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On $\wedge\mathfrak{g}$ for a Semisimple Lie Algebra $\mathfrak{g}$, as an Equivariant Module over the Symmetric Algebra $S(\mathfrak{g})$

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§1. Introduction

1.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathcal{C}$ be the set of all commutative Lie subalgebras $\mathfrak{a}$ of $\mathfrak{g}$. If $\mathfrak{a} \in \mathcal{C}$ and $k = dim\,\mathfrak{a}$ let $[\mathfrak{a}] = \wedge^k\mathfrak{a}$. Regard $[\mathfrak{a}]$ as a 1-dimensional subspace of $\wedge^k\mathfrak{g}$ and let $C \subset \wedge\mathfrak{g}$ be the span of all $[\mathfrak{a}]$ for all $\mathfrak{a} \in \mathcal{C}$. The exterior algebra $\wedge\mathfrak{g}$ is a $\mathfrak{g}$-module with respect to the extension, θ, of the adjoint representation, defined so that $\theta(x)$ is a derivation for any $x \in \mathfrak{g}$. It is obvious that $C = \sum_{k=1}^{n} C^k$ is a graded $\mathfrak{g}$-submodule of $\wedge\mathfrak{g}$. Of course $C^k = 0$ for $k > n_{abel}$ where n_{abel} is the maximal dimension of an abelian Lie subalgebra of $\mathfrak{g}$. The paper [4] initiated a study of the $\mathfrak{g}$-module C. It was motivated by a result of Malcev giving the value of n_{abel} for all complex simple Lie subalgebras. For example, for the exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8, the value of n_{abel}, respectively, is $3, 9, 16, 27$ and 36. See [10].

One of the results in [4] is that C (denoted by A in [4]) is a multiplicity free $\mathfrak{g}$-module. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. If Ξ is an index set for the set of all abelian ideals $\{\mathfrak{a}_\xi\}$, $\xi \in \Xi$, of $\mathfrak{b}$, then the irreducible components of C may also be indexed by Ξ. The irreducible components, written as C_ξ, $\xi \in \Xi$, are characterized by the property that $[\mathfrak{a}_\xi]$ is the highest weight space of C_ξ. One therefore has the unique decomposition

$$C = \sum_{\xi \in \Xi} C_\xi$$

into irreducible components. Sometime after [4] was published, Dale Peterson established the striking result that the cardinality of Ξ was 2^l. His ingenious proof, using the affine Weyl group, sets up a natural bijection between Ξ and the set of elements of order 2 (and the identity) in a maximal torus of a simply-connected Lie group G with Lie

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algebra $\mathfrak{g}$. An outline of Peterson's theory is given in [8]. Peterson's result suggested to us that there should be some interesting connection between the set of abelian ideals $\{\mathfrak{a}_\xi\}$ of $\mathfrak{b}$ and the theory of symmetric spaces of inner type (i.e. where the corresponding Cartan involution is an inner automorphism). By Harish-Chandra theory, the corresponding inner real forms $G_{\mathbb{R}}$ of G are exactly the real forms which admit discrete series representations. In fact we have obtained results giving a construction of the abelian ideals $\mathfrak{a}_\xi$ in terms of the Cartan decompositions corresponding to such real forms. In addition we have set up natural bijections between the families of discrete series for such groups and the 2^l-element set $\{\mathfrak{a}_\xi\}$ of abelian ideals in $\mathfrak{b}$. In fact, using W. Schmid's construction of the discrete series (see [11]), we establish a direct connection between, on the one hand, minimal "K-types" and the cohomological degree in which the discrete series appears and, on the other hand, the dimension of the corresponding abelian ideal $\{\mathfrak{a}_\xi\}$ and the highest weight of C_ξ.

A summary of the above results (for $\mathfrak{g}$ simple) will appear in [8]. Another result, stated as Theorem 1.5 in [8], is a theorem on the role C plays in the full structure of $\wedge\mathfrak{g}$. The present paper is an elaboration and proof of this result.

In more detail let $B_\mathfrak{g}$ be the Killing form on $\mathfrak{g}$ and let $B_{\wedge\mathfrak{g}}$ be its natural extension to $\wedge\mathfrak{g}$. Identify $\mathfrak{g}$ with its dual space $\mathfrak{g}^*$ so that $\wedge\mathfrak{g}$ has the structure of a cochain complex with respect to the usual, degree 1, Lie algebra coboundary operator. The coboundary operator is denoted by d. In particular $d\mathfrak{g} \subset \wedge^2\mathfrak{g}$. The subspace $d\mathfrak{g}$ is a $\mathfrak{g}$-submodule and, as such, is equivalent to $\mathfrak{g}$ itself. For any $u \in \wedge\mathfrak{g}$, let $\iota(u)$ be the operator on $\wedge\mathfrak{g}$ of interior product by u. Let $\mathcal{A}$ be the ideal in $\wedge\mathfrak{g}$ generated by the subspace $d\mathfrak{g}$. One of the main results in the present paper is the following completely different characterization of the submodule $C \subset \wedge\mathfrak{g}$.

Theorem A. *One has*

$$C = \{u \in \wedge\mathfrak{g} \mid \iota(dx)u = 0,\ \forall x \in \mathfrak{g}\}$$

Moreover $B_{\wedge\mathfrak{g}}$ is non-singular on C and

$$\wedge\mathfrak{g} = \mathcal{A} \oplus C$$

is a $B_{\wedge\mathfrak{g}}$-orthogonal direct sum.

