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# On the Theta Lift for the Trivial Representation

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**Abstract.**

We describe the Howe quotient and theta lift for the trivial representation of $Sp(2n,\mathbb{R})$ for the dual pairs $(O(p,q), Sp(2n,\mathbb{R}))$, within the stable range (i.e., min $(p,q) > 2n$), by explicitly constructing the Howe quotients.

## §1. Introduction

Let $Sp(2k,\mathbb{R})$ be the symplectic group on $\mathbb{R}^{2k}$ and $\widetilde{Sp}(2k,\mathbb{R})$ be the metaplectic group. If $H$ is a subgroup of $Sp(2k,\mathbb{R})$, we shall let $\widetilde{H}$ be the pullback of $H$ by the covering map from $\widetilde{Sp}(2k,\mathbb{R})$ to $Sp(2k,\mathbb{R})$. The oscillator representation $\omega$ of $\widetilde{Sp}(2k,\mathbb{R})$ may be realized on a space of holomorphic functions on $\mathbb{C}^k$, using the Fock model.

Let $(G,G')$ be a reductive dual pair in $Sp(2k,\mathbb{R})$. A maximal compact subgroup of $\widetilde{Sp}(2k,\mathbb{R})$ is $\widetilde{U}(k)$, the half-determinant cover of $U(k)$. In the Fock model, the space of $\widetilde{U}(k)$-finite vectors of the oscillator representation is $\mathcal{P} = \mathcal{P}(\mathbb{C}^k)$, the space of complex-valued polynomials on $\mathbb{C}^k$. We can assume that $K = U(k)\cap G$ and $K' = U(k)\cap G'$ are maximal compact subgroups in $G$ and $G'$ respectively. We shall let lower gothic symbols denote Lie algebras of Lie groups, e.g., $\mathfrak{g}$ and $\mathfrak{g}'$ will be the Lie algebras of $G$ and $G'$ respectively.

For a reductive subgroup $H$ (with maximal compact subgroup $K_H = U(k)\cap H$) of $Sp(2k,\mathbb{R})$, we denote by $\mathcal{R}(\mathfrak{h},\widetilde{K}_H,\omega)$ the set of infinitesimal equivalence classes of irreducible $(\mathfrak{h},\widetilde{K}_H)$ modules realizable as quotients of $\mathcal{P}$. Consider the dual pair $(G,G')$. For $\rho\in\mathcal{R}(\mathfrak{g}',\widetilde{K'},\omega)$, the Howe

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quotient corresponding to $\rho$ is defined by (see [Ho2])

$$\Omega(\rho) = \mathcal{P}/\mathcal{N}_\rho,$$

where $\mathcal{N}_\rho$ is the intersection of all $(\mathfrak{g}', \widetilde{K'})$-invariant subspaces $\mathcal{N}$ of $\mathcal{P}$ such that $\mathcal{P}/\mathcal{N} \cong \rho$ as $(\mathfrak{g}', \widetilde{K'})$ modules. It is known (see [Ho2]) that

$$\Omega(\rho) \simeq \rho' \otimes \rho,$$

where $\rho'$ is a quasi-simple $(\mathfrak{g}, \widetilde{K})$ module of finite length, with a unique irreducible quotient $\theta(\rho)$. The correspondence

$$\rho \longmapsto \theta(\rho)$$

is commonly known as the (local) theta correspondence, and $\theta(\rho)$ is often called the theta lift of $\rho$.

The pullback $\widetilde{H}$ of a Lie subgroup of $Sp(2k, \mathbb{R})$ is a split or non-split extension by $\mathbb{Z}/2\mathbb{Z}$ depending on the dual pair under consideration. The representations which occur in the theta correspondence are genuine, i.e., they do not factor to $H$. In the case where the cover of $H$ is split, this just means that they are of the form $\pi \otimes sgn$, where $\pi$ is a representation of $H$, and $sgn$ is the non-trivial character of $\mathbb{Z}/2\mathbb{Z}$. If $H \doteq O(p,q)$, the non-split cover $\widetilde{H}$ may be realized as $H \times \mathbb{Z}/2\mathbb{Z}$ with group law $(g,\epsilon)(h,\delta) = (gh, \epsilon\delta(det(g), det(h))_\mathbb{R})$, where $(\cdot,\cdot)_\mathbb{R}$ is the Hilbert symbol of $\mathbb{R}$. The character $\chi$ of $\widetilde{O}(p,q)$ given by

$$\chi(g,\epsilon) = \epsilon \cdot \begin{cases} \sqrt{-1} & \text{if } det(g) = -1, \\ 1 & \text{otherwise} \end{cases}$$

is genuine, and a genuine representation of $\widetilde{O}(p,q)$ is of the form $\pi \otimes \chi$ for $\pi \in \widehat{O}(p,q)$. In either case we will only refer to $\pi$.

Consider the dual pairs $(O(p,q), Sp(2n,\mathbb{R}))$. It is easy to see that the theta lift of the trivial representation of $Sp(2n,\mathbb{R})$ (which we shall denote by $\theta(\mathbb{1}; p, q)$) exist only if $p+q$ is even: For the dual pair $(O(p,q), Sp(2n,\mathbb{R}))$, $\widetilde{Sp}(2n,\mathbb{R})$ is split if $p+q$ is even and non-split if $p+q$ is odd. So $\mathbb{1} \in \mathcal{R}(\mathfrak{sp}(2n,\mathbb{R}), \widetilde{U}(n), \omega)$ only if $p+q$ is even. It is also known that $\theta(\mathbb{1}; p, q)$ is irreducible and unitary (see [HLi], [Li] and [ZH]) when min $(p,q) \geq 2n$, i.e., in the stable range. Beyond the stable range, nothing much is known. For $n = 1$, these representations of $O(p,q)$ are known as ladder representations (see [AFR], [BZ] and [HT]) and are studied in [BZ], [Ko1] and [Ko2]. For all $n$, these representations are "small" in the sense that they have small Gelfand-Kirillov dimensions and small rank (in the sense of [Ho4]). They "should" arise from appropriate quantization of nilpotent orbits (see [HLi] and [ZH]). In fact, in [BZ] (the case where $n = 1$), it was shown that these are minimal representations, i.e., their annihilators in the universal enveloping algebra are the Joseph ideals.

The aim of this paper is to provide a basis of $O(p) \times O(q)$ highest weight vectors for $\Omega(\mathbb{1})$ when min $(p, q) > 2n$. With such a basis, irreduciblity of $\Omega(\mathbb{1})$'s results from similar considerations as in [HT] (and hence $\theta(\mathbb{1}; p, q) = \Omega(\mathbb{1})$). As we have noted, irreducibility and unitarity for $\Omega(\mathbb{1})$ follow from [Li]'s results (see Theorem 2.2 of [ZH] for the argument). But our technique has invariant-theoretic flavour and has the advantage of providing a model of the representation space which might be useful to those who would like to make explicit calculations on these representations. Technical difficulties have prevented a result for the entire stable range, i.e., to include the cases min $(p, q) = 2n$. It would seem that our techniques would still apply to these cases when $p \neq q$. Unfortunately, we still find it difficult to extend the results to the case when $p = q = 2n$ for $n > 1$. (In this particular case, by results of [HLi], the theta lift of the trivial representation of $Sp(2n, \mathbb{R})$ is precisely the same as the theta lift of the trivial representation of $Sp(2n-2, \mathbb{R})$ for the dual pair $(O(2n, 2n), Sp(2n-2, \mathbb{R}))$).

Although we do not discuss it here, the construction works for the dual pairs $(U(p,q), U(n,n))$, $(Sp(p,q), O^*(4n))$, $(Sp(2m, \mathbb{R}), O(n,n))$ and $(O^*(2m), Sp(n,n))$ with appropriate condtions on $p, q, m$ and $n$.

Another reason for the construction is to study the Howe quotient. We believe that in the stable range (see [H4]), the Howe quotient (corresponding to a unitary representation or "small" representation) is irreducible. Evidence in support of this can be found in this paper as well as [LZ1], [LZ2] and [PT]. The Howe quotient also features prominently in many applications; see [KV2] and [Zh] (and the references therein) for applicatons to invariant distributions and [KR] (and the references therein) for applications to the construction of automorphic forms. The setup used here enables one to have control on the Howe quotients and it is our hope to try to extend it to other dual pairs.

