisting of 12 lines passing each through 3 of 9 inflection points of a smooth cubic.

The following example from [5] can be used to show that the exceptional finitely many cosets of elements of R^1 from Theorem 4.2 can indeed exist.

Example [5]. The arrangement consists of 7 lines that are the zero loci of the following forms α_i (ordered from left to right): x, $x+y+z$, $x+y-z$, y, $x-y-z$, $x-y+z$, z. These lines define 3 double and 6 triple points of intersection with the latter (viewed as the sets of indices of lines passing through them) being

$$X_1 = \{1,2,5\},\ X_2 = \{1,3,6\},\ X_3 = \{2,3,7\},$$
$$X_4 = \{2,4,6\},\ X_5 = \{3,4,5\},\ X_6 = \{5,6,7\}.$$

The resonance variety $R^1 = R^1_1$ has 3 irreducible components W_1, W_2, and W_3 of dimension 2 defined by the collections

$$\mathcal{X}_1 = \{X_3, X_4, X_5, X_6\},\ \mathcal{X}_2 = \{X_1, X_2, X_3, X_6\},\ \mathcal{X}_3 = \{X_1, X_2, X_4, X_5\}$$

respectively. The pencil of quadrics corresponding to, say $\mathcal{W}_1$, is generated by $x^2 - y^2 - z^2$ and yz.

Consider $a = 1/2(-e_2 + e_3 - e_5 + e_6) \in W_1$. Then $a + N_1 \in W_2$ and $a + N_2 \in W_3$ with $N_1 = e_2 - e_6$ and $N_2 = e_3 - e_6$. Thus $\mathcal{V}(a) = \mathcal{V}(a + N_i)$. It is not hard to see that $\dim H^1(M, \mathcal{V}(a)) = 2$, i.e., $\mathcal{V}(a) \in \Sigma^1_2$. In fact, $\mathcal{V}(a)$ together with the constant system forms a discrete component of Σ^1_2 that is a group of order 2.

§5. Problems

The emerging picture of the cohomology of rank one local systems is far from being complete. We suggest several problems as an attempt to clarify it.

The case of arrangements of lines in the projective plane seems to be the most promising and a majority of our problems is devoted to this case. In them, M is the complement of the union of lines (cf. section 4).

Problem 5.1. *Is it true that every positive dimensional irreducible component of* $\mathrm{Char}^1(M)$ *contains* 1 (*whence is covered by a component of the resonance variety* R^1)*? The above example shows that it is not true for discrete components.*

Problem 5.2. *For* $a \in \bar{S}_1$, *is it possible to compute* $H^1(M, \mathcal{V}(a))$ *knowing* $H^1(\bar{S}, a+N)$ *for all vectors* $N \in \mathbb{Z}^n$*?*

Theorem 4.2 gives the positive answer for almost all $a \in R^1$. Even for the exceptional a in the above example the cocycles with coefficients in $\mathcal{V}(a)$ are generated by differential forms corresponding to cocycles for $a+N$ in $\bar{S}_1$.

This problem may split depending on which a one considers, from R^1 or not. In particular the following particular case of the problem might be easier.

Problem 5.3. *Can there exist* $a \in \bar{S}_1$ *such that* $H^1(\bar{S}, a+N) = 0$ *for all* $N \in \mathbb{Z}^n$ *but* $H^1(X, \mathcal{V}(a)) \neq 0$*?*

Problem 5.4 (Combinatorial invariance of characteristic varieties). *Are the characteristic varieties combinatorial invariants of arrangements, i.e., can one reconstruct them from the lattice of an arrangement?*

It is known that the fundamental groups of the complements to arrangements are *not* invariants of the lattice ([28]). On the other hand, the results of [5], [22], and [23] show that components of positive dimension containing the identity character are. For algebraic curves, Alexander polynomials can not be determined just from the degrees of the curve and the local type of singularities. This follows from seminal example of two sextics with six cusps on and not on a conic. Recent results on Zariski's pairs and triples are discussed in [26].

Problem 5.5 (Realization and classification). *How many components can the characteristic variety have? Can this number be arbitrary large? Can one bound dimensions and the number of components in terms of lattice of arrangement? Can one classify arrangements for* *which the characteristic varieties have components with positive dimension or sufficiently large dimension?*

Results of [23] show that the dimension of the characteristic variety imposes a bound on the number of lines in the arrangement. Similar realization problem for characteristic varieties and Alexander polynomials of algebraic curves (e.g. which polynomials can appear as the Alexander polynomials of algebraic curves or algebraic curves of given degree) seems to be also open. More concretly: how large can the degree of the Alexander polynomial be for a curve with nodes and cusps? For the sextic dual to a non singular plane cubic the Alexander polynomial is equal to $(t^2 - t + 1)^3$. Are there the curves, with nodes and cusps only, for which the degree of Alexander polynomial is bigger than 6? The Alexander polynomial of the complement to an algebraic curve divides the product of local Alexander polynomials and the Alexander polynomial at infinity (cf. [17] and references there). This gives a bound for the degree of the Alexander polynomial in terms of the degree of the curve. For example for a curve with singularities not worse than ordinary cusps we obtain $2(d-2)$. For calculations of Alexander polynomials for curves with more complicated singularites we refer to [6].

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