Commutator Algebra and Resolvent Estimates

Christian Gérard
Hiroshi Isozaki
Erik Skibsted

§1. Introduction

In studying the detailed properties of Schrödinger operators, the method of micro-localization seems to be indispensable. For the many-body problem, this point of view was introduced by Enss [3], Mourre [11] and then by Sigal-Soffer [13] to investigate the propagation properties of the unitary group. These sorts of estimates not only lead us to a deep understanding of the space-time behavior of the solution to the Schrödinger equation, but also give us many applications. The aim of this paper is to prove a certain variation of these kinds of estimates for the resolvent of the N-body Schrödinger operator.

We consider a system of N-particles moving in $\mathbf{R}^\nu$ with mass m_i and position $x^i \in \mathbf{R}^\nu (1 \le i \le N)$. Let $\mathcal{X}$ be defined by

$$\mathcal{X} = \{(x^1, \cdots, x^N); \sum_{i=1}^N m_i x^i = 0\},$$

and consider the Schrödinger operator

$$H = H_0 + \sum_{i<j} V_{ij},$$

where $-H_0$ is the Laplace-Beltrami operator on $\mathcal{X}$ equipped with the Riemannian metric induced from $ds^2 = 2\sum_{i=1}^N m_i (dx^i)^2$ on $\mathbf{R}^{N\nu}$. Each pair potential $V_{ij} = V_{ij}(x^i - x^j)$ is assumed to be a real-valued C^∞-function on $\mathbf{R}^\nu$ and satisfies for some constant $\rho > 0$

$$(1.1) \qquad |\partial_y^m V_{ij}(y)| \le C_m <y>^{-m-\rho},$$

for any $m = 0, 1, 2, \cdots$, where ∂_y^m denotes an arbitrary derivative of order m and $<y> = (1 + |y|^2)^{1/2}$. Let $R(z) = (H - z)^{-1}$. Let Λ be the set of thresholds of H. For $\lambda \in \sigma_{\text{ess}}(H) \cap \sigma_p(H)^c \cap \Lambda^c$, we define

$$(1.2) \qquad a(\lambda) = \inf\{\lambda - \mu; \mu \in \Lambda, \mu < \lambda\}.$$

Note that $a(\lambda) = \lambda$ if $\lambda > 0$, which follows from the absence of positive eigenvalues of Schrödinger operators (see [5]). We consider a pseudo-differential operator (Ps.D.Op.) P with symbol $p(x, \xi)$ belonging to the following class. For a positive integer k and $a \in \mathbf{R}$, let $\mathcal{R}^k(a)$ be the set of C^∞-functions $p(x, \xi)$ having the following estimates:

$$(1.3) \qquad |\partial_x^m \partial_\xi^n p(x, \xi)| \le C <x>^{-m} <\xi>^{-k}, \quad 0 \le m, n \le k,$$

and also satisfying

$$(1.4) \qquad \sup_{x, \xi} \frac{x \cdot \xi}{<x>} < a \quad \text{on} \quad \text{supp } p(x, \xi).$$

A typical example of the element of $\mathcal{R}^k(a)$ is given as follows. We take $\rho(t) \in C^\infty(\mathbf{R})$ such that $\rho(t) = 1$ if $t < a - 2\epsilon$, $\rho(t) = 0$ if $t > a - \epsilon$, ϵ being a small positive constant. Then

$$\rho(\frac{x \cdot \xi}{<x>}) <\xi>^{-2k} \in \mathcal{R}^k(a).$$

For a Ps.D.Op. P, $P \in \mathcal{R}^k(a)$ means that the symbol of P belongs to $\mathcal{R}^k(a)$. As is well-known, for a sufficiently large k, $P \in \mathcal{R}^k(a)$ is L^2-bounded. Let $\mathbf{B}$ denote the totality of bounded operators on $L^2(\mathcal{X})$. The main result of this paper is the following

Theorem 1.1. *For any $s > -1/2$ and $t > 1$, there exists $k = k(s) > 0$ such that*

$$<x>^s P R(\lambda + i0) <x>^{-s-t} \in \mathbf{B}$$

for any $P \in \mathcal{R}^k(\sqrt{a(\lambda)})$.

Although the above theorem is formulated by Ps.D.Op.'s, the main part of the proof consists in the calculus of commutators in an algebra consisting of functions of several operators, which is one of the interesting features of the many-body problem. This commutator calculus has its origin in the work of Mourre [11], was developed by Sigal-Soffer [13], [14] with great success and is now considered as a basic tool for the many-body problem.