## §2. Preliminaries

Consider the dual pair $(O(p), Sp(2n, \mathbb{R}))$ acting on the $\widetilde{U}(np)$-finite vectors of the associated Fock space $\mathbb{C}[x_{11}, \ldots, x_{1p}, \ldots, x_{n1}, \ldots, x_{np}]$ as follows:

(a) Action of $\mathfrak{so}(p)_{\mathbb{C}}$: $\displaystyle\sum_{s=1}^{n}\left(x_{si}\frac{\partial}{\partial x_{sj}} - x_{sj}\frac{\partial}{\partial x_{si}}\right), \quad 1 \le i < j \le p.$

(b) Action of $\mathfrak{sp}(2n,\mathbb{R})_{\mathbb{C}} = \mathfrak{u}(n)_{\mathbb{C}} \oplus \text{Span}\,\{r_{ij}^2\} \oplus \text{Span}\,\{\Delta_{ij}\}$ :

$$\text{(2.1)(i)}\ \mathfrak{u}(n)_{\mathbb{C}} = \text{Span}\left\{E_{ij}^{x} = \sum_{t=1}^{p} x_{it}\frac{\partial}{\partial x_{jt}} + \delta_{i,j}\frac{p}{2} \;\middle|\; 1 \le i \le j \le n\right\};$$

$$\text{(ii)}\ r_{ij}^2 = \sum_{t=1}^{p} x_{it}x_{jt}, \qquad 1 \le i \le j \le n,$$

$$\text{(iii)}\ \Delta_{ij} = \sum_{t=1}^{p}\frac{\partial^2}{\partial x_{it}\partial x_{jt}}, \qquad 1 \le i \le j \le n.$$

Observe that the actions of $O(p)$ and $U(n)$ arise from the natural right and left multiplication on the polynomial algebra $\mathbb{C}[x_{11},\ldots,x_{np}]$. We shall choose a Cartan subalgebra of $\mathfrak{u}(n)_{\mathbb{C}} \simeq \mathfrak{gl}(n)_{\mathbb{C}}$ as

$$\mathfrak{h}' = \{\text{diag}\,(a_1,\ldots,a_n) \mid a_1,\ldots,a_n \in \mathbb{C}\},$$

and parametrize (as it is usually done) irreducible representations of $U(n)$ by $n$-tuples of integers $(\alpha_1,\ldots,\alpha_n)$ where $\alpha_1 \ge \alpha_2 \ge \ldots \ge \alpha_n$.

We shall now describe a root space decomposition for $\mathfrak{so}(p)_{\mathbb{C}}$. Choose the Cartan subalgebra as

$$\mathfrak{h} = \begin{cases} \{H = \text{diag}\,(H_1,H_2,\ldots,H_l)\} & \text{if } p = 2l, \\ \{H = \text{diag}\,(H_1,H_2,\ldots,H_l,0)\} & \text{if } p = 2l+1, \end{cases}$$

where for each $1 \le j \le l$, $H_j$ is a block of $2 \times 2$ submatrix of the form

$$H_j = \begin{pmatrix} 0 & \sqrt{-1}h_j \\ -\sqrt{-1}h_j & 0 \end{pmatrix}, \qquad h_j \in \mathbb{C}.$$

For $1 \le j \le l$, let $e_j : \mathfrak{h} \longrightarrow \mathbb{C}$ be the functional defined by

$$e_j(H) = h_j, \qquad H \in \mathfrak{h}.$$

The root system of $\mathfrak{so}(p)_{\mathbb{C}}$ with respect to $\mathfrak{h}$ is given by

$$\Phi = \begin{cases} \{\pm(e_a \pm e_b) \mid 1 \le a < b \le l\} & \text{if } p = 2l, \\ \{\pm(e_a \pm e_b) \mid 1 \le a < b \le l\} \cup \{\pm e_a \mid 1 \le a \le l\} & \text{if } p = 2l+1. \end{cases}$$

For $\alpha \in \Phi$, let $\mathfrak{so}(p)_\alpha$ denote the root space of $\mathfrak{so}(p)_\mathbb{C}$ corresponding to $\alpha$. Then

$$\mathfrak{so}(p)_\mathbb{C} = \mathfrak{h} \oplus \sum_{\alpha\in\Phi} \mathfrak{so}(p)_\alpha$$

is a root space decomposition for $\mathfrak{so}(p)_\mathbb{C}$. Note that the subalgebra

$$\mathfrak{h} \oplus \sum_{1\le a\neq b\le l} \mathfrak{so}(p)_{e_a-e_b}$$

of $\mathfrak{so}(p)$ is isomorphic to $\mathfrak{gl}(l)_\mathbb{C}$.

If $A$ is a $p\times p$ matrix, then it contains $[\frac{p}{2}]^2$ blocks of $2\times 2$ submatrices. For $1 \le s,t \le [\frac{p}{2}]$, we denote by $(A)_{st}$ the $(s,t)$ $2\times 2$-block of $A$. If $p$ is odd, we shall allow $2\times 1$ blocks on the rightmost column and $1\times 2$ blocks in the bottom row. With this notation, we define a basis for $\mathfrak{h}$ and the root vectors $E_{a,-b}$, $E_{-a,b}$, $E_{a,b}$ and $E_{-a,-b}$ (and $E_{\pm a}$ if $p$ is odd) in $\mathfrak{so}(p)_\mathbb{C}$ (where $1 \le a \le [\frac{p}{2}]$ for (1) and $1 \le |a| < |b| \le [\frac{p}{2}]$ for (2) and (3)) as follows:

$$(1)\qquad (H_a)_{st} = \begin{cases} \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} & \text{if } s=t=a \\ 0 & \text{otherwise} \end{cases}$$

$$(2)\qquad (E_{a,b})_{st} = \begin{cases} X_{\mathrm{sgn}(a),\mathrm{sgn}(b)} & \text{if } s=|a| \text{ and } t=|b| \\ -X^t_{\mathrm{sgn}(a),\mathrm{sgn}(b)} & \text{if } t=|a| \text{ and } s=|b| \\ 0 & \text{otherwise} \end{cases}$$

$$(3)\qquad (E_a)_{st} = \begin{cases} X_{\mathrm{sgn}(a)} & \text{if } s=|a| \text{ and } t=\frac{p+1}{2} \\ -X^t_{\mathrm{sgn}(a)} & \text{if } t=|a| \text{ and } s=\frac{p+1}{2} \\ 0 & \text{otherwise} \end{cases}$$

where
$$X_{+,-} = \begin{pmatrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix}, \qquad X_{-,+} = \begin{pmatrix} 1 & -\sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix},$$

$$(2.2)\quad X_{+,+} = \begin{pmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & -1 \end{pmatrix}, \qquad X_{-,-} = \begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & -1 \end{pmatrix},$$

$$X_+ = \begin{pmatrix} 1 \\ -\sqrt{-1} \end{pmatrix}, \qquad X_- = \begin{pmatrix} 1 \\ \sqrt{-1} \end{pmatrix}.$$

Then $E_{a,-b} \in \mathfrak{so}(p)_{e_a-e_b}$, $E_{-a,b} \in \mathfrak{so}(p)_{-e_a+e_b}$, $E_{a,b} \in \mathfrak{so}(p)_{e_a+e_b}$, $E_{-a,-b} \in \mathfrak{so}(p)_{-e_a-e_b}$, $E_a \in \mathfrak{so}(p)_{e_a}$ and $E_{-a} \in \mathfrak{so}(p)_{-e_a}$.

