Advanced Studies in Pure Mathematics 22, 1993
Progress in Differential Geometry
pp. 321–331

On the Poles of Riemannian Manifolds of Nonnegative Curvature

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Dedicated to Professor Masahisa Adachi on his 60th birthday

Abstract.

The diameter of the set of poles on Riemannian manifolds of nonnegative sectional curvature is estimated by a constant defined by Maeda. We study the constant for elliptic paraboloids and show that our estimate is sharp.

§1. Introduction

Let M be a noncompact complete Riemannian manifold. In [2] M. Maeda defined a constant $d_o(M)$ which describes how M expands at infinity. For a point p of M let $S_t(p) = \{q \in M \ ; \ d(p,q) = t\}$ denote the metric sphere centered at p with radius $t \geq 0$ and $D_t(p)$ the diameter $\operatorname{diam} S_t(p)$ of $S_t(p)$. He defined

$$d_o(M) = \limsup_{t \to \infty} \frac{D_t(p)^2}{t}$$

and showed that d_0 does not depend on the choice of p and the distance between two poles does not exceed $d_o(M)$ if M is of nonnegative sectional curvature, where a point q of M is said to be a pole if the exponential mapping $\exp_q : T_qM \to M$ is a diffeomorphism. In this paper we shall improve his estimate as follows:

Theorem 1.1. *Let M be a noncompact and complete Riemannian manifold of nonnegative sectional curvature. Then the distance between two poles does not exceed $d_o(M)/8$.*

Received March 12, 1991.
Revised May 29, 1991.

The distance of two poles of an elliptic paraboloid defined by

$$x_0^2/a_0 + x_1^2/a_1 = 2x_2$$

with $0 < a_0 < a_1$ goes towards $d_0/8$ as $a_0 \to 0$. Hence our estimate is sharp.

We note that elliptic paraboloids are Liouville surfaces. So, by deforming elliptic paraboloids through Liouville surfaces, we can construct various surfaces of nonnegative curvature with two poles and $d_o < \infty$.

On the other hand, M. Tanaka [4] studied the poles on surfaces of revolution and showed that the center of revolution is the only pole if and only if d_o is finite. Hence we conjecture

Conjecture 1.2. *If the constant $d_o(M)$ is finite for a Riemannian manifold M of nonnegative sectional curvature, then the number of poles of M is finite or at most two.*

In §2 we shall give a proof of Theorem 1.1. In §3 we shall study the behavior of geodesics on elliptic paraboloids using the elliptic coordinates to show that two umbilic points are the poles. In §4 we shall give the exact value of d_0 for an elliptic paraboloid and show that our estimate is sharp.

§2. The proof of Theorem 1.1

In this section let M denote a Riemannian manifold of nonnegative sectional curvature and all geodesics of M are assumed to be parametrized by arc length.

Lemma 2.1. *Let $\gamma : [0,\infty) \to M$ be a ray emanating from p, i.e., $\gamma|[0,t]$ is minimizing for any $t > 0$. Let $\alpha : [0,s] \to M$ be a geodesic from $\gamma(t_o)$ to q and θ the angle $-\dot\gamma(t_o)$ and $\dot\alpha(0)$ make. Then*

$$t_o - s\cos\theta \le d(p,q).$$

Proof. First we assume that α is a minimizing geodesic. Toponogov's comparison theorem for a triangle $\triangle\gamma(t_0)\gamma(t)q$ with $t > t_0$ implies

$$\begin{aligned} d(q,\gamma(t))^2 &\le d(q,\gamma(t_0))^2 + d(\gamma(t),\gamma(t_0))^2 \\ &\qquad - 2d(q,\gamma(t_0))d(\gamma(t),\gamma(t_0))\cos(\pi-\theta) \\ &= s^2 + (t-t_0)^2 + 2s(t-t_0)\cos\theta \\ &= ((t-t_0) + s\cos\theta)^2 + s^2(1-\cos^2\theta). \end{aligned}$$