One of the authors proved a slightly weaker theorem in [15] and we should note that the above Theorem 1.1 is implicitly suggested in [16], where the commutator of H and

$$\tilde{A} = \frac{1}{2i}(x \cdot \nabla_x + \nabla_x \cdot x) - C<x>$$

was used. In this paper, we shall explain a method which treats directly the resolvent. Our idea is very close to those of Sigal-Soffer [13] and Derezinski [2]. One of the applications of the above theorem is the study of the detailed structure of the S-matrix ([9], [10], [16]). Other applications will be given elsewhere.

Finally, we remark that throughout the paper we neglect the domain question and treat freely the product of unbounded operators. This is justified by defining them by quadratic forms on $\mathcal{S} \times \mathcal{S}$, where $\mathcal{S}$ is the space of rapidly decreasing functions.

§2. Commutator Algebra

For two operators P and A, we introduce their multiple commutators by

$$\mathrm{ad}_0(P, A) = P,$$
$$\mathrm{ad}_n(P, A) = [\mathrm{ad}_{n-1}(P, A), A], \quad n \geq 1.$$

The fundamental formulas to calculate the commutators are as follows:

$$(\mathrm{ad}_n(P, A))^* = (-1)^n \, \mathrm{ad}_n(P^*, A^*),$$

$$\mathrm{ad}_n(PQ, A) = \sum_{k=0}^{n} \binom{n}{k} \mathrm{ad}_{n-k}(P, A) \, \mathrm{ad}_k(Q, A),$$

$$[P, A^n] = \sum_{k=1}^{n} c_{n,k} \, \mathrm{ad}_k(P, A) A^{n-k},$$

$c_{n,k}$ being constants.

We choose the coordinates $x = (x_1, \cdots, x_{(N-1)\nu})$ on $\mathcal{X}$ such that

$$H_0 = - \sum_{i=1}^{(N-1)\nu} (\partial/\partial x_i)^2.$$

As in [2] and [13], an important role is played by the self-adjoint operator B defined by

$$(2.1) \qquad B = \frac{1}{2i}\left(\frac{x}{<x>} \cdot \nabla_x + \nabla_x \cdot \frac{x}{<x>}\right).$$

We first consider the commutation relations between H, B and $X = <x>$. Let L_0 be the differential operator defined by

$$L_0 = \sum_{i=1}^{(N-1)\nu} x_i \frac{\partial}{\partial x_i}.$$

Let $\mathcal{V}$ be the set of C^∞-functions v on $\mathcal{X}$ such that $L_0^n v$ is bounded on $\mathcal{X}$ for any $n \geq 0$. This set $\mathcal{V}$ forms an algebra and is independent of the choice of the Jacobi coordinates.

Example. If $v \in C^\infty(\mathbf{R}^\nu)$ satisfies $|\partial_y^m v(y)| \leq C_m < y >^{-m}$, $\forall m \geq 0$, then $v(x^i - x^j) \in \mathcal{V}$. In particular, each two-body potential $V_{ij}(x^i - x^j)$ belongs to $\mathcal{V}$.

Let $\mathcal{V}_m = X^m \mathcal{V}$. Let $\mathcal{P}_{k,m}$ be the set of differential operators of order k with coefficients $\in \mathcal{V}_m$. $\mathcal{V}_m$ is invariant by the action of L_0, which implies that, if $L \in \mathcal{P}_{k,m}$, $[L, B] \in \mathcal{P}_{k,m-1}$. We have, therefore,

Lemma 2.1. *For $n \geq 1$, we have*
(1) $\mathrm{ad}_n(X, B) \in \mathcal{P}_{0,1-n}$.
(2) $\mathrm{ad}_n(H, B) \in \mathcal{P}_{2,-n}$.
(3) $\mathrm{ad}_n(B, H) \in \mathcal{P}_{n+1,-1}$.

These commutation relations suggest us to introduce the following

Definition 2.2. $P \in \mathcal{O}p^m(X)$ $(m \in \mathbf{R}) \Longleftrightarrow$ $X^\alpha \mathrm{ad}_n(P, B) X^\beta \in \mathbf{B}$, for any $\alpha, \beta \in \mathbf{R}$ and $n \geq 0$ such that $\alpha + \beta = n - m$.