Fix a non-zero element $\mu \in \wedge^n\mathfrak{g}$. For any $v \in \wedge\mathfrak{g}$, let $\widetilde{v} = \iota(v)\mu$ and $\widetilde{C} = \{\widetilde{v} \mid v \in C\}$. It is immediate that $C \to \widetilde{C}$, $v \mapsto \widetilde{v}$ is a $\mathfrak{g}$-module isomorphism. An easy consequence of Theorem A is

Theorem B. *One has*

$$\widetilde{C} = \{v \in \wedge\mathfrak{g} \mid dx \wedge v = 0, \forall x \in \mathfrak{g}\}$$

1.2. We will express Theorems A and B in a "functorial" way. Consider the symmetric algebra $S(\mathfrak{g})$ over $\mathfrak{g}$. Since the elements of $\wedge^2\mathfrak{g}$ commute with each other, there exists a unique homomorphism

$$s : S(\mathfrak{g}) \to \wedge\mathfrak{g}$$

where $s(x) = dx$ for $x \in \mathfrak{g}$. The homomorphism s of course defines the structure of an $S(\mathfrak{g})$ module on $\wedge\mathfrak{g}$. Furthermore since s is a $\mathfrak{g}$-map with respect to the adjoint action, this $S(\mathfrak{g})$-module structure is equivariant with respect to the adjoint action.

The homomorphism s arises in a number of contexts. For example, if K is a compact Lie group corresponding to the compact form $\mathfrak{k}$ of $\mathfrak{g}$ and P is a principal K-bundle, with connection, then s arises from Chern-Weil theory if one considers the fiber instead of the base. Along these lines the map s is the main tool used in Chevalley's well known construction of the "transgression" map, of invariants, $S(\mathfrak{g})^{\mathfrak{g}} \to (\wedge\mathfrak{g})^{\mathfrak{g}}$. See e.g. [2] and in more detail §6 in [7]. The map s also plays a key role in the Lie algebra generalization of the Amitsur-Levitski theorem as formulated in [6].

The functors $Ext^j_{S(\mathfrak{g})}(\mathbb{C}, \wedge\mathfrak{g})$ clearly have the structure of $\mathfrak{g}$-modules. Considering only the two extreme values of j, one has $\mathfrak{g}$-module maps

$$Ext^0_{S(\mathfrak{g})}(\mathbb{C}, \wedge\mathfrak{g}) \to \wedge\mathfrak{g} \tag{a}$$

and

$$\wedge\mathfrak{g} \to Ext^n_{S(\mathfrak{g})}(\mathbb{C}, \wedge\mathfrak{g}) \tag{b}$$

Recalling the definitions of *Ext* at these two extremes, Theorems A and B immediately translate to

Theorem C. *The map (a) defines a $\mathfrak{g}$-module isomorphism*

$$Ext^0_{S(\mathfrak{g})}(\mathbb{C}, \wedge\mathfrak{g}) \to \widetilde{C}$$

and the map (b) restricts to a $\mathfrak{g}$-module isomorphism

$$C \to Ext^n_{S(\mathfrak{g})}(\mathbb{C}, \wedge\mathfrak{g})$$

1.3. An element $u \in \wedge\mathfrak{g}$ is called totally exact if it is the sum of products of elements of the form dx, $x \in \mathfrak{g}$. Let A be the image of s so that A is the algebra of all totally exact elements in $\wedge\mathfrak{g}$. See Theorem 1.4 in [6] for a characterization of A. Some features of the $\mathfrak{g}$-module structure of A were studied and used in [7]. See Theorem 69 in [7]. Of course the $S(\mathfrak{g})$-module structure on $\wedge\mathfrak{g}$ can be regarded as defining an A-module structure on $\wedge\mathfrak{g}$. Consider the question of determining generators for this module. A subspace $C_o \subset \wedge\mathfrak{g}$ will be said to be A-generating if C_o is a graded $\mathfrak{g}$-submodule of $\wedge\mathfrak{g}$ such that $\wedge\mathfrak{g} = A \wedge C_o$.

Theorem D. *The subspace C is A-generating so that*

$$\wedge g = A \wedge C$$

Moreover it is minimal among all A-generating subspaces in $\wedge\mathfrak{g}$. In fact if C_o is any graded $\mathfrak{g}$-submodule of $\wedge\mathfrak{g}$, then C_o is A-generating if and only if $C \subset C_o$.

Note that Theorem D implies that the set of elements of the form $y_1 \wedge \cdots \wedge y_k \wedge dx_1 \wedge \cdots \wedge dx_m$ spans $\wedge\mathfrak{g}$, where $x_i, y_j \in \mathfrak{g}$ and the $\{y_j\}$ pairwise commute.

