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Orbits on Homogeneous Spaces of Arithmetic Origin and Approximations

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Abstract.

We prove an S-arithmetic version, in the context of algebraic groups defined over number fields, of Ratner's theorem for closures of orbits of subgroups generated by unipotent elements. We apply this result in order to obtain a generalization of results of Margulis and of Borel-Prasad about values of irrational quadratic forms at integral points to the general setting of hermitian forms over division algebras with involutions of first or second kind. As a byproduct of our considerations we obtain another proof of the strong approximation theorem for algebraic groups defined over number fields.

Introduction

Many problems from number theory and, in particular, in Diophantine approximations can be reformulated in terms of dynamics of actions of subgroups on homogeneous spaces. In this way, ideas and methods from dynamical systems can be successfully used in number theory and, vice versa, problems from number theory stimulate the study of certain kinds of dynamical systems. One of the most impressing example is provided by the following conjecture formulated by Oppenheim in 1929 [Op 1,2]: If f is a real nondegenerate indefinite quadratic form of $n \geq 5$ variables and f is not a multiple of a quadratic form with rational coefficients, then for any real $\varepsilon > 0$ there exists a nonzero vector $z \in \mathbb{Z}^n$ such that $\mid f(z) \mid < \varepsilon$. Let us briefly recall some of the conjectures and the subsequent results connected with Oppenheim's conjecture. (For detail, we refer the reader to the exhaustive review paper [M6], as well as to the earlier review papers [B2],[D4],[M5],[Rat4] and [S2].) In the

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mid-seventies, M.S.Raghunathan formulated a conjecture that the closure of an orbit of a unipotent subgroup on a homogeneous space G/Γ, where G is a Lie group and Γ is a lattice in G, is an orbit of a larger subgroup, and also noted that this statement implies the Oppenheim conjecture. In 1981, S.G.Dani [D2] formulated the metric version of the Raghunathan conjecture, often called "measure rigidity" : any U-invariant U-ergodic Borel probability measure on G/Γ, where U is a unipotent subgroup of G, coincides with the Haar measure on a closed orbit of a connected subgroup containing U. In [M2,3], using the homogeneous space approach, G.A.Margulis proved that $f(\mathbb{Z}^n)$ is dense in $\mathbb{R}$ which, in particular, confirms Oppenheim's conjecture . The conjectures of Dani and Raghunathan have been proved in full generality by M.Ratner in [Rat1] and [Rat2], respectively. (We will refer to the first result as to "the measure classification theorem" and to the second one as to "the theorem for orbit closures.") The main part in [Rat2] is dedicated to the proof of a theorem about uniform distribution of unipotent flows on the homogeneous space G/Γ. Later Dani and Margulis [DM2] applied other methods in order to prove a refined version of Ratner's uniform distribution theorem.

In [BPr] Borel and Prasad obtained the following generalization of the Margulis theorem. Let S be a finite set of normalized valuations of a number field K containing the set S_∞ of archimedean ones, K_S the direct sum of the completions K_v of K at $v \in S$, and $\mathcal{O}$ the ring of S-integers of K. Let f_S be a nondegenerate quadratic form on K_S^n; equivalently, let f_S be a collection f_v, $v \in S$, where f_v is a nondegenerate quadratic form on K_v^n. Assume that f_S is K-irrational (i.e. f_S is not a multiple of a quadratic form on K^n), f_v is isotropic for all $v \in S$ and $n \geq 3$. Then $f_S(\mathcal{O}^n)$ is dense in K_S. (Note that K (respectively, $\mathcal{O}^n$) is diagonally embedded in K_S (respectively, in K_S^n). Under these embeddings, K_S is a K-algebra and $\mathcal{O}^n$ is a lattice in K_S^n.) The Borel-Prasad result can be regarded as an analog for the irrational quadratic forms of the local-global principle for quadratic forms over number fields (that is, of the Hasse-Minkowski theorem [Se, ch. 4, Theorem 8]). In fact, Landherr, Kneser and Springer proved the local-global principle for K-rational hermitian forms over finite-dimensional division algebras with involutions of first or second kind(cf. [L], [K1] and [Sch,ch. 10]). This suggests the problem about the extension of the results [M2,3] and [BPr] to the general framework of the hermitian forms over division algebras. In [BPr] and [Pr2] Borel and Prasad raised the problem of generalizing Ratner's theorems to the case when G is a finite direct product of real and p-adic Lie groups. The generalizations of these theorems were obtained by Ratner herself [Rat3] and, independently, the generalization of the measure classification theorem was obtained by Margulis and the author [MTo1,2].

Note that for the arithmetic applications of the theorem for orbit closures we need only the S-arithmetic version of this theorem when $\mathbf{G}$ is a K-algebraic group, $G = \mathbf{G}(K_S)$ and Γ is an S*-arithmetic subgroup* of G in the sense that Γ and $\mathbf{G}(\mathcal{O})$ are commensurable subgroups of $\mathbf{G}(K)$. (Here and further on S and K_S are as above and G is identified with the direct product $\prod_{v \in S} \mathbf{G}(K_v)$.) The proof of the theorem in the S-arithmetic case allows to avoid some technical complications although the main ideas from the general case remain involved. In the present paper we give a proof of the theorem for orbit closures in the S-arithmetic case using the approach and methods from Dani-Margulis paper [DM2] and, subsequently, we apply this theorem in order to obtain the analogs for the hermitian forms over division algebras of the result of Borel and Prasad [BPr]. Our versions in the S-arithmetic case of both the theorem for orbit closures and the measure classification theorem (see Theorem 1 and Theorem 2 below) are somewhat more precise than in the general case of direct products of real and p-adic Lie groups (cf.[Rat3], [MTo1,2]). Some of our arguments are applied in order to give a short proof of the strong approximation theorem for the simply connected algebraic groups defined over number fields (see [Pl] and also [Pr1] where the strong approximation theorem is proved for any global field K).

Let us fix the following notions. We say that a connected K-algebraic subgroup $\mathbf{P}$ of $\mathbf{G}$ is a subgroup of class $\mathcal{F}$ (relatively to S) if for each proper normal K-algebraic subgroup $\mathbf{Q}$ of $\mathbf{P}$ there exists $v \in S$ such that $(\mathbf{P}/\mathbf{Q})(K_v)$ contains a unipotent element different from the identity. Recall that according to [B1], an S-arithmetic subgroup Γ of G is a lattice (i.e. Γ has finite covolume in G) if and only if the connected component of $\mathbf{G}$ does not admit nontrivial K-rational characters. In the latter case, Γ is called S*-arithmetic lattice.* Note that if $\mathbf{P}$ is a subgroup of class $\mathcal{F}$ in $\mathbf{G}$ then $P' \cap \Gamma$ is an S-arithmetic lattice in P' for any subgroup of finite index P' in $\mathbf{P}(K_S)$ and any S-arithmetic subgroup Γ in G. Given a subgroup $H \subset G$, we will denote by H_u the subgroup generated by all 1-parameter unipotent subgroups of H (see 1.5).

Theorem 1. *Let* $\mathbf{G}$ *be a* K*-algebraic group,* Γ *an* S*-arithmetic lattice in* G*,* H *a subgroup of* G *such that* $H = H_u$ *and* $x = g\Gamma$ *a point in* G/Γ*. Then there exists a subgroup* $\mathbf{P} \subset \mathbf{G}$ *of class* $\mathcal{F}$ *and a subgroup of finite index* P' *in* $\mathbf{P}(K_S)$ *such that* $gP'g^{-1}$ *contains* H *and the closure of* Hx *in* G/Γ *coincides with* $gP'g^{-1}x$*.*

In sections 3 and 4 we will give a direct proof of Theorem 1. (It can also be deduced from [Rat3, Theorem 2] and Theorem 3 below.) The proof of Theorem 1 essentially uses the following measure classification theorem, very important by itself.

Theorem 2. *Let* $\mathbf{G}$ *be a* K*-algebraic group,* Γ *an* S*-arithmetic subgroup of* G*,* H *a subgroup of* $\mathbf{G}$ *such that* $H = H_u$ *and* μ *an* H*-invariant* H*-ergodic Borel probability measure on* G/Γ*. Then there exist a subgroup* $\mathbf{P} \subset \mathbf{G}$ *of class* $\mathcal{F}$*, a subgroup of finite index* P' *in* $\mathbf{P}(K_S)$ *and a point* $x = g\Gamma$ *in* G/Γ *such that* $gP'g^{-1}$ *contains* H*,* $gP'g^{-1}x$ *is closed in* G/Γ *and the measure* μ *is* $gP'g^{-1}$*-invariant and concentrated on* $gP'g^{-1}x$*.*

Recall that in the usual formulations of the above theorems (see [Rat3] and [MTo1]) P' is a closed subgroup of G such that $P' \cap \Gamma$ has finite covolume in P' (without the additional specification that P' has finite index in $\mathbf{P}(K_S)$, where $\mathbf{P}$ is a subgroup of class $\mathcal{F}$). Theorem 2 follows from [MTo1,Theorem 2] or [Rat3. Theorem 1] and from the next theorem.

Theorem 3. *Let* $\mathbf{G}$ *be a* K*-algebraic group,* Γ *an* S*-arithmetic subgroup of* G *,* M *a closed subgroup of* G*,* $x = g\Gamma$ *a point in* G/Γ *and* $M_x = \{a \in G \mid ax = x\}$*. Assume that* Mx *is closed,* $M_u x$ *is dense in* Mx *and* Mx *admits* M*-invariant Borel probability measure* μ*. Let* $P = \mathbf{P}(K_S)$ *where* $\mathbf{P}$ *is the connected component of the Zariski closure of* $g^{-1}M_x g$ *in* $\mathbf{G}$*. Then* $P' = \{a \in P \mid ag^{-1}\mu = g^{-1}\mu\}$ *is a subgroup of finite index in* P*,* $gP'g^{-1} = Mx$*, and* $g^{-1}Mg \cap P$ *is an open subgroup in* P' *containing* $g^{-1}M_u g$*. Furthermore,* $\mathbf{P}$ *is an algebraic subgroup of class* $\mathcal{F}$ *in* $\mathbf{G}$ and it is uniquely defined by μ.

Theorem 3 will be proved in section 2. Also in section 2, we easily derive from Theorem 3 the following strong approximation theorem.

Theorem 4. (cf. [Pl]) *Let* $\mathbf{G}$ *be a connected, simply connected, algebraic group defined over a number field* K*. Let* $\mathcal{V}$ *be the adele ring of* K *and* T *be a finite set of normalized valuations of* K*. Assume that for any proper* K*-algebraic subgroup* $\mathbf{N}$ *of* $\mathbf{G}$ *there exists a valuation* $v \in T$ *such that* $(\mathbf{G}/\mathbf{N})(K_v)$ *is not compact. Then* $\mathbf{G}(K_T)\mathbf{G}(K)$ *is dense in* $\mathbf{G}(\mathcal{V})$*.*

Note that the group $\mathbf{G}$ in the formulation of Theorem 4 is actually a group of class $\mathcal{F}$. Indeed, since $\mathbf{G}$ is a simply connected algebraic group, the solvable radical of $\mathbf{G}$ coincides with its unipotent radical. This implies that $\mathbf{G}$ is of class $\mathcal{F}$ relatively to any finite set S of valuations of K containing T and S_∞.