Hence we get

$$d(q,\gamma(t))-((t-t_0)+s\cos\theta)\leq\frac{s^2(1-\cos^2\theta)}{d(q,\gamma(t))+((t-t_0)+s\cos\theta)}=O(1/t).$$

If $d(p,q) < t_0 - s\cos\theta$, then there is a positive constant ϵ such that $d(p,q) < t_0 - s\cos\theta - \epsilon$. Hence we get

$$\begin{aligned} t = d(p,\gamma(t)) &\leq d(p,q) + d(q,\gamma(t)) \\ &< (t_0 - s\cos\theta - \epsilon) + (t - t_0 + s\cos\theta) + O(1/t) \\ &= t - \epsilon + O(1/t) < t \end{aligned}$$

for large t, which contradicts the assumption that γ is a ray.

If α is not minimizing, then we divide α into minimizing arcs $\alpha|[s_{i-1}, s_i]$ $(i = 1\dots k)$ with $0 = s_0 < s_1 < \dots < s_k = s$. We consider a polygon $\bar\gamma(t)\bar\alpha(s_0)\dots\bar\alpha(s_k)$ in the two-dimensional Euclidean space which corresponds to $\gamma(t)\alpha(s_0)\dots\alpha(s_k)$ with

$$\begin{aligned} d(\bar\gamma(t),\bar\alpha(s_i)) &= d(\gamma(t),\alpha(s_i)) \qquad (i = 0\dots k) \\ d(\bar\alpha(s_{i-1}),\bar\alpha(s_i)) &= d(\alpha(s_{i-1}),\alpha(s_i)) \ (i = 1\dots k). \end{aligned}$$

Then Toponogov's comparison theorem implies that the polygon is convex. Therefore we easily get

$$\begin{aligned} &\lim_{t\to\infty}\{t - t_0 + s\cos\theta - d(\alpha(s),\gamma(t))\} \\ &\quad = \lim_{t\to\infty}\{t - t_0 + s\cos\theta - d(\bar\alpha(s),\bar\gamma(t))\} \geq 0. \end{aligned}$$

Hence the assertion is clear.

Let p_1 and p_2 be poles of M and $a = d(p_1,p_2)$. Let $\gamma_1,\gamma_2 : [0,\infty) \to M$ be two rays with $\gamma_1(0) = p_1$, $\gamma_1(a) = p_2$, $\gamma_2(0) = p_2$ and $\gamma_2(a) = p_1$. Let $q_1 = \gamma_1(t)$, $q_2 = \gamma_2(t+a)$ and $v = d(q_1,q_2)$. Then $v \leq D_{p_1}(t)$ because $d(p_1,q_1) = d(p_1,q_2) = t$. Let q be the middle point of a minimizing geodesic between q_1 and q_2. Let $\theta_i = \angle p_1q_iq$ $(i = 1,2)$. Then Toponogov's comparison theorem for a triangle $\triangle p_1q_2q$ implies

$$d(p_1,q)^2 \leq t^2 + v^2/4 - tv\cos\theta_2. \tag{2.1}$$

And from Lemma 2.1, we get

$$t - (v/2)\cos\theta_1 \leq d(p_1,q) \tag{2.2}$$

$$(t+a) - v\cos\theta_2 \leq d(p_2,q_1) = t - a.$$

Hence we have

$$2a \le v \cos\theta_2.$$

Since $\limsup v^2/t \le d_o < \infty$, we may assume the left side of (2.2) is positive. Therefore (2.1) combined with (2.2) yields

$$(t - (v/2)\cos\theta_1)^2 \le t^2 + v^2/4 - tv\cos\theta_2,$$

which is reduced to

$$v^2\cos^2\theta_1 - 4tv\cos\theta_1 + 4tv\cos\theta_2 - v^2 \le 0. \tag{2.3}$$

Toponogov's comparison theorem for a triangle $\Delta p_2 q_1 q$ gives

$$d(p_2, q)^2 \le (t-a)^2 + v^2/4 - (t-a)v\cos\theta_1. \tag{2.4}$$

And from Lemma 2.1 we get

$$(t+a) - (v/2)\cos\theta_2 \le d(p_2, q). \tag{2.5}$$

Hence (2.4) combined with (2.5) yields

$$((t+a) - (v/2)\cos\theta_2)^2 \le (t-a)^2 + v^2/4 - (t-a)v\cos\theta_1,$$

which is reduced to

$$v^2\cos^2\theta_2 - 4(t+a)v\cos\theta_2 + 4(t-a)v\cos\theta_1 + 16at - v^2 \le 0.$$

