Advanced Studies in Pure Mathematics 22, 1993
Progress in Differential Geometry
pp. 333–346

Bubbling of Minimizing Sequences for Prescribed Scalar Curvature Problem

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§1. Introduction

Let (M, g) be a compact Riemannian manifold of dimension n (≥ 3) and K be a smooth function on M. In this paper we consider the problem of finding a metric conformal to g having the scalar curvature K. Any conformal metric to g can be written $\tilde{g} = u^{2/(n-2)}g$ where u is a positive smooth function on M. From the transformation law for the scalar curvature, this problem is equivalent to solve the nonlinear partial differential equation

$$
\begin{gathered}
L_g u := -\kappa\Delta_g u + Ru = Ku^{N-1}\,, \qquad u > 0 \qquad \text{in } M\,, \\
\kappa = \frac{4(n-1)}{n-2}\,, \qquad N = \frac{2n}{n-2}\,,
\end{gathered}
\tag{1.1}
$$

where Δ_g denotes the negative definite Laplacian and R is the scalar curvature of g. The linear elliptic operator L_g is called the conformal Laplacian of (M, g). In the case K is a constant the problem was first studied by Yamabe [26]. For general K the problem was presented by Kazdan-Warner [16], [17]. Since their pioneer work, the problem has drawn attentions of both geometers and analysts (for example, see [3], [11], [14]).

As proved in [15], the problem can be reduced to the case where scalar curvature R is everywhere either positive, zero or negative. Here we consider only the case that R is positive everywhere. In this case, we easily see that a necessary condition for the solvability of (1.1) is that K is positive somewhere. For such function K, the problem has the variational formulation. We consider the functional

$$E(u) = \int_M (\kappa|\nabla u|^2 + Ru^2)\, dV\,,$$

Received April 5, 1991.
Revised April 26, 1991.

on the constraint set $\mathcal{C}_K = \{ u \in H^1(M) \mid F_K(u) = 1 \}$ where

$$F_K(u) = \int_M K|u|^N dV ,$$

and dV is the volume element of (M, g). Here $H^1(M)$ is the Sobolev space of L^2 functions whose first derivatives are in $L^2(M)$. The condition that K is positive somewhere guarantees that $\mathcal{C}_K$ is not empty. We set

$$\lambda_K = \lambda_K(M, g) = \inf\{ E(u) \mid u \in \mathcal{C}_K \} .$$

From the Sobolev inequality we see that λ_K is a positive constant. As stated in [11], [14], if a function u of $\mathcal{C}_K$ achieves the infimum λ_K, then $u/\lambda_K^{(n-2)/4}$ is a smooth solution of (1.1).

Take a minimizing sequence $\{u_j\}$ of $\mathcal{C}_K$ for E, that is, $E(u_j)$ tends to λ_K as $j \to \infty$. We may assume that each u_j is non-negative almost everywhere. In fact, if $\{u_j\}$ is a minimizing sequence, then so is $\{|u_j|\}$. Since $\mathcal{C}_K$ is closed in $H^1(M)$, the infimum is achieved if $\{u_j\}$ is compact in $H^1(M)$. Aubin [2] showed that any minimizing sequence is compact in $H^1(M)$ if the strict inequality

$$\lambda_K < \Lambda/(\max K)^{2/N}$$
$$\text{where} \quad \Lambda = \lambda_{K=1}(S^n, g_0) = n(n-1)\,\mathrm{vol}\,(S^n)^{2/n} ,$$

holds (also, see [7], [14]). However, the non-existence results of Kazdan-Warner [17] and Bourguignon-Ezin [4] for the equation (1.1) lead to the fact that no minimizing sequence is compact in $H^1(M)$.

The purpose of this paper is to describe how a minimizing sequence behaves if its compactness in $H^1(M)$ fails. In section 2 we prove the following result.