We shall use a more convenient system of coordinates. Define, for $j = 1, \dots, n$,

$$z_{jk} = x_{j,2k-1} - \sqrt{-1}x_{j,2k}, \quad \bar{z}_{jk} = x_{j,2k-1} + \sqrt{-1}x_{j,2k}, \quad k = 1, \dots, [\frac{p}{2}],$$
$$z_{j\frac{p+1}{2}} = x_{j,\frac{p+1}{2}}, \qquad \text{if } p \text{ is odd.}$$

**Lemma 2.1.** *The actions by the root vectors of $\mathfrak{so}(p)_{\mathbb{C}}$ on $\mathbb{C}[x_{11}, \dots, x_{np}]$ are given by the following differential operators:*

$$H_a = \sum_{s=1}^{n} \left( z_{sa}\frac{\partial}{\partial z_{sa}} - \bar{z}_{sa}\frac{\partial}{\partial \bar{z}_{sa}} \right),$$

$$E_{a,-b} = \sum_{s=1}^{n} \left( z_{sa}\frac{\partial}{\partial z_{sb}} - \bar{z}_{sb}\frac{\partial}{\partial \bar{z}_{sa}} \right), \quad E_{-a,b} = \sum_{s=1}^{n} \left( \bar{z}_{sa}\frac{\partial}{\partial \bar{z}_{sb}} - z_{sb}\frac{\partial}{\partial z_{sa}} \right),$$

$$E_{a,b} = \sum_{s=1}^{n} \left( z_{sa}\frac{\partial}{\partial \bar{z}_{sb}} - z_{sb}\frac{\partial}{\partial \bar{z}_{sa}} \right), \quad E_{-a,-b} = \sum_{s=1}^{n} \left( \bar{z}_{sa}\frac{\partial}{\partial z_{sb}} - \bar{z}_{sb}\frac{\partial}{\partial z_{sa}} \right),$$

$$E_a = \sum_{s=1}^{n} \left( z_{sa}\frac{\partial}{\partial z_{s\frac{p+1}{2}}} - z_{s\frac{p+1}{2}}\frac{\partial}{\partial \bar{z}_{sa}} \right), \quad E_{-a} = \sum_{s=1}^{n} \left( \bar{z}_{sa}\frac{\partial}{\partial z_{s\frac{p+1}{2}}} - z_{s\frac{p+1}{2}}\frac{\partial}{\partial z_{sa}} \right).$$

*Proof.* Omitted. ∎

Using the above setup, we shall parametrize representations of $SO(p)$ by tuples $(\alpha_1, \alpha_2, \dots, \alpha_{[\frac{p}{2}]})$ of integers, satisfying

$$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{[\frac{p}{2}]} \geq 0 \qquad \text{if } p \text{ is odd,}$$
$$\alpha_1 \geq \alpha_2 \geq \dots \geq |\alpha_{[\frac{p}{2}]}| \qquad \text{if } p \text{ is even.}$$

We shall require some results from the theory of spherical harmonics. First, define the following determinants:

$$C_i = \begin{vmatrix} z_{11} & z_{12} & \dots & z_{1i} \\ z_{21} & z_{22} & \dots & z_{2i} \\ \vdots & \vdots & \dots & \vdots \\ z_{i1} & z_{i2} & \dots & z_{ii} \end{vmatrix}, \qquad i = 1, \dots, n \tag{2.3}$$

It is not difficult to check that $C_i$ are $SO(p) \times U(n)$ highest weight vectors of weight

$$(1, \dots (i \text{ copies}) \dots, 1, 0, \dots, 0) \otimes (1, \dots (i \text{ copies}) \dots, 1, 0, \dots, 0).$$

Define the $O(p)$ invariants $\mathcal{I}_1$ and pluri-harmonics $\mathcal{H}_1$ as follows:

(a) $\mathcal{I}_1 = \mathbb{C}[x_{11}, \dots, x_{np}]^{O(p)}$;

(b) $\mathcal{H}_1 = \{f \in \mathbb{C}[x_{11}, \dots, x_{np}] \mid \Delta_{ij} f = 0, \quad 1 \leq i \leq j \leq n\}$.

The results of [DT], [To], [We] and [KV1] say that:

**Theorem 2.2.** *(a)([DT] and [To]) If $p > 2n$, then*

$$\mathbb{C}[x_{11}, \dots, x_{np}] = \mathcal{I}_1 \otimes \mathcal{H}_1.$$

*(b)([We]) The $O(p)$ highest weight vectors in $\mathcal{I}_1$ are generated freely by $r_{ij}^2$.*
*(c)([KV1]) If $p \geq 2n$, the $O(p) \times U(n)$ highest weight vectors in $\mathcal{H}_1$ are generated freely by $C_i$, $i = 1, \dots, n$.*

**Remark.** *To some extent, it is possible to consider the cases for $p \leq 2n$ when $n$ is small, but these cases are a little messy (see [PT]). We shall omit them in this paper.*

Consider the action of the dual pair $(O(q), Sp(2n, \mathbb{R}))$ on $\mathbb{C}[y_{11}, \dots, y_{nq}]$. We shall use the more convenient coordinate system: for $j = 1, \dots, n$,

$$w_{jk} = y_{j,2k-1} - \sqrt{-1} y_{j,2k}, \quad \bar{w}_{jk} = y_{j,2k-1} + \sqrt{-1} y_{j,2k}, \quad k = 1, \dots, [\tfrac{q}{2}],$$

$$w_{j\frac{q+1}{2}} = y_{j,\frac{q+1}{2}} \qquad \text{if } q \text{ is odd}$$

All of what we have mentioned for $(O(p), Sp(2n, \mathbb{R}))$ carries over and we shall use the superscripts $y$ to denote the corresponding operators from $\mathfrak{u}(n)_{\mathbb{C}}$ and tildes to differentiate the $r_{ij}$'s and $\Delta_{ij}$'s. The description for the root space decomposition for $\mathfrak{so}(q)_{\mathbb{C}}$ is similar and we shall use small $f$'s to denote the roots and capital $F$'s to denote the root vectors of $\mathfrak{so}(q)_{\mathbb{C}}$. We will omit the discussion.

From here on we shall assume $\min(p, q) > 2n$. Let $\mathcal{P} = \mathbb{C}[x_{11}, \dots, x_{1p}, y_{11}, \dots, y_{1q}, \dots, x_{n1}, \dots, x_{np}, y_{n1}, \dots, y_{nq}]$. This is the space of $\widetilde{U}(np + nq)$-finite vectors of the associated Fock model for the dual pair $(O(p, q), Sp(2n, \mathbb{R}))$, and the actions of the complexified Lie algebras of $O(p, q)$ and $Sp(2n, \mathbb{R})$ can be described as follows:

(a) Action of $\mathfrak{o}(p, q)_{\mathbb{C}} = \mathfrak{o}(p)_{\mathbb{C}} \oplus \mathfrak{o}(q)_{\mathbb{C}} \oplus \mathfrak{p}$ :

(i) Action of $\mathfrak{o}(p)_{\mathbb{C}}$: as in (2.1);

(ii) Action of $\mathfrak{o}(q)_{\mathbb{C}}$: similar to (2.1);

(2.4) (iii) Action of $\mathfrak{p}$: $\sum_{s=1}^{n}\left(x_{si}y_{sj} - \frac{\partial^2}{\partial x_{si}\partial y_{sj}}\right) \quad 1 \le i \le p,\ 1 \le j \le q,$

(b) Action of $\mathfrak{sp}(2n,\mathbb{R})_{\mathbb{C}} = \mathfrak{u}(n)_{\mathbb{C}} \oplus \text{Span}\ \{X_{ij}\} \oplus \text{Span}\ \{Y_{ij}\}$ :

(i) $\mathfrak{u}(n)_{\mathbb{C}} = \text{Span}\left\{E_{ij} = E^x_{ij} - E^y_{ji} + \delta_{i,j}\frac{p-q}{2} \;\middle|\; 1 \le i \le j \le n\right\},$

(ii) $X_{ij} = r^2_{ij} - \tilde{\Delta}_{ij}, \qquad 1 \le i \le j \le n,$

(iii) $Y_{ij} = \tilde{r}^2_{ij} - \Delta_{ij}, \qquad 1 \le i \le j \le n.$

## §3. The Dual Pairs $(O(p,q), Sp(2n,\mathbb{R}))$, for min $(p,q) > 2n$

Let $\Omega(\mathbb{1}) = \mathcal{P}/\mathcal{N}_{\mathbb{1}}$ be the Howe quotient corresponding to the trivial representation $\mathbb{1}$ of $Sp(2n,\mathbb{R})$. It is not difficult to see that

(3.1) $$\mathcal{N}_{\mathbb{1}} = \{Xf \mid X \in \mathfrak{sp}(2n,\mathbb{R})_{\mathbb{C}}, f \in \mathcal{P}\}.$$

Our next result provides a description of the $O(p) \times O(q)$ structure of this quotient space. Recall that the space of pluri-harmonics is simply $\mathcal{H}_1 \otimes \mathcal{H}_2$ (see Theorem 2.2).

