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Absolute Continuity of the Essential Spectrum for some Linearized MHD Operator

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§1. Introduction

The magnetohydrodynamic (MHD) motion of plasma is described by the system of equations which consist of the compressible Euler equation and the reduced Maxwell equation with the mutual interaction terms given by the Lorenz force and Ohm's low. Related to the plasma confinement experiment, the study of the behavior of plasma motion around the equilibrium is very important. The MHD motion in the vicinity of the equilibrium is described by the following linearized MHD equation:

$$\text{(1.1)}\qquad \rho_0 \frac{\partial^2 \xi}{\partial t^2} = -K\xi \equiv \operatorname{grad}\{\gamma P_0(\operatorname{div}\xi) + (\operatorname{grad} P_0)\cdot\xi\} + B_0 \times \operatorname{rot}(\operatorname{rot}(B_0\times\xi)) - (\operatorname{rot} B_0)\times \operatorname{rot}(B_0\times\xi),$$

for the Lagrangian displacement vector field $\xi : \Omega \subset R^3 \to R^3$. Here, the equilibrium quantities ρ_0 (=density), P_0 (=pressure), B_0 (=magnetic field), are given bounded smooth functions which satisfy the equilibrium condition:

$$\text{(1.2)}\qquad \begin{aligned} &\operatorname{grad} P_0 = j_0 \times B_0, \ \ \operatorname{div} B_0 = 0, \\ &\text{with } j_0 = \operatorname{rot} B_0 \text{ (=electric current density)}, \\ &P_0 \geq c_P > 0, \ \ \rho_0 \geq c_\rho > 0 : \text{arbitrary}. \end{aligned}$$

We assume in (1.1) that the specific heat ratio γ is a positive constant. We impose a slip condition: $\xi \cdot n = 0$ on the boundary $\partial\Omega$ where n is the unit normal on the boundary.

In this paper, we shall study some spectral properties of the operator $\rho_0^{-1}K$ in a Hilbert space $L^2(\Omega;\rho_0 dr)^3$. In particular, we shall prove the absolute continuity of the essential spectrum and the discreteness of the embedded eigenvalues in the continuum under some assumptions on the

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shape of the region Ω and the symmetry of the equilibrium. We assume hereafter that Ω is a flat torus in R^3:

$$\Omega = \{(x, y, z) : x, y, z \in S \equiv R/2\pi Z\} = S^3.$$

We consider the equilibrium where the quantities B_0 and P_0 depend only on one variable x. Then, these one-dimensional equilibrium quantities are given as:

$$B_0 = (0, b(x)\sin\phi(x), b(x)\cos\phi(x))$$
$$P_0 = c - \frac{1}{2}b(x)^2,$$

where $b(x)$ and $\phi(x)$ are arbitrary smooth functions with the property:

$$b(0) = b(2\pi),\ \phi(0) = \phi(2\pi) \mod 2\pi,$$

and c is a sufficiently large positive constant. Due to the symmetry of the coefficients, we can decompose ξ into (m,n) Fourier modes:

$$e^{imy+inz}\xi(x),\quad m, n : \text{integers}$$

and the force operator ${\rho_0}^{-1}K$ is realized in the decomposed space as a selfadjoint operator with a form (see Kako [1]):

(1.3)
$$K = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

where A, B, B^* and C are differential/multiplication operators given as

(1.4)
$$A = -\frac{d}{dx}(b^2 + \gamma P_0)\frac{d}{dx} + b^2{n_\phi}^2$$
$$B = \left(-i\frac{d}{dx}(b^2 + \gamma P_0)m_\phi, -i\frac{d}{dx}\gamma P_0 n_\phi\right)$$
$$B^* = \begin{pmatrix} -i(b^2 + \gamma P_0)m_\phi\frac{d}{dx} \\ \\ -i\gamma P_0 n_\phi\frac{d}{dx} \end{pmatrix}$$
$$C = \begin{pmatrix} m_\phi^2(b^2 + \gamma P_0) + b^2 n_\phi^2 & m_\phi n_\phi \gamma P_0 \\ \\ m_\phi n_\phi \gamma P_0 & n_\phi^2\gamma P_0 \end{pmatrix}$$

with

(1.5)
$$n_\phi = n(\cos\phi) + m(\sin\phi),\quad m_\phi = m(\cos\phi) - n(\sin\phi).$$

We can construct a selfadjoint operator ${\rho_0}^{-1}K$ in $L^2(S;\rho_0 dx)^3$ with the resolvent expression:

$$(1.6)\quad (\rho_0^{-1}K+\lambda)^{-1} = \begin{pmatrix} E_\lambda^{-1} & -E_\lambda^{-1}BC_\lambda^{-1} \\ -C_\lambda^{-1}B^*E_\lambda^{-1} & C_\lambda^{-1}+C_\lambda^{-1}B^*E_\lambda^{-1}BC_\lambda^{-1} \end{pmatrix}\rho_0,$$

where $E_\lambda = A_\lambda - BC_\lambda^{-1}B^*$ with $A_\lambda = A+\lambda\rho_0$ and $C_\lambda = C+\lambda\rho_0$ (see Kako [1]).