Deleting $v^2\cos^2\theta_2$, we get

$$\frac{4(t-a)v\cos\theta_1 + 16at - v^2}{4(t+a)} \le v\cos\theta_2.$$

We substitute this inequality to (2.3). Then (2.3) becomes

$$v^2\cos^2\theta_1 - \frac{8atv}{t+a}\cos\theta_1 + \frac{16at^2 - 2tv^2 - av^2}{t+a} \le 0.$$

Deleting $v^2\cos^2\theta_1$, we get

$$2t - \frac{(2t+a)v^2}{8at} \le v\cos\theta_1. \tag{2.6}$$

Applying Toponogov's comparison theorem to a triangle $\Delta p_2 q_1 q_2$, we get

$$\begin{aligned}(t+a)^2 &\le (t-a)^2 + v^2 - 2(t-a)v\cos\theta_1.\\ 4at &\le v^2 - 2(t-a)v\cos\theta_1.\end{aligned}$$

Substituting (2.6) to this inequality, we get

$$4at \leq v^2 - 2(t-a)\left(2t - \frac{(2t+a)v^2}{8at}\right)$$

$$4at + 4t(t-a) \leq v^2\left(1 + \frac{(t-a)(2t+a)}{4at}\right).$$

Dividing both sides by t^2 and letting $t \to \infty$, we get

$$4 \leq \frac{d_o}{2a},$$

i.e.,

$$a \leq \frac{d_o}{8}.$$

§3. Geodesics on elliptic paraboloids

H. von Mangoldt studied the behavior of geodesics of hyperboloids in [3] and stated that his method could be applied to show that two umbilic points of an elliptic paraboloid are the only poles. In this section we study the behavior of geodesics of elliptic paraboloids and prove his assertion. Our argument mainly relies on [1, §3.5]. Let us consider an elliptic paraboloid

$$M = \{(x_0, x_1, x_2) \in \mathbf{R}^3 \ ; \ x_0^2/a_0 + x_1^2/a_1 = 2x_2\}$$

with $0 < a_0 < a_1$.

We introduce the elliptic coordinates $(u_1, u_2) \in]a_0, a_1[\times]a_1, \infty[$:

$$x_0^2 = \frac{a_0(a_0-u_1)(a_0-u_2)}{a_1-a_0}$$
$$x_1^2 = \frac{a_1(a_1-u_1)(a_1-u_2)}{a_0-a_1}$$
$$x_2 = \frac{u_1+u_2-a_0-a_1}{2}.$$

Note that $u_1 = u_2 = a_1$ corresponds to the umbilic points

$$(\pm\sqrt{a_0}\sqrt{a_1-a_0}, 0, \frac{a_1-a_0}{2})$$

and the distance between two umbilic points of M is equal to

$$\sqrt{a_1-a_0}\sqrt{a_1} + a_0 \log\left|\sqrt{\frac{a_1-a_0}{a_0}} + \sqrt{\frac{a_1}{a_0}}\right|.$$

The first fundamental form is expressed in the elliptic coordinates as follows:

$$ds^2 = (-u_1 + u_2)(U_1 du_1^2 + U_2 du_2^2),$$

where

$$U_i = \frac{(-1)^i u_i}{f(u_i)} \ ; \ f(u_i) = 4(a_0 - u_i)(a_1 - u_i).$$

For a real number γ, $a_0 < \gamma < a_1$ or $a_1 < \gamma$, we consider a coordinate change

$$\begin{aligned} du_1' &= \sqrt{-u_1+\gamma}\sqrt{U_1}\, du_1 \pm \sqrt{u_2-\gamma}\sqrt{U_2}\, du_2 \\ du_2' &= \frac{\sqrt{U_1}}{\sqrt{-u_1+\gamma}}\, du_1 \mp \frac{\sqrt{U_2}}{\sqrt{u_2-\gamma}}\, du_2. \end{aligned} \tag{3.1}$$