**Proposition 3.1.** *Assume that* $\min(p,q) > 2n$. *A set of representatives for* $O(p) \times O(q)$ *highest weight vectors in* $\mathcal{P}/\mathcal{N}_{\mathbb{1}}$ *could be chosen from the set of* $O(p) \times O(q)$ *highest weight vectors in the space of pluri-harmonics.*

*Proof.* We shall first note the following formula, which follows from a straightforward computation:

$$[\Delta_{ij}, r^2_{st}] = \delta_{i,t}E^x_{sj} + \delta_{j,s}E^x_{ti} + \delta_{i,s}E^x_{tj} + \delta_{j,t}E^x_{si} + p(\delta_{i,t}\delta_{j,s} + \delta_{i,s}\delta_{j,t})$$

Define

$$R_{\tilde{s}}\tilde{R}_{\tilde{t}} = \Pi_{1\le i\le j\le n}(r^2_{ij})^{s_{ij}}(\tilde{r}^2_{ij})^{t_{ij}}$$

where $\tilde{s}$ and $\tilde{t}$ are $\frac{n(n+1)}{2}$-tuples of non-negative integers $(s_{ij})$ and $(t_{ij})$ respectively. Let $\psi$ be any pluri-harmonic polynomial and consider the polynomial $R_{\tilde{s}}\tilde{R}_{\tilde{t}}\psi$. Since $X_{ij}R_{\tilde{s}}\tilde{R}_{\tilde{t}}\psi \in \mathcal{N}_{\mathbb{1}}$ and $Y_{ij}R_{\tilde{s}}\tilde{R}_{\tilde{t}}\psi \in \mathcal{N}_{\mathbb{1}}$ (see (2.4) for the definition of $X_{ij}$ and $Y_{ij}$), application of the above formula shows that $R_{\tilde{s}}\tilde{R}_{\tilde{t}}\psi$ is expressible as a linear combination of

$$R_{\tilde{s}'}\tilde{R}_{\tilde{t}'}\psi'$$

where $\sum_{i,j} s'_{ij} = \sum_{i,j} s_{ij} - 1$, $\sum_{i,j} t'_{ij} = \sum_{i,j} t_{ij} - 1$ and $\psi'$ is another pluri-harmonic polynomial. We conclude that

$$R_{\tilde{s}} \tilde{R}_{\tilde{t}} \psi \equiv 0 \mod \mathcal{N}_{\mathbb{1}} \qquad \text{if} \qquad \sum_{i,j} s_{ij} \neq \sum_{i,j} t_{ij}.$$

If $\sum_{i,j} s_{ij} = \sum_{i,j} t_{ij}$, then in the quotient space $\mathcal{P}/\mathcal{N}_{\mathbb{1}}$, $R_{\tilde{s}} \tilde{R}_{\tilde{t}} \psi$ is expressible as a linear combination of the following polynomials

$$\{r_{ij}^2 \tilde{r}_{kl}^2 \psi_{i,j,k,l} \mid 1 \leq i \leq j \leq n, 1 \leq k \leq l \leq n, \psi_{i,j,k,l} \in \mathcal{H}\}.$$

These can again be simplified, i.e., expressed as elements in $\mathcal{H}_1 \otimes \mathcal{H}_2$. For instance, take

$$\begin{aligned} r_{11}^2 \tilde{r}_{11}^2 \psi &\equiv [\tilde{\Delta}_{11}, \tilde{r}_{11}^2]\psi \equiv (4E_{11}^y + 2q)\psi \mod \mathcal{N}_{\mathbb{1}} \\ &\equiv [\Delta_{11}, r_{11}^2]\psi \equiv (4E_{11}^x + 2p)\psi \mod \mathcal{N}_{\mathbb{1}}. \end{aligned} \tag{3.2}$$

Thus $r_{11}^2 \tilde{r}_{11}^2 \psi$ is congruent to an element in the space of pluri-harmonics. This completes the proof of the proposition. ∎

**Corollary 3.2.** *Assume that* $\min(p, q) > 2n$. *The trivial representation* $\mathbb{1}$ *of* $Sp(2n, \mathbb{R})$ *does not belong to* $\mathcal{R}(\mathfrak{sp}(2n, \mathbb{R}), \widetilde{U}(n), \omega)$ *unless* $\frac{p-q}{2} \in \mathbb{Z}$.

*Proof.* For $\mathbb{1} \in \mathcal{R}(\mathfrak{sp}(2n, \mathbb{R}), \widetilde{U}(n), \omega)$, there must be some pluri-harmonic $\psi \not\equiv 0 \mod \mathcal{N}_{\mathbb{1}}$. Choose one such $\psi$ which is homogeneous in the $x$ and $y$ coordinates. From (3.2), we observe that

$$0 \equiv (4E_{11}^x - 4E_{11}^y + 2p - 2q)\psi \equiv (4\deg_x \psi - 4\deg_y \psi + 2p - 2q)\psi \mod \mathcal{N}_{\mathbb{1}},$$

where $\deg_x \psi$ and $\deg_y \psi$ are the degrees of $\psi$ in the $x$ and $y$ coordinates respectively. Since $\deg_x \psi$ and $\deg_y \psi$ are integers, we conclude that $\frac{p-q}{2} \in \mathbb{Z}$. ∎

From here on, we shall assume that $\frac{p-q}{2} \in \mathbb{Z}$ and without loss of generality, we shall also assume that $p \geq q$. Our next task is to extract the $O(p) \times O(q) \times U(n)$ highest weight vectors in $\mathcal{P}/\mathcal{N}_{\mathbb{1}}$. Let $S_i$ be the permutation group on $i$ symbols and for a permutation $\sigma$, denote its signature by sgn $\sigma$. Define for $i, j = 1, \ldots, n$,

$$D_1(i; j) = \sum_{s=1}^{n} z_{si} \bar{w}_{sj},$$

$$D_i = \sum_{\sigma \in S_i} (-1)^{\mathrm{sgn}\,\sigma} D_1(1;\sigma(1)) D_1(2;\sigma(2)) \ldots D_1(i;\sigma(i)), i \neq n$$

(3.3) $D_n = C_n$ (see (2.3))

$$\tilde{C}_i = \begin{vmatrix} \bar{w}_{1,1} & \bar{w}_{1,2} & \cdots & \bar{w}_{1,i} \\ \bar{w}_{2,1} & \bar{w}_{2,2} & \cdots & \bar{w}_{2,i} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{w}_{i,1} & \bar{w}_{i,2} & \cdots & \bar{w}_{i,i} \end{vmatrix}, \text{ and } \tilde{D}_n = \tilde{C}_n \qquad i = 1, \ldots, n,$$

The last theorem has enable us to look at the pluri-harmonics in the study of $\mathcal{P}/\mathcal{N}_{\mathbb{1}}$. Basically, we have to look at the $O(p) \times O(q)$ highest weight vectors which are also $U(n)$-invariants in $\mathcal{H}$. The following theorem describes the situation for $n = 2$. Although one could ignore this result and proceed straightaway to the general case, the computations nevertheless are interesting and provide the $U(n)$-invariant polynomial $D$ (for $n = 2$).