§2. V_ρ and the "spin" of the adjoint representation

2.1. Let V be a complex finite dimensional vector space endowed with some fixed non-singular symmetric bilinear form B_V. Let $n = dim\, V$. The bilinear form B_V extends to a non-singular symmetric bilinear form $B_{\wedge V}$ on the exterior algebra $\wedge V$ where $\wedge^p V$ is orthogonal to $\wedge^q V$ for $p \neq q$ and for $x_i, y_j \in V$, $i, j = 1, \ldots, k$,

$$(x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k) = det\,(x_i, y_j)$$

where (u, v) denotes the value of the $B_{\wedge V}$ on $u, v \in \wedge V$. For any $u \in \wedge V$ let $\epsilon(u) \in End \wedge V$ be the operator of left exterior multiplication by u and let $\iota(u) \in End \wedge V$ be the transpose of $\epsilon(u)$ with respect to $B_{\wedge V}$. Regarding $\wedge V$ as a $\mathbb{Z}$-graded super commutative associative algebra, let $Der \wedge V = \sum_{j=-1}^{n-1} Der^j \wedge V$ be the $\mathbb{Z}$-graded super Lie algebra of all super derivations of $\wedge V$. If $y \in V$ one has $\iota(y) \in Der^{-1} \wedge V$ and if also $x \in V$ then

$$\epsilon(x)\iota(y) + \iota(y)\epsilon(x) = (x, y)I \tag{2.1}$$

where I is the identity operator on $\wedge V$.

Let $Lie\, SO(V) \subset End\, V$ be the Lie algebra of all skew-symmetric operators on V with respect to B_V. One defines a linear isomorphism $\tau : \wedge^2 V \to Lie\, SO(V)$ so that if $\omega \in \wedge^2 V$ and $x \in V$, then $\tau(\omega)x = -2\iota(x)\omega$. (See §2.3 in [7].) The introduction of the factor -2 is motivated by Clifford algebra considerations.) Let $\omega \in \wedge^2 V$ be arbitrary. Proposition 2.1 below gives a formula for the commutator $[\epsilon(\omega), \iota(\omega)]$. In the special case where V is the complexified tangent space at a point p of a Kahler manifold and ω is the Kahler form at p one knows, e.g. from Hodge theory, that the Lie algebra generated by $\epsilon(\omega)$ and $\iota(\omega)$ is isomorphic to $Lie\, Sl(2,\mathbb{C})$. See Chapter 1 in Weil's book [12] for formulas involving the action of this Lie algebra on $\wedge V$. See [9] for other recent results in this area.

Returning to the general case, for any $\alpha \in End\, V$ let D_α be the unique element in $Der^0 \wedge V$ such that $D_\alpha | V = \alpha$.

Proposition 2.1. *Let $\omega \in \wedge^2 V$. Let $\alpha = -\frac{1}{4}\tau(\omega)^2$. Then*

$$[\epsilon(\omega), \iota(\omega)] = D_\alpha - \frac{tr\,\alpha}{2} I \tag{2.2}$$

Proof. We may assume $n \geq 2$. Let $x, y \in V$ be such that $(x,x) = (y,y) = 1$ and $(x,y) = 0$. It follows immediately from (2.1) that

$$[\epsilon(x \wedge y), \iota(x \wedge y)] = \epsilon(x)\iota(x) + \epsilon(y)\iota(y) - I$$

However if $W = \mathbb{C}x + \mathbb{C}y$ and $\pi : V \to W$ is the B_V-orthogonal projection then one readily has that $\epsilon(x)\iota(x) + \epsilon(y)\iota(y) = D_\pi$. Thus

$$[\epsilon(x \wedge y), \iota(x \wedge y)] = D_\pi - I \tag{2.3}$$

Assume that ω is such that $\tau(\omega)$ is a semisimple element of $Lie\, SO(V)$. Then from the normal form of such elements there exists a subset $\{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ of an orthonormal basis of V and scalars $\mu_i \in \mathbb{C}$, $i = 1, \ldots, k$ such that $\omega = \sum_{i=1}^k \omega_i$ where $\omega_i = \mu_i x_i \wedge y_i$. But clearly $[\epsilon(\omega_i), \iota(\omega_j)] = 0$ for $i \neq j$ by (2.1) so that

$$[\epsilon(\omega), \iota(\omega)] = \sum_{i=1}^{k} [\epsilon(\omega_i), \iota(\omega_i)]$$

But then

$$[\epsilon(\omega), \iota(\omega)] = (\sum_{i=1}^{k} \mu_i^2 D_{\pi_i}) - (\sum_{i=1}^{k} \mu_i^2) I \tag{2.4}$$

by (2.3) where W_i is the span of x_i and y_i and $\pi_i : V \to W_i$ is the orthogonal projection. Now let $\beta_i = \frac{1}{2}\tau(\omega_i)$. One notes that the 2-plane W_i is stable under β_i and that β_i vanishes on the B_V orthocomplement of W_i in V. Clearly then $\beta_i\beta_j = 0$ for $i \neq j$ so that

$$\alpha = -\sum_{i=1}^{k} \beta_i^2$$

But it is also immediate that $-\beta_i^2 = \mu_i^2\pi_i$. Hence

$$\alpha = \sum_{i=1}^{k} \mu_i^2\pi_i \tag{2.5}$$

But since $tr\,\pi_i = 2$ the equality (2.2) follows from (2.4) and (2.5). Thus the proposition has been established for any $\omega \in U$ where U is the set of all $\omega \in \wedge^2 V$ such that $\tau(\omega)$ is semisimple. But then note that, by continuity, (2.2) follows for all elements in $\wedge^2 V$ since U is Zariski open and dense in $\wedge^2 V$. QED