Theorem A. *Let $\{u_j\} \subset \mathcal{C}_K$ be a minimizing sequence for E with $u_j \geq 0$ almost everywhere. If $\{u_j\}$ is not compact in $H^1(M)$, then there exist*

(i) *a subsequence $\{k\} \subset \{j\}$,*
(ii) *a point $a \in M$,*
(iii) *a sequence $\{r_k\}$ of $\mathbb{R}_+$ with $r_k \to 0$ as $k \to \infty$, and*
(iv) *a sequence $\{a_k\}$ of M with $a_k \to a$ as $k \to \infty$,*

satisfying the following conditions :

(1) *u_k converges to 0 in $H^1_{\mathrm{loc}}(M\backslash\{a\})$.*
(2) *The measure $u_k^N\, dV$ converges to $K(a)^{-1}\delta_a$ weakly in the sense of measures on M where δ_a denotes Dirac measure.*

(3) *The renormalized sequence* $\tilde{u}_k(x) = r_k^{(n-2)/2} u_k(\exp_{a_k}(r_k x))$ *converges to the function*

$$v(x) = \left(\frac{2^n}{\mathrm{vol}(S^n)K(a)^2}\right)^{(n-2)/2n}\left(\frac{\rho}{\rho^2+|x-b|^2}\right)^{(n-2)/2},$$

in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$ *for some* $\rho > 0$, $b \in \mathbb{R}^n$. *Here,* $\exp_{a_k}$ *denotes the exponential map of* M *at* a_k.

(4) $\lambda_K = \Lambda/(\max K)^{(n-2)/n}$.

(5) *The point* a *attains the maximum of* K.

A similar phenomenon to Theorem A has been observed in various nonlinear problems and called *bubbling* or *concentration* (for example, see [5], [9], [20], [22], [23]). P. L. Lions obtained the same results of (2) and (5) in Theorem A by using his theory of concentration-compactness principle [19]. Our proof differs from his. We only use the notion of the concentration function introduced in [19]. the statement (1), (3) and (4) of our result give a more precise description of the behavior of minimizing sequences. In the case $K \equiv 1$ the results corresponding to the statements (1)–(4) are proved in [24]. We note that the above mentioned result of Aubin can also be derived from Theorem A (4).

In section 3 we consider the case $(M, g) = (S^n, g_0)$ where g_0 denotes the standard metric of the sphere. We prove the following result.

Theorem B. *Let* $\{u_j\} \subset C_K$ *be a minimizing sequence for* E *with* $u_j \geq 0$ *almost everywheree. If* $\{u_j\}$ *is not relatively compact in* $H^1(S^n)$, *then there exist*

(i) *a subsequence* $\{k\} \subset \{j\}$,

(ii) *a sequence* $\{\psi_k\}$ *of conformal transformations on* S^n,

(iii) *a conformal transformation* ψ *on* S^n,

such that

(1) *the renormalized sequence* $\{\tilde{u}_k\}$ *defined by*
$\tilde{u}_k^{N-2} g_0 = \psi_k^*(u_k^{N-2} g_0)$
converges to a positive smooth function u_0 *in* $H^1(S^n)$, *and*

(2) *the function* u_0 *is determined by the equality*
$u_0^{N-2} g_0 = (\max K)^{-2/n} \psi^*(g_0)$.

This theorem states that on the sphere we are able to take the globally defined renormalized sequence by using conformal transformations. In the case $K \equiv 1$, Lee-Parker [18] proved an analogous result for the special minimizing sequence of approximate solutions for (1.1).

Finally, in Section 4 we state some results related to Theorems A and B.

§2. Proof of Theorem A

We first recall several known facts about the minimizing problem. Take a minimizing sequence $\{u_j\}$ of $\mathcal{C}_K$ for E with $u_j \geq 0$ almost everywhere. Since the quantity $\sqrt{E(\cdot)}$ is equivalent to the Sobolev norm $\|\cdot\|_{H^1}$, we see that $\{u_j\}$ is bounded in $H^1(M)$. Therefore, $\{u_j\}$ is compact in the weak topology of $H^1(M)$. Using the Rellich compactness theorem, we may assume

$$u_j \longrightarrow u \qquad \begin{aligned} &\text{weakly in } H^1(M), \\ &\text{strongly in } L^2(M), \\ &\text{almost everywhere on } M, \end{aligned}$$

for some $u \in H^1(M)$ with $u \geq 0$ a.e. From the general theory of calculus of variation we obtain

$$L_g u_j - \lambda_K K(x) u_j^{N-1} \longrightarrow 0 \qquad \text{in } H^{-1}(M) = (H^1(M))^*, \tag{2.1}$$

(for example, see [10], [13]).