The analogy of the class $\mathcal{O}p^m(X)$ to that of Ps.D.Op.'s is apparent when one thinks of Beals' characterization of the standard class of Ps.D.Op.'s ([1]). The basic properties of $\mathcal{O}p^m(X)$ are summarized in the following lemma whose proof follows easily from the definition.

Lemma 2.3. (1) $P \in \mathcal{O}p^m(X) \Longleftrightarrow$ *There exists* $P_0 \in \mathcal{O}p^0(X)$ *such that* $P = X^m P_0$.
(2) $P \in \mathcal{O}p^m(X) \Longrightarrow [P, B] \in \mathcal{O}p^{m-1}(X)$.
(3) $P \in \mathcal{O}p^m(X) \Longrightarrow X^k P X^l \in \mathcal{O}p^{m+k+l}(X)$, $\forall k, l \in \mathbf{R}$.
(4) $P \in \mathcal{O}p^m(X) \Longrightarrow P^* \in \mathcal{O}p^m(X)$.
(5) $P \in \mathcal{O}p^m(X)$, $Q \in \mathcal{O}p^n(X) \Longrightarrow PQ \in \mathcal{O}p^{m+n}(X)$.

Therefore, $\cup_m \mathcal{O}p^m(X)$ forms an algebra which is our basic tool in this paper.

The basic subject of this section is to calculate the commutators of functions of operators. For $m \in \mathbf{R}$, let $\mathcal{F}^m$ be the set of C^∞-functions on $\mathbf{R}$ such that

$$|f^{(k)}(x)| \leq C_k(1+|x|)^{m-k}, \quad \forall k \geq 0.$$

Then for $f \in \mathcal{F}^m$ ($m \in \mathbf{R}$), there exists $F(z) \in C^\infty(\mathbf{C})$, called an almost analytic extension of f, having the following properties:

$$F(x) = f(x), \quad x \in \mathbf{R},$$

$$|\overline{\partial_z}F(z)| \leq C_N <z>^{m-1-N} |\mathrm{Im}z|^N, \quad \forall N \geq 0,$$

$$\mathrm{supp}F(z) \subset \{|\mathrm{Im}z| \leq \epsilon(1+|\mathrm{Re}z|)\}, \quad 0 < \epsilon \ll 1.$$

Furthermore, $\partial_x^k F(z)$ is an almost analytic extension of $f^{(k)}(x)$ (see [6]). Let $f \in \mathcal{F}^{-\epsilon}$ ($\epsilon > 0$) and F be its almost analytic extension. Then for any self-adjoint operator A we have

$$(2.2) \qquad f(A) = \frac{1}{2\pi i} \int_{\mathbf{C}} \overline{\partial_z}F(z)(z-A)^{-1}dz \wedge d\bar{z}$$

(see [8]). One can also prove the following formula of the asymptotic expansion of the commutator: If $f \in \mathcal{F}^m$ ($m \in \mathbf{R}$) and A is self-adjoint, we have

$$(2.3) \qquad [P, f(A)] = \sum_{n=1}^{N-1} (-1)^{n-1}/n! \; \mathrm{ad}_n(P, A)f^{(n)}(A) + R_N,$$

$$(2.4) \qquad R_N = \frac{1}{2\pi i} \int_{\mathbf{C}} \overline{\partial_z}F(z)(A-z)^{-1} \mathrm{ad}_N(P, A)(A-z)^{-N}dz \wedge d\bar{z}.$$

R_N is bounded if there exists k such that $m+k < N$ and $\mathrm{ad}_N(P, A)(A+i)^{-k} \in \mathbf{B}$. This commutator expansion formula turns out to be a powerful tool of analysis (see also [6], [7]).

An important example of the element of $\mathcal{O}p^m(X)$ is given by

Lemma 2.4. $f(H), \; f(B) \in \mathcal{O}p^0(X)$ *if* $f \in \mathcal{F}^{-\epsilon}$, $\epsilon > 0$.