2.2. We now consider the case $V = \mathfrak{g}$ where $\mathfrak{g}$ is a complex semisimple Lie algebra and $B_{\mathfrak{g}}$ is the Killing form on $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{h}^*$ be the dual space to $\mathfrak{h}$. Let $l = dim\,\mathfrak{h}$ and let $\Delta \subset \mathfrak{h}^*$ be the set of roots for the pair $\{\mathfrak{h}, \mathfrak{g}\}$. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ which contains $\mathfrak{h}$. Let Δ_+ be the set of roots for $\{\mathfrak{h}, \mathfrak{b}\}$ so that $\Delta_+ \subset \Delta$ is a choice of a system of positive roots. Let $\Lambda_+ \subset \mathfrak{h}^*$ be the semigroup of integral linear forms on $\mathfrak{h}$ which are dominant with respect to $\mathfrak{b}$. In particular $\rho \in \Lambda_+$ where, as usual, $\rho = \frac{1}{2}\sum_{\varphi\in\Delta_+}\varphi$. The restriction $B_{\mathfrak{g}}|\mathfrak{h}$ induces a symmetric non-singular bilinear form on $\mathfrak{h}^*$. Its value on $\mu, \nu \in \mathfrak{h}^*$ is denoted by (μ, ν). This bilinear form is positive definite on the real span $\mathfrak{h}^*_{\mathbb{R}}$ of Λ_+ and we put $|\nu| = \sqrt{(\nu,\nu)}$ for $\nu \in \mathfrak{h}^*_{\mathbb{R}}$.

For any $\lambda \in \Lambda_+$ let $\pi_\lambda : \mathfrak{g} \to End\,V_\lambda$ be some fixed irreducible representation with highest weight λ. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. If M is a $\mathfrak{g}$-module with respect to a representation π of $\mathfrak{g}$ we will also use π to denote the extension $U(\mathfrak{g}) \to End\,M$ of the representation to $U(\mathfrak{g})$. Let $Q \in Cent\,U(\mathfrak{g})$ be the Casimir element corresponding to the Killing form $B_{\mathfrak{g}}$. Thus if $\{x_i\}$ and $\{y_j\}$ are dual bases of $\mathfrak{g}$ with respect to $B_{\mathfrak{g}}$ and π is a representation of $\mathfrak{g}$ then

$$\pi(Q) = \sum_{i=1}^{n} \pi(x_i)\pi(y_i) \tag{2.6}$$

Let $\lambda \in \Lambda_+$ and let $\pi : \mathfrak{g} \to End\, M$ be a finite dimensional representation. The representation π is said to be primary of type π_λ if every irreducible component of π is equivalent to π_λ. One knows that $\pi_\lambda(Q)$ is a scalar operator where the scalar is $|\lambda+\rho|^2 - |\rho|^2$. It follows therefore that if π is primary of type π_λ, then

$$\pi(Q) = (|\lambda+\rho|^2 - |\rho|^2)\, I \tag{2.7}$$

where I here is the identity operator on M.

2.3. The adjoint representation of $\mathfrak{g}$ on itself will be denoted by $ad_{\mathfrak{g}}$. Let

$$\theta : \mathfrak{g} \to End \wedge \mathfrak{g}$$

be the representation of $\mathfrak{g}$ on $\wedge\mathfrak{g}$ defined so that $\theta(x) = D_{ad_{\mathfrak{g}}\, x}$ for any $x \in \mathfrak{g}$. Identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ using the Killing form $B_{\mathfrak{g}}$. Then $(\wedge\mathfrak{g}, d)$ is a cochain complex with respect to the usual Lie algebra coboundary operator d. We recall that explicitly,

$$d = \frac{1}{2}\sum_{i=1}^{n} \epsilon(x_i)\theta(y_i) \tag{2.8}$$

using notation in (2.6). One readily establishes that $d \in Der^1 \wedge \mathfrak{g}$ and $d^2 = 0$. In particular

$$d : \mathfrak{g} \to \wedge^2\mathfrak{g}$$

and one notes that d is equivariant with the action defined by θ. The derived cohomology is Lie algebra cohomology $H^*(\mathfrak{g})$. The following result was implicitly established in [7].

Theorem 2.2. *For any $x \in \mathfrak{g}$ let $\pi(x) \in End \wedge \mathfrak{g}$ be defined by putting*

$$\pi(x) = \frac{1}{2}(\epsilon(dx) - \iota(dx) + \theta(x)) \tag{2.9}$$

Then

$$\pi : \mathfrak{g} \to End \wedge \mathfrak{g}, \quad x \mapsto \pi(x)$$

is representation of $\mathfrak{g}$. Furthermore π is primary of type π_ρ.

Proof. The Clifford algebra over $\mathfrak{g}$ is denoted by $C(\mathfrak{g})$ in [7]. Following Chevalley in his treatment of Clifford algebras, the underlying vector spaces of $C(\mathfrak{g})$ and $\wedge\mathfrak{g}$ are identified in [7]. Consequently there are two multiplicative structures on $\wedge\mathfrak{g}$. If $u, v \in \wedge\mathfrak{g}$ then $uv \in \wedge\mathfrak{g}$ denotes the Clifford product and $u \wedge v \in \wedge\mathfrak{g}$ is the original exterior product. Using Clifford commutation, an operator $ad\, u$ on $\wedge\mathfrak{g}$ was defined in §2.3 of [7]. If $u \in \wedge^2\mathfrak{g}$ (or more generally if u is even) then $(ad\, u)(w) = uw - wu$ for any $w \in \wedge\mathfrak{g}$. By (71) and (106) in [7] one has

$$\theta(x) = ad\,\frac{dx}{2} \tag{2.10}$$

for any $x \in \mathfrak{g}$. But then if $\gamma(u) \in End \wedge \mathfrak{g}$, for $u \in \wedge\mathfrak{g}$, is the operator of left Clifford multiplication by u, it follows from (19) in [7] that

$$\gamma(\frac{dx}{2}) = \pi(x) \tag{2.11}$$

where $\pi(x)$ is defined by (2.9) above. On the other hand by (66) and (106) in [7] the map $\delta : \mathfrak{g} \to \wedge^2\mathfrak{g},\ x \mapsto \frac{1}{2}dx$ is a Lie algebra homomorphism, using Clifford commutation in $\wedge^2\mathfrak{g}$. But then π is a representation by (2.11) (above). Furthermore π is primary of type π_ρ by Theorem 39 in [7], recalling the definitions at the beginning of §5.2 in [7]. QED