In order to formulate the results about the hermitian forms, we need to fix some standard algebraic notions and concepts (cf. [Sch.], ch.8,10]). We denote by D a central division algebra over a number field L and of finite degree r i.e. $\dim_L D = r^2$. We fix a subfield K of L such that either $L = K$ or L is a quadratic extension of K. We let S, S_∞, $\mathcal{O}$, K_v and K_S be the same as before. In addition, we denote by Λ an $\mathcal{O}$-order in D (i.e. Λ is an $\mathcal{O}$-algebra of finite type such that $D = \Lambda \bigotimes_{\mathcal{O}} K$). Tensoring with K_v gives the topological K_v-algebras $D_v = D \bigotimes_K K_v$ and $L_v = L \bigotimes_K K_v$. Let T be a subset of S. The direct sums $D_T = \bigoplus_{v\in T} D_v$, $L_T = \bigoplus_{v \in T} L_v$ and $K_T = \bigoplus_{v\in T} K_v$ are endowed with the product topology. We will identify D, L and K with their diagonal embeddings in D_T, L_T and K_T, respectively. It is well known that each of these embeddings is dense. Furthermore, Λ is a discrete cocompact abelian subgroup of D_S. Note that L_S (resp. L_v) coincides with the center of D_S (resp. D_v).Let $v \in S$ and τ_v be a L_v/K_v-involution on D_v, i.e. τ_v is an antiautomorphism on D_v such that $\tau_v^2 = id$ and $K_v = \{x \in L_v \mid {}^{\tau_v}x = x\}$. Clearly, $\tau_T = \bigoplus_{v\in T} \tau_v$ is a L_T/K_T-involution on D_T for any $T \in S$ and, conversely, every L_T/K_T-involution on D_T is a direct sum of L_v/K_v-involution on D_v, $v \in T$. Note that any L/K-involution τ on D extends in a unique way to a L_v/K_v-involution on D_v (resp. L_T/K_T-involution on D_T). If $L = K$ (resp. L is a quadratic extension of K), then τ_v, τ_T and τ are involutions of *first* (resp. *second*) kind (cf.[Sch]).

Let $\lambda_T = (\lambda_v)_{v\in T} \in L_T$ and h_T be a nondegenerate λ_T-hermitian form on D_T^n (see 5.1). Then h_T can be equivalently viewed as a collection h_v, $v \in T$, where h_v is a λ_v-hermitian form (with respect to τ_v) on D_v^n.(Note that if τ_T =id and $\lambda_T = 1$ (resp. $\lambda_T = -1$) then $D_T = K_T$ and each h_v , $v \in T$, is a bilinear symmetric (resp. symplectic) form. In the first case $h_v(x, x)$ is a quadratic form which is the object under investigation in [M2,3] and [BPr].)

The hermitian form h_T is called *K-rational* if there exists an invertible element a in D_T and a λ-hermitian form h on D^n such that $h_T = ah$ and h_T is called *K-irrational* in the opposite case. If h_S is a nondegenerate hermitian form on D_S^n, we denote by S_0 the set of all $v \in S$ such that h_v is isotropic (i.e. the K_v-algebraic group $\mathbf{SU}(h_v)$ corresponding to h_v is K_v-isotropic, see 5.1). The form h_S is called *isotropic* if $S_0 \neq \emptyset$. We will denote by h_{S_0} the hermitian form on $D_{S_0}^n$ given by all h_v, $v \in S_0$. The hermitian forms h_S and h'_S are *properly equivalent* if there exists $g \in \mathrm{SL}_n(D_S)$ such that $h'_S = h_S^g$ (where $\mathrm{SL}_n(D_S) = \prod_{v\in S} \mathrm{SL}_n(D_v)$ acts in the usual way on the hermitian forms). We will say that h_S and h'_S

are *almost S-integer equivalent* if for any $\varepsilon > 0$ and any $\mathcal{O}$-order Λ in D there exists $g \in \mathrm{SL}_n(\Lambda)$ such that

$$\| h_S^g - h'_S \| < \varepsilon,$$

where $\| \ \|$ is a norm on the space of all hermitian forms on D_S^n comparable with the topology on this space induced by the topology on K_S.

It is easy to prove that the almost S-integer equivalence between two hermitian forms implies their proper equivalence (see 5.6). The next theorem shows in particular that the converse is true for all K-irrational isotropic hermitian forms of dimension $n \geq 3$.

Theorem 5. *With the above notations, let h_S be a nondegenerate isotropic hermitian form on D_S^n. Assume that (a) $rn \geq 3$ if τ_S is of first kind and $\tau_S \neq id$, (b) $rn \geq 2$ if τ_S is of second kind, and (c) $n \geq 3$ if $\tau_S = id$. Then the following conditions are equivalent:*

(i) h_{S_0} is K-irrational hermitian form;
(ii) h_S is properly equivalent to an hermitian form h'_S if and only if h_S is almost S-integer equivalent to h'_S.

Theorem 5 and the next corollary will be proved in section 5. (Also in §5 we give examples which show that the restrictions in the formulation of the theorem can not be weekened.)

Corollary 1. *Let $n \geq 2$, h_S be as in Theorem 1 and h_{S_0} be K-irrational. Then for any $\varepsilon > 0$, any $\mathcal{O}$-order Λ in D and any $x_1, x_2, ..., x_{n-1} \in D_S^n$ there exist $z_1, ..., z_{n-1} \in \Lambda^n$ such that*

$$(1) \qquad | h_S(x_i, x_j) - h_S(z_i, z_j) | < \varepsilon,$$

for all i, j (here $| \ |$ stands for a norm on D_S comparable with the topology on D_S). In particular, the closure of $\{ h_S(z,z) \mid z \in \Lambda^n\}$ in D_S coincides with $\{ h_S(x,x) \mid x \in D_S^n\}$.

In the case of quadratic forms (i.e. $\tau_S = \mathrm{id}$ and $\lambda_S = 1$), the above corollary was proved by Borel and Prasad (see [BPr], [B2]) for primitive vectors $z_1, ... z_{n-1} \in \mathcal{O}^n$ assuming that $S = S_0$. (For 3-dimensional real quadratic forms the result had been earlier proved by Dani and Margulis [DM1].) The observation in the case of quadratic forms that $S_0 \subset S$ may be taken arbitrary nonempty belongs to Margulis.

In contrast to the case of quadratic forms, it is generally possible that all nondegenerate hermitian forms on D_S^n are K-irrational. This occurs when D is not abelian and D does not admit L/K-involutions.

Recall that D admits a nontrivial involution of first kind if and only if D is a quaternion division algebra. The conditions for the existence on D of an L/K-involution of second kind is given by relations between the local invariants of D (cf. [Sch., 10.2.4]). Our theorem implies immediately the following criterion.

Corollary 2. *Let D be non-abelian and $L = K$ (resp. L is a quadratic extension of K). Then the following assertions are equivalent :*

(i) *D does not admit an involution of first (resp. an L/K-involution of second) kind;*
(ii) *If $rn \geq 3$ (resp. $rn \geq 2$) and two nondegenerate isotropic hermitian forms on D_S^n are properly equivalent, then they are almost S-integer equivalent.*

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§1. Notation and terminology

1.1 As usual, $\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Q}_p, \mathbb{Z}, \mathbb{N}$ denote the complex, real, rational, rational p-adic, integer and natural numbers. Furthermore, $\mathbb{R}^+$ will be the set of all strictly positive real numbers. K will denote a number field, i.e.a finite extension of $\mathbb{Q}$. All valuations of K under consideration will be *normalized* valuations (see [CaF, ch.1]). In particular, if v is a valuation of K and K_v is the completion of K with respect to v then $| \ |_v$ and θ_v denote the normalized norm and the normalized Haar measure on K_v, respectively. The field K_v contains a unique completion $\mathbb{Q}_{p_v}$ of $\mathbb{Q}$, where p_v is a prime number if v is nonarchimedean and $p_v = \infty$ and $\mathbb{Q}_\infty = \mathbb{R}$ if v is archimedean.

If v is a nonarchimedean valuation we denote by $\mathcal{O}_v$ the ring of integers of K_v. We will denote by $\mathcal{V}$ the adele ring of K (i.e. $\mathcal{V}$ is the restricted topological product of all completions K_v of K relative to $\mathcal{O}_v$, $v \notin S_\infty$ [CaF, ch.2]).

1.2 Throughout the paper, we fix a finite set S of valuations of K containing the set S_∞. We denote by $\mathcal{O}$ the ring of S-integers of K and by $\mathcal{O}^*$ the group of units in $\mathcal{O}$.

Given a subset T of S, we denote by $\theta_T = \prod_{v\in T} \theta_v$ the Haar measure on K_T and by $\| \ \|_T = \sup_{v\in T} | \ |_v$, the norm on K_T.

Let $r : T \to \mathbb{R}^+$ and $a = (a_v)$ be a point in K_T. By an interval with radius r and center a in K_T, we mean the set

$$I(r,a) = \{(x_v) \in K_T \mid | x_v - a_v |_v \leq r(v) \text{ for all } v \in T\}.$$

If $a = 0$, we write $I(r)$ instead of $I(r,0)$.

1.3 By a *K_T-algebraic variety* $\mathbf{M}$, we mean a (formal) direct product $\prod_{v\in T} \mathbf{M}_v$ of K_v-algebraic varieties $\mathbf{M}_v$. A map $f : \mathbf{M} \to \mathbf{M}'$, where $\mathbf{M}$ and $\mathbf{M}'$ are K_T-algebraic varieties, is called *K_S-rational* (resp. *K_S-regular*) if f is a product of K_v-rational (resp. K_v-regular) maps $f_v : \mathbf{M}_v \to \mathbf{M}'_v$, $v \in S$. If in the above definition of $\mathbf{M}$ all $\mathbf{M}_v$, $v \in S$, are K_v-algebraic groups we say that $\mathbf{M}$ is a *K_S-algebraic group*. Analogously, we define the notions of *linear K_S-space*, *K_S-rational representation* of a *K_S-algebraic group* etc. The set $\prod_{v\in S} \mathbf{M}_v(K_v)$ will be denoted by $\mathbf{M}(K_S)$ or simply M and called *set of K_S-rational points of* $\mathbf{M}$ (respectively, *group of K_S-rational points*, in the case of a K_S-algebraic group $\mathbf{M}$). Naturally if $\mathbf{V}$ is a K-algebraic variety, we associate to it a K_S-algebraic variety, also denoted by $\mathbf{V}$, such that $\mathbf{V}_v = \mathbf{V}$ for all $v \in S$.

1.4 By *Zariski topology* on a K_S-algebraic variety $\mathbf{M}$, we mean the product of the Zariski topologies on $\mathbf{M}_v$, $v \in S$. $\mathbf{M}$ is called connected if every $\mathbf{M}_v$, $v \in S$, is connected for the Zariski topology.

On $\mathbf{M}(K_S)$ we have two topologies : one induced by the Zariski topology on $\mathbf{M}$ and another which is a product of the locally compact Hausdorff topologies on $\mathbf{M}(K_v)$, $v \in S$. In order to distinguish the two topologies, all topological notions connected with the first one will be used with the prefix "Zariski". (We will say : Zariski closure, Zariski closed, etc.) Given a subset X in $\mathbf{M}(K_S)$, we will denote by $\overline{X}$ the Zariski closure of X in $\mathbf{M}(K_S)$. By a K_S-algebraic subvariety of $\mathbf{M}(K_S)$, we mean a Zariski closed subset of $\mathbf{M}(K_S)$.

1.5 Let $\mathbf{G}$ be an algebraic group defined over K. Every $\mathbf{G}(K_v)$ is naturally embedded in G. By a 1-parameter unipotent K_v-subgroup $U_v = \{u_v(t)\}$ of G, we mean a nontrivial K_v-rational homomorphism $u_v : K_v \to \mathbf{G}(K_v)$. Given a subgroup M of G, we will denote by M_u the subgroup of M generated by all 1-parameter unipotent K_v-subgroups for all $v \in S$.

Let $T \subset S$ and for each $v \in T$ let $U_v = \{u_v(t_v) \mid t_v \in K_v\}$ be a 1-parameter unipotent K_v-subgroup. Then the homomorphism u_T :

$K_T \to G$, $(t_v)_{v\in T} \to (u_v(t_v))_{v\in T}$, defines a 1-parameter unipotent K_T-subgroup $U_T = \prod_{v\in T} U_v$ of G.

Finally, given a 1-parameter unipotent K_v-subgroup $U_v = \{u_v(t)\}$ of G and a 1-dimensional over $\mathbb{Q}_{p_v}$ linear subspace l in K_v the restriction of u_v on l is called 1-parameter unipotent $\mathbb{Q}_{p_v}$-subgroup of G.

§2. Proof of Theorem 3 and of the strong approximation theorem.

Theorem 3 will be proved in 2.1-2.4. We preserve the notations from its announcement.

2.1 Denote by μ the M-invariant Borel probability measure on Mx. Note that μ is M_u-ergodic and $M'x = Mx$ for each open subgroup $M' \subset M$ containing M_u. Replacing, if necessary, M by its subgroup of finite index, we will assume that M satisfies the following conditions:

(*) *Every open subgroup of M is Zariski dense in $\overline{M}$ and $M = \prod_{v\in S} M_v$ where $M_v \subset \mathbf{G}(K_v)$.*

We need the following version of the Borel density theorem.