Proof. By (2.2), we have

$$\mathrm{ad}_n(f(H), B) = \frac{1}{2\pi i} \int_{\mathbf{C}} \overline{\partial_z}F(z) \, \mathrm{ad}_n((z-H)^{-1}, B)dz \wedge d\bar{z}.$$

For α, $\beta \in \mathbf{R}$ such that $\alpha + \beta = n$, one can show

$$\|X^\alpha \operatorname{ad}_n((H-z)^{-1}, B)X^\beta\| \le C <z>^\gamma |\operatorname{Im}z|^{-\gamma-1},$$

with $\gamma = \gamma(\alpha, \beta) > 0$. The above mentioned properties of the almost analytic extensions then prove that $f(H) \in \mathcal{O}p^0(X)$. To prove the lemma for $f(B)$, we have only to note that

$$\|X^n(B-z)^{-1}X^{-n}\| \le C_n |\operatorname{Im}z|^{-n-1}, \quad \forall n \ge 0. \qquad \square$$

It is convenient to introduce the following notation : Let $P_n \in \mathcal{O}p^{k(n)}(X)$, $k(1) > k(2) > \cdots \to -\infty$. Then an operator P is said to have the asymptotic expansion $\sum_{n\ge 1} P_n$, written as $P \sim \sum_{n\ge 1} P_n$, if and only if

$$P - \sum_{n=1}^{N-1} P_n \in \mathcal{O}p^{k(N)}(X), \quad \forall N \ge 2.$$

Using (2.3), one can show the following

Lemma 2.5. *Let* $P \in \mathcal{O}p^m(X)$, $f \in \mathcal{F}^n$, $m, n \in \mathbf{R}$. *Then*

$$[P, f(B)] \sim \sum_{k\ge 1} P_k f^{(k)}(B), \quad P_k \in \mathcal{O}p^{m-k}(X).$$

By the same methods as above, we can also show

Lemma 2.6. *Let* $\varphi \in C_0^\infty(\mathbf{R})$ *and* $f \in \mathcal{F}^m, m \in \mathbf{R}$. *Then we have:*

(1) $\quad \operatorname{ad}_n(\varphi(H), X) \in \mathcal{O}p^0(X), \quad n \ge 0.$

(2) $\quad [\varphi(H), f(X)] \sim \sum_{n\ge 1}(-1)^{n-1}/n! \operatorname{ad}_n(\varphi(H), X)f^{(n)}(X).$

Lemma 2.7. *Let* $f \in \mathcal{F}^m, g \in \mathcal{F}^n, m, n \in \mathbf{R}$. *Then we have:*

(1) $\quad \operatorname{ad}_k(g(X), B) \in \mathcal{O}p^{n-k}(X), \quad k \ge 0.$

(2) $\quad [g(X), f(B)] \sim \sum_{k\ge 1}(-1)^{k-1}/k! \operatorname{ad}_k(g(X), B)f^{(k)}(B).$

§3. Resolvent Estimates (1)

We fix $\lambda \in \sigma_{\mathrm{ess}}(H) \cap \sigma_p(H)^c \cap \Lambda^c$ and let $C_0(\lambda) = a(\lambda) - \epsilon$ for small $\epsilon > 0$. Let $\varphi \in C_0^\infty(\mathbf{R})$ be such that $\varphi(t) = 1$ if $|t - \lambda| < \delta$, $\varphi(t) = 0$ if $|t - \lambda| > 2\delta$. Our starting point is the following Mourre type estimate which holds for small $\delta > 0$ ([4]):

$$(3.1) \qquad \varphi(H)i[H, A]\varphi(H) \geq 2C_0(\lambda)\varphi(H)^2,$$

where

$$A = \frac{1}{2i}(x \cdot \nabla_x + \nabla_x \cdot x).$$

We now introduce

Definition 3.1. $f \in \mathcal{F}_-^m(\lambda)$, $m \in \mathbf{R} \iff f \in \mathcal{F}^m$, supp $f \subset (-\infty, \sqrt{a(\lambda)})$.

For a small $\epsilon_0 > 0$, we take $F_0(t) \in \mathcal{F}_-^0(\lambda)$ such that

$$\begin{cases} F_0(t) = 0 & \text{if } t > \sqrt{C_0(\lambda) - \epsilon_0}, \\ F_0(t) = 1 & \text{if } t < \sqrt{C_0(\lambda) - 2\epsilon_0}, \\ F_0(t) \geq 0, & \sqrt{F_0(t)} \in \mathcal{F}_-^0(\lambda), \\ F_0'(t) \leq 0, & \sqrt{-F_0'(t)} \in \mathcal{F}_-^0(\lambda). \end{cases}$$