**Proposition 3.3.** *Assume that $n = 2$, $p \geq q > 4$ and $p + q$ is even. A set of $O(p) \times O(q)$ highest weight vectors in $\mathcal{P}/\mathcal{N}_{\mathbb{1}}$ could be chosen* **from**

$$T_2 = \{D_1^{\xi_1} D_2^{\xi_2} \tilde{D}_2^{\xi_2 + \frac{p-q}{2}} \mid \xi_1, \xi_2 \in \mathbb{N}\}.$$

**Remark.** *The $SO(p) \times SO(q)$ weight of $D_1^{\xi_1} D_2^{\xi_2} \tilde{D}_2^{\xi_2 + \frac{p-q}{2}}$ is*

$$(\xi_1 + \xi_2, \xi_2, 0, \ldots, 0) \otimes (\xi_1 + \xi_2 + \frac{p-q}{2}, \xi_2 + \frac{p-q}{2}, 0, \ldots, 0).$$

*We have yet to show that elements in $T_2$ are non-zero. This will be done in Theorem 3.9. It is easy to see that two elements with distinct $SO(p) \times SO(q)$ weights in $T_2$ are not equal unless they are both in $\mathcal{N}_{\mathbb{1}}$.*

*Proof.* We shall adopt the same notations as in the proof of Proposition 3.1. Continuing from the computation in (3.2), we observe that if $\psi \not\equiv 0 \mod \mathcal{N}_{\mathbb{1}}$, we must have

$$(2E_{11}^x - 2E_{11}^y)\psi \equiv -(p-q)\psi \mod \mathcal{N}_{\mathbb{1}}.$$

We note the other 3 relations arising from similar considerations as above:

(a) $(2E_{22}^x - 2E_{22}^y)\psi \equiv -(p-q)\psi \mod \mathcal{N}_{\mathbb{1}}$

(b) $E_{12}^x \psi \equiv E_{21}^y \psi \mod \mathcal{N}_{\mathbb{1}}$

(c) $E_{21}^x \psi \equiv E_{12}^y \psi \mod \mathcal{N}_{\mathbb{1}}$

Basically, these relations boil down to the fact that we only need to look at the $U(2)$-invariant highest weight $O(p) \times O(q)$ vectors in $\mathcal{H}$.

Fix an $SO(p) \times SO(q)$ highest weight:

$$(\xi_1 + \xi_2, \xi_2, 0, \ldots, 0) \otimes (\eta_1 + \eta_2, \eta_2, 0, \ldots, 0). \tag{3.4}$$

For $i = 1, \ldots, \xi_1$ and $j = 1, \ldots, \eta_1$, we define (see (2.3) and (3.3) for the definitions of $D_2$ and $\tilde{D}_2$)

$$\phi_{ij} = z_{11}^i z_{21}^{\xi_1 - i} D_2^{\xi_2} \bar{w}_{11}^j \bar{w}_{21}^{\eta_1 - j} \tilde{D}_2^{\eta_2}.$$

Then an $O(p) \times O(q)$ highest weight vector in the space of pluri-harmonics of weight (3.4) is given by

$$v = \sum_{j=0,1,\ldots,\eta_1} \sum_{i=0,1,\ldots,\xi_1} \lambda_{ij} \phi_{ij}, \qquad \lambda_{ij} \in \mathbb{C}.$$

If such a vector lives non-trivially in $\mathcal{P}/\mathcal{N}_{\mathbb{1}}$, it must be an $U(2)$-invariant, i.e.,

$$(E_{12}^x - E_{21}^y)v = 0 \qquad \text{and} \qquad (E_{21}^x - E_{12}^y)v = 0.$$

We compute that

$$(a) \qquad (E_{12}^x - E_{21}^y)v = \sum_{i,j} \left\{ (\xi_1 - i + 1)\lambda_{i-1,j} - (j+1)\lambda_{i,j+1} \right\} \phi_{ij},$$

$$(b) \qquad (E_{21}^x - E_{12}^y)v = \sum_{i,j} \left\{ (i+1)\lambda_{i+1,j} - (\eta_1 - j + 1)\lambda_{i,j-1} \right\} \phi_{ij}.$$

This gives rise to two recursion relations:

$$\begin{aligned} &(a) \qquad j\lambda_{i,j} = (\xi_1 - i + 1)\lambda_{i-1,j-1} \\ &(b) \qquad i\lambda_{i,j} = (\eta_1 - j + 1)\lambda_{i-1,j-1} \end{aligned}$$

For a non-trivial solution to $v$, we must have $\xi_1 = \eta_1$. If we assume this, there is only one non-trivial solution (up to multiples) given by

$$\lambda_{i,j} = \begin{cases} \binom{\xi_1}{i} \lambda_{0,0} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

This translates to

$$v = \lambda_{0,0} D_1^{\xi_1} D_2^{\xi_2} \tilde{D}_2^{\eta_2}.$$

We also require

$$(E_{11}^x - E_{11}^y + \frac{p-q}{2})v = 0 = (E_{22}^x - E_{22}^y + \frac{p-q}{2})v$$

which forces $\eta_2 = \xi_2 + \frac{p-q}{2}$. This concludes the proof. ∎

**Proposition 3.4.** *Assume that $p \geq q > 2n$ and $p+q$ is even. A set of $O(p) \times O(q)$ highest weight vectors in $\mathcal{P}/\mathcal{N}_{\mathbb{1}}$ could be chosen* **from**

$$T_n = \left\{ \left(\Pi_{i=1}^{n-1} D_i^{\xi_i}\right) D_n^{\xi_n} \tilde{D}_n^{\xi_n + \frac{p-q}{2}} \;\middle|\; \xi_1, \ldots, \xi_n \in \mathbb{N} \right\}.$$

**Remark.** *See (2.3) and (3.3) for the definitions of $D_i$ and $\tilde{D}_n$. The $SO(p) \times SO(q)$ weight of $\left(\Pi_{i=1}^{n-1} D_i^{\xi_i}\right) D_n^{\xi_n} \tilde{D}_n^{\xi_n + \frac{p-q}{2}}$ is*

$$\begin{aligned}(3.5)\quad &(\xi_1 + \ldots + \xi_n, \xi_2 + \ldots + \xi_n, \ldots, \xi_{n-1} + \xi_n, \xi_n, 0, \ldots, 0) \otimes \\ &(\xi_1 + \ldots + \xi_n + \frac{p-q}{2}, \ldots, \xi_{n-1} + \xi_n + \frac{p-q}{2}, \xi_n + \frac{p-q}{2}, 0, \ldots, 0).\end{aligned}$$

*As in Proposition 3.3, we will show that elements in $T_n$ are non-zero in Theorem 3.9.*

*Proof.* Recall from Theorem 2.2(c) that $\phi = \Pi_{i=1}^n C_i^{\xi_i}$ and $\tilde{\phi} = \Pi_{i=1}^n \tilde{C}_i^{\eta_i}$ are $O(p) \times U(n)$ and $O(q) \times U(n)$ highest weight vectors in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Look at the tensor product of the $O(p) \times U(n)$ and $O(q) \times U(n)$ modules generated by $\phi$ and $\tilde{\phi}$ respectively. Our objective is to extract the $U(n)$-invariants, since these are the only ones which are possibly not killed in the quotient space $\mathcal{P}/\mathcal{N}_{\mathbb{1}}$. The respective $U(n)$ highest weights of $\phi$ and $\tilde{\phi}$ are

$$(\xi_1 + \ldots + \xi_n + \frac{p}{2}, \xi_2 + \ldots + \xi_n + \frac{p}{2}, \ldots, \xi_{n-1} + \xi_n + \frac{p}{2}, \xi_n + \frac{p}{2}) \text{ and}$$
$$(-\eta_n - \frac{q}{2}, -\eta_{n-1} - \eta_n - \frac{q}{2}, \ldots, -\eta_2 - \ldots - \eta_n - \frac{q}{2}, -\eta_1 - \ldots - \eta_n - \frac{q}{2}).$$

The tensor product of these two $U(n)$ modules yields an $U(n)$ invariant if and only if

$$\eta_i = \xi_i, \qquad i = 1, \ldots, n-1, \qquad \text{and} \qquad \eta_n = \xi_n + \frac{p-q}{2},$$

and there is at most one $U(n)$ invariant (up to scalars). The candidate in our situation is

$$\left(\Pi_{i=1}^{n-1} D_i^{\xi_i}\right) D_n^{\xi_n} \tilde{D}_n^{\xi_n + \frac{p-q}{2}}.$$