Proposition 2.1. *If the weak limit $u \not\equiv 0$, then*

(1) *u belongs to $\mathcal{C}_K$ and achieves the infimum λ_K.*
(2) *$\{u_j\}$ converges to u in the strong topology of $H^1(M)$.*

Proof. Passing to the limit in (2.1), we get

$$L_g u = \lambda_K K(x) u^{N-1} \in H^{-1}(M). \tag{2.2}$$

From the regularity result of Brezis-Kato [6] and Trudinger [26] we see that u is a smooth function. Multiplying the both side of (2.2) by u and integrating over M, we have $E(u) = \lambda_K F_K(u)$. By the assumption that $u \not\equiv 0$, we get $F_K(u) > 0$. From the definition of λ_K we have

$$\lambda_K \leq E(u/F_K(u)^{1/N}) = E(u)/F_K(u)^{2/N} = \lambda_K F_K(u)^{2/n}.$$

This shows $F_K(u) \geq 1$. On the other hand, since E is weakly lower semi-continuous, we have

$$\lambda_K F_K(u) = E(u) \leq \liminf_{j\to\infty} E(u_j) = \lambda_K.$$

Then, we obtain $F_K(u) = 1$ and $E(u) = \lambda_K$. Since u_j converges to u weakly in $H^1(M)$, we get

$$E(u_j) = \kappa \int_M |\nabla(u_j - u)|^2 dV + E(u) = \lambda_K + o(1).$$

Hence, we have

$$\int_M |\nabla(u_j - u)|^2 dV = o(1).$$

The proof is completed.

The next theorem plays a crucial role in the proof of our main results.

Theorem 2.2. (Local convergence theorem). *Let Ω be a domain in M. Suppose that sequences $\{u_j\} \subset H^1(\Omega)$ and $\{\lambda_j\} \subset \mathbb{R}$ satisfy*

(1) $u_j \longrightarrow u$ *weakly in* $H^1(\Omega)$,
(2) $\lambda_j \longrightarrow \lambda > 0$,
(3) $L_g u_j - \lambda_j K(x)|u_j|^{N-2}u_j \longrightarrow 0$ *in* $H^{-1}(\Omega) = (H_0^1(\Omega))^*$.

If each u_j satisfies

$$(2.3) \qquad \int_\Omega K^+|u_j|^N dV \leq \epsilon \qquad \text{for some } \epsilon < \left(\frac{\Lambda}{\lambda}\right)^{n/2}(\max K)^{-(n-2)/2},$$

where $K^+(x) = \max\{K(x), 0\}$, then $u_j \longrightarrow u$ in $H^1_{\mathrm{loc}}(\Omega)$.

This theorem was proved in [24] in case K is a constant. For general K the proof in [24] also works with a slight modification.

Remark 2.3. We may replace the condition (2.3) by

$$(2.4) \qquad \int_\Omega |u_j|^N dV \leq \epsilon/(\max K).$$

We now give a proof of Theorem A. From Proposition 2.1 we know the weak limit u is identically zero if $\{u_j\}$ is not compact in $H^1(M)$.

Proof of statement (1). We take ϵ as in Theorem 2.2. We define the set $\mathcal{S}$ as

$$(2.5) \qquad \mathcal{S} = \bigcap_{r>0} \left\{ x \in M \mid \liminf_{j\to\infty} \int_{B(x,r)} K^+|u_j|^N dV \geq \epsilon \right\},$$

where $B(x,r)$ is the open geodesic ball with center x and radius r. As proved in [23], $\mathcal{S}$ is a compact subset of M. We first show that $K(x) > 0$ for any $x \in \mathcal{S}$.

Take any point y in M with $K(y) \leq 0$. Since $K^+(x)$ is Lipschitz continuous, we have

$$\begin{aligned}
&\int_{B(y,r)} K^+ u_j^N dV \\
&\qquad \leq K^+(y) \int_{B(y,r)} u_j^N dV + \max_{B(y,r)} |K^+(\cdot) - K^+(y)| \int_{B(y,r)} u_j^N dV \\
&\qquad \leq O(r)\,.
\end{aligned}$$

This leads to $y \notin \mathcal{S}$.

We next show that a subsequence $\{u_k\}$ of $\{u_j\}$ converges to 0 in $H^1_{\mathrm{loc}}(M \setminus \mathcal{S})$. If y in $M \setminus \mathcal{S}$, then there exist $r > 0$ and infinitely many j such that the inequality

$$\int_{B(y,r)} K^+ u_j^N \, dV \leq \epsilon$$

holds. By Theorem 2.2 we show that such u_j converges to 0 strongly in $H^1(B(y, r/2))$. By a diagonal subsequence argument, a subsequence $\{u_k\}$ of $\{u_j\}$ converges to 0 strongly in $H^1(\Omega)$ for each $\Omega \Subset M \backslash \mathcal{S}$.