§3. The operators $\Box$ and $\Box'$

3.1. We now introduce two operators $\Box$ and $\Box'$ on $\wedge\mathfrak{g}$, both of degree 0. Let $\{x_i\}$ be a basis of $\mathfrak{g}$ and let $\{y_j\}$ be the $B_{\mathfrak{g}}$-dual basis. Put

$$\Box = \sum_{i=1}^{n} \epsilon(dx_i)\iota(dy_i)$$

and reversing the order of multiplication,

$$\Box' = \sum_{i=1}^{n} \iota(dx_i)\epsilon(dy_i)$$

It is clear that the definitions are independent of the choice of basis $\{x_i\}$. Recall that the identity operator on $\wedge\mathfrak{g}$ is denoted by I.

Theorem 3.1. *One has*

$$3|\rho|^2\, I = \frac{1}{4}(\theta(Q) - \Box - \Box') \tag{3.1}$$

Proof. Obviously $|\rho+\rho|^2-|\rho|^2=3|\rho|^2$. Using the notation of (2.6) it then follows from (2.7) and Theorem 2.2 that

$$3|\rho|^2\, I=\frac{1}{4}(\theta(Q)-\square-\square')+\beta_4+\beta_2+\beta_{-2}+\beta_{-4} \tag{3.2}$$

where

$$\beta_4=\frac{1}{4}\sum_{i=1}^{n}\epsilon(dx_i)\epsilon(dy_i)$$

$$\beta_{-4}=\frac{1}{4}\sum_{i=1}^{n}\iota(dx_i)\iota(dy_i)$$

$$\beta_2=\frac{1}{4}\sum_{i=1}^{n}(\epsilon(dx_i)\theta(dy_i)+\theta(dx_i)\epsilon(dy_i))$$

$$\beta_{-2}=-\frac{1}{4}\sum_{i=1}^{n}(\iota(dx_i)\theta(dy_i)+\theta(dx_i)\iota(dy_i))$$

But now both sides of (3.1) are operators of degree 0. On the other hand β_i, for $i\in\{4,2,-2,-4\}$, is an operator of degree i. Thus all β_i vanish by (3.2). But then (3.1) follows from (3.2). QED

3.2. We can further simplify (3.1) by applying the "strange formula"

$$|\rho|^2=\frac{dim\,\mathfrak{g}}{24} \tag{3.3}$$

of Freudental-de Vries. See p. 243 in [1]. But in fact (3.1) yields a proof of the "strange formula" (3.3). Indeed first note that for any $x,y\in\mathfrak{g}$ one has

$$(dx,dy)=-\frac{1}{2}(x,y) \tag{3.4}$$

To establish (3.4) let ∂ be the negative transpose of d with respect to $B_{\wedge\mathfrak{g}}$. Then $L=d\partial+\partial d$ is the "Hodge" Laplacian. But it is an easy consequence of (2.8) that $L=\frac{1}{2}\theta(Q)$. See e.g. (2.1.7) in [4]. But, by definition of the Killing form, $\theta(Q)$ reduces to the identity on $\mathfrak{g}$. Thus $(dx,dy)=-(x,Ly)=\frac{1}{2}(x,y)$ proving (3.4). Now apply both sides of (3.1) to $1\in\wedge^0\mathfrak{g}$. Then by (3.4)

$$3|\rho|^2=-\frac{1}{4}\sum_{i=1}^{n}(dx_i,dy_i)$$
$$=\frac{n}{8}$$

But this proves the "strange formula" (3.3) of Freudental-de Vries. Applying the formula to (3.1) one has the refinement

$$\frac{n}{2}I = \theta(Q) - \Box - \Box' \tag{3.5}$$

3.3. Now by applying the commutation formula (2.2) we can separate out the terms and solve individually for $\Box$ and $\Box'$. We first make a better choice of basis for $\mathfrak{g}$. Let $\mathfrak{k}$ be a compact real form of $\mathfrak{g}$ and let $\mathfrak{q} = i\mathfrak{k}$. Then $B_{\mathfrak{g}}$ is positive definite on $\mathfrak{q}$. Let $\{z_i\}$ be an orthonormal basis of $\mathfrak{q}$. Then in the definition of $\Box$ and $\Box'$ we can choose

$$x_i = y_i = z_i \tag{3.6}$$

Let ε be the identity operator on $\mathfrak{g}$ so that the derivation D_ε is the Euler operator on $\wedge\mathfrak{g}$. That is

$$D_\varepsilon = k \text{ on } \wedge^k \mathfrak{g} \tag{3.7}$$

Theorem 3.2. *One has*

$$\begin{aligned} \Box &= \frac{1}{2}(\theta(Q) - D_\varepsilon) \\ \Box' &= \frac{1}{2}(\theta(Q) - (nI - D_\varepsilon)) \end{aligned} \tag{3.8}$$