Lemma *M_x is Zariski dense in M.*

Proof. Since Mx is closed Mx is homeomorphic to M/M_x and, therefore, μ can be considered as a M-invariant M_u-ergodic Borel probability measure on M/M_x. Let μ_1 be the image of the measure μ on $M/M \cap \overline{M}_x$ under the natural map $M/M_x \to M/M \cap \overline{M}_x$. Clearly the measure μ_1 is M_u-invariant and M_u-ergodic. On the other hand, $M/M \cap \overline{M}_x$ can be regarded as a Borel subset of a K_S- algebraic variety on which M_u acts rationally. This in view of [MTo1, 3.1] implies that μ_1 is concentrated in a point. The lemma is proved.

2.2 We will denote by $R(\overline{M})$ the solvable radical of $\overline{M}$ (i.e. $R(\overline{M})$ is the maximal connected in the Zariski topology solvable normal subgroup of $\overline{M}$). Clearly $\overline{M} = \prod_{v\in S} \overline{M}_v$ and $R(\overline{M}) = \prod_{v\in S} R(\overline{M}_v)$ where $R(\overline{M}_v)$ is the solvable radical of $\overline{M}_v$.

Lemma *Assume that $R(\overline{M}) = R(\overline{M})_u$. Then M is open in the Hausdorff topology of $\overline{M}$.*

Proof. It is enough to prove the lemma when $S = \{v\}$. Let $\mathcal{G}$ be the Lie algebra of M_v. It is well known that the commutator of $\mathcal{G}$ coincides with the Lie algebra of the commutator $\mathcal{D}^1(\overline{M}_v)$ of $\overline{M}_v$. (cf. [C, ch.2, theorem 13]). Therefore the commutator $\mathcal{D}^1(M_v)$ of M_v is open in $\mathcal{D}^1(\overline{M}_v)$. This allows to reduce the proof of the lemma to the case when M_v is abelian. In this case $\overline{M}_v$ is isomorphic to a vector space K_v^n. Now the lemma follows from the assumption (*) and from the fact that if v is archimedean (respectively, nonarchimedean) then the connected component of M_v is Zariski closed (respectively, $\{1\}$ is the only discrete subgroup of M_v).

2.3 Lemma *Let $R(\overline{M}) = R(\overline{M})_u$ and also let $\overline{M}x$ be closed and admit $\overline{M}$-invariant Borel probability measure $\overline{\mu}$. Then $M' = \{a \in \overline{M} \mid aMx = Mx\}$ is a subgroup of finite index in $\overline{M}$.*

Proof. Since M_u is a normal subgroup of $\overline{M}$ and $M_u x$ is dense in Mx we get that for any $a \in \overline{M}$ either $aMx = Mx$ or $aMx \cap Mx = \emptyset$. In view of 2.2 $\overline{\mu}(aMx) = \overline{\mu}(Mx) > 0$ for all $a \in \overline{M}$. Therefore there exists a finite subset $\{a_1, a_2, ..., a_r\} \subset M$ such that $\overline{M}x = a_1Mx \cup a_2Mx \cup ... \cup a_rMx$ and the multiplication from the left by an element from $\overline{M}$ permutes the subsets $a_iMx, i = 1, 2, ..., r$. This implies the lemma.

2.4 Proof of Theorem 3. Replacing μ by $g^{-1}\mu$ and M by $g^{-1}Mg$ we may (and will) assume that $g = e$. Let $\mathbf{P}_1$ be the largest normal subgroup of class $\mathcal{F}$ in $\mathbf{P}$ and $P_1 = \mathbf{P}_1(K_S)$. In view of Lemma 2.1 M is contained in P and, therefore, $M_u \subset P_1$. Since $P_1 \cap \Gamma$ is a lattice in P_1 the orbit P_1x is closed and contains Mx. In particular, P_1 contains an open subgroup of M. In view of the assumption (*), Lemma 2.1 and the definition of $\mathbf{P}$ we get that $\mathbf{P} = \mathbf{P}_1$ and $P = \overline{M}$. It follows from 2.2 and 2.3 that M is open in P and $P' = \{a \in P \mid a\mu = \mu\}$ has finite index in P. The uniqueness of $\mathbf{P}$ follows from the fact that the Lie algebras of $\mathbf{P}(K_v)$ and M_v, $v \in S$, coincide. The theorem is proved.

2.5 Proof of Theorem 4. Let us first consider the case when $\mathbf{G}$ is semisimple. In view of Weil's restriction of scalars functor [W2, ch.1] and the result of Borel and Tits [BTi2, 6.21(ii)] we may (and will) assume that $\mathbf{G}$ is absolutely almost simple. Also, it is easy to see that it is enough to prove the theorem for $T = \{v_o\}$. Assuming all this, denote by S_o the (finite) set of all nonarchimedean valuations v of K such that $\mathbf{G}$ is K_v-anisotropic. Let S_1 be any finite set of valuations such that $S_1 \cap (S_o \cup \{v_o\}) = \emptyset$ and $S_1 \cup \{v_o\}$ contains all archimedean valuations of K. For each $v \in S_o$, fix an open subgroup R_v in $\mathbf{G}(K_v)$. Define an open subgroup A in $\mathbf{G}(\mathcal{V})$ as follows:

$$A = \prod_{v \in S_1 \cup \{v_o\}} \mathbf{G}(K_v) \times \prod_{v \in S_o} R_v \times \prod_{v \notin S_o \cup S_1 \cup \{v_o\}} \mathbf{G}(\mathcal{O}_v).$$

The group $\Gamma = A \cap \mathbf{G}(K)$ is an S-arithmetic subgroup, where $S = S_o \cup S_1 \cup \{v_o\}$. Since $\mathbf{G}$ is simply connected, $\mathbf{G}(K_{v_o}) = \mathbf{G}(K_{v_o})_u$ (cf.[Pl]).

Let G' be the closure of $\mathbf{G}(K_{v_o})\Gamma$ in $\mathbf{G}(K_S)$. In view of Theorem 3, G' is a subgroup of finite index in $\mathbf{G}(K_S)$. Note that $\mathbf{G}(K_v)$ does not contain a subgroup of finite index if v is archimedean (Cartan) and, also, if v is nonarchimedean and $\mathbf{G}$ is K_v-isotropic (cf. [Ti1],[Pl]). Therefore $G' \supset \mathbf{G}(K_{S_1 \cup \{v_o\}})$. Since the subgroups R_v, $v \in S_o$, can be chosen arbitrary small and the finite set S_1 arbitrary large, we obtain that the closure of $\mathbf{G}(K_{v_o})\mathbf{G}(K)$ in $\mathbf{G}(\mathcal{V})$ contains all adeles $x = (x_v)$ with $x_v = 1$ for each $v \in S_o$. If $S_o \neq \emptyset$ then $\mathbf{G}$ is a group of type A_n and, for every $v \in S_o$, $\mathbf{G}(K_v)$ is the group of elements with reduced norm 1 in a central division algebra over K_v. By a result of Kneser [K2] the diagonal embedding of $\mathbf{G}(K)$ in $\prod_{v \in S_o} \mathbf{G}(K_v)$ is dense. Therefore $\mathbf{G}(K_{v_o})\mathbf{G}(K)$ is dense in $\mathbf{G}(\mathcal{V})$.

For a non-semisimple K-algebraic group $\mathbf{G}$, we apply the following standard argument. The group $\mathbf{G}$ is a semidirect product over K of its unipotent radical $\mathbf{U}$ and its semisimple K-algebraic subgroup $\mathbf{L}$. Therefore $\mathbf{G}(\mathcal{V})\mathbf{L}(\mathcal{V})\mathbf{U}(\mathcal{V})$ and $\mathbf{G}(K) = \mathbf{L}(K)\mathbf{U}(K)$. Now, the theorem follows from the validity of the strong approximation for both $\mathbf{L}$ and $\mathbf{U}$. This completes the proof.

§3. Ratner's uniform distribution theorem

3.1 Similarly to [Rat2,3], we will deduce Theorem 1 from its stronger version for 1-parameter unipotent K_T-subgroups, the so-called uniform distribution theorem.

Theorem. *Let G, Γ and $x = g\Gamma$ be as in Theorem 1. Furthermore, let T be a nonempty subset of S and $U = \{u(t) \mid t \in K_T\}$ be a 1-parameter unipotent K_T-subgroup of G and let $\{I(r_i)\}$ be an increasing sequence of intervals in K_T such that $\cup_i I(r_i) = K_T$. Then there exists a subgroup $\mathbf{P} \subset \mathbf{G}$ of class $\mathcal{F}$ and a subgroup of finite index P' in $P = \mathbf{P}(K_S)$ such that the closure of the orbit Ux coincides with $gP'g^{-1}x$ and*

$$\lim_{i\to\infty} \frac{1}{\theta_T(I(r_i))} \int_{I(r_i)} f(u(t)x)d\theta_T(t) = \int_{gP'g^{-1}x} f(y)d\mu(y)$$

for any bounded continuous function f on G/Γ, where μ is the Haar measure on $gP'g^{-1}x$.

For 1-parameter real and p-adic subgroups $U = \{u(t)\}$ the uniform distribution theorem was proved in the general context of direct products of real and p-adic Lie groups in [Rat3, Theorem 3]. (In fact in [Rat3] and, earlier, in [DM2] for real Lie groups, a stronger version of this theorem is proved : the point x is replaced by a sequence of points converging to a generic point x (cf [Rat3, Theorem 4] and [DM2, Theorem 2]. In the present paper we do not treat this more technical case.) Our proof of Theorem 3.1 uses methods, with some modifications, of the proof in [DM2].

3.2 Deduction of Theorem 1 from Theorem 3.1. It is enough to prove Theorem 1 for $x = \Gamma$. Let $H = H_u$ as in the formulation of Theorem 1. Denote by M the subset of all unipotent elements in H. It is clear that M is a K_S-algebraic subvariety of G. Since each element in M is contained in a maximal unipotent subgroup of H and any two maximal unipotent subgroups of H are conjugated [B3,15.9], M is Zariski connected. On the other hand, every element $a \in M$ is contained in a 1-parameter unipotent K_T-subgroup $U(a)$ of H. In view of Theorem 3.1, there exists a K-algebraic subgroup $\mathbf{P}(a) \subset \mathbf{G}$ of class $\mathcal{F}$ and a subgroup of finite index $P(a)'$ in the group of K_S-rational points of $\mathbf{P}(a)$ such that the closure of $U(a)x$ coincides with $P(a)'x$. Let A be an open subset in M homeomorphic to a neighbourhood of 0 in some linear K_S-space. Then since $\{P(a)' \mid a \in M\}$ is a countable set, there exists $a_o \in M$ such that $P(a_o)' \cap A$ has positive Lebesque measure. As M is Zariski connected, we get that $P(a_o)' \cap A$ is Zariski dense in M. Therefore, $H \subset P(a_o)'$, which implies Theorem 1.

3.3 The following result is important for the proof of Theorem 3.1.

Theorem. *With G, Γ and U as in Theorem 3.1, let $\varepsilon > 0$ and $\mathcal{K} \subset G/\Gamma$ be a compact. Then there exists a compact $\mathcal{K}_1$ in G/Γ such that for any $x \in \mathcal{K}_l$ and any interval $I(r)$ in K_T,*

$$\frac{1}{\theta_T(I(r))}\theta_T\{t \in I(r) \mid u(t)x \in \mathcal{K}_1\} \geq 1 - \varepsilon.$$

In the case when $T = \{v\}$, the above theorem was announced in [MT1,11.4] with indications about the proof. (The details will appear elsewhere.) The general case follows from this one by a simple application of the Fubini theorem. In the real case the theorem is proved in [DM2,6.1] using earlier results [M1], [D1,3,4] and the arithmeticity theorem [M4,ch.9].