We finally show that $\mathcal{S}$ consists of a single point. Note that we may take $\epsilon = 1 - \delta$ for any sufficiently small $\delta > 0$. For $r > 0$, we take a maximal family $\{\, B(x_1, r)\,, \cdots, B(x_I, r)\,\}$ of $I = I(r)$ disjoint geodesic balls of radius r with center $x_i \in \mathcal{S}$. By maximality $\mathcal{S}$ is covered by $B(x_1, 2r)\,, \cdots, B(x_I, 2r)$. Since each x_i lies in $\mathcal{S}$, for any $\delta > 0$

$$\int_{B(x_i,r)} K^+ u_k^N dV \geq (1 - \delta)^2\,,$$

holds if k is sufficiently large. Summing up these, we get

$$\begin{aligned}
I &\leq \frac{1}{(1-\delta)^2} \sum_{i=1}^{I} \int_{B(x_i,r)} K^+ u_k^N dV \leq \frac{1}{(1-\delta)^2} \int_M K^+ u_k^N dV \\
&\leq \frac{1}{(1-\delta)^2} (\, 1 + \int_M K^- u_k^N dV\,) \\
&\leq \frac{1}{(1-\delta)^2} (\, 1 + \|K\|_\infty \int_{M^-} u_k^N dV\,)\,,
\end{aligned}$$

where $M^- = \{\, x \in M \bigm| K \leq 0 \,\}$. Since M^- is a compact set in $M \backslash \mathcal{S}$, u_k converges to 0 in the strong H^1 topology on some neighborhood of

M^-. Thus, by taking k sufficiently large, we have

$$I \leq \frac{1+\delta}{(1-\delta)^2}.$$

This shows $\mathcal{H}^0(\mathcal{S}) \leq (1+\delta)/(1-\delta)^2$ where $\mathcal{H}^0$ denotes the 0-dimensional Hausdorff measure on M. If we choose δ small enough, we have $\mathcal{H}^0(\mathcal{S}) < 2$. Since the 0-dimensional Hausdorff measure coincides with the counting measure, either $\mathcal{S} = \{a\}$ for some $a \in M$ or $\mathcal{S}$ is empty. If $\mathcal{S}$ is empty, $\{u_k\}$ converges strongly in $H^1(M)$ because of Theorem 2.2 and the compactness of M. Thus, we obtain the desired result.

Proof of statement (2). For each k, we define the Radon measure μ_k on M by

$$\mu_k(A) = \int_A u_k^N \, dV \qquad \text{for } A \subset M.$$

Since $\{u_k\}$ is bounded in $L^N(M)$, the total variation of μ_k is uniformly bounded. Then, taking a subsequence if necessary, μ_k converges to some Radon measure μ weakly. Since $\{u_k\}$ converges to 0 in $H^1_{\mathrm{loc}}(M \backslash \{a\})$, the support of the measure μ is contained in $\{a\}$. Thus, we have $\mu = \alpha\delta_a$ for some $\alpha \geq 0$. Since each u_k lies in $\mathcal{C}_K$, we have

$$1 = \lim_{k\to\infty} \int_M K u_k^N dV = \lim_{k\to\infty} \int_M K \, d\mu_k = K(a)\alpha.$$

The proof is completed.

Proof of statement (3)–(5). We take a normal coordinate neighborhood W of a and a normal coordinate system x of M centered at a. Through this coordinate W can be regarded as a neighborhood of the origin 0 in $\mathbb{R}^n$. So we note that the metric g satisfies $g_{\alpha\beta} = \delta_{\alpha\beta} + O(|x|^2)$ in the x-coordinate. Let $B(x,r)$ be the open ball with center x and radius r and let $B(r) = B(0,r)$. We choose $R > 0$ small enough. As in [19] and [24], we introduce the concentration function

$$Q_j(t) = \sup_{y \in B(R)} \int_{B(y,t)} u_j^N \, dx \qquad \text{for } 0 \leq t \leq R.$$