Then

$$ds^2 = du_1'^2 + (-u_1+\gamma)(u_2-\gamma)du_2'^2.$$

From this expression of the first fundamental form, we see that u_1'-parameter curves are geodesics. Hence we get

Theorem 3.1 ([1, Theorem 3.5.5]). *In the elliptic coordinates geodesics of M are characterized by*

$$\frac{\sqrt{U_1}}{\sqrt{-u_1+\gamma}}\, \dot{u}_1 \mp \frac{\sqrt{U_2}}{\sqrt{u_2-\gamma}}\, \dot{u}_2 = 0,$$

together with the condition $E(u, \dot{u}) = \text{const}$, where $E = ds^2/2$ is the energy function. Here γ is a constant with value in $]a_0, a_1[$ or $]a_1, \infty[$.

The constant γ is called the parameter of the geodesic.

Corollary 3.2 (cf. [1, Corollary 3.5.6]). *Denote by $(T_1M)'$ the open and dense subset of the unit tangent bundle T_1M formed by those unit tangent vectors which are tangent to a geodesic with parameter γ, $\gamma \in]a_0, a_1[$ or $\gamma \in]a_1, \infty[$. Define $F : (T_1M)' \to \mathbf{R}$ in elliptic tangent coordinates $(u, \dot{u})$ by*

$$F(u, \dot{u}) = (-u_1 + u_2)(u_2 U_1 \dot{u}_1^2 + u_1 U_2 \dot{u}_2^2).$$

Then F is a first integral of the geodesic flow on T_1M. And if $u(t) = (u_1(t), u_2(t))$ is a geodesic parametrized by arc length with parameter γ, then $F(u(t), \dot{u}(t)) = \gamma$.

If we denote by $\mu(X)$ the angle between $X \in (T_1M)'$ and the u_1-parameter line through $\tau_M X$, then we may also write

$$F(X) = u_1(\tau_M X)\sin^2 \mu(X) + u_2(\tau_M X)\cos^2 \mu(X),$$

where $\tau_M : T_1M \to M$ denotes the canonical projection.

We now go to the co-geodesic flow ϕ_t on the cotangent bundle T^*M. The cotangent coordinates (u,v) are related to the tangent coordinates $(u,\dot{u})$ by

$$\dot{u}_i = g^{ij}(u)v_j = \frac{v_i}{(-u_1+u_2)U_i},\ i=1,2.$$

The functions E, F correspond to the following functions on $(T^*M)'$:

$$E^*(u,v) = \frac{1}{2(-u_1+u_2)}\left(\frac{1}{U_1}v_1^2 + \frac{1}{U_2}v_2^2\right)$$
$$F^*(u,v) = \frac{1}{(-u_1+u_2)}\left(\frac{u_2}{U_1}v_1^2 + \frac{u_1}{U_2}v_2^2\right).$$

Theorem 3.3 (cf. [1, Theorem 3.5.7]). *For $\gamma \in]a_0,a_1[$ or $\gamma \in]a_1,\infty[$ the ϕ_t-invariant set $\{F^* = \gamma\}$ in the total unit cotangent space T_1^*M consists of two embedded 2-dimensional cylinders which we denote by $T_\gamma^\pm$.*

We distinguish the cases $\gamma \in]a_1,\infty[$ or $\gamma \in]a_0,a_1[$ as type I *and* II*, respectively.*

The flow lines on the cylinder $T_\gamma^\pm$ of type I *correspond, under the projection $\tau_M^* : T_1^*M \to M$, to geodesics which monotonously wind x_2-axis, while descending to tangent to a u_1-parameter line $\{u_2 = \gamma\}$ then ascending to $x_2 = \infty$. The cylinder of Type* I *corresponds, under τ_M^*, to $\{(u_1,u_2)\ ;\ a_0 \le u_1 \le a_1, \gamma \le u_2\}$.*