For $0 < \epsilon_1 < \epsilon_0$, let $C_1(\lambda) = \sqrt{C_0(\lambda) - \epsilon_1}$ and define

$$F_m(t) = (C_1(\lambda) - t)^m F_0(t),$$

$$\widetilde{F_{2m+1}}(t) = (C_1(\lambda) - t)F_m(t)^2.$$

In the following arguments, $(*)$ denotes an operator having the asymptotic expansion:

$$\sum_{n \geq 2} P_n f_n(B), \quad P_n \in \mathcal{O}p^{2m+1-n}(X),$$

$$f_n \in \mathcal{F}_-^{2m+1-n}(\lambda), \quad \text{supp } f_n \subset \text{supp } F_0.$$

The crucial step is the following lemma.

Lemma 3.2. *Let* $m > -1/2$. *With* $F_m(t)$ *and* $\varphi(t)$ *introduced above, we define* $P_m = X^m F_m(B)\varphi(H)$. *Then there exists a constant* $C_0 > 0$ *such that*

$$- \operatorname{Re} \varphi(H)i[H, X^{2m+1}\widetilde{F_{2m+1}}(B)]\varphi(H) \geq C_0 P_m^* P_m + (*).$$

Proof.   To calculate the commutator $i[H, X^{2m+1}\widetilde{F_{2m+1}}(B)]$ in the category of the algebra explained in §2, we make the following device. Let $\varphi_1(t) \in C_0^\infty(\mathbf{R})$ be such that $\varphi_1(t) = 1$ on supp φ, and put $\psi(t) = t\varphi_1(t)$. Then

$$\varphi(H)i[H, X^{2m+1}\widetilde{F_{2m+1}}(B)]\varphi(H)$$
$$= \varphi(H)i[\psi(H), X^{2m+1}\widetilde{F_{2m+1}}(B)]\varphi(H)$$
$$= \varphi(H)i[\psi(H), X^{2m+1}]\widetilde{F_{2m+1}}(B)\varphi(H)$$
$$+ \varphi(H)X^{2m+1}i[\psi(H), \widetilde{F_{2m+1}}(B)]\varphi(H).$$

We first show that

(3.2) $$- \operatorname{Re} \varphi(H)X^{2m+1}i[\psi(H), \widetilde{F_{2m+1}}(B)]\varphi(H)$$
$$\geq (2m + 1)P_m^*(2C_0(\lambda) - 2B^2 - \epsilon_2)P_m + (*),$$

ϵ_2 being a sufficiently small positive constant. In fact, we have

$$\frac{d}{dt}\widetilde{F_{2m+1}}(t) = -(2m + 1)F_m(t)^2 - G(t),$$

where

$$G(t) = -2(C_1(\lambda) - t)^{2m+1}F_0(t)F_0'(t).$$

Then using (2.3), we see that the left-hand side of (3.2) is written as

$$(2m + 1) \operatorname{Re} \varphi(H)X^{2m+1}i[\psi(H), B]F_m(B)^2\varphi(H)$$
$$+ \operatorname{Re} \varphi(H)X^{2m+1}i[\psi(H), B]G(B)\varphi(H) + (*).$$

Taking note of the relation,

$$\varphi(H)X^{1/2}i[H, B]X^{1/2}\varphi(H)$$
$$= \varphi(H)(i[H, A] - 2B^2 + K)\varphi(H),$$

K being a compact operator, we have

$$\operatorname{Re} \varphi(H)X^{2m+1}i[\psi(H), B]G(B)\varphi(H)$$
$$= X^m\sqrt{G(B)}\varphi(H)X^{1/2}i[H, B]X^{1/2}\varphi(H)\sqrt{G(B)}X^m + (*)$$
$$\geq X^m\sqrt{G(B)}\varphi(H)(2C_0(\lambda) - 2B^2 + K)\varphi(H)\sqrt{G(B)}X^m + (*)$$
$$\geq (*),$$

where we have used Lemmas 2.5, 2.6 and 2.7 in the first line, (3.1) in the second line and the fact that $-2t^2 \geq -2(C_0(\lambda) - \epsilon_0)$ on supp $G(t)$ in the

third line. We can then see that the left-hand side of (3.2) is estimated from below by

$$(2m + 1)F_m(B)X^m\varphi(H)X^{1/2}i[H, B]X^{1/2}\varphi(H)X^m F_m(B) + (*)$$
$$\geq (2m + 1)P_m^*(2C_0(\lambda) - 2B^2 - \epsilon_2)P_m + (*).$$