Each function Q_j is continuous and non-decreasing in t, and $Q_j(0) = 0$. We fix an arbitrary small $\delta > 0$. By the definition of the point a,

$$Q_j(R) \geq \int_{B(R)} u_j^N \, dx \geq (1-\delta)/(\max K)$$

holds for sufficiently large j. By continuity of Q_j, there exist $0 < r_j < R$ and $a_j \in \overline{B(R)}$ such that

$$Q_j(r_j) = \int_{B(a_j, r_j)} u_j^N \, dx = \epsilon(1 - 2\delta)/(\max K).$$

Then we easily see that

$$r_j \longrightarrow 0 \qquad \text{and} \qquad a_j \longrightarrow 0 \qquad \text{as} \qquad j \longrightarrow \infty.$$

We set $U(j) = B(a_j/r_j, 2R/r_j) \subset \mathbb{R}^n$ and

$$\tilde{u}_j(x) = r_j^{(n-2)/2} u_j(a_j + r_j x).$$

Since a_j lies in $B(R/2)$ for sufficiently large j, we have $B(R/r_j) \subset U(j)$ which leads to $U(j) \longrightarrow \mathbb{R}^n$ as $j \to \infty$. We fix any bounded domain Ω of $\mathbb{R}^n$. Then we have

$$\int_\Omega |\nabla \tilde{u}_j|^2 dx \le (1 + C_1 R^2) \int_M |\nabla u_j|^2 dV \le C_2 < \infty,$$
$$\int_\Omega \tilde{u}_j^N dx \le (1 + C_1 R^2) \int_M u_j^N dV \le C_3 < \infty.$$

From (2.1) we have

$$\kappa \Delta_j \tilde{u}_j - R(a_j + r_j \cdot) r_j^2 \tilde{u}_j + \lambda_j K(a_j + r_j \cdot) \tilde{u}_j^{N-1} \longrightarrow 0 \qquad \text{in} \quad H^{-1}(\Omega),$$

where Δ_j is the Laplacian with respect to the metric $g_j = g(a_j + r_j)$. Since g is the standard Euclidean metric up to second order, we have

$$\kappa \Delta \tilde{u}_j - \lambda_K K(a) \tilde{u}_j^{N-1} \longrightarrow 0 \qquad \text{in} \quad H^{-1}(\Omega). \tag{2.6}$$

Using the diagonal subsequence argument, we can take a subsequence $\{k\} \subset \{j\}$ so that for each domain $\Omega \Subset \mathbb{R}^n$,

$$\tilde{u}_k \longrightarrow v \qquad \text{weakly in } H^1(\Omega),$$
$$\tilde{u}_k(x) \longrightarrow v(x) \qquad \text{almost everywhere on } \mathbb{R}^n,$$

for some $v \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $v \ge 0$ almost everywhere. Passing to the limit in (2.2), we know that v is a weak solution of

$$\kappa \Delta v + \lambda_K K(a) v^{N-1} = 0. \tag{2.7}$$

By the regularity theorem in [6], [26] and the maximum principle, v is either a positive smooth function or identically zero.

We prove that $\{\tilde{u}_k\}$ converges in $H^1_{\rm loc}(\mathbb{R}^n)$. Fix any $z \in \mathbb{R}^n$. By the definition of a_k, r_k, we have

$$\int_{B(z,1)} \tilde{u}_k^N \, dx \le \int_{B(1)} \tilde{u}_k^N \, dx = (1-2\delta)/(\max K) < (1-\delta)/(\max K).$$

By Theorem 2.2 and Remark 2.3, $\tilde{u}_k$ converges to v strongly in $H^1(B(z,1/2))$. Also, we obtain $v \not\equiv 0$, that is, v is positive everywhere.

From the result of Gidas-Ni-Nirenberg [12], all positive solutions of (2.7) are completely determined. Hence, we have

$$v(x) = (\frac{4n(n-1)}{\lambda_K K(a)})^{(n-2)/4} (\frac{\rho}{\rho^2 + |x-b|^2})^{(n-2)/2},$$

for some $\rho > 0$ and $b \in \mathbb{R}^n$. By the result of Talenti [25] on the Sobolev inequality such v satisfies the equality

$$(\int_{\mathbb{R}^n} v^N dx)^{2/N} = \frac{\kappa}{\Lambda} \int_{\mathbb{R}^n} |\nabla v|^2 \, dx.$$