3.4 Singular and generic points. Given a subgroup U of G and a proper subgroup $\mathbf{P} \subset \mathbf{G}$ of class $\mathcal{F}$, we put $X(P, U) = \{g \in G \mid Ug \subset gP\}$. It is clear that $X(P, U)$ is a K_S-algebraic subvariety of G. We denote $\mathcal{S}(U) = \cup_{\mathbf{P}\in\mathcal{F},\mathbf{P}\neq\mathbf{G}} X(P, U)\Gamma/\Gamma$ and $\mathcal{G}(U) = G/\Gamma - \mathcal{S}(U)$. As in [DM2], the points from $\mathcal{S}(U)$ (resp. $\mathcal{G}(U)$) are called *singular* (resp. *generic*) points with respect to U.

Theorem 3.1 will be derived from the following

Proposition. *Let $G, \Gamma, U, \varepsilon$ and $\mathcal{K}$ be as in Theorem 3.3. Also let* $\mathbf{P}$ *be a proper subgroup of* $\mathbf{G}$ *of class* $\mathcal{F}$ *and* $C = \prod_{v \in S} C_v$ *be a compact subset of* $X(P, U)$*, where* $C_v \subset \mathbf{G}(K_v)$ *for each* $v \in S$*. Then there exists a compact* $D = \prod_{v \in S} D_v$ *in* $X(P, U)$ *such that* $D_v \supset C_v$ *for all* $v \in S$*,* $D_v = C_v$ *for* $v \notin T$ *and the following holds: For any neighbourhood* Φ_o *of* D *in* G *there exists a neighbourhood* Φ *of* C *in* G*, such that for any* $x \in \mathcal{K} - \Phi_o\Gamma/\Gamma$ *and any interval* $I(r)$ *in* K_T*,*

$$\frac{1}{\theta_T(I(r))}\theta_T\{t \in I(r) \mid u(t)x \in \Phi\Gamma/\Gamma\} < \varepsilon. \tag{2}$$

3.5 Deduction of the theorem for uniform distribution from Proposition 3.4. Let $\mathbf{P}$ be the smallest subgroup of class $\mathcal{F}$ in $\mathbf{G}$ such that $Ug \subset gP$ (if x is singular then $\mathbf{P} \neq \mathbf{G}$). Put $U_1 = g^{-1}Ug$ and $\Delta = \Gamma \cap P$. Then Δ is an S- arithmetic lattice in P, $y = \Delta$ is a generic point in P/Δ with respect to $U_1 \subset P$, and $P\Gamma/\Gamma$ is closed in G/Γ and homeomorphic to P/Δ. This reduces the proof of the theorem to the case when x is generic, which we will assume from now on.

For any interval $I(r)$ define a probability measure μ_r on G/Γ by the formula

$$\int_{G/\Gamma} f(y)d\mu_r(y) = \frac{1}{\theta_T(I(r))}\int_{I(r)} f(u(t)x)d\theta_T(t),$$

where f is a bounded continuous function on G/Γ. We denote by $\widetilde{G/\Gamma}$ the one-point compactification of G/Γ if G/Γ is not compact, and G/Γ itself if G/Γ is compact. It is well-known that, given a compact metrizable topological space Y, the space $\mathcal{P}(Y)$ of all Borel probability measures on Y is compact with respect to the weak * topology. Let $\{I(r_i)\}$ be a sequence of intervals as in the formulation of Theorem 3.1. Put $\mu_i = \mu_{r_i}$ for every i. The sequence of measures $\{\mu_i\}$ is naturally embedded in $\mathcal{P}(\widetilde{G/\Gamma})$. Let λ be a limit point of $\{\mu_i\}$ in $\mathcal{P}(\widetilde{G/\Gamma})$. It follows from 3.3 that λ is concentrated on G/Γ. Using the fact that f is bounded and performing the linear substitution $s = t_o + t$ in the integrals

$$\int_{I(r_i)} f(u(t_o + t)x)d\theta_T(t),$$

a simple argument shows that

$$\lambda(u(t_o)f) = (u(-t_o)\lambda)(f) = \lambda(f),$$

for any bounded continuous function f on G/Γ. This means that λ is U-invariant. On the other hand, since $X(P, U)$ is second countable topological space and $\mathcal{F}$ is countable it follows from Proposition 3.4 applied for $\mathcal{K} = \{x\}$ that $\lambda(\mathcal{S}(G/\Gamma)) = 0$. In view of Theorem 2, every U-invariant U-ergodic measure on G/Γ which is not supported by an orbit of an open subgroup of G is supported by $\mathcal{S}(G/\Gamma)$. This, in view of the decomposition of λ into a continuous sum of its U-ergodic components implies that λ is U-ergodic and coincides with the Haar measure on $G'x$ where G' is an open subgroup of G. So, assuming the validity of Proposition 3.4, the proof of Theorem 3.1 is complete.

§4. Polynomial-like behaviour of the unipotent orbits and proof of Proposition 3.4

4.0 Let us make the following simple observation : Given a field F and a 1-parameter unipotent subgroup $U = \{u(t)\}$ in $GL_n(F)$, the map $t \to u(t)x$ is polynomial of degree less that or equal to n for each $x \in F^n$. This fact is in the root of the phenomenon that the dynamics of the actions of subgroups on homogeneous spaces are much easier to be understood when the subgroups are generated by unipotent elements than, say, when they are generated by split semisimple elements. (In more geometrical terms, the first type of actions corresponds to a horosperical flow and the second one to a geodesic flow on a Riemannian manifold with constant negative curvature.) The above observation will be used in the proof of the key Proposition 4.2. For the proof of 4.2, we need a property of polynomial maps given by the following lemma. (Note that Lemma 4.1 plays also a crucial role in the proof of Theorem 3.3.)

4.1 Lemma. *Let v be either a real or a nonarchimedean valuation of K and ϵ and α be positive reals. Also, let $f = (f_1, ..., f_s)$ be a polynomial map $K_v \to K_v^s$ of degree not greater than $n \in \mathbb{N}$ (i.e. $deg f_i \leq n$ for all i). Put $\delta = \frac{\varepsilon^n}{(n+1)^{n+1}}$ and denote by $\| \ \|_v$ the norm sup on K_v^s. Then for any interval $I \subset K_v$ which contains a number t_o with $\|f(t_o)\|_v \geq \alpha$,*

$$\theta_v\{t \in I \mid \|f(t_o\|_v \leq \delta\alpha\} \leq \varepsilon\theta_v\{t \in I \mid \|f(t_o\| \leq \alpha\}.$$

Proof. We will omit the case when $K_v = \mathbb{R}$ which is considered in detail in [DM2,4.1]. Without loss of generality we may assume that α is a value of the norm $| \ |_v$. It is easy to see that, given two intervals in K_v with nonempty intersection, one of them contains the other. So, for any $x \in I$ there exists a unique interval $J(x)$ in I which contains x and is maximal with the property : $\|f(y)\|_v \leq \alpha$ for all $y \in J(x)$. Therefore, it is enough to consider the case $I = J(x)$. Clearly, I contains an element y_o with $\|f(y_o)\|_v = \alpha$. Since there exists an i such that $| f_i(y_o) |_v = \alpha$, the proof is reduced to the case when $s = 1$. Furthermore, replacing f by af, where $a \in K_v$ and $| a |_v = \alpha^{-1}$, and doing a linear substitution $t \to t + t_o$, we reduce the proof of the lemma to the following case : $I = \{t \in K_v \mid | t |_v \le 1\}$, $t_o = 0$, $| f(0) |_v = \alpha = 1, \varepsilon \le 1$, and $| f(t) |_v \le 1$ for all $t \in I$. Assume that the statement of the lemma is false, that is,

$$\theta_v(A) > \varepsilon,$$

where $A = \{t \in I \mid | f(t) |_v \le \delta\}$. Denote $I_1 = \{t \in K_v \mid | t |_v \le \frac{\varepsilon}{n+1}\}$. Since the norm $| \ |_v$ and the measure θ_v on K_v are normalized, we obtain

$$\theta_v(I_1) \le \frac{\varepsilon}{n+1},$$

cf. [CaF, ch.1]. Therefore, there exist points $\alpha_1, ..., \alpha_{n+1}$ in A such that $\{\alpha_i + I_1\} \cap \{\alpha_j + I_1\} = \emptyset$ for all $i \neq j$. (Note that I_1 is an ideal in the ring I.) In particular,

(3) $$| \alpha_i - \alpha_j |_v > \frac{\varepsilon}{n+1},$$

for all $i \neq j$. Let us write the Lagrange interpolation formula for f at the points $\alpha_1, \alpha_2, ..., \alpha_{n+1}$:

$$f(t) = \sum_{i=1}^{n+1} f(\alpha_i) \frac{(x-\alpha_1)...(x-\alpha_{i-1})(x-\alpha_{i+1})...(x-\alpha_{n+1})}{(\alpha_i-\alpha_1)...(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})...(\alpha_i-\alpha_{n+1})}.$$

The substitution $t = 0$, the inequality (5), and the ultrametric inequality in K_v imply that

$$| f(0) |_v < \frac{\delta(n+1)^n}{\varepsilon^n} < 1.$$

Contradiction. The lemma is proved.

4.2 Proposition. *Let M be a Zariski closed subset in K_v^m. Then for any compact subset A of M and any $\varepsilon > 0$ there exists a compact B in M containing A such that the following holds: given a compact neighbourhood W_o of B in K_v^m, there exists a neighbourhood W of A in K_v^m such that for any 1-parameter unipotent subgroup $\{u(t)\}$ in $GL_m(K_v)$, any $a \in K_v^m - W_o$ and any interval I in K_v containing* 0, *we have*

(4) $$\theta_v\{t \in I \mid u(t)a \in W\} \le \varepsilon\{t \in I \mid u(t)a \in W_o\}.$$

Proof. Note that the case $K_v = \mathbb{C}$ can be easily reduced to the real one by embedding M in $\mathbb{R}^{2m}$ via Weil's restriction of scalars and by using Lemma 4.4 below. The real case itself is considered in [DM2, 4.2]. So, we will assume that v is nonarchimedean. First note that if the proposition is true for a compact subset of M containing A, then it is also true for A. Therefore, it is enough to consider the case $A = \{x \in M \mid \|x\| \leq R_o\}$ where $R_o > 0$ is a constant and $\| \ \|$ is the norm sup on K_v^m. Let $f_1, f_2, ..., f_r \in K_v[x_1, .., x_m]$ be such that $M = \{x \in K_v^m \mid f_i(x) = 0 \text{ for all } i = 1, 2, ..., r\}$. Let $n \in \mathbb{N}$ be such that the degree of each polynomial f_i is $\leq n$. Put $\delta = \frac{\varepsilon^{mn}}{(mn+1)^{mn+1}}$. Let R be a real number such that $R \geq R_o\delta^{-1}$ and R be a value of the norm $\| \ \|$. Denote,

$$B = \{x \in M \mid \|x\| \leq R\}.$$

Let W_o be a neighbourhood of B, $a \notin W_o$ and $u(t)$ be a 1-parameter unipotent subgroup of $GL_m(K_v)$. Denote $g(t) = u(t)a$, $t \in K_v$. Then $g(t) = (g_1(t), ..., g_m(t))$ where $g_i(t)$ are polynomials of degree $\leq m$. Also denote $F(t) = (F_1(t), ..., F_r(t))$ where $F_i(t) = f_i(g_1(t), ..., g_m(t))$ for all i. It is easy to see (for example, by an argument from the contrary and using the compactness of B) that there exists an $\alpha > 0$ which is a value of the norm $| \ |_v$ and such that

$$W_1 = \{x \in K_v^m \mid \|x\| \leq R \text{ and } | f_i(x) |_v \leq \alpha \text{ for all } i\}$$

is a neighbourhood of B contained in W_o. We will prove that

$$W = \{x \in K_v^m \mid \|x\| \leq R_o \text{ and } | f_i(x) |_v \leq \alpha\delta \text{ for all } i\}$$

is the required neighbourhood of A. Let $I \subset K_v$ be an interval containing 0. Put $J = \{t \in I \mid u(t)a \in W_1\}$. For each $t \in J$, we denote by $J(t)$ the maximal (closed) subinterval of I such that $u(J(t))a \subset W_1$. Since $a \notin W_1$ and $0 \in I$, for every $t \in J$ there exists $t' \in J(t)$ such that either $\|u(t')a\| = R$ or $\|F(t')\| = \alpha$. Using 4.1 we get

$$\varepsilon\theta_v(J(t)) \geq \theta_v\{y \in J(t) \mid \|u(y)a\| \leq R_o \text{ and } \|F(y)\| \leq \alpha\delta\}.$$

Since the intervals $J(t)$ form a partition of J, the above formula implies (4).