We next show that

$$(3.3) \qquad - \operatorname{Re} \varphi(H)i[\psi(H), X^{2m+1}]\widetilde{F_{2m+1}}(B)\varphi(H)$$
$$\geq (2m + 1)P_m^*(2B^2 - 2C_1(\lambda)^2)P_m + (*).$$

In fact, the left-hand side of (3.3) is written as

$$- \operatorname{Re} \varphi(H)i[H, X^{2m+1}]\widetilde{F_{2m+1}}(B)\varphi(H) + (*)$$
$$= - \operatorname{Re} 2(2m + 1)\varphi(H)X^{2m}B(C_1(\lambda) - B)F_m(B)^2\varphi(H) + (*).$$

Since $t \leq C_1(\lambda)$ on supp $F_m(t)$, we have

$$-B(C_1(\lambda) - B)F_m(B)^2 \geq (B^2 - C_1(\lambda)^2)F_m(B)^2,$$

which proves (3.3).

The lemma now follows from (3.2) and (3.3). $\qquad \square$

Let $F_m(t)$ be as above. We call $X^m F_m(B)$ the operator of canonical type.

Lemma 3.3. *Let $m \in \mathbf{R}$, $P \in \mathcal{O}p^{2m}(X)$ and $f \in \mathcal{F}_-^{2m}(\lambda)$. Take $n > m$. Then for any $N \geq 1$, there exist the operators of canonical type $X^{n-k/2}F_{n-k/2}(B)$ ($k = 1, \cdots, N-1$), $P_N \in \mathcal{O}p^{2n-N}(X)$ and a constant $C > 0$ such that*

$$Re\, Pf(B) \leq C \sum_{k=0}^{N-1} F_{n-k/2}(B)X^{2n-k}F_{n-k/2}(B) + P_N.$$

Proof. By enlarging the support of $F_n(t)$ suitably, we see that $\psi(t) = f(t)F_n(t)^{-2} \in \mathcal{F}_-^{-\epsilon}(\lambda)$, $\epsilon > 0$. Then we have

$$Pf(B) = P\psi(B)F_n(B)^2$$
$$= F_n(B)P\psi(B)F_n(B) + [P\psi(B), F_n(B)]F_n(B).$$

One can then see that

$$\operatorname{Re} F_n(B)P\psi(B)F_n(B) = F_n(B)X^n P_0 X^n F_n(B),$$

where $P_0 = P_0^* \in \mathcal{O}p^0(X)$. Therefore, for a suitable constant $C > 0$,

$$\mathrm{Re}\, F_n(B)P\psi(B)F_n(B) \le CF_n(B)X^{2n}F_n(B),$$

$X^n F_n(B)$ being the operator of canonical type. Since $[P\psi(B), F_n(B)]$ has an asymptotic expansion:

$$[P\psi(B), F_n(B)] \sim \sum_{k \ge 1} P_k F_n^{(k)}(B), \; P_k \in \mathcal{O}p^{2m-k}(X),$$

we repeat the above procedure to conclude the lemma.   □

The main purpose of this section is the following

Theorem 3.4.   *Let* $m > -1/2, t > 1$ *and* $F \in \mathcal{F}_-^m(\lambda)$. *Then we have*

$$X^m F(B)\varphi(H)R(\lambda + i0)X^{-m-t} \in \mathbf{B}.$$

Proof.   We take $\psi \in C_0^\infty(\mathbf{R})$ such that $\psi = 1$ on supp φ. Let $u = \psi(H)R(\lambda + i\epsilon)f, \epsilon > 0$. By Lemma 3.3, we have only to consider the case where $X^m F(B)$ is the operator of canonical type $X^m F_m(B)$.