One could check immediately that it is an $O(p) \times O(q)$ highest weight vector which is also an $U(n)$ invariant of the correct weight if one observes that

$$D_i = \sum_{|\{s_1,\ldots,s_i\}|=i,\ 1\le s_1\le\ldots\le s_i\le n} C(s_1,\ldots,s_i)\tilde{C}(s_1,\ldots,s_i),$$

where

$$C(s_1,\ldots,s_i) = \begin{vmatrix} z_{s_1,1} & z_{s_1,2} & \cdots & z_{s_1,i} \\ z_{s_2,1} & z_{s_2,2} & \cdots & z_{s_2,i} \\ \vdots & \vdots & \cdots & \vdots \\ z_{s_i,1} & z_{s_i,2} & \cdots & z_{s_i,i} \end{vmatrix}, \quad \text{and}$$

$$\tilde{C}(s_1,\ldots,s_i) = \begin{vmatrix} \bar{w}_{s_1,1} & \bar{w}_{s_1,2} & \cdots & \bar{w}_{s_1,i} \\ \bar{w}_{s_2,1} & \bar{w}_{s_2,2} & \cdots & \bar{w}_{s_2,i} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{w}_{s_i,1} & \bar{w}_{s_i,2} & \cdots & \bar{w}_{s_i,i} \end{vmatrix}.$$

This concludes our proof. ∎

Next, we shall give a brief description of the $\mathfrak{so}(p)_{\mathbb{C}}\times\mathfrak{so}(q)_{\mathbb{C}}$ structure of $\mathfrak{p}$. Recall from (2.4) that

$$\mathfrak{o}(p,q)_{\mathbb{C}} = \mathfrak{o}(p)_{\mathbb{C}} \oplus \mathfrak{o}(q)_{\mathbb{C}} \oplus \mathfrak{p}$$

where $\mathfrak{p} \simeq \mathbb{C}^p \otimes \mathbb{C}^q$ as an $\mathfrak{so}(p)_{\mathbb{C}} \times \mathfrak{so}(q)_{\mathbb{C}}$ module with highest weight $(1,0,\ldots,0)\otimes(1,0,\ldots,0)$.

Identify $\mathfrak{p}$ with the space of $p\times q$ matrices over $\mathbb{C}$ and write a $p\times q$ matrix in blocks of $2\times 2$ matrices. If $p$ is odd, we allow $1\times 2$ matrices in the bottom row and if $q$ is odd, we allow $2\times 1$ blocks in the rightmost column. For $1\le |a|\le [\frac{p}{2}]$ and $1\le |b|\le [\frac{q}{2}]$, define the following matrices (see (2.2) for the definitions of $X_{\pm,\pm}$ and $X_{\pm}$):

$$(Z_{a,b})_{st} = \begin{cases} X_{\mathrm{sgn}(a),-\mathrm{sgn}(b)} & \text{if } s=|a| \text{ and } t=|b|, \\ 0 & \text{otherwise,} \end{cases}$$

$$(Z_{a,0})_{st} = \begin{cases} X_{\mathrm{sgn}(a)} & \text{if } s=|a| \text{ and } t=\frac{q+1}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

$$(Z_{0,b})_{st} = \begin{cases} X^t_{-\mathrm{sgn}(a)} & \text{if } s=\frac{p+1}{2} \text{ and } t=|b|, \\ 0 & \text{otherwise,} \end{cases}$$

The vectors denoted by $Z$'s are all weight vectors of $\mathfrak{p}$. We note that $Z_{a,b}$ is of weight $\mathrm{sgn}(a)e_{|a|}\otimes\mathrm{sgn}(b)f_{|b|}$, $Z_{a,0}$ is of weight $\mathrm{sgn}(a)e_{|a|}\otimes 0$ and $Z_{0,b}$ is of weight $0\otimes\mathrm{sgn}(b)f_{|b|}$.

The following lemma gives the actions of the weight vectors of $\mathfrak{p}$ on $\mathcal{P}$:

**Lemma 3.5.** *Let* $1 \leq a \leq [\frac{p}{2}]$ *and* $1 \leq b \leq [\frac{q}{2}]$. *The actions of the weight vectors of* $\mathfrak{p}$ *on* $\mathcal{P}$ *are given as follows:*

$$Z_{a,b} = \sum_{s=1}^{n} \left( z_{sa}\bar{w}_{sb} - 4\frac{\partial^2}{\partial \bar{z}_{sa}\partial w_{sb}} \right), \quad Z_{a,-b} = \sum_{s=1}^{n} \left( z_{sa}w_{sb} - 4\frac{\partial^2}{\partial \bar{z}_{sa}\partial \bar{w}_{sb}} \right),$$

$$Z_{-a,b} = \sum_{s=1}^{n} \left( \bar{z}_{sa}\bar{w}_{sb} - 4\frac{\partial^2}{\partial z_{sa}\partial w_{sb}} \right), \; Z_{-a,-b} = \sum_{s=1}^{n} \left( \bar{z}_{sa}w_{sb} - 4\frac{\partial^2}{\partial z_{sa}\partial \bar{w}_{sb}} \right),$$

$$Z_{a,0} = \sum_{s=1}^{n} \left( z_{sa}w_{s\frac{q+1}{2}} - 4\frac{\partial^2}{\partial \bar{z}_{sa}\partial w_{s\frac{q+1}{2}}} \right),$$

$$Z_{-a,0} = \sum_{s=1}^{n} \left( \bar{z}_{sa}w_{s\frac{q+1}{2}} - 4\frac{\partial^2}{\partial z_{sa}\partial w_{s\frac{q+1}{2}}} \right),$$

$$Z_{0,b} = \sum_{s=1}^{n} \left( z_{s\frac{p+1}{2}}\bar{w}_{sb} - 4\frac{\partial^2}{\partial z_{s\frac{p+1}{2}}\partial w_{sb}} \right),$$

$$Z_{0,-b} = \sum_{s=1}^{n} \left( z_{s\frac{p+1}{2}}w_{sb} - 4\frac{\partial^2}{\partial z_{s\frac{p+1}{2}}\partial \bar{w}_{sb}} \right).$$

*Proof.* Omitted. ∎

We would be interested in the action of $\mathfrak{o}(p,q)_{\mathbb{C}}$ on the set of vectors in $T_n$ (see Proposition 3.4). Algebraically, we should look at the tensor product

$$\begin{aligned} m : \mathfrak{p} \otimes V_{\xi_1,\ldots,\xi_n} &\longrightarrow \mathcal{P}^{U(n)} \\ m(X \otimes f) &= X(f) \end{aligned} \tag{3.6}$$

where $V_{\xi_1,\ldots,\xi_n}$ is a $O(p) \times O(q)$ module of highest $\mathfrak{so}(p) \times \mathfrak{so}(q)$ weight given by (3.5). The tensor product $\mathfrak{p} \otimes V_{\xi_1,\ldots,\xi_n}$ could possibly have any one of the $2^{2n}$ components with $\mathfrak{so}(p) \times \mathfrak{so}(q)$ highest weight as follows, and they can only appear with multiplicity at most one:

$$((\xi_1 + \ldots + \xi_n, \xi_2 + \ldots + \xi_n, \ldots, \xi_{n-1} + \xi_n, \xi_n, 0, \ldots, 0) \pm e_s) \otimes$$
$$((\xi_1 + \ldots + \xi_n + \frac{p-q}{2}, \ldots, \xi_{n-1} + \xi_n + \frac{p-q}{2}, \xi_n + \frac{p-q}{2}, 0, \ldots, 0) \pm f_t).$$

where $s, t \in \{1, \ldots, n\}$. We shall let $P_{\pm e_s, \pm f_t}$ denote the image of the above component under the map in (3.6). Note that $P_{\pm e_s, \pm f_t}$ is either zero or a highest weight $O(p) \times O(q)$ vector in $\mathcal{P}^{U(n)}$.