Proof. For $x \in \mathfrak{g}$ note that (see §2.1)

$$\tau(\frac{dx}{2}) = ad_{\mathfrak{g}} x \tag{3.9}$$

Indeed (3.9) is implied by (18), (71) and (106) in [7]. Let $\{z_i\}$ be as in (3.6) and let $\alpha_i = -(ad_{\mathfrak{g}} z_i)^2 \in End\,\mathfrak{g}$. By definition of the Killing form, one has $tr\,\alpha_i = -1$. Thus

$$\epsilon(dz_i)\iota(dz_i) - \iota(dz_i)\epsilon(dz_i) = D_{\alpha_i} + \frac{1}{2}I \tag{3.10}$$

by (2.2) and (3.9). But $\theta(Q)$ reduces to the identity ε on $\mathfrak{g}$ so that $\sum_{i=1}^n \alpha_i = -\varepsilon$. Hence, by linearity, $\sum_{i=1}^n D_{\alpha_i} = -D_\varepsilon$. Thus

$$\begin{aligned} \Box - \Box' &= \sum_{i=1}^n (\epsilon(dz_i)\iota(dz_i) - \iota(dz_i)\epsilon(dz_i)) \\ &= \frac{n}{2}I - D_\varepsilon \end{aligned} \tag{3.11}$$

But

$$\square + \square' = \theta(Q) - \frac{n}{2}I \tag{3.12}$$

by (3.5). But then adding and subtracting (3.11) and (3.12) yields (3.8). QED

3.4. We define a Hilbert space structure H in $\wedge\mathfrak{g}$. The inner product of $u, v \in \wedge\mathfrak{g}$ will be denoted by $\{u, v\}$. This structure has been defined in §3.2 of [3] and we refer to that reference for a more comprehensive treatment of H. The discussion here will be limited to what will be needed in this paper. The real exterior algebra $\wedge_{\mathbb{R}}\mathfrak{q}$ is a real form of $\mathfrak{g}$. Furthermore $B_{\wedge\mathfrak{g}}$ is positive definite on $\wedge_{\mathbb{R}}\mathfrak{q}$. One defines a $*$-operation in $\wedge\mathfrak{g}$ by defining $(u + iv)^* = u - iv$ for $u, v \in \wedge_{\mathbb{R}}\mathfrak{q}$. Then H is given by defining $\{u, v\} = (u, v^*)$ for $u, v \in \wedge\mathfrak{g}$. For any $\beta \in End \wedge \mathfrak{g}$, let $\beta^* \in End \wedge \mathfrak{g}$ be the Hermitian adjoint of β with respect to H. By (3.9.3) in [3] one has

$$\iota(u)^* = \epsilon(u^*) \tag{3.13}$$

for any $u \in \wedge\mathfrak{g}$. But for $z \in \mathfrak{q}$ one has

$$dz^* = -dz \tag{3.14}$$

Indeed since $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{k} = i\mathfrak{q}$, the equation (3.14) clearly follows from (2.8) when we make the choice given in (3.6).

Proposition 3.3. *Let $\{z_i\}$ be the basis of $\mathfrak{g}$ defined as in (3.6). Then the operators $\epsilon(dz_i)\iota(dz_i)$ and $\iota(dz_i)\epsilon(dz_i)$ are negative semidefinite with respect to H for all i. In particular $\square = \sum_{i=1}^{n} \epsilon(dz_i)\iota(dz_i)$ and $\square' = \sum_{i=1}^{n} \iota(dz_i)\epsilon(dz_i)$ are negative semidefinite with respect to H.*

Proof. This is immediate from (3.13) and (3.14). QED

§4. The main results

4.1. Let $\mathcal{C}$ be the set of all commutative Lie subalgebras $\mathfrak{a}$ of $\mathfrak{g}$. If $\mathfrak{a} \in \mathcal{C}$ and $k = dim\, \mathfrak{a}$ let $[\mathfrak{a}] = \wedge^k\mathfrak{a}$. Regard $[\mathfrak{a}]$ as a 1-dimensional subspace of $\wedge^k\mathfrak{g}$ and let $C \subset \wedge \mathfrak{g}$ be the span of all $[\mathfrak{a}]$ for all $\mathfrak{a} \in \mathcal{C}$. It is obvious that $C = \sum_{k=1}^{n} C^k$ is a graded $\mathfrak{g}$-submodule (with respect to θ) of $\wedge\mathfrak{g}$. Of course $C^k = 0$ for $k > n_{abel}$ where n_{abel} is the maximal dimension of an abelian Lie subalgebra of $\mathfrak{g}$.

One of the results in [4] is that C (denoted by A in [4]) is a multiplicity free $\mathfrak{g}$-module. See Theorem (8) in [4]. If Ξ is an index set for the set of all abelian ideals $\{\mathfrak{a}_\xi\}$, $\xi \in \Xi$, of $\mathfrak{b}$ then the irreducible components of C may also be indexed by Ξ. The irreducible components, written as C_ξ, $\xi \in \Xi$, are characterized by the property that $[\mathfrak{a}_\xi]$ is the highest weight space of C_ξ. One therefore has the unique decomposition

$$C = \sum_{\xi \in \Xi} C_\xi \tag{4.1}$$

into irreducible components. For Peterson's results and our subsequent new results about C, see [8].