4.3 Lemma. *Assume that Proposition 3.4 is valid if T is a singleton. Then it is valid for any T.*

Proof. The proof is by induction on the cardinality of T. Let $T = T_1 \cup T_2$, where both T_1 and T_2 are nonempty and $U = U_1 \times U_2$, where $U_i = \{u_i(t) \mid t \in K_{T_i}\}$, $i = 1, 2$, are 1-parameter K_{T_i}-subgroups of G. Assume that Proposition 3.4 is true for both U_1 and U_2. Let $\mathcal{K}$, C and ε be as in 3.4. We have to prove that a compact D, as in the formulation of Proposition 3.4, exists. Because of 3.3, there exists a compact subset $\mathcal{K}_1$ in G/Γ such that for any $x \in \mathcal{K}$ and any interval I_1' in K_{T_1} containing 0,

$$\text{(5)} \qquad \frac{1}{\theta_1(I_1')}\theta_1\{t \in I_1' \mid u_1(t)x \in \mathcal{K}_1\} \geq 1 - \frac{\varepsilon}{3},$$

where $\theta_1 = \theta_{T_1}$ (see 1.2). Note that $X(P,U) = X(P,U_1) \cap X(P,U_2)$. Applying Proposition 3.4 for U_2, we get a compact $D' = \prod_{v \in S} D_v'$ in $X(P,U)$ with $D_v' = C_v$ for all $v \notin T_2$ and such that if Ψ is a neighbourhood of D' then there exists a neighbourhood Φ of C such that for any $y \in \mathcal{K}_1 - \Psi\Gamma/\Gamma$ and any interval I_2 in K_{T_2} containing 0, we have

$$\text{(6)} \qquad \frac{1}{\theta_2(I_2)}\theta_2\{t \in I_2 \mid u_2(t)y \notin \Phi\Gamma/\Gamma\} \geq 1 - \frac{\varepsilon}{3},$$

where $\theta_2 = \theta_{T_2}$. Applying again Proposition 3.4 (this time for U_1), we get a compact $D = \prod_{v \in S} D_v$ in $X(P,U)$ with $D_v = D_v'$ for all $v \notin T_1$ and such that if Φ_o is a neighbourhood of D then there exists a neighbourhood Ψ of D' with

$$\text{(7)} \qquad \frac{1}{\theta_1(I_1)}\theta_1\{t \in I_1 \mid u_1(t)x \notin \Psi\Gamma/\Gamma\} \geq 1 - \frac{\varepsilon}{3}$$

for any interval I_1 in K_{T_1} containing 0 and any $x \in \mathcal{K} - \Phi_o\Gamma/\Gamma$.

Let $I = I_1 \times I_2$ be an interval in K_T containing 0, Φ_o a neighbourhood of D, Ψ a neighbourhood of D' as given by (7) and Φ a neighbourhood of C as given by (6). Now it follows from (5)-(7) and the Fubini theorem that

$$\frac{1}{\theta_T(I)}\theta_T\{t \in I \mid u(t)x \notin \Phi\Gamma/\Gamma\} \geq$$

$$\frac{1}{\theta_T(I)}\theta_T\{t = (t_1,t_2) \in I \mid u_1(t_1)x \in \mathcal{K}_1 - \Psi\Gamma/\Gamma \text{ and } u_1(t_1)u_2(t_2)x \notin \Phi\Gamma/\Gamma\} \geq$$

$$\left(1 - \frac{\varepsilon}{3}\right)^3 \geq 1 - \varepsilon$$

for any $x \in \mathcal{K} - \Phi_o\Gamma/\Gamma$. The lemma is proved.

4.4 Let $T = \{v\}$, $K_v = \mathbb{C}$ and $U = \{u(t)\}$ a 1-parameter unipotent $\mathbb{C}$-subgroup of G. In this case the proof of Proposition 3.4 can be reduced to the case of actions of 1-parameter unipotent real subgroup of G as follows. For every 1-dimensional real subspace $l \subset \mathbb{C}$ we denote by u_l the restriction of u on l. Put $U_l = \{u_l(t)\}$. In order to prove that (2) holds it is enough to show that for all $l \subset \mathbb{C}$ and all intervals $I(r) = [-r, r]$ in $\mathbb{R}$ we have,

$$\frac{1}{\theta_o(I(r))}\theta_o\{t \in I(r) \mid u_l(t)x \in \Phi\Gamma/\Gamma\} < \varepsilon, \tag{8}$$

where θ_o is the Lebesques measure on $\mathbb{R}$. The fact that the fulfilment of (8) for all l implies (2) follows from the elementary lemma below.

Lemma. *Let $I = \{t \in \mathbb{C} \mid \mid t \mid \leq 1\}$, $\varepsilon > 0$ and A be a measurable subset of I such that for any $x \in I$ we have,*

$$\varepsilon\theta_o\{a \in \mathbb{R} \mid ax \in I\} \geq \theta_o\{a \in \mathbb{R} \mid ax \in I \cap A\}.$$

Then $\theta_v(A) < \varepsilon\pi$.

4.5 Up to the end of section 4, we preserve the notations from 3.4 and suppose that $T = \{v\}$. Denote by $\mathcal{U} = \{u(t)\}$ a 1-parameter unipotent $\mathbb{R}$-subgroup if v is archimedean and put $\mathcal{U} = U$ if v is nonarchimedean. We let $F = \mathbb{R}$ in the former and $F = K_v$ in the latter case. Note that $X(P, U) = X(P, \mathcal{U})$.

Let us fix a K-rational representation $\varrho : \mathbf{G} \to \mathbf{GL}(\mathbf{V})$ such that the normalizer $\mathcal{N}_{\mathbf{G}}(\mathbf{P})$ of $\mathbf{P}$ in $\mathbf{G}$ coincides with the stabilizer in $\mathbf{G}$ of a 1-dimensional subspace of $\mathbf{V}$ spanned by a vector $m \in \mathbf{V}(K)$. (The existence of such a representation follows from the Chevalley theorem [B3,5.1].) Let χ be the K-rational character of $\mathcal{N}_{\mathbf{G}}(\mathbf{P})$ given by $gm = \chi(g)m$, $g \in \mathcal{N}_{\mathbf{G}}(\mathbf{P})$. We denote $\mathbf{N} = \{g \in \mathbf{G} \mid gm = m\}$, $N = \mathbf{N}(K_S), \Gamma_N = \Gamma \cap N$ and $\Gamma_P = \Gamma \cap \mathcal{N}_{\mathbf{G}}(\mathbf{P})$. Let $\eta : \mathbf{G} \to \mathbf{G}m$, $g \to gm$. (We will denote in the same way the map $G \to V$, $g \to gv$, where $V = \mathbf{V}(K_S)$.) Note that $\mathbf{G}m$ is open in its Zariski closure, $\mathbf{G}m$ is isomorphic to $\mathbf{G}/\mathbf{N}$ and η is a quotient map [B3, 6.7]. Let $\mathbf{X} = \{g \in \mathbf{G} \mid Ug \subset g\mathbf{P}\}$. Clealy, $\mathbf{X}$ is a K_S-algebraic variety and $\mathbf{X}(K_S) = X(P, U)$. We will denote $X_v = \mathbf{X}_v(K_v)$. Since $\mathbf{X}\mathcal{N}_{\mathbf{G}}(\mathbf{P}) = \mathbf{X}$, $\mathbf{N} \subset \mathcal{N}_{\mathbf{G}}(\mathbf{P})$ and η is a Zariski open map, we get that $\eta(\mathbf{X})$ is Zariski closed in $\mathbf{G}m$. This implies that

$$\eta^{-1}(\overline{\eta(X(P, U))} = X(P, U). \tag{9}$$

(In the above formula the Zariski closure is taken in V.)

4.6 In the following remarks we use some standard facts from algebraic number theory (cf.[CaF],[W1]).

(a) Since χ is a K-character, $\chi(\Gamma) \cap \mathcal{O}^*$ has finite index in $\chi(\Gamma)$. In view of the facts that $\mathcal{O}$ is integrally closed and $\chi(\Gamma) \subset K$, we get that $\chi(\Gamma) \subset \mathcal{O}^*$. Also since $\mathbf{V}(\mathcal{O})$ is discrete in V and η is K-rational, Γm is discrete in V.

(b) Denote by K_S^1 the set of all $x = \{x_w\}_{w \in S} \in K_S$ such that

$$\prod_{w \in S} \mid x_w \mid_w = 1.$$

The group $\mathcal{O}^*$ is diagonally embedded in K_S^1. For each $w \in S$ let $\lambda_w : K_w^* \to \mathbb{R}$ be the map $x \to \log |x|_w$. Put $R_w = \mathrm{Im}(\lambda_w)$. So, R_w coincides with $\mathbb{R}$ if v is archimedean and R_w is a cyclic subgroup of $\mathbb{R}$ if v is nonarchimedean. Let $\lambda : K_S^1 \to \prod_{w \in S} R_w$ be the direct sum of all λ_w and let $R = \mathrm{Im}(\lambda)$. Then R is a locally compact abelian group,$\lambda(\mathcal{O}^*)$ is a lattice in R and $\mathrm{Ker}(\lambda) \cap \mathcal{O}^*$ is the group of roots of unity in K [CaF,ch.2, 18.1]. *Therefore there exists $\delta > 0$ such that if $\xi \in \mathcal{O}^*$ and $|1 - \xi|_w < \delta$ for all $w \neq v$ then ξ is a root of unity in K.*

(c) Let $A = \prod_{w \in S} A_w$ be a subset of G. We will say that A is $S(v)$-*small* if for every $w \neq v$ in S the following holds : if $c \in K_w^*$ is such that

$$c(A_w m) \cap A_w m \neq \emptyset$$

then $\mid c - 1 \mid_w < \delta$. In particular, if $c \in \mathcal{O}^*$ then, in view of (b), c is a root of unity.

(d) Clearly, every element $g \in G$ is contained in a $S(v)$-suffitiently small neighbourhood.

The consideration of $S(v)$-sufficiently small subsets for $\mathbb{Q}$-algebraic varieties was suggested by G.A.Margulis.

4.7 Proposition. *Let $\phi : G/\Gamma_N \to G/\Gamma \times V$, $\phi(g\Gamma_N) = (g\Gamma, gm)$. Then ϕ is a proper map.*

Proof. Let $\{g_i\Gamma_N\}$ be a sequence in G/Γ_N such that $\phi(g_i\Gamma_N)$ converges to $(c\Gamma, q) \in G/\Gamma \times V$. Fix $c_i \in G$ and $\gamma_i \in \Gamma$ such that $g_i = c_i\gamma_i$ for all i and $\lim_i c_i = c$. As $\{\gamma_i m\} \subset V$ is discrete (cf. 4.6(a)), there exists i_o such that $\gamma_i m = \gamma_{i_o} m$ for all $i \geq i_o$. So, $\gamma_i\Gamma_N = \gamma_{i_o}\Gamma_N$ for all $i \geq i_o$. Therefore $\{g_i\Gamma_N\}$ is bounded in G/Γ_N which proves that ϕ is a proper map.

4.8 The above proposition implies the following

Corollary. *Let D_o and L be compact subsets in G. Then there exists a compact D in G such that $D_o \subset D \subset D_oN$ and*

$$DN \cap L\Gamma \subset D\Gamma_N. \tag{10}$$

Furthermore, if Ω is a neighbourhood of D then Ω contains a neighbourhood Ψ of D_o such that

$$\Psi N \cap L\Gamma \subset \Psi\Gamma_N.$$

According to 4.7, $\phi^{-1}(L\Gamma/\Gamma, D_o m)$ is a compact subset of D_oN/Γ_N. Now, the existence of D satisfying (10) follows by a simple continuity argument. The second part can be proved in a similar way.

4.9 Let A be a subset of G. Following [DM2], a point $x \in A$ will be called a *point of* (P,Γ)*-self-intersection* in A if there exists $\gamma \in \Gamma - \Gamma_P$ such that $x\gamma \in A$. The next proposition corresponds to Corollary 3.5 in [DM2].