§2. The essential spectrum of $\rho_0^{-1}K$

From the resolvent expression (1.6), we can extract some spectral properties of the operator $\rho_0^{-1}K$ such as the range of the essential spectrum.

Theorem 2.1 (Kako [1]). *The operator $\rho_0^{-1}K$ has a natural self-adjoint realization in the Hilbert space $L^2(S;\rho_0 dx)^3$, and the essential spectrum of $\rho_0^{-1}K$ consists of σ_A and σ_S with*

$$\sigma_A = \{\lambda : \lambda = \omega_A(x),\ 0 \le x \le 2\pi\}$$

and

$$(2.1)\qquad \sigma_S = \{\lambda : \lambda = \omega_S(x),\ 0 \le x \le 2\pi\},$$

where

$$\omega_A \equiv b^2{n_\phi}^2/\rho_0 \ (Alfvén\ frequency)$$

and

$$(2.2)\qquad \omega_S \equiv \omega_A\gamma P_0/(b^2+\gamma P_0)\ (slow\ magnetosonic\ frequency).$$

The proof of this theorem is based on the following expression of the resolvent:

$$(2.3)\qquad (\rho_0^{-1}K+\lambda_0)^{-1} = \begin{pmatrix} 0 & -GF_{\lambda_0}^{-1} \\ -F_{\lambda_0}^{-1}G^* & F_{\lambda_0}^{-1} \end{pmatrix}\rho_0 + R_1.$$

with $G = A_{\lambda_0}^{-1}B$ and $G^* = B^*A_{\lambda_0}^{-1}$. Where the remainder R_1 is a trace class operator in $L^2(S;\rho_0 dx)^3$ and G is a Hilbert-Schmidt class operator from $L^2(S;\rho_0 dx)^2$ to $L^2(S;\rho_0 dx)$, and F_{λ_0} is a multiplication operator:

$$(2.4)\qquad F_{\lambda_0} = \rho_0\begin{pmatrix} \omega_A(x)+\lambda_0 & 0 \\ 0 & \omega_S(x)+\lambda_0 \end{pmatrix}.$$

Introducing unitary operators U and U^* in $L^2(S;\rho_0 dx)^3$ as

$$(2.5) \qquad U=\rho_0^{-1}\exp\begin{pmatrix}0 & G\\ -G^* & 0\end{pmatrix} \text{ and } U^*=\rho_0^{-1}\exp\begin{pmatrix}0 & -G\\ G^* & 0\end{pmatrix},$$

we have

$$(2.6) \qquad U(\rho_0^{-1}K+\lambda_0)^{-1}U^*=\begin{pmatrix}0 & 0\\ 0 & F_{\lambda_0}^{-1}\end{pmatrix}\rho_0+R_2,$$

where R_2 is a trace class operator which maps $L^2(S;\rho_0 dx)^3$ to the Sobolev space of order two: $H^2(S;\rho_0 dx)^3$. Applying the trace class perturbation theory (see Kato [2]), we can prove that there exists an absolutely continuous spectrum which consists of the union of the ranges of functions $\omega_A(x)$ and $\omega_S(x)$ (see Kako [1]).

§3. Application of Mourre's estimate

We shall apply Mourre's commutator estimate to the present problem and prove the discreteness of embedded eigenvalues in the continuum as well as the absolute continuity of the continuous spectrum in the complement of eigenvalues under the following assumption.

Assumption. The functions ω_A and ω_S are smooth and a number of critical points $x_A^c(k), k=1,2,...,M$ and $x_S^c(l), l=1,2,...,N$:

$$\omega_A'(x_A^c(k))=\omega_S'(x_S^c(l))=0$$

are finite.