For $k = 0, \ldots, n$, let m_k be the maximal eigenvalue of the Casimir operator $\theta(Q)$ on $\wedge^k \mathfrak{g}$. The following result is a restatement of Theorem (5) in [4] (noting that m_k in [4] is one half its value here).

Theorem 4.1. *For $k = 0, \ldots, n$, one has*

$$m_k \leq k \tag{4.2}$$

Furthermore one has equality $m_k = k$ if and only if there exists an abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}$ such that $\dim \mathfrak{a} = k$, that is, if and only if $k \leq n_{abel}$. Moveover in such a case the eigenspace for the eigenvalue k of $\theta(Q)$ in $\wedge^k \mathfrak{g}$ is exactly C^k. In particular $\theta(Q)$ has integral (and consecutive) eigenvalues on C.

As an example, illustrating the first part of Theorem 4.1, if $\mathfrak{g} \simeq E_8$, then, since $n_{abel} = 36$, one has $m_k = k$ for $k \leq 36$. But $m_k < k$ for $k \geq 37$.

Remark 4.2. Note that (4.2) also follows from Theorem 3.2, using Proposition 3.3, since the Proposition 3.3 implies that the spectrum of $\Box$ is non-negative.

4.2. We can now establish one of the main results of the paper. Let $\mathcal{A}$ be the ideal in $\wedge \mathfrak{g}$ generated by all $dx \in \wedge^2 \mathfrak{g}$ for $x \in \mathfrak{g}$. Corollary (5.1) in [4] asserts that

$$\wedge^2 \mathfrak{g} = d\mathfrak{g} \oplus C^2 \tag{4.3}$$

is a direct sum. Note that (4.5) below in Theorem 4.3 is a generalization of (4.3). Recall the definition $\Box$ in §3.

Theorem 4.3. *One has $C = Ker\,\square$. In addition*

$$C = \{u \in \wedge\mathfrak{g} \mid \iota(dx)u = 0,\ \forall x \in \mathfrak{g}\} \tag{4.4}$$

Moreover $B_{\wedge\mathfrak{g}}$ is non-singular on C and

$$\wedge\mathfrak{g} = \mathcal{A} \oplus C \tag{4.5}$$

is a $B_{\wedge\mathfrak{g}}$-orthogonal direct sum.

Proof. The statement that $C = Ker\,\square$ is an immediate consequence of Theorem 4.1 and (3.8) in Theorem 3.2. On the one hand since, using the notation of Proposition 3.3, the operators $\epsilon(dz_i)\iota(dz_i)$ are negative semidefinite with respect to H, one has

$$Ker\,\square = \bigcap_{i=1}^{n} Ker\,\epsilon(dz_i)\iota(dz_i) \tag{4.6}$$

On the other hand $\epsilon(dz_i) = -\iota(dz_i)^*$ by (3.13) and (3.14). Hence

$$Ker\,\epsilon(dz_i)\iota(dz_i) = Ker\,\iota(dz_i) \tag{4.7}$$

This establishes (4.4). Of course $B_{\wedge\mathfrak{g}}$ is non-singular on $\wedge^k\mathfrak{g}$ for any $k = 0, \ldots, n_{abel}$. But then since $\square$ is diagonalizable and symmetric with respect to $B_{\wedge\mathfrak{g}}$ it follows that the restriction of $B_{\wedge\mathfrak{g}}$ to the eigenspace C_k of $\square$ in $\wedge^k\mathfrak{g}$ is non-singular. Hence $B_{\wedge\mathfrak{g}}$ is non-singular on C. But then (4.5) follows immediately from (4.4) and the equality $(dx \wedge v, u) = (v, \iota(dx)u)$ for any $u, v \in \wedge\mathfrak{g}$ and $x \in \mathfrak{g}$. QED

4.3. Fix an element $\mu \in \wedge^n\mathfrak{g}$ where $(\mu, \mu) = 1$. For any $v \in \wedge\mathfrak{g}$ let $\widetilde{v} = \iota(v)\mu$. Also if $M \subset \wedge\mathfrak{g}$ is a subspace let $\widetilde{M} = \{\widetilde{v} \mid v \in M\}$. It is a simple fact that if $M \subset \wedge\mathfrak{g}$ is a graded subspace, then

$$\widetilde{\widetilde{M}} = M \tag{4.8}$$

Furthermore since $\theta(\mathfrak{g})$ annihilates μ it is clear that if M is a $\mathfrak{g}$ submodule (with respect to θ) then $M \to \widetilde{M},\ v \mapsto \widetilde{v}$ is a $\mathfrak{g}$-module isomorphism. In particular $\widetilde{C}$ is isomorphic to C as a $\mathfrak{g}$-module.