Recall the root system that we have described for $\mathfrak{so}(p)_{\mathbb{C}}$. Let

$$H_{ab} = H_a - H_b$$

and define

$$U_{jj} = 1, \qquad j = 1, \ldots, n.$$

Define for $1 \leq t < j \leq n$, the determinant of a $(j-t) \times (j-t)$ matrix of elements from the universal enveloping algebra of $\mathfrak{so}(p)_{\mathbb{C}}$ (see Lemma 2.1 for the relevant definitions),

$$U_{tj} = \begin{vmatrix} E_{-(j-1),j} & H_{j-1,j}+1 & 0 & \cdots & \cdots & 0 \\ E_{-(j-2),j} & E_{-(j-2),j-1} & H_{j-2,j-1}+2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ E_{-(t+1),j} & E_{-(t+1),j-1} & E_{-(t+1),j-2} & \cdots & \cdots & H_{t+1,j}+j-t-1 \\ E_{-t,j} & E_{-t,j-1} & E_{-t,j-2} & \cdots & \cdots & E_{-t,t+1} \end{vmatrix}.$$

Since we are dealing with a non-commutative algebra, care has to be taken to define the determinant. Recall that $S_k$ is the permutation group on $k$ symbols. We shall use the following definition of the determinant:

$$\begin{vmatrix} d_{11} & \cdots & d_{1k} \\ \vdots & \vdots & \vdots \\ d_{k1} & \cdots & d_{kk} \end{vmatrix} = \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \, d_{\sigma(1)1} \cdots d_{\sigma(k)k}$$

Thus, as an element in the universal enveloping algebra of $\mathfrak{so}(p)_{\mathbb{C}}$, $U_{tj} =$

$$\sum_{l=0}^{j-t-1} \sum_{I_l} (-1)^{j-t-l+1} E_{-i_l,j} E_{-i_{l-1},i_l} \cdots E_{-t,i_1} \Pi_{a \notin I_l, t+1 \leq a \leq j-1} (H_{aj}+j-a).$$

where $I_l = \{i_s \mid s = 1, \ldots, l, t+1 \leq i_1 < i_2 < \ldots < i_l \leq j-1\}$. Finally, define

$$H(j; a, b) = \Pi_{s=a}^{b} (H_s - H_j + j - s), \qquad 1 \leq a \leq b < j,$$
$$H^{-}(j; a, b) = \Pi_{s=a}^{b} (H_s - H_j + j - s - 1), \qquad 1 \leq a \leq b < j.$$

**Theorem 3.6.** *([HLe],[Le1]) Suppose $p \geq 2n$. Let $W_{\xi_1,\ldots,\xi_n}$ be an irreducible $SO(p)$ module of highest weight*

$$\lambda = (\xi_1 + \ldots + \xi_n, \xi_2 + \ldots + \xi_n, \ldots, \xi_{n-1} + \xi_n, \xi_n, 0, \ldots, 0).$$

*Let $u$ and $v$ denote the highest weight vectors in $\mathbb{C}^p$ and $W_{\xi_1,\ldots,\xi_n}$ respectively. The following operators from $\mathcal{U}(\mathfrak{so}(p))$ gives the projection of $u \otimes v$ to each of the possible $n$ highest weights $\lambda + e_j$, $j = 1, \ldots, n$, (in the positive direction) of irreducible $SO(p)$ components:*

(1) $Q_{e_1} = 1 \otimes 1$

(2) $Q_{e_j} = 1 \otimes U_{1j} + \sum_{t=2}^{j} (-1)^{t+1} (E_{-1,t} \otimes U_{tj} H^-(j; 1, t-1)),\ j = 2, \ldots, n$

**Remark.** *It would be nice to have the formulae for the transition in the negative directions, even for $Q_{-e_1}$ and $Q_{-e_2}$.*

*Proof.* These formulae are all in [Le1]. They are the projections if the modules were $\mathfrak{gl}([\frac{p}{2}])_{\mathbb{C}}$ modules. Recall that there is an embedding of $\mathfrak{gl}([\frac{p}{2}])_{\mathbb{C}}$ in $\mathfrak{so}(p)_{\mathbb{C}}$. It therefore suffices to check that $Q_{e_i}(u \otimes v)$s are killed by another positive simple root. The tedious computations are essentially done in [HLe]. ∎

Let

$$\psi^{\xi} = \psi^{(\xi_1,\ldots,\xi_n)} = \left(\Pi_{i=1}^{n-1} D_i^{\xi_i}\right) D_n^{\xi_n} \tilde{D}_n^{\xi_n + \frac{p-q}{2}} \tag{3.7}$$

With the above theorem, we note that (see Lemma 3.5 for the definition of $Z_{1,1}$)

$$P_{\pm e_s, \pm f_t} = m((Q_{\pm e_s} Q_{\pm f_t})(Z_{1,1} \otimes \psi^{\xi})). \tag{3.8}$$

Define the constants,

$$\begin{aligned} \lambda(j; a, b)\psi^{\xi} &= H(j; a, b)\psi^{\xi} \\ &= \Pi_{s=a}^{b} (\xi_s + \ldots + \xi_{j-1} + j - s)\psi^{\xi} \end{aligned} \tag{3.9}$$

(3.10)

$$\begin{aligned} \lambda^-(j; a, b)\psi^{\xi} &= H^-(j; a, b)\psi^{\xi} \\ &= \Pi_{s=a}^{b} (\xi_s + \ldots + \xi_{j-1} + j - s - 1)\psi^{\xi} \end{aligned}$$

With the formulae for $Q_{e_s}$ and $Q_{f_t}$, (3.8) simplifies as follows:

$$\begin{aligned}
&P_{e_i,f_j}\\
&=\sum_{s=1}^{i}\sum_{t=1}^{j}(-1)^{i+j+s+t}m\,(E_{-1,s}F_{-1,t}Z_{1,1}\\
&\qquad\qquad\otimes\lambda^-(i,1,s-1)\lambda^-(j,1,t-1)U_{si}\tilde{U}_{tj}\psi^{\xi}\Big),\\
&=\sum_{s=1}^{i}\sum_{t=1}^{j}(-1)^{i+j+s+t}m\left(Z_{s,t}\otimes\lambda^-(i;1,s-1)\lambda^-(j;1,t-1)U_{si}\tilde{U}_{tj}\psi^{\xi}\right),\\
&=\sum_{s=1}^{i}\sum_{t=1}^{j}(-1)^{i+j+s+t}\lambda^-(i;1,s-1)\lambda^-(j;1,t-1)D_1(s;t)U_{si}\tilde{U}_{tj}\psi^{\xi}.
\end{aligned}$$

We will only be interested in the case when $i=j$.

**Proposition 3.7.** *Take* $E_{-i,i}=F_{-i,i}=-1$.

*(a)* $\sum_{t=1}^{i}D_1(s;t)(F_{-ti}D_{i-1})=-\delta_{si}D_i,\qquad s=1,\dots,i;$

*(b)* $U_{si}=\sum_{a=s+1}^{i}(-1)^{a+s+1}E_{-sa}U_{ai}H^-(i;s+1,a-1),\qquad s=1,\dots,i-1;$

*(c)* $U_{si}\psi^{\xi}=(-1)^{i+s+1}\lambda^-(i;s,i-1)(E_{-si}D_{i-1})\psi^{\xi-e_{i-1}},\quad s=1,\dots,i-1;$

*(d) Suppose $f$ is a polynomial with*

$$H^-(i;s+1,a-1)f=\lambda^-(i;s+1,a-1)f,$$

*then for* $s=1,\dots,i-1$,

$$\begin{aligned}
&U_{si}(D_1(s;t)f)\\
&=\sum_{a=s+1}^{i}(-1)^{a+s+1}\lambda^-(i;s+1,a-1)D_1(a;t)U_{ai}f+D_1(s;t)U_{si}f.
\end{aligned}$$