The flow lines on the cylinders of type II *correspond, under τ_M^*, to geodesics which oscillate between the two u_2-parameter lines $\{u_1 = \gamma\}$. The cylinder of type* II *corresponds, under τ_M^*, to $\{(u_1,u_2)\ ;\ a_0 \le u_1 \le \gamma, a_1 \le u_2\}$.*

As γ goes towards a_0, the cylinders $T_\gamma^\pm$ become degenerate, i.e., we get two embedded curves given by the unit tangent vectors to the curve $M \cap \{x_1 = 0\}$.

Proof. Let

$$\begin{aligned} u_1 &= a_0\cos^2\psi_1 + a_1\sin^2\psi_1 \\ u_2 &= a_1 + \psi_2^2 \end{aligned} \tag{3.2}$$

with $(\psi_1,\psi_2) \in \mathbf{R}/2\pi \times \mathbf{R}$. Then equations $2E^* = 1$ and $F^* = \gamma$ yield

$$v_1^2 = U_1(\gamma - u_1)\ ;\ v_2^2 = U_2(u_2-\gamma).$$

For the cotangent coordinates (Ψ_1, Ψ_2) corresponding to (ψ_1, ψ_2) we get

$$\Psi_1 = v_1 \frac{\partial u_1}{\partial \psi_1} = 2(a_1 - a_0) v_1 \sin \psi_1 \cos \psi_1$$
$$\Psi_2 = v_2 \frac{\partial u_2}{\partial \psi_2} = 2 v_2 \psi_2.$$

Hence

$$\begin{aligned} \Psi_1^2 &= (\gamma - u_1) u_1 \\ \Psi_2^2 &= (u_2 - \gamma) u_2 / (u_2 - a_0). \end{aligned} \tag{3.3}$$

With $u_i = u_i(\psi_i)$ as in (3.2), we get $\Psi_i = \Psi_i(\psi_i)$.

Consider now type I, i.e., $a_1 < \gamma$. Then $\Psi_2 = \Psi_2(\psi_2)$ describes a simple non-closed curve in the (ψ_2, Ψ_2)-plane. $\Psi_1 = \Psi_1(\psi_1)$ yields two non-closed curves in the (ψ_1, Ψ_1)-plane, one with $\Psi_1 > 0$, the other with $\Psi_1 < 0$, since $\Psi_1(\psi_1)$ is always $\neq 0$. However, in $T_1^* M$, $\Psi_1 = \Psi_1(\psi_1)$, $\psi_1 \in S^1$, describes two closed curves, since the (u, v) are periodic in ψ_1. Thus, $T_1^* M \cap \{F^* = \gamma\}$ consists of two embedded cylinders.

The discussion of type II, i.e., $a_0 < \gamma < a_1$, is similar.

Let $P(t, \gamma) = (-t)(\gamma - t)(a_0 - t)(a_1 - t)$. For $\gamma \in]a_0, a_1[$ define $\omega_2 = (\omega_{12}, \omega_{22})$ with

$$\omega_{12} = 4 \int_{a_0}^{\gamma} \frac{-t(\gamma - t)}{\sqrt{P(t, \gamma)}} \, dt \; ; \quad \omega_{22} = 4 \int_{a_0}^{\gamma} \frac{-t}{\sqrt{P(t, \gamma)}} \, dt.$$

For $\gamma \in]a_1, \infty[$ define $\omega_2 = (\omega_{12}, \omega_{22})$ with

$$\omega_{12} = 4 \int_{a_0}^{a_1} \frac{-t(\gamma - t)}{\sqrt{P(t, \gamma)}} \, dt \; ; \quad \omega_{22} = 4 \int_{a_0}^{a_1} \frac{-t}{\sqrt{P(t, \gamma)}} \, dt.$$

In each case, put $-\omega_{21} : \omega_{22} = \omega(\gamma)$.