Proposition. *Let D_o and L be compact subsets of G and Y be the (closed) subset of all points of (P,Γ)-self-intersections in D_o. Assume that $D_oN \cap L\Gamma \subset D_o\Gamma_N$. Then for every relatively compact neighbourhood Ψ of Y there exists an open neighbourhood Ω of D_o such that*

$$(\Omega - \Psi\Gamma_P) \cap L\Gamma$$

does not contain points of (P,Γ)-self-intersections.

Proof. Assume the contrary, that is, there exists a sequence of neighbourhoods $\{\Omega_i\}$ of D_o such that $\Omega_i \supset \Omega_{i+1}$, $\bigcap_i \Omega_i = D_o$ and there exist $g_i, g_i' \in (\Omega_i - \Psi\Gamma_P) \cap L\Gamma$ with $g_i = c_i\gamma_i$, $g_i' = c_i\gamma_i'$, where $c_i \in L$, γ_i and $\gamma_i' \in \Gamma$, and $\gamma_i^{-1}\gamma_i' \notin \Gamma_P$. Passing to subsequences, we may (and will) assume that each of the sequences $\{c_i\}$, $\{g_i m\}$ and $\{g_i' m\}$ converges. Since Γm is discrete, there exists i_o such that $\gamma_i m = \gamma_{i_o} m$ and $\gamma_i' m = \gamma_{i_o}' m$ for all $i \geq i_o$. Put $c = \lim_i c_i$. Then $c\gamma_{i_o} m$ and $c\gamma_{i_o}' m \in D_o m$. Therefore $c\gamma_{i_o}, c\gamma_{i_o}' \in D_oN \cap L\Gamma \subset D_o\Gamma_N$. As $\gamma_{i_o}^{-1}\gamma_{i_o}' \notin \Gamma_P$, we get that $c\gamma_{i_o} \in Y\Gamma_P$. The latter contradicts the fact that $c\gamma_{i_o} \notin \Psi\Gamma_P$. The proposition is proved.

4.10 Proof of Proposition 3.4 Let $\mathcal{K} \subset G/\Gamma$, $\varepsilon > 0$ and C be as in the formulation of Proposition 3.4. According to 4.3 and 4.4, it is enough to prove (2) for $\mathcal{U}$ as defined in 4.5. Also, in view 4.6(d) we can (as we will) suppose that C is $S(v)$ - small subset of $X(P,U)$.

The proposition will be proved by induction on dim$\mathbf{P}$. (The proof is trivial for dim$\mathbf{P} = 0$.)

Using 4.2, we can find a compact $B = \prod_{w\in S} B_w$ in $\overline{X(P,U)m} \subset V$ such that $B_w = C_w m$ for all $w \neq v$, $B_v \supset C_v m$, and for any neighbourhood $\mathcal{A}_o$ of B_v in $\mathbf{V}(K_v)$ there exists a neighbourhood $\mathcal{A}$ of $C_v m$ in $\mathbf{V}(K_v)$ such that

$$(11) \qquad \theta\{t \in I(r) \mid u(t)a \in \mathcal{A}\} \leq \frac{\varepsilon}{2k}\theta\{t \in I(r) \mid u(t)a \in \mathcal{A}_o\},$$

for all $a \in X_v - \mathcal{A}_o$, where k is the order of the group of roots of unity in K. (Here and later on θ is the Haar measure on F and $I(r)$ is an interval in F with radius r centered at 0.)

Applying 3.3, we fix a compact $L \subset G$ such that $\mathcal{K} \subset L\Gamma/\Gamma$ and

$$(12) \qquad \frac{1}{\theta(I(r))}\theta\{t \in I(r) \mid u(t)x \in L\Gamma/\Gamma\} \geq 1 - \frac{\varepsilon}{4},$$

for all $x \in \mathcal{K}$.

In view of 4.7, 4.8 and (9), there exists a compact $D_o \subset X(P,U)$ which satisfies

$$\phi^{-1}(L\Gamma \times B) \subset D_o\Gamma_N/\Gamma_N,$$

and

$$D_o N \cap L\Gamma \subset D_o\Gamma_N.$$

Denote by Y the subset of all points of (P,Γ)-self- intersection in D_o. If $y \in Y$ there exists $\gamma \in \Gamma - \Gamma_P$ such that $y\gamma \in Y$. This implies that $\mathcal{U}y \subset Qy$ where $Q = \mathbf{Q}(K_S)$ and $\mathbf{Q}$ is a group from the class $\mathcal{F}$ contained in $\mathbf{P} \cap \gamma\mathbf{P}\gamma^{-1}$, in particular, dim$\mathbf{Q}$ < dim$\mathbf{P}$.Since D_o is compact and Γ is discrete in G, there are finitely many proper algebraic subgroups $\mathbf{P}_1, ..., \mathbf{P}_s$ of $\mathbf{P}$ such that $\mathbf{P}_i \in \mathcal{F}$ for all i and $\bigcup_{i\geq 1} X(P_i,\mathcal{U}) \supset Y$. Denote $C_i = X(P_i,\mathcal{U}) \cap Y$, $i = 1, 2, ..., s$. By the induction hypothesis there exists for every i a compact $D_i \subset X(P_i,\mathcal{U})$ so that if Φ_o is any neighbourhood of $\bigcup_{i\geq 1} D_i$ then we have an open neighbourhood Ψ of $\bigcup_{i\geq 1} C_i$ such that

$$(13) \qquad \frac{1}{\theta(I(r))}\theta\{t \in I(r) \mid \ u(t)x \in \Psi\Gamma/\Gamma\} \leq \frac{\varepsilon}{4},$$

for all $x \in \mathcal{K} - \Phi_o\Gamma/\Gamma$ and any interval $I(r)$.

Put $D = \bigcup_{i \geq 0} D_i$. Now, let us fix a neighbourhood Φ_o of D. We will prove that there exists a neighbourhood Φ of C which satisfies (2). Let Ψ be a neighbourhood of $\bigcup_{i \geq 1} C_i$ which satisfies (13) for the last choice of Φ_o. Using 4.9, 4.8 and the definition of D_o, one can find a neighbourhood Ω of D_o such that $\Omega \subset \Phi_o$, the set $(\Omega - \Psi\Gamma_P) \cap L\Gamma$ is without points of (P, Γ)-self-intersections, and

$$\Omega N \cap L\Gamma \subset \Omega\Gamma_N.$$

This, together with (9) and the fact that B is $S(v)$-small, implies that there exists a compact $S(v)$-small neighbourhood W_o of B in V such that

$$\phi^{-1}(W_o \times L\Gamma/\Gamma) \subset \Omega\Gamma_N/\Gamma_N. \tag{14}$$

Using the property (11) of B, as well as the fact that $\mathcal{U}$ acts trivially on $\mathbf{V}(K_w)$ for all $w \neq v$, we fix a neighbourhood W of C in V such that if $a \in X(P, \mathcal{U})m - W_o$ and I is a maximal subinterval of $I(r)$ with $u(I)a \subset W_o$ then

$$\theta\{t \in I \mid u(t)a \in W\} \leq \frac{\varepsilon}{2k}\theta(I). \tag{15}$$

We will prove that $\Phi = \eta^{-1}(W)$ is the neighbourhood of C which we need. Let $x = g\Gamma \in \mathcal{K} - \Phi_o\Gamma/\Gamma$. Denote

$$J^{(1)} = \{t \in I(r) \mid u(t)x \notin L\Gamma/\Gamma \text{ or } u(t)x \in \Psi\Gamma/\Gamma\}$$

and

$$J^{(2)} = \{t \in I(r) \mid u(t)x \in (\Phi\Gamma/\Gamma \cap L\Gamma/\Gamma) - \Psi\Gamma/\Gamma\}.$$

It is clear that

$$J^{(1)} \cup J^{(2)} \supset \{t \in I(r) \mid u(t)x \in \Phi\Gamma/\Gamma\}. \tag{16}$$

In view of (12) and (13)

$$\theta(J^{(1)}) \leq \frac{\varepsilon}{2}\theta(I(r)). \tag{17}$$

Assume that there exists $\gamma \in \Gamma$ with $g\gamma m \in W_o$. Then since $x \in L\Gamma/\Gamma$, it follows from (14) that $g\gamma \in \Omega\Gamma_N$ which, in view of the inclusion $\Omega \subset \Phi_o$, implies that $x \in \Phi_o\Gamma/\Gamma$. Contradiction. Therefore,

(18) $$g\gamma m \notin W_o,$$

for all $\gamma \in \Gamma$.

Next, for every $q \in g\Gamma m$, we define a subset J_q in $I(r)$ in the following way: (i) if v is nonarchimedean then $t \in J_q$ iff t is contained by a subinterval I of $I(r)$ such that $u(I)g\gamma m \subset W_o$ and $u(t')x \in L\Gamma/\Gamma - \Psi\Gamma/\Gamma$ for some $t' \in I$, and (ii) if v is archimedean then $t \in J_q$ iff t is contained by a subinterval $[\alpha, \beta]$ in $I(r)$ such that $u([\alpha,\beta])g\gamma m \subset W_o$ and $u(\beta)x \in L\Gamma/\Gamma - \Psi\Gamma/\Gamma$. Let $t \in J_q \cap J_{q'}$ where $q = g\gamma m$ and $q' = g\gamma' m$. Denote by $J_q(t)$ (resp. $J_{q'}(t)$) the maximal interval in J_q (resp. $J_{q'}$) containing t. It follows from the definition of J_q and $J_{q'}$ (and from the fact that in the nonarchimedean case if two intervals have nonempty intersection then one of them contains the other) that there exists $t_o \in J_q(t) \cap J_{q'}(t)$ such that $u(t_o)x \in L\Gamma/\Gamma - \Psi\Gamma/\Gamma$. It follows from (14) that $u(t_o)g\gamma$ and $u(t_o)g\gamma'$ belong to $\Omega\Gamma_N$. Since $(\Omega\Gamma_N - \Psi\Gamma_P) \cap L\Gamma$ is a set without (P,Γ)-self-intersections, we obtain that $\gamma' = \gamma\delta$, where $\delta \in \Gamma_P$. Therefore $u(t_o)g\gamma' m = \chi(\delta)u(t_o)g\gamma m$.

Since $\chi(\delta) \in \mathcal{O}^*$, $u(t_o)g\gamma m$ and $u(t_o)g\gamma' m$ belong to W_o, and W_o is $S(v)$-small set, using 4.6(c) we obtain that

(19) $$q' = \xi q,$$

where ξ is a root of unity in K.

Applying (15) and (18), we get

$$\frac{\varepsilon}{2k}\theta(J_q(t)) \geq \theta(J_q(t) \cap J^{(2)}).$$

Since for any t and t' in J_q either $J_q(t) = J_q(t')$ or $J_q(t) \cap J_q(t') = \emptyset$, we obtain

(20) $$\frac{\varepsilon}{2k}\theta(J_q) \geq \theta(J_q \cap J^{(2)}).$$

Now since $\bigcup_q J_q \supset J^{(2)}$, it follows from (19) and (20) that

$$\frac{\varepsilon}{2}\theta(I(r)) \geq \frac{\varepsilon}{2}\theta(\cup_q J_q) \geq \frac{\varepsilon}{2k}\sum_q \theta(J_q) \geq \sum_q \theta(J_q \cap J^{(2)}) \geq \theta(J^{(2)}).$$

This, in view of (16) and (17), completes the proof.

§5. Applications to the Hermitian forms.

5.0 In this section we prove Theorem 5 and its corollaries, after first developping the necessary algebraic background for the irrational hermitian forms. We conclude the section by giving some examples and making some remarks about possible generalizations and strenghtenings of Theorem 5.