Then we have

$$\int_{\mathbb{R}^n} v^N \, dx = (\frac{\Lambda}{\lambda_K K(a)})^{n/2}.$$

From the result of Aubin [2], we have the upper estimate of λ_K as

$$\lambda_K \le \Lambda/(\max K)^{2/N}.$$

Thus we have

$$\begin{aligned}
1 &\le (\frac{\Lambda}{\lambda_K (\max K)^{2/N}})^{n/2} \le (\frac{\Lambda}{\lambda_K K(a)^{2/N}})^{n/2} = K(a) \int_{\mathbb{R}^n} v^N dx, \\
&\le \liminf_{k\to\infty} \int_{\mathbb{R}^n} K(\exp_{a_k}(r_k x)) \tilde{u}_k(x)^N \, dx, \\
&\le \liminf_{k\to\infty} \int_M K|u_k|^N dV + C_1 R^2 \le 1 + C_1 R^2.
\end{aligned}$$

Hence we obtain

$$\lambda_K = \Lambda/(\max K)^{2/N}, \qquad K(a) = \max K.$$

The proof is completed.

§3. Proof of Theorem B

Since $\{u_j\}$ is not compact in $H^1(M)$, Theorem A implies that the bubbling phenomenon occurs at a point a of S^n. We may assume that a is the south pole. Let $P = (0, \cdots, 0, 1)$ be the north pole and $\pi : S^n \backslash \{P\} \longrightarrow \mathbb{R}^n$ be the stereographic projection. We take the local coordinate system defined by π. Using the similar argument to the proof of (3) in Theorem A, we can choose

(a) a subsequence $\{k\} \subset \{j\}$,
(b) a sequence $\{r_k\}$ of $\mathbb{R}_+$ with $r_k \to 0$ as $k \to \infty$, and
(c) a sequence $\{a_k\}$ of $\mathbb{R}^n$ with $a_k \to a$ as $k \to \infty$,

so that the sequence $\{r_k^{(n-2)/2} u_k(\pi^{-1}(r_k \cdot + a_k))\}$ converges in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$.

We define the mapping $\psi_k : S^n \longrightarrow S^n$ by $\psi_k(x) = \pi^{-1}(r_k \pi(x) + a_k)$. Then we easily see that each ψ_k is a conformal transformation of S^n. We set the renormalized sequence $\{\tilde{u}_k\}$ by the relation $\tilde{u}_k^{N-2} g_0 = \psi_k^*(u_k^{N-2} g_0)$. We easily obtain

$$\tilde{u}_k(x) = r_k^{(n-2)/2} u_k(\pi^{-1}(r_k \pi(x) + a_k)) .$$

Thus, we get

$$\tilde{u}_k \longrightarrow u_0 \qquad \begin{aligned} &\text{weakly in } H^1(S^n) , \\ &\text{strongly in } H^1_{\mathrm{loc}}(S^n \backslash \{P\}) , \\ &\text{almost everywhere on } S^n , \end{aligned}$$

for some $u_0 \in H^1(S^n)$ with $u_0 \geq 0$, $u_0 \not\equiv 0$.

We show the statement (2). Using the same argument as the proof of Theorem A (3), we see that the sequence $\{\tilde{u}_k\}$ satisfies

$$L_g \tilde{u}_k - \lambda_K (\max K) \tilde{u}_k^{N-1} \longrightarrow 0 \qquad \text{in} \quad H^{-1}_{\mathrm{loc}}(S^n \backslash \{P\}) . \tag{3.1}$$

Passing to the limit in (3.1), we have

$$L_g u_0 + \lambda_K (\max K) u_0^{N-1} = 0 \in H^{-1}_{\mathrm{loc}}(S^n \backslash \{P\}) .$$

By the regularity theorem in [6], [26] and the maximum principle, u_0 is a positive smooth function on $S^n \backslash \{P\}$ satisfying

$$L_g u_0 = \lambda_K (\max K) u_0^{N-1} , \tag{3.2}$$

in $S^n \backslash \{P\}$. The result of Caffarelli-Gidas-Spruck [8] implies that u_0 can be extended to a positive function defined on the whole of S^n and satisfies (3.2) on S^n. This means that the conformal metric $u_0^{N-2} g_0$ on S^n has constant scalar curvature. From the result of Obata [21], we can take a conformal transformation ψ so that the statement (2) holds.