Theorem 3.4 (cf. [1, Theorem 3.5.10]). *The geodesic flow on each of the invariant cylinders $T_\gamma^\pm$ in appropriate coordinates, is equivalent to the linear flow of slope $\omega(\gamma)$ on the flat cylinder.*

Proof. Let $\gamma \in]a_0, a_1[$. The differentials du_1', du_2' in (3.1) determine functions $u_1'(u_1, u_2), u_2'(u_1, u_2)$ on $T_\gamma^\pm$, i.e.,

$$u_1' = \int_{a_0}^{u_1} \sqrt{-u_1 + \gamma} \sqrt{U_1} \, du_1 \pm \int_{a_1}^{u_2} \sqrt{u_2 - \gamma} \sqrt{U_2} \, du_2 \; ;$$
$$u_2' = \int_{a_0}^{u_1} \frac{\sqrt{U_1}}{\sqrt{-u_1 + \gamma}} \, du_1 \mp \int_{a_1}^{u_2} \frac{\sqrt{U_2}}{\sqrt{u_2 - \gamma}} \, du_2.$$

Denote by T_ω the flat cylinder $\mathbf{R}^2/\mathbf{Z}\omega_2$. Then the functions $u' = u'(u)$ give a transformation from $T_\gamma^\pm$ to T_ω. The geodesic lines go into the u_1'-parameter lines.

The case $\gamma \in]a_1, \infty[$ is treated in exactly the same manner.

Theorem 3.5 (cf. [1, Theorem 3.5.16]). *The flow-invariant set $\{F^* = a_1\} \cap T_1^*M$ is formed by those flow lines which, when projected into M, yield the geodesics which pass through the umbilic points. And the umbilic points are the only poles of M.*

Proof. Solve equations

$$E^* = 1/2 \ ; \ F^* = a_1 \tag{3.4}$$

at a point $p \in M$ which does not lie on the x_0x_2-plane. Since (3.4) is equivalent to

$$\Psi_1^2 = (a_1 - u_1)u_1 \ ; \ \Psi_2^2 = (u_2 - a_1)u_2/(u_2 - a_0), \tag{3.5}$$

we see that there are four solutions of the equation in T_1^*M. On the other hand there are at least four geodesics between p and the umbilic points even if we take the directions of geodesics in consideration. If $\gamma \neq a_1$, the equations (3.3) and (3.5) have no common solutions. Hence each solution of (3.5) corresponds to a geodesic between p and an umbilic point and there is only one geodesic between p and each umbilic point. Therefore umbilic points are poles. From Theorem 3.3 we easily see any geodesic half-lines with $F^* \neq a_1$ are not rays.

§4. The constant d_o for an elliptic paraboloid

In this section we give the exact value of the constant d_o for a paraboloid M in $\mathbf{R}^3$ defined by an equation

$$x_0^2/a_0 + x_1^2/a_1 = 2x_2$$

with $0 < a_0 < a_1$ in §3.

Let $M(t) = \{(x_0, x_1, x_2) \in M \ ; \ x_2 = t\}$ and let $p = (0,0,0), q_0(t) = (\sqrt{2a_0t}, 0, t)$ and $q_1(t) = (0, \sqrt{2a_1t}, t)$. Let $\ell_0(t)$ (resp. $\ell_1(t)$) denote the distance between p and $q_0(t)$ (resp. $q_1(t)$) along $M \cap \{x_1 = 0\}$ (resp. $\{x_0 = 0\}$). Then

$$\ell_i(t) = \sqrt{t^2 + \frac{a_it}{2}} + \frac{a_i}{2}\log\left|\sqrt{\frac{2t}{a_i}} + \sqrt{\frac{2t}{a_i} + 1}\right| \ (i = 1, 2).$$

And

$$d(p, M(t)) = \ell_0(t).$$

Let $\ell_0(t) = \ell_1(t')$. Then the metric circle $S_{\ell_0(t)}(p)$ is located between two planes $\{x_2 = t\}$ and $\{x_2 = t'\}$ and

$$\left|\operatorname{diam} S_{\ell_0(t)}(p) - \operatorname{diam} M(t)\right| \leq 2(\ell_1(t) - \ell_1(t')). \tag{4.1}$$