**Proof.** Let us abbreviate

$$\begin{aligned}
&D_{i-1}(1,\dots,i-1;k_1,\dots,k_{i-1})\\
&=\begin{vmatrix}D_1(1;k_1)&D_1(1;k_2)&\dots&D_1(1;k_{i-1})\\ \vdots&\vdots&\vdots&\vdots\\ D_1(i-1;k_1)&D_1(i-1;k_2)&\dots&D_1(i-1;k_{i-1})\end{vmatrix}.
\end{aligned}$$

$$
\begin{aligned}
\text{(a)} \sum_{t=1}^{i} & D_1(s;t)(F_{-ti}D_{i-1}) \\
&= \sum_{t=1}^{i-1} D_1(s;t)D_{i-1}(1,\ldots,i-1;1,\ldots,t-1,i,t+1,\ldots,i-1) \\
&\qquad\qquad - D_1(s;i)D_{i-1} \\
&= \sum_{t=1}^{i-1}(-1)^{i-t-1}D_1(s;t)D_{i-1}(1,\ldots,i-1;1,\ldots,t-1,t+1,\ldots,i-1,i) \\
&\qquad\qquad - D_1(s;i)D_{i-1} \\
&= (-1)^i \begin{vmatrix} D_1(s;1) & D_1(s;2) & \ldots & D_1(s;i) \\ D_1(1;1) & D_1(1;2) & \ldots & D_1(1;i) \\ \vdots & \vdots & \vdots & \vdots \\ D_1(i-1;1) & D_1(i-1;2) & \ldots & D_1(i-1;i) \end{vmatrix} \\
&= -\delta_{si}D_i.
\end{aligned}
$$

We refer the reader to Lemma 4.4 of [Le1] for the proof of (b) and Proposition 4.8 of [Le1] for the proof of (c). The proof of statement (d) which uses (b), is as follows:

$$
\begin{aligned}
& U_{si}(D_1(s;t)f) \\
&= \sum_{a=s+1}^{i} (-1)^{a+s+1}\lambda^-(i;s+1,a-1)E_{-sa}U_{ai}(D_1(s;t)f)
\end{aligned}
$$

(since $D_1(s;t)$ does not contribute to $H^-(i;s+1,a-1)$)

$$
= \sum_{a=s+1}^{i} (-1)^{a+s+1}\lambda^-(i;s+1,a-1)E_{-sa}D_1(s;t)U_{ai}f
$$

(since $U_{ai}$ kills $D_1(s;t)$ for $a = s+1,\ldots,i$)

$$
= \sum_{a=s+1}^{i} (-1)^{a+s+1}\lambda^-(i;s+1,a-1)D_1(a;t)U_{ai}f + D_1(s;t)U_{si}f. \quad \blacksquare
$$

For convenience, we shall also use the notation

$$
\psi^{\xi\pm e_s} = \psi^{(\xi_1,\ldots,\xi_{s-1},\xi_s\pm 1,\xi_{s+1},\ldots,\xi_n)}.
$$

**Theorem 3.8.** *Assume that $p \geq q > 2n$ and $p+q$ is even. Let $V_{\xi_1,\ldots,\xi_n}$ be the $O(p)\times O(q)$ module generated by $\psi^\xi$ (see (3.7)). Then projections in the "positive directions" of the image of $\psi^\xi$ (in $\mathcal{P}/\mathcal{N}_{\mathbb{1}}$) to various irreducible $O(p)\times O(q)$ components under the map $m$ in (3.6) are as follows:*

$$\text{(1)} \qquad P_{e_1,f_1} = \psi^{\xi+e_1}$$

$$\text{(2)} \qquad P_{e_j,f_j} = \lambda(j;1,j-1)\lambda^-(j;1,j-1)\psi^{\xi-e_{j-1}+e_j},\ j=2,\ldots,n.$$

*Proof.* Using (c) of Proposition 3.7, we have

$$\begin{aligned}
P_{e_j,f_j}
&= \sum_{s=1}^{j}\sum_{t=1}^{j}(-1)^{s+t}\lambda^-(j;1,s-1)\lambda^-(j;1,t-1)D_1(s;t)U_{sj}\tilde{U}_{tj}\psi^\xi \\
&= \sum_{s=1}^{j}\sum_{t=1}^{j}(-1)^{s+j+1}\lambda^-(j,1,s-1)\lambda^-(j;1,t-1)\lambda^-(j;t,j-1) \\
&\qquad\qquad D_1(s;t)U_{sj}\{(F_{-tj}D_{j-1})\psi^{\xi-e_{j-1}}\} \\
&= \sum_{s=1}^{j}(-1)^{s+j+1}\lambda^-(j;1,s-1)\lambda^-(j;1,j-1)\times \\
&\qquad\qquad \sum_{t=1}^{j}D_1(s;t)U_{sj}\{(F_{-tj}D_{j-1})\psi^{\xi-e_{j-1}}\} \\
&\qquad\qquad (\text{since } \lambda^-(j;1,j-1)=\lambda^-(j;1,t-1)\lambda^-(j;t,j-1)) \\
&= \sum_{s=1}^{j}(-1)^{s+j+1}\lambda^-(j;1,s-1)\lambda^-(j;1,j-1)\psi_s
\end{aligned}$$

where $\psi_s$ (which is dependent on $j$) is defined as follows:

$$\psi_s = \sum_{t=1}^{j}D_1(s;t)U_{sj}\{(F_{-tj}D_{j-1})\psi^{\xi-e_{j-1}}\}.$$

Using Proposition 3.7 (a) and (d), one can show that for $s=1,\ldots,j-1$,

$$\psi_s = \sum_{a=s+1}^{j}(-1)^{a+s}\lambda^-(j;s+1,a-1)\psi_a.$$

By induction, it follows that for $s = 1, \ldots, j-2$,

$$\begin{aligned}\psi_s &= (-1)^{j+s+1}\left(\Pi_{k=s+1}^{j-1}\lambda^-(j;k,k)\right)\psi^{\xi-e_{j-1}+e_j}\\ &= (-1)^{j+s+1}\lambda(j;s+1,j-1)\psi^{\xi-e_{j-1}+e_j}.\end{aligned}$$

Another induction gives the formula for $P_{e_j,f_j}$. ∎

**Theorem 3.9.** *Assume that $p \geq q > 2n$ and $p+q$ even. A set of $O(p)\times O(q)$ highest weight vectors in $\mathcal{P}/\mathcal{N}_{\mathbb{1}}$ could be chosen as*

$$T_n = \left\{\psi^\xi = \left(\Pi_{i=1}^{n-1}D_i^{\xi_i}\right)D_n^{\xi_n}\tilde{D}_n^{\xi_n+\frac{p-q}{2}} \;\middle|\; \xi_1,\ldots,\xi_n \in \mathbb{N}\right\}.$$

*The $SO(p)\times SO(q)$ weight of $\left(\Pi_{i=1}^{n-1}D_i^{\xi_i}\right)D_n^{\xi_n}\tilde{D}_n^{\xi_n+\frac{p-q}{2}}$ is*

$$(\xi_1+\ldots+\xi_n, \xi_2+\ldots+\xi_n, \ldots, \xi_{n-1}+\xi_n, \xi_n, 0, \ldots, 0)\otimes$$
$$(\xi_1+\ldots+\xi_n+\frac{p-q}{2}, \ldots, \xi_{n-1}+\xi_n+\frac{p-q}{2}, \xi_n+\frac{p-q}{2}, 0, \ldots, 0).$$

*The Howe quotient $\mathcal{P}/\mathcal{N}_{\mathbb{1}}$ is irreducible and $\theta(\mathbb{1};p,q)$ is the $(\mathfrak{o}(p,q), \widetilde{O}(p)\times \widetilde{O}(q))$ module $M_{p,q,n}$ generated by the vectors $\psi^\xi$.*

**Remark.** *The $(\mathfrak{o}(p,q), \widetilde{O}(p)\times\widetilde{O}(q))$ module $M_{p,q,n}$ (for $p+q$ even) have Gelfand-Kirillov dimension $n(p+q-2n-1)$ (see [ZH]).*

*Proof.* Everything except the irreducibility under $O(p,q)$ has been shown in Proposition 3.4. To get the irreduciblity, we note that the lowest joint harmonic (which sits in the unique irreducible quotient of $M_{p,q,n}$) is just the constant polynomial. The formulae in Theorem 3.8 shows that we can move from the constant polynomial (trivial $O(p)\times O(q)$-type) to any other $\widetilde{O}(p)\times\widetilde{O}(q)$-type in $M_{p,q,n}$. Hence, $M_{p,q,n}$ must be irreducible. ∎

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