5.1 Let R be a ring with center Z and an involution σ (i.e. σ is an antiautomorphism of R of order two). Also let $\lambda \in Z$ be such that ${}^{\sigma}\lambda\lambda = 1$. A λ-hermitian form (relatively to the involution σ) on the right free R-module R^n is a sesquilinear map $h : R^n \times R^n \to R$ such that

$$h(x, y) = \lambda^{\sigma} h(y, x) \tag{21}$$

for all $x, y \in R^n$. The hermitian form h is *nondegenerate* if the map $\hat{h} : R^n \to \mathrm{Hom}_R(R^n, R)$, $(\hat{h}x)(y) = h(x, y)$, is an isomorphism of abelian groups [Sch,7.1.3]. *Further on, by an hermitian form we will mean always a nondegenerate hermitian form.*

5.2 Unless something else is specified, in the subsections 5.2 - 5.5 we will denote by D a central division algebra of degree r over an *arbitrary* infinite field L of characteristic $\neq 2$. As in the Introduction, K is a subfield of L such that either $L = K$ or L is a quadratic extension of K. Let K_1 be any field extension of K, $D_1 = D \bigotimes_K K_1$ and $L_1 = L \bigotimes_K K_1$. (In the applications, D will be a division algebra over a number field L, K_1 will stand for the completion K_v of K at a valuation v of K, $L_1 = L_v$ and $D_1 = D_v$.) We will assume that D_1 admits an involution τ which is a L_1/K_1-involution, that is, $K_1 = \{x \in L_1 \mid {}^{\tau}x = x\}$. (Recall that τ is an involution of first (respectively, second) kind if $L_1 = K_1$ (respectively, $L_1 \neq K_1$).)

There are two possibilities : either L_1 is a field or $L_1 = K_1 \bigoplus K_1$. Let first L_1 be a field. Then D_1 coincides with a matrix algebra $\mathrm{M}_s(\Delta)$ with entries from a central division algebra Δ over L_1. It is known [K1, Theorem, p.37] that Δ admits an involution $^- : \Delta \to \Delta$ which is of the same kind as τ. We can define a *standard* involution σ on D_1 as follows: ${}^{\sigma}(a_{ij}) = (\bar{a}_{ji})$ for all $(a_{ij}) \in \mathrm{M}_s(\Delta)$.

Now let $L_1 = K_1 \bigoplus K_1$. Then the restriction of τ on L_1 transposes the direct summands of L_1. This implies that $D_1 = \mathrm{M}_s(\Delta) \bigoplus \mathrm{M}_s(\Delta^o)$ where Δ is a division algebra with center K_1 and Δ^o is the division algebra opposite to Δ (i.e. Δ^o coincides with Δ as abelian group and has the multiplication $x.y = yx$). In this case ${}^{\tau}(x, y) = (y, x)$ for all $(x, y) \in D_1$.

The relation between the different involutions of the same kind on D_1 is given by the following Proposition. Its proof is similar to [Sch, 8.7.4].

Proposition. *Let σ and τ be L_1/K_1-involutions on D_1. Then $\tau = \sigma\circ$ Int(d), where d is an invertible element of D_1 such that ${}^{\sigma}d = \pm d$ when σ and τ are involutions of first kind, and ${}^{\sigma}d = d$ when σ and τ are involutions of second kind.*

Proof. By the Scolem-Noether theorem, $\tau = \sigma\circ\mathrm{Int}(b)$ where $b \in D_1$. A simple direct argument shows that

$$id = \tau^2 = \mathrm{Int}(({}^{\sigma}b^{-1})b).$$

So, ${}^{\sigma}b = lb$ where $l \in L_1$. This implies that ${}^{\sigma}ll = 1$. Hence if σ is of first kind then $l = \pm 1$ and we can choose $d = b$. Otherwise there exists $c \in L_1$ such that $l = \dfrac{c}{{}^{\sigma}c}$, $c \in L$. (The existence of c follows from the Hilbert 90 theorem if L_1 is a field. If $L_1 = K_1 \bigoplus K_1$ and $l = (s, s^{-1})$ then we can choose $c = (s, 1)$ since σ acts on L by interchanging the two coordinates.) Put $d = cb$. It is easy to check that ${}^{\sigma}d = d$. The proposition is proved.

5.3 Let h be a λ-hermitian form on D_1^n, $n \geq 1$, with respect to τ. Since ${}^{\tau}\lambda\lambda = 1$, we get that $\lambda = \pm 1$ if τ is of first kind. Let τ be of second kind and let c be an invertible element in L_1. It follows from (21) that ch is a λ'-hermitian form with respect to τ where $\lambda' = \lambda\left(\dfrac{c}{{}^{\tau}c}\right)$. By Hilbert 90 theorem (and its simple analog when $L_1 = K_1 \bigoplus K_1$) c can be chosen in such a way that $\lambda = \dfrac{{}^{\tau}c}{c}$. In this case, ch is a 1-hermitian form.

Let $d \in D_1$ be such that ${}^{\tau}d = \varepsilon d$ where $\varepsilon = \pm 1$. An easy computation shows that $\tau' = \tau\circ\mathrm{Int}(d)$ is an involution on D_1 and $h' = dh$ is an $\varepsilon\lambda$-hermitian form with respect to τ' [Sch, 7.6.7]. The converse of this assertion is given by the following proposition.

Proposition. *Let h be a λ-hermitian form on D_1^n with respect to a L_1/K_1-involution τ and h' be a λ'-hermitian form on D_1^n with respect to a L_1/K_1-involution τ'. Assume that $h' = ah$ where $a \in D_1$. Then there exist $\alpha \in L_1$ and $d \in D_1$ such that $a = \alpha d$, $\tau' = \tau\circ$Int(d), $\lambda' = \lambda\left(\dfrac{\alpha}{{}^{\tau}\alpha}\right)$ and ${}^{\tau}d = \pm d$ if τ is of first kind and ${}^{\tau}d = d$ if τ is of second kind.*

Proof. Since h and h' are nondegenerate hermitian forms a is an invertible element in D_1. Let $x, y \in D_1^n$ and $c \in D_1$. Then

$$({}^{\tau'}c)h'(x,y) = a({}^{\tau}c)h(x,y) = a({}^{\tau}c)a^{-1}h'(x,y).$$

Hence

$$(22) \qquad {}^{\tau'}c = a({}^{\tau}c)a^{-1}.$$

On the other hand, $\tau' = \tau \circ \mathrm{Int}(d)$ with d as in Proposition 5.2. Then dh is a λ-hermitian form with respect to τ'. Applying (22) with dh instead of h we get that ad^{-1} is in the center of D_1 i.e. $a = \alpha d$ where $\alpha \in L_1$. The fact that $\lambda' = \lambda\left(\frac{\alpha}{{}^{\tau}\alpha}\right)$ follows from (21). The proposition is proved.

5.4 Let h be a λ-hermitian form on D_1^n with respect to an involution τ such that $\lambda = \pm 1$ if τ is of first kind and $\lambda = 1$ if τ is of second kind (see 5.3). Denote by $\mathrm{Nrd} : \mathrm{M}_n(D_1) \to L_1$ the usual reduced norm on $\mathrm{M}_n(D_1)$ if D_1 is a simple algebra and the direct sum of the reduced norms on $\mathrm{M}_{ns}(\Delta)$ and $\mathrm{M}_{ns}(\Delta^o)$ if $D_1 = \mathrm{M}_s(\Delta) \bigoplus \mathrm{M}_s(\Delta^o)$ (see 5.1). The special unitary group corresponding to h is defined by $\mathrm{SU}(h) = \{g \in \mathrm{M}_n(D_1) \mid \mathrm{Nrd}(g) = 1$ and $h(x,y) = h(gx, gy)$ for all $x, y \in D_1^n\}$. The group $\mathrm{SU}(h)$ coincides with the group of K_1-rational points of a K_1-algebraic group $\mathbf{SU}(h)$. The hermitian form h is *isotropic* if $\mathbf{SU}(h)$ is K_1-isotropic, equivalently, if $\mathrm{SU}(h)$ contains a diagonizable over K_1 infinite subgroup. It is known (and follows easily from the classification results in [Ti2] and [K1]) that *any* classical algebraic group defined over an infinite field K_1 of characteristic $\neq 2$ can be realized as $\mathbf{SU}(h)$ for certain h. If τ is of first kind then $\mathbf{SU}(h)$ gives all K_1-algebraic groups of types B_m, C_m and D_m. If τ is of second kind then we get all K_1-algebraic groups of type A_m. Note that if L_1 is a field then h is isotropic if and only if h represents nontrivially 0. If $L_1 = K_1 \bigoplus K_1$ (i.e. $D_1 = \mathrm{M}_s(\Delta) \bigoplus \mathrm{M}_s(\Delta^o)$) then the description of τ in 5.2 implies that $\mathrm{SU}(h)$ coincides with the image of $\mathrm{SL}_{ns}(\Delta)$ in $\mathrm{M}_{ns}(\Delta) \bigoplus \mathrm{M}_{ns}(\Delta^o)$ under the embedding $g \to (g, g^{-1})$. In particular, $\mathbf{SU}(h)$ is K_1-isotropic if and only if $ns > 1$.

Let us summarize the last observations in the following lemma.

Lemma. *Assume that either $n \neq 1$ or $D_1 \not\cong \Delta \bigoplus \Delta^o$ where Δ is a division algebra. Then h is isotropic if and only if h represents nontrivially zero.*

Remark. Let K be a number field, $K_1 = K_v$ be the completion of K at a valuation v of K and h be isotropic. Recall that if $\mathbf{G}$ is a simple K_v-isotropic K_v-algebraic group then the subgroup of $\mathbf{G}(K_v)$ generated by its unipotent elements has finite index [BTi1, 6.14]. Also, if L_v is a field then the only central division algebras over L_v with an involution of the first kind (respectively, an L_v/K_v-involution of second kind) are L_v itself and the unique quaternion division algebra (respectively, L_v itself) [Sch, 10.2.2]. Using these facts and the above description of $\mathbf{SU}(h)$, one can easily see that the unipotent elements in $\mathrm{SU}(h)$ generate a subgroup of finite index (and, therefore, Zariski dense subgroup) if and only if (a) $rn \geq 3$ if τ is of first kind and $\tau \neq \mathrm{id}$, (b) $rn \geq 2$ if τ is of second kind, and (c) $n \geq 3$ if $\tau = \mathrm{id}$. If the inequality in some of the cases (a)-(c) is not fulfilled then $\mathrm{SU}(h)$ is abelian, it consists of semisimple elements, and Theorem 5 (as well as Theorem 1) is not true (see 5.8).

5.5 Proposition. *With the notations from 5.4, let Σ be a subgroup of $SU(h)\cap SL_n(D)$ which is Zariski dense in $\mathbf{SU}(h)$. Assume that (a) $rn \geq 3$ if τ is of first kind and $\tau \neq id$, (b) $rn \geq 2$ if τ is of second kind, and (c) $n \geq 3$ if $\tau = id$. Then there exist, defined by Σ, an involution σ on D of the same kind as τ and an hermitian form $h_\circ$ on D^n with respect to σ such that $h = ah_\circ$, $a \in D_1$. In particular, $SU(h)\cap SL_n(D) = SU(h_\circ)$.*

Proof. Let $M = (h(e_i, e_j))$ be the matrix of h relatively to the standard basis $e_1, ..., e_n$ of D_1^n. Denote by $\rho : \mathrm{M}_n(D_1) \to \mathrm{M}_n(D_1)$ the involution ${}^\rho(a_{ij}) = ({}^\tau a_{ji})$, $(a_{ij}) \in \mathrm{M}_n(D_1)$. Then ${}^\rho M = \lambda M$ with $\lambda \in L_1$ and $\mathrm{SU}(h) = \{g \in \mathrm{SL}_n(D_1) \mid {}^\rho g M g = M\}$. Let $I : \mathrm{M}_n(D_1) \to \mathrm{M}_n(D_1)$ where ${}^I a = M^{-1}({}^\rho a)M$ for all $a \in \mathrm{M}_n(D_1)$. Since L_1 coincides with the center of D_1, we get that I is an involution of the same kind as ρ. Let G be the Zariski closure of Σ in $\mathrm{SL}_n(D)$ and $L[\Sigma]$ be the L-subalgebra of $\mathrm{M}_n(D)$ generated by Σ over L. It is easy to see that L is I-invariant and ${}^I g = g^{-1}$ for each $g \in \Sigma$. Therefore the restriction of I on $L[\Sigma]$ induces an involution which will be denoted also by I. It follows from the assumptions (a)-(c) in the formulation of Theorem 5 (see also 5.4) that the algebra $\mathrm{M}_n(D_1)$ is generated by $\mathrm{SU}(h)$ over L_1. Since Σ is Zariski dense in $\mathrm{SU}(h)$, this implies that $\mathrm{M}_n(D_1)$ is generated by Σ over L_1. Therefore $L[\Sigma] = \mathrm{M}_n(D)$. Now, the existence of the involution I on $\mathrm{M}_n(D)$ implies the existence of an involution σ on D of the same kind as ρ (and τ)[K1,Theorem, p.37]. Let $J : \mathrm{M}_n(D) \to \mathrm{M}_n(D)$, ${}^J(a_{ij}) = ({}^\sigma a_{ji})$. According to Proposition 5.2, $I = J\circ\mathrm{Int}(N)$ where $N \in \mathrm{GL}_n(D)$ and ${}^J N = \lambda N$, $\lambda = \pm 1$. Since $\mathrm{SU}(h) = \{g \in \mathrm{SL}_n(D_1) \mid {}^I g = g^{-1}\}$, we have that $G = \{g \in \mathrm{SL}_n(D) \mid {}^J g N g = N\}$. Therefore $G = \mathrm{SU}(h_\circ)$ where $h_\circ$ is the λ-hermitian form with respect to σ having matrix N.