Lemma 4.1. $\displaystyle\lim_{t\to\infty} \frac{2 \operatorname{diam} M(t)}{\operatorname{length} M(t)} = 1.$

Proof. Let c be a minimizing geodesic of M from $q_0(t)$ to $-q_0(t)$. Let $t_2 = \min_c x_2$ and $t_1 = \sqrt{2a_1t_2}$. Let

$$C_1 = \{(x_0, x_1, x_2) \in M \ ; \ x_1 = t_1 \text{ and } x_2 \leq t\}$$
$$C_2 = \{(x_0, x_1, x_2) \in M \ ; \ x_2 = t_2 \text{ and } x_1 \geq 0\}$$

Since c satisfies $x_1 \circ c \leq t_1$ and $x_2 \circ c \geq t_2$ (cf. §3),

$$\operatorname{length}(c) \geq \operatorname{length} C_i \ (i = 1, 2).$$

We note

$$\begin{aligned}
\operatorname{length} M(t) &= \sqrt{2t}\int_0^{2\pi} \sqrt{a_0 \sin^2\theta + a_1 \cos^2\theta}\, d\theta, \\
\operatorname{length} C_1 &= \sqrt{t^2 + \frac{a_1 t}{2}} + \frac{a_1}{2} \log\left|\sqrt{\frac{2t}{a_1}} + \sqrt{\frac{2t}{a_1} + 1}\right| \\
&\quad - \sqrt{t_2^2 + \frac{a_1 t_2}{2}} - \frac{a_1}{2} \log\left|\sqrt{\frac{2t_2}{a_1}} + \sqrt{\frac{2t_2}{a_1} + 1}\right| \\
\operatorname{length} C_2 &= \sqrt{\frac{t_2}{2}} \int_0^{2\pi} \sqrt{a_0 \sin^2\theta + a_1 \cos^2\theta}\, d\theta.
\end{aligned}$$

If $\limsup t_2/t = 1$, then

$$\begin{aligned}
1 \geq \limsup_{t\to\infty} \frac{2 \operatorname{diam} M(t)}{\operatorname{length} M(t)} &= \limsup_{t\to\infty} \frac{2 \operatorname{length}(c)}{\operatorname{length} M(t)} \\
&\geq \limsup_{t\to\infty} \frac{2 \operatorname{length} C_2}{\operatorname{length} M(t)} = 1.
\end{aligned}$$

Suppose $\limsup t_2/t < 1$. Then

$$\operatorname{length} C_1 \sim \text{const.} t \gg \text{const.} \sqrt{t} \sim \frac{1}{2} \operatorname{length} M(t).$$

as $t \to \infty$. Hence

$$\operatorname{diam} M(t) \geq \operatorname{length}(c) \geq \operatorname{length} C_1 > \frac{1}{2} \operatorname{length} M(t)$$

for large t, which contradicts $\operatorname{diam} M(t) \leq \frac{1}{2} \operatorname{length} M(t)$.

Since $\ell_1(t) - \ell_1(t') \sim \text{const.} \log t$ and $\operatorname{diam} M(t) \sim \text{const.}\sqrt{t}$ as $t \to \infty$, the inequality (4.1) combined with Lemma 4.1 yields

Lemma 4.2. $\displaystyle\lim_{t\to\infty} \frac{\operatorname{diam} S_{\ell_0(t)}(p)}{\operatorname{diam} M(t)} = 1.$

From Lemma 4.2 we easily get

Proposition 4.3. $\displaystyle d_o(M) = \frac{1}{2}\left(\int_0^{2\pi} \sqrt{a_0 \sin^2\theta + a_1 \cos^2\theta}\, d\theta\right)^2.$

Hence the distance between two umbilic points goes towards

$$a_1 = \lim_{a_0 \to 0} d_0/8$$

as $a_0 \to 0$, so the estimate in Theorem 1.1 is sharp.

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