Finally we prove that $\{\tilde{u}_k\}$ converges in $H^1(S^n)$. The result of Obata [21] leads to

$$\Lambda = \inf\{\, E(u)/\|u\|_N^2 \mid u \in H^1(S^n),\ \ u \not\equiv 0 \,\} = E(u_0)/\|u_0\|_N^2 .$$

Multiplying (3.2) by u_0 and integrating over S^n, we have

$$E(u_0) = \lambda_K (\max K) \|u_0\|_N^N .$$

Noting that the relation $\lambda_K (\max K)^{2/N} = \Lambda$, we obtain

$$\|u_0\|_N = (\max K)^{-1/N}, \qquad E(u_0) = \lambda_K .$$

Since the functional E is conformally invariant, we have

$$E(\tilde{u}_k) = E(u_k) = \lambda_K + o(1) .$$

Thus, we get

$$\int_{S^n} |\nabla(\tilde{u}_k - u_0)|^2 dV = \frac{1}{\kappa}\big(E(\tilde{u}_k) - E(u_0) \big) = o(1) .$$

The proof is completed.

§4. Some remarks

We first remark that the bubbling phenomenon in Theorem A may occur at each point where K achieves the maximum.

Proposition 4.1. *Suppose (M, g) and K satisfy the equality $\lambda_K = \Lambda/(\max K)^{2/N}$. For each $p \in M$ with $K(p) = \max K$, there exists a minimizing sequence $\{u_j\} \subset \mathcal{C}_K$ satisfying*

(1) *each u_j is a non-negative smooth function on S^n,*
(2) *$u_j^N \, dV \longrightarrow (\max K)^{-1} \delta_p$ weakly in the sense of measures on M.*

Proof. We take a radial cutoff function $\eta \in C_0^\infty(\mathbb{R}^n)$ satisfying

$$\eta(x) = \begin{cases} 1 & \text{if } \ |x| \le 1, \\ 0 & \text{if } \ |x| \ge 2, \end{cases}$$

$$0 \le \eta \le 1, \qquad |\nabla \eta| = |\partial \eta / \partial r| \le 2.$$

Take a normal coordinate of M centered at p. For small $\epsilon > 0$ and $\rho > 0$ we define

$$u_{\epsilon,\rho}(x) = \eta(\frac{x}{\epsilon})(\frac{\rho}{\rho^2 + |x|^2})^{(n-2)/2}.$$

If we choose ϵ small enough, we have $F_K(u_{\epsilon,\rho}) > 0$ for any $\lambda > 0$. The calculation in [1], [14] gives

$$\lambda_K \leq E(u_{\epsilon,\rho})/F_K(u_{\epsilon,\rho})^{2/N} \leq \Lambda(1 + C\epsilon)(1 + C\rho)/(\max K)^{2/N}.$$

Taking sequences $\epsilon_j \to 0$ and $\rho_j \to 0$ as $j \to \infty$, we obtain the sequence $u_j(x) = u_{\epsilon_j,\rho_j}(x)/F_K(u_{\epsilon_j,\rho_j})^{1/N}$ having the desired properties.

We next consider the case that (M, g) is the sphere S^n with the standard metric g_0. Consider the case that K is a constant. We remark that the functional F_K is conformally invariant if K is a constant. This implies that renormalized sequence $\{\tilde{u}_k\}$ in Theorem B is also a minimizing sequence of E. Thus we obtain the following theorem as a corollary of Theorem B.

Theorem 4.2. *If $(M, g) = (S^n, g_0)$ and K is a constant, then every minimizing sequence of $\mathcal{C}_K$ for E can be renormalized to converge in the strong topology of $H^1(S^n)$ by conformal transformations.*

On the other hand, if K is not a constant, then the following nonexistence result was proved.

Theorem 4.3 (Kazdan [15]). *If $(M, g) = (S^n, g_0)$ and K is not a constant, then the infimum λ_K is never achieved.*

Proof. Suppose that there exists a function $u \in \mathcal{C}_K$ with $E(u) = \lambda_K$. We may assume that u is a positive smooth function on S^n. From the definition of Λ, we have

$$\lambda_K = \frac{E(u)}{F_K(u)^{2/N}} \geq \frac{E(u)}{(\max K)^{2/N}\|u\|_N^2} \geq \frac{\Lambda}{(\max K)^{2/N}} = \lambda_K .$$

This leads to

$$F_K(u) = \int_{S^n} K|u|^N dV = \max K \int_{S^n} |u|^N dV .$$

Since u is positive everywhere, K is a constant. the proof is completed.

Thus we obtain the following.

Corollary 4.4. *If K is not a constant, then no minimizing sequence is compact in $H^1(S^n)$.*

Also, we observe that the renormalized sequence in Theorem B is not a minimizing one.

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