In view of 5.3, having replaced h by a suitable multiple, we may (and will) assume that $\tau = \sigma$ (equivalently, $I = J$). Therefore

$$g^{-1} = N^{-1}({}^I g)N = M^{-1}({}^I g)M$$

for all $g \in G$. So, MN^{-1} commutes with each $g \in G$. Therefore MN^{-1} is in the center of $M_n(D_1)$ i.e. $h = ah_\circ$ where $a \in D_1$. The proposition is proved.

5.6 Proof of Theorem 5. We will use the notations from the formulation of Theorem 5 in the Introduction. Let $\mathbf{G}_1$ be the L-algebraic group corresponding to $\mathrm{SL}_n(D)$, i.e $\mathbf{G}_1(L)$ =$\mathrm{SL}_n(D)$. Let $\mathbf{G} = R_{L/K}\mathbf{G}_1$ where $R_{L/K}$ is Weil's restriction of scalars. Then $\mathbf{G}(K_v) = \mathrm{SL}_n(D_v)$ for each $v \in S$. Put $G = \mathbf{G}(K_S)$, $\Gamma = \mathrm{SL}_n(\Lambda)$ and $H = \prod_{v \in S}\mathrm{SU}(h_v)$. It follows from [BTi1, 3.18] that, under the natural action of G on the space of all hermitian forms on D_S^n, the orbit Gh is closed and, therefore, homeomorphic to G/H. Hence, the almost S-integer equivalence implies the proper equivalence and , also, the assertion (ii) from the formulation of Theorem 5 is equivalent to the density of $H\Gamma$ in G.

Let us prove that (i) implies (ii). In view of the above remark, it is enough to show that $H_u\Gamma$ is dense in G. Since Γ is a lattice in G, it follows from Theorem 1 that there exists a connected K-algebraic group $\mathbf{L}$ of $\mathbf{G}$ and a subgroup of finite index L' in $L = \mathbf{L}(K_S)$ such that the closure of $H_u\Gamma/\Gamma$ in G/Γ coincides with $L'\Gamma/\Gamma$, $\Sigma = L' \cap \Gamma$ is Zariski dense in $\mathbf{L}$, and L' contains H_u. Note that $\mathbf{SU}(h_v)$ is a maximal connected algebraic subgroup of $\mathbf{SL}_n(D)$ and $L' \cap \mathrm{SU}(h_v)$ is Zariski dense in $\mathrm{SU}(h_v)$ for all $v \in S_o$. This implies that either $\mathbf{L}(K_v) = \mathrm{SU}(h_v)$ for all $v \in S_o$ or $\mathbf{L} = \mathbf{G}$. In the first case, it follows from Proposition 5.5 that there exists an hermitian form h_o on D^n determined by Σ and such that h_{S_o} is multiple of h_o. This contradicts our hypothesis. Let $\mathbf{L} = \mathbf{G}$. We will show that $L'\Gamma = G$. This is clear when $n > 1$ because $\mathbf{G}(K_v)$, $v \in S$, does not contain subgroups of finite index and, therefore, $L' = G$. Let $n = 1$. It follows from the general description of the orders [W1, ch. 5] that Γ is the intersection of G with an open compact subgroup of $\mathbf{G}(\mathcal{V}_S)$ where $\mathcal{V}_S$ is the S-adele ring (i.e. $\mathcal{V}_S$ is the restricted topological product of all fields K_v, $v \notin S$, relative to the rings of integers $\mathcal{O}_v \subset K_v$, $v \notin S$). By the strong approximation theorem, the projection of Γ into $\mathbf{G}(K_{S-S_o})$ is dense. Therefore $L'\Gamma = G$, which proves the implication.

Next we will prove that (ii) implies (i). Assume the contrary, that is , h_{S_o} is a multiple of a rational form h_o on D^n. Let $\mathbf{L}_1 = \mathbf{SU}(h_o)$, $L_1 = \mathbf{L}_1(K_S)$ and $C = \prod_{v \in S-S_o}\mathrm{SU}(h_v)$. Since $H_u\Gamma$ is dense in G, $L_1\Gamma$ is closed, $L_1 \supset H_u$ and C is compact, we get that $G = CL_1\Gamma$. Note that H_u commutes elementwise with C. Therefore every H_u-ergodic component of the Haar measure on G/Γ is concentrated on $gL_1\Gamma/\Gamma$ for some $g \in C$. In particular, H_u does not act ergodically on G/Γ . On the other hand, by a generalization of a theorem of Moore about the Mautner property [MTo2, 2.1], every H_u-invariant L^2-function on G/Γ is invariant under the action of the smallest normal subgroup $G_o \subset G$ containing H_u. It is clear that $G_o = \mathbf{G}(K_{S_o})$. Since $G_o\Gamma$ is dense in G by the strong approximation, G_o acts ergodically on G/Γ. Therefore, the action of H_u is ergodic. Contradiction. The theorem is proved.

5.7 Proof of Corollary 1 Since D^n is dense in D^n_S (weak approximation), we can approximate each x_i by a vector $y_i \in D^n$ in such a way that $y_1, y_2, ..., y_{n-1}$ are linearly independent over D and $\mid h_S(x_i, x_j) - h_S(y_i, y_j) \mid < \frac{\varepsilon}{2}$ for all $i, j = 1, 2, ..., n-1$. Denote by $e_1, e_2, ..., e_n$ the standard basis of D^n. There exists $g \in \mathrm{SL}_n(D)$ such that $ge_i = y_i$ for all $i = 1, 2, ..., n-1$. Put $y_n = ge_n$ and $h' = h^g_S$. In view of Theorem 1, h_S is almost S-integer equivalent to h'. Hence there exists $\gamma \in \mathrm{SL}_n(\Lambda)$ such that $\mid h_S(x_i, x_j) - h_S(\gamma e_i, \gamma e_j) \mid < \frac{\varepsilon}{2}$ for all $i, j = 1, 2, ..., n-1$. This implies (1) with $z_i = \gamma e_i$. The Corollary is proved.

5.8 Examples and concluding remarks. 1. Let us show that the assumptions in the formulation of Theorem 5 are essential and can not be relaxed. Let $a, b \in \mathbb{Z} - \{0\}$ and $D = \{a, b\}$ be the quaternion algebra over $\mathbb{Q}$ defined by a and b, i.e. $D = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$ where $i^2 = a, j^2 = b$ and $k = ij - ji$. Assume that $D_\infty = D \bigotimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $\mathrm{M}_2(\mathbb{R})$. Let $\tau : D \to D$ be the standard involution of D, i.e. ${}^\tau(x + yi + zj + tk) = x - yi - zj - tk)$, and let $\Lambda = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$. Recall that τ acts on D_∞ as follows :

$$ {}^\tau\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} t & -y \\ -z & x \end{pmatrix}, $$

[Sch, p.361]. Denote by $\mathbf{G}$ the $\mathbb{Q}$-algebraic group corresponding to $\mathrm{SL}_1(D)$ (i.e $\mathbf{G}(\mathbb{Q}) = \mathrm{SL}_1(D)$) and put $\Gamma = \mathrm{SL}_1(D) \cap \Lambda$. Then $\mathbf{G}(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ and, in view of a classical result of Borel and Harish-Chandra [M4, I.3.2.4], $\mathrm{SL}_2(\mathbb{R})/\Gamma$ is compact if and only if D is a division algebra. Let T be the subgroup of all diagonal matrices in $\mathrm{SL}_2(\mathbb{R})$ and $K = \mathrm{SO}_2(\mathbb{R})$. Then $X = K \backslash \mathrm{SL}_2(\mathbb{R})/\Gamma$ can be regarded as a Riemannian surface with constant curvature -1, and the action of T by left transformations on $\mathrm{SL}_2(\mathbb{R})/\Gamma$ induces the geodesic flow on X. It is a standard fact that there exists a relatively compact, non-compact, and non-dense T-orbit on X. (We refer to [St1, Lemma 2] for a more general result due to Margulis.) Thus, there exists a $g \in \mathrm{SL}_2(\mathbb{R})$ such that $\Gamma g T$ is neither dense nor closed in $\mathrm{SL}_2(\mathbb{R})$. Put $\alpha_o = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\alpha = g\alpha_o g^{-1}$. Then ${}^{\tau}\alpha = -\alpha$ and $h(x,y) = {}^{\tau}x\alpha y$ is an isotropic -1-hermitian form on D_∞. Since $\Gamma g T$ is not closed, h is not rational, and since $\Gamma g T$ is not dense in $\mathrm{SL}_2(\mathbb{R})$, there are hermitian forms which are properly but not almost S-integer equivalent to h. In the case when D is a division algebra (say, $\{a,b\} = \{-1,3\}$), we get an example showing that if $n=1$ and $r=2$ (i.e. the assumption (a) in the formulation of Theorem 1 is not fulfilled) then (i) does not imply (ii). If $D = \mathrm{M}_2(\mathbb{Q})$ and $\Lambda = \mathrm{M}_2(\mathbb{Z})$ then $\alpha = \begin{pmatrix} -\gamma & -\delta \\ -\beta & \gamma \end{pmatrix}$, where $\beta, \gamma, \delta \in \mathbb{R}$. It is easy to see that if the quadratic form $f(x,y) = \beta x^2 + 2\gamma xy + \delta y^2$ is isotropic and irrational then the closure of $f(\mathbb{Z}^2)$ in $\mathbb{R}$ does not contain 0 and there exists a properly equivalent to f quadratic form which is not almost S-integer equivalent to f. This shows that the assumption (c) in Theorem 5 is essential. (We refer to [M3, 1.2] and [G, 4.2] for explicit examples of quadratic forms with the same properties.) Concerning (b), note that if $r = n = 1$ then h_S is always rational.

2. We use the notations from Corollary 1. It is easy to see that if $n \geq 2$ then h_S is anisotropic if and only if the map $D_S^n \to D_S$, $x \to h_S(x,x)$, is proper. This implies that if h_S is anisotropic then the subset $\{h_S(z,z) \mid z \in \Lambda^n\} \subset D_S$ is discrete. Let $n=1$. Then $\mathrm{Nrd}\{h_S(z,z) \mid z \in \Lambda^n\}$ is discrete in L_S. This means that the assertion analogous to Corollary 1 is not true for $n=1$. Similar arguments show that Theorem 5 can not be modified to be true for "equivalent" instead of "properly equivalent" hermitian forms. (Two hermitian forms h_S and h'_S are equivalent if they are conjugated by an element from $\mathrm{GL}_n(D_S) = \prod_{v \in S} \mathrm{GL}_n(D_v)$.)

3. Almost the same proofs allow to establish similar results to Theorem 5 and its corollaries when considering finite dimensional central simple algebras with involutions τ_S of "mixed" type (i.e. $\tau_S = \bigoplus_{v \in S} \tau_v$ where $\tau_v, v \in S$, is an involution of first *or* second type).

4. Recently Eskin, Margulis and Mozes proved the quantitative version of the Oppenheim conjecture for real quadratic forms [EMM]. It is plausible to obtain quantitative results in the general framework of the hermitian forms over division algebras in the S-arithmetic case.

5. Another very interesting application of the dynamical approach to the number theory is the recent proof by Kleinbock and Margulis [KM] of conjectures of Bauer and Sprindzhuk from the theory of the Diophantine approximation on manifolds. It is of interest to generalize these results to the S-arithmetic setting as well.

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