Corollary 4.4. *One has $\widetilde{C} = Ker\,\square'$. In addition*

$$\widetilde{C} = \{v \in \wedge\mathfrak{g} \mid dx \wedge v = 0,\ \forall x \in \mathfrak{g}\} \tag{4.9}$$

Proof. If $v \in \wedge\mathfrak{g}$ and $x \in \mathfrak{g}$ note that

$$\widetilde{dx \wedge v} = \iota(dx)\widetilde{v} \tag{4.10}$$

But then (4.9) follows from (4.4) and (4.8). The argument in the proof of Theorem 4.3 establishing the equivalence of the equation $C = Ker\,\Box$ with (4.4) likewise, upon interchanging $\epsilon(dz_i)$ with $\iota(dz_i)$, clearly establishes the equivalence of the equation $\widetilde{C} = Ker\,\Box'$ with (4.9). QED

4.4. We will express (4.4) and (4.9) in a "functorial" way. Consider the symmetric algebra $S(\mathfrak{g})$ over $\mathfrak{g}$. Since the elements of $\wedge^2\mathfrak{g}$ commute with each other there exists a unique homomorphism

$$s : S(\mathfrak{g}) \to \wedge\mathfrak{g} \tag{4.11}$$

where $s(x) = dx$ for $x \in \mathfrak{g}$. The homomorphism s of course defines the structure of an $S(\mathfrak{g})$ module on $\wedge\mathfrak{g}$. Furthermore since s is a $\mathfrak{g}$-map with respect to the adjoint action, this $S(\mathfrak{g})$-module structure is equivariant with respect to the adjoint action.

The functors $Ext^j_{S(\mathfrak{g})}(\mathbb{C}, \wedge\mathfrak{g})$ clearly have the structure of $\mathfrak{g}$-modules. Considering only the two extreme values of j, one has $\mathfrak{g}$-module maps

$$Ext^0_{S(\mathfrak{g})}(\mathbb{C}, \wedge\mathfrak{g}) \to \wedge\mathfrak{g} \tag{4.12}$$

and

$$\wedge\mathfrak{g} \to Ext^n_{S(\mathfrak{g})}(\mathbb{C}, \wedge\mathfrak{g}) \tag{4.13}$$

Recalling the definitions of Ext at these two extremes, (4.4) and (4.9) immediately translate to

Theorem 4.10. *The map (4.12) defines a $\mathfrak{g}$-module isomorphism*

$$Ext^0_{S(\mathfrak{g})}(\mathbb{C}, \wedge\mathfrak{g}) \to \widetilde{C} \tag{4.14}$$

and the map (4.13) restricts to a $\mathfrak{g}$-module isomorphism

$$C \to Ext^n_{S(\mathfrak{g})}(\mathbb{C}, \wedge\mathfrak{g}) \tag{4.15}$$

4.5. An element $u \in \wedge\mathfrak{g}$ is called totally exact if it is the sum of products of elements of the form dx, $x \in \mathfrak{g}$. Let A be the image of s so that A is the algebra of all totally exact elements in $\wedge\mathfrak{g}$. See Theorem 1.4 in [6] for a characterization of A. Some features of the $\mathfrak{g}$-module structure of A were studied and used in [7]. See Theorem 69 in [7]. Of course the $S(\mathfrak{g})$ module structure on $\wedge\mathfrak{g}$ can be regarded as defining an A-module structure on $\wedge\mathfrak{g}$. Consider the question of determining generators for this module. A subspace $C_o \subset \wedge\mathfrak{g}$ will be said to be A-generating if C_o is a graded $\mathfrak{g}$-submodule (with respect to θ) of $\wedge\mathfrak{g}$ such that $\wedge\mathfrak{g} = A \wedge C_o$.

Theorem 4.6. *The subspace C is A-generating so that*

$$\wedge g = A \wedge C \tag{4.16}$$

Moreover it is minimal among all A-generating subspaces in $\wedge\mathfrak{g}$. In fact if C_o is any graded $\mathfrak{g}$-submodule (with respect to θ) of $\wedge\mathfrak{g}$ then C_o is A-generating if and only if $C \subset C_o$.

Proof. The proof that C is A-generating is a standard exercise using (4.5). Assume inductively that for $k \geq 1$, $\wedge^j\mathfrak{g} \subset A \wedge C$ for all $j \leq k$. Obviously $\wedge^0\mathfrak{g} = C^0 \subset A\wedge C$ and $\wedge^1\mathfrak{g} = C^1 \subset A\wedge C$. Let $u \in \wedge^{k+1}\mathfrak{g}$. By (4.5) we may write $u = v + w$ where $v \in \mathcal{A}^{k+1}$ and $w \in C^{k+1}$. Then $w \in A \wedge C$. But $v \in d\mathfrak{g} \wedge (\wedge^{k-1}\mathfrak{g})$. But $\wedge^{k-1}\mathfrak{g} \subset A \wedge C$ by induction. Hence $v \in A \wedge C$. Thus C is A-generating.

Obviously if $C \subset C_o$ one has $\wedge\mathfrak{g} = A \wedge C_o$. Now for $k = 0, \ldots, n$, let $p_k : \wedge^k\mathfrak{g} \to C^k$ be the projection defined by (4.5). Obviously p_k is a $\mathfrak{g}$-map. Assume $C_o \subset \wedge\mathfrak{g}$ is A-generating. Then clearly the restriction $p_k : C_o^k \to C^k$ is a surjective $\mathfrak{g}$-map. However by Theorem 4.1 the irreducible representations $\mathfrak{g}$ occurring in C^k do not occur in $\mathcal{A}^k$. Thus one must have $C^k \subset C_o^k$. Hence $C_o \subset C$. QED

Remark 4.7. Note that (4.15) implies that the set of elements of the form $y_1 \wedge \cdots \wedge y_k \wedge dx_1 \wedge \cdots \wedge dx_m$ span $\wedge\mathfrak{g}$, where $x_i, y_j \in \mathfrak{g}$ and the $\{y_j\}$ pairwise commute. QED

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