We introduce a notation here: $Q \in \mathcal{O}p_-^m(\lambda; X)$ if and only if $Q = Pf(B)$ for some $P \in \mathcal{O}p^m(X)$ and $f \in \mathcal{F}_-^m(\lambda)$.
By Lemma 3.2, we have

$$
(3.4) \quad
\begin{aligned}
C_0\|X^m F_m(B)\varphi(H)u\|^2 \le{}& - \mathrm{Re}\, (i[H, Q]\varphi(H)u, \varphi(H)u) \\
&+ \mathrm{Re}\, \sum_{n=2}^{N-1}(Q_n u, u) + (Q_N u, u),
\end{aligned}
$$

where $Q = X^{2m+1}\widetilde{F_{2m+1}}(B)$, $Q_n \in \mathcal{O}p_-^{2m+1-n}(\lambda; X)$ and $Q_N \in \mathcal{O}p^{2m+1-N}(X)$. Note that

$$
\begin{aligned}
&- \mathrm{Re}\, (i[H, Q]\varphi(H)u, \varphi(H)u) \\
={}& \mathrm{Im}\, \{(Q\varphi(H)f, \varphi(H)u) - (Q\varphi(H)u, \varphi(H)f)\} \\
&- 2\epsilon\, \mathrm{Re}\, (Q\varphi(H)u, \varphi(H)u).
\end{aligned}
$$

Let $\delta = t - 1$. Since Q is written as

$$Q = \sum_{i=0}^{N-1} \widetilde{P_i}^* P_i + Q_N,$$

where $P_i \in \mathcal{O}p_-^{m-i-\delta}(\lambda; X)$, $\widetilde{P}_i \in \mathcal{O}p_-^{m+t}(\lambda; X)$ and $Q_N \in \mathcal{O}p^{2m+1-N}(X)$, we have

$$|(Q\varphi(H)u, \varphi(H)f)| \leq \sum_{i=0}^{N-1} \|P_i\varphi(H)u\|^2 + C\|X^{m+t}f\|^2.$$

Here and in the sequel C denotes a constant independent of $\epsilon > 0$. $|(Q\varphi(H)f, \varphi(H)u)|$ is estimated from above in the same way. Since Q can be written as

$$Q = \sqrt{\widetilde{F_{2m+1}(B)}} X^{2m+1} \sqrt{\widetilde{F_{2m+1}(B)}}$$
$$+ [\sqrt{\widetilde{F_{2m+1}(B)}}, [\sqrt{\widetilde{F_{2m+1}(B)}}, X^{2m+1}]],$$

one can show that

$$- \operatorname{Re} \varphi(H)Q\varphi(H) \leq \sum_{i \geq 0} P_i^* P_i + Q_N,$$

with a finite number of $P_i \in \mathcal{O}p_-^{m-1/2-i}(\lambda; X)$, and $Q_N \in \mathcal{O}p^{-N}(X)$. Therefore

$$- \operatorname{Re} (Q\varphi(H)u, \varphi(H)u) \leq \sum_{i \geq 0} \|P_i u\|^2 + C\|X^{m+t}f\|^2.$$

$\operatorname{Re} (Q_n u, u)$ in (3.4) is estimated from above similarly. We then arrive at

$$(3.5) \qquad \|X^m F_m(B)\varphi(H)u\|^2 \leq \sum_{i \geq 0} \|P_i u\|^2 + C\|X^{m+t}f\|^2,$$

with a finite number of $P_i \in \mathcal{O}p_-^{m-\delta}(\lambda; X)$. In view of Lemma 3.3, one can use (3.5) with m replaced by $m - \delta$ to estimate $\|P_i u\|^2$. We repeat this procedure and finally obtain

$$\|X^m F_m(B)\varphi(H)u\|^2 \leq C(\|X^{-s}u\|^2 + \|X^{m+t}f\|^2),$$

with $s > 1/2$. The limiting absorption principle then implies the theorem (see [12]). □

§4. Resolvent Estimates (2)

In this section, we shall give the proof of Theorem 1.1 which consists in translating Theorem 3.4 in terms of Ps.D.Op.'s. Let $\varphi(H)$ be as in

the previous section. Then by Lemma 2.4,

$$X^m(1 - \varphi(H))R(\lambda + i0)X^{-m} \in \mathbf{B}, \quad \forall m \in \mathbf{R}.$$

Therefore to prove Theorem 1.1, we have only to consider $\varphi(H)R(\lambda+i0)$. For a small $\epsilon_0 > 0$, we define $C(\lambda) = \sqrt{a(\lambda) - \epsilon} + 3\epsilon_0$ so that $C(\lambda) < \sqrt{a(\lambda)}$. We take $F_-(t) \in \mathcal{F}^0$ such that $F_-(t) = 1$ if $t < C(\lambda) - \epsilon_0$, $F_-(t) = 0$ if $t > C(\lambda)$. Let $F_+(t) = 1 - F_-(t)$. Throughout this section, we shall use the Weyl calculus of Ps.D.Op.'s.