We define functions $H_A(x)$ and $H_S(x)$ as

$$H_A(x)=(\omega_A(x)+\lambda_0)^{-1} \text{ and } H_S(x)=(\omega_S(x)+\lambda_0)^{-1}.$$

Let T be an unitary operator from $L^2(S;\rho_0 dx)^3$ to $L^2(S)^3$:

$$(3.1) \qquad T: L^2(S;\rho_0 dx)^3 \ni f \mapsto \rho_0^{1/2} f \in L^2(S)^3.$$

Then the operator $T\rho_0^{-1}KT^{-1}=\rho_0^{-1/2}K\rho_0^{-1/2}$ is unitarily equivalent to $\rho_0^{-1}K$. We denote this selfadjoint operator in $L^2(S)^3$ by K'. We introduce a conjugate operator H to K' as

$$(3.2) \qquad H\equiv\begin{pmatrix}0 & 0 & 0\\ 0 & H_A'(x)\frac{d}{dx}+\frac{d}{dx}H_A'(x) & 0\\ 0 & 0 & H_S'(x)\frac{d}{dx}+\frac{d}{dx}H_S'(x)\end{pmatrix}.$$

Proposition 1. *Under the assumption of the smoothness of* ω_A *and* ω_S*, the operator* iH *with domain:*

$$\mathcal{D}(iH) = \{f : f = (f_1, f_2, f_3)^t, f_k \in L^2(S), k = 1,2,3, \quad H'_A(x)\frac{d}{dx}f_2, H'_S(x)\frac{d}{dx}f_3 \in L^2(S)\}$$

is skew selfadjoint in $L^2(S)^3$

Proof. Let $a(x)$ be a real valued continuously differentiable function. We claim that an operator A defined as

$$\begin{aligned}\mathcal{D}(A) =& \{f : f \in L^2(S), a(x)\frac{d}{dx}f \in L^2(S)\}\\ Af =& i(a(x)\frac{d}{dx}f + \frac{d}{dx}a(x)f)\end{aligned}$$

is selfadjoint. In fact, for $f, g \in L^2(S)$ with the property that $a(x)\frac{d}{dx}g$, $a(x)\frac{d}{dx}f \in L^2(S)$, we have

$$\int_S \frac{d}{dx}(a(x)g(x)\overline{f(x)})dx = 0. \tag{3.3}$$

Using this identity, we can prove that A is closed and symmetric. The denseness of the range of $A \pm i$ can be shown in the standard way.

Q.E.D.

Let $E(\cdot)$ and $E_0(\cdot)$ be spectral resolutions of $D \equiv (K' + \lambda_0)^{-1}$ and

$$D_0 \equiv \begin{pmatrix} 0 & 0 \\ 0 & F_{\lambda_0}^{-1} \end{pmatrix} \rho_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & H_A(x) & 0 \\ 0 & 0 & H_S(x) \end{pmatrix}.$$

Then the commutator $[H, D_0] \equiv HD_0 - D_0H$ between H and D_0 can be calculated as

$$HD_0 - D_0H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2H'_A(x)^2 & 0 \\ 0 & 0 & 2H'_S(x)^2 \end{pmatrix}. \tag{3.4}$$

This operator is nonnegative. Using Proposition 1 and this expression of the commutator $[H, D_0]$, we can prove the following lemma.

Lemma 2. *Let $\Delta \subset R$ be such that*

$$(H_A^{-1}(\Delta) \cup H_S^{-1}(\Delta)) \cap \{x_A^c(k)\}_{k=1}^N \cap \{x_S^c(l)\}_{l=1}^M = \emptyset$$

and also let the intersection between Δ and the point spectrum of $(K' + \lambda_0)^{-1}$ be empty. Then we have the following Mourre type estimate:

$$E(\Delta)[H, D]E(\Delta) \geq \alpha E(\Delta) + Q, \ \alpha > 0, \tag{3.5}$$

where Q is a compact operator.

Proof. Since $D - D_0$ is compact, $E(\Delta) - E_0(\Delta)$ is also compact. Furthermore, $(D - D_0)H$ is a compact operator in $L^2(S)^3$, since the difference $D - D_0$ is bounded from $L^2(S)^3$ to $H^2(S)^3$. Hence we have that the operator $E(\Delta)[H, D]E(\Delta) - E_0(\Delta)[H, D_0]E_0(\Delta)$ is compact. Using the non-negativity of the commutator $[H, D_0]$ and the assumption for the interval Δ, we have the estimate (3.5). Q.E.D.

From this lemma, applying the results of Mourre (see [4 , Theorem 4.7 and Theorem 4.9] and [3]), we have the follwing theorem.

Theorem 3. *Let Δ be as in Lemma 2. Then the operator $(K' + \lambda_0)^{-1}$ restricted to the subspace $E(\Delta)L^2(S)^3$ is absolutely continuous except for some discrete set. The absolutely continuous part is unitarily equivalent to a part of the multiplication operator $F_{\lambda_0}^{-1}\rho_0$.*

From this theorem, we can have the corresponding results for the absolute continuity of the continuous spectrum of $\rho_0^{-1}K$ and the unitary equivalence between the absolutely continuous part of the operator $\rho_0^{-1}K$ and the multiplication opetator $\rho_0^{-1}F_0$.

References

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