Let $P \in \mathcal{R}^k(\sqrt{a(\lambda)})$. Then for $s > -1/2$ one can take k large enough so that $X^s P <B>^{-s} X^{-s} \in \mathbf{B}$. Therefore by Theorem 3.4,

$$X^s P F_-(B)\varphi(H)R(\lambda + i0)X^{-s-t}$$

$$= X^s P <B>^{-s} X^{-s} \cdot X^s <B>^s F_-(B)\varphi(H)R(\lambda + i0)X^{-s-t} \in \mathbf{B}$$

for $s > -1/2$ and $t > 1$.

The proof of Theorem 1.1 is thus completed if we show the following assertion : For any $s > 0$, there exists $k = k(s) > 0$ such that

$$(4.1) \qquad X^s P F_+(B)X \in \mathbf{B}, \ \forall P \in \mathcal{R}^k(\sqrt{a(\lambda)}).$$

Applying Lemma 2.7 to $[X, F_+(B)]$, we see that (4.1) follows from the following assertion: For any $s > 0$, there exists $k = k(s) > 0$ such that

$$(4.2) \qquad X^s P F_+(B) \in \mathbf{B}, \ \forall P \in \mathcal{R}^k(\sqrt{a(\lambda)}).$$

Suppose (4.2) is proved for some $s \geq 0$. Let $C_1(\lambda) = \sqrt{a(\lambda) - \epsilon} + \epsilon_0$. Then by taking ϵ and ϵ_0 small enough we have

$$\frac{x \cdot \xi}{<x>} \leq C_1(\lambda) - \epsilon_0$$

on supp $p(x, \xi)$ and $t \geq C_1(\lambda) + \epsilon_0$ on supp $F_+(t)$. Let $B_1 = B - C_1(\lambda)$ and consider

$$P(t) = e^{-tB_1} F_+(B)P^* X^{2s+1} P F_+(B)e^{-tB_1}, \quad t \geq 0.$$

Let $b_1(x, \xi)$ be the symbol of B_1. Namely,

$$b_1(x, \xi) = \frac{x \cdot \xi}{<x>} - C_1(\lambda).$$

Then on supp $p(x, \xi)$, $b_1(x, \xi) < - \epsilon_0$. Let P_0 be the Ps.D.Op. with symbol

$$p_0(x, \xi) = (-b_1(x, \xi))^{1/2}p(x, \xi).$$

As is easily seen $P_0 \in \mathcal{R}^{k-1}(\sqrt{a(\lambda)})$. We now take k large enough and apply the standard symbolic calculus to obtain

$$2P_0^* X^{2s+1} P_0 = -B_1 P^* X^{2s+1} P - P^* X^{2s+1} P B_1$$

$$+ \operatorname{Re} \overset{finite}{\underset{i}{\sum}} \tilde{P}_i^* X^{2s} P_i + Q,$$

where $P_i, \tilde{P}_i \in \mathcal{R}^l(\sqrt{a(\lambda)})$, $l = l(k,s)$ satisfies $l(k,s) \to \infty$ as $k \to \infty$, and the symbol of Q is rapidly decreasing in x. We have, therefore,

$$B_1 P^* X^{2s+1} P + P^* X^{2s+1} P B_1$$

$$\le \operatorname{Re} \sum_i \tilde{P}_i^* X^{2s} P_i + Q.$$

Hence by the induction hypothesis

$$-\frac{d}{dt} P(t) \le e^{-tB_1} F_+(B)(\operatorname{Re} \sum_i \tilde{P}_i^* X^{2s} P_i + Q) F_+(B) e^{-tB_1}$$

$$\le C e^{-t\epsilon_0},$$

with some constant $C > 0$, if k is chosen large enough. Since

$$F_+(B) P^* X^{2s+1} P F_+(B) = P(0) = - \int_0^\infty \frac{d}{dt} P(t) dt,$$

one can see that $X^{s+1/2} P F_+(B) \in \mathbf{B}$, which completes the proof of Theorem 1.1.

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Christian Gérard
Centre de Mathématiques
Ecole Polytechnique
91128, Palaiseau, Cedex
France

Hiroshi Isozaki
Department of Mathematics
Osaka University
Toyonaka 560
Japan

Erik Skibsted
Matematisk Institut
Aarhus Universitet
8000 Aarhus C
Denmark