Advanced Studies in Pure Mathematics 22, 1993
Progress in Differential Geometry
pp. 299–308

The Length Function of Geodesic Parallel Circles

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Dedicated to Professor T. Otsuki on his 75th birthday

§0. Introduction

The isoperimetric inequalities for a simply closed curve C on a Riemannian plane Π (i.e., a complete Riemannian manifold homeomorphic to $\mathbf{R}^2$) was first investigated by Fiala in [1] and later by Hartman in [2]. These inequalities were generalized by the first named author in [3],[4] for a simply closed curve on a finitely connected complete open surface and by both authors in [5] for a simply closed curve on an infinitely connected complete open surface. Here a noncompact complete and open Riemannian 2-manifold M is called *finitely connected* if it is homeomorphic to a compact 2-manifold without boundary from which finitely many points are removed, and otherwise M is called *infinitely connected.* Fiala and Hartman investigated certain properties of geodesic parallel circles $S(t) := \{x \in \Pi \ ; \ d(x, C) = t\}$, $t \geq 0$ around C of a Riemannian plane Π in order to prove the isoperimetric inequalities, where d denotes the Riemannian distance function. Fiala proved in [1] that if a Riemannian plane Π and a simple closed curve C on Π are *analytic*, then $S(t)$ is a finite union of piecewise smooth simple closed curves except for t in a discrete subset of $[0, \infty)$ and its length $L(t)$ is *continuous* on $[0, \infty)$. If Π and C are *not analytic but smooth*, then $L(t)$ is *not always continuous* as pointed out by Hartman in [2]. What is worse is that $S(t)$ does not always admit its length. Under the assumption of low differentiability of Π and C, Hartman proved that $S(t)$ is a finite union of piecewise smooth simple closed curves except for t in a closed subset of Lebesgue measure zero in $[0, \infty)$. This result was recently extended by the authors [5] to an arbitrary given simply closed curve C in an arbitrary given complete, connected, oriented and noncompact Riemannian 2-manifold M.

Received January 7, 1991.
Revised April 5, 1991.

The normal exponential map along C induces a local chart and a function $L(t)$ for all $t \geq 0$ is well defined with the aid of this local chart. As mentioned above, $L(t)$ for all $t \geq 0$ defines the length of $S(t)$ whenever $S(t)$ is a finite union of piecewise smooth simple closed curves. However we do not know the geometric meaning of $L(t)$ for the other t-values. Hartman introduced a certain monotone function $J: [0, \infty) \to \mathbf{R}$ by using this local chart and proved in Theorem 6.2 ; [2] that the following function

$$(*) \qquad H(t) := J(t) + L(t)$$

is absolutely continuous on every compact interval of $[0, \infty)$.

The purpose of the present article is to extend the absolute continuity of H as defined in $(*)$ for an arbitrary given simple closed curve C in an arbitrary given connected, complete, noncompact and oriented Riemannian 2-manifold M. The cut locus and focal locus to C are essential in our discussion. In §1 we introduce the notations concerning with the cut points and focal points to C as used in [2],[5]. Under our situation $M \setminus C$ has at most two components. The type of cut locus and focal locus changes as the number of components of $M \setminus C$. In §2 we deal with the simpler case where $M \setminus C$ has two components and prove the absolute continuity of $(*)$ in this case (see Theorem 2.2). We also need to modify the definition of $J(t)$ in the case where $M \setminus C$ is connected. In §3 we prove the absolute continuity of $(*)$ in the case where $M \setminus C$ is connected (see Theorem 3.2).

§1. Preliminaries

From now on let M be a connected, oriented, complete and noncompact Riemannian 2-manifold and C a smooth simply closed curve on M. Since our discussion proceeds in the same manner as developed by Hartman , we shall employ the same terminologies as used in [2],[5]. Let L_0 be the length of C. A point on C is expressed as $z_0(s)$ with respect to the arclength parameter $s \in [0, L_0]$. $z_0(s)$ and other functions of s will be considered periodic of period of L_0 for convenience. Let g be the Riemannian metric on M and N a unit normal field along C with $N_0 = N_{L_0}$. A map $z : \mathbf{R} \times [0, L_0] \to M$ is defined by

$$z(t, s) := \exp_{z_0(s)} tN_s$$

where $\exp_p$ is the exponential map of M at p. If $|t|$ is sufficiently small, then z gives a coordinate system (t, s) and $g\left(\frac{\partial z}{\partial t}, \frac{\partial z}{\partial t}\right) = 1$ holds around

C and $g\left(\frac{\partial z}{\partial t}, \frac{\partial z}{\partial s}\right) = 0$ follows from Gauss Lemma. For every $s \in [0, L_0]$ let $\gamma_s: R \to M$ be a geodesic with $\gamma_s(t) = z(t,s)$ and $e_s(t)$ a unit parallel vector field along γ_s with $e_s(0) = \frac{\partial z}{\partial s}(0,s)$. For each s let $Y_s(t)$ denote the Jacobi field along γ_s with $Y_s(0) = e_s(0)$, $g(Y_s(t), \gamma_s'(t)) = 0$. By setting $f(t,s) = g(Y_s(t), e_s(t))$, we have $f(0,s) = 1$, $f_t(0,s) = \kappa(s)$ and $g\left(\frac{\partial z}{\partial s}, \frac{\partial z}{\partial s}\right) = f^2(t,s)$, where $\kappa(s)$ is the geodesic curvature of C at $z_0(s)$ and $f_t = \frac{\partial f}{\partial t}$. Since Y_s is a Jacobi field we have $f_{tt}(t,s) + G(z(t,s))f(t,s) = 0$, where $f_{tt} = \frac{\partial^2 f}{\partial t^2}$.

Let $P(s)$ (respectively $N(s)$) denote the least positive (respectively the largest negative) t with $f(s,t) = 0$, or $P(s) = +\infty$ (respectively $N(s) = -\infty$) if there is no such zero. If $P(s_0) < +\infty$ (respectively $N(s_0) > -\infty$), then P (respectively N) is smooth around s_o and $z(P(s_0), s_0)$, (respectively $z(N(s_0), s_0)$ is called the first positive (respectively negative) focal point to C along γ_{s_0}.

A unit speed geodesic $\sigma: [0,\ell] \to M$ is called a *C-segment* iff $\sigma(0) \in C$ and $d(\sigma(t),\mathrm{C}) = t$ holds for all $t \in [0,\ell]$. Every C-segment is a subarc of some γ_s . Let $\rho(s) := \sup\{t > 0\ ;\ d(\gamma_s(t), C) = t\}$ and $\nu(s) := \inf\{t < 0\ ;\ d(\gamma_s(t), C) = -t\}$. $\rho(s)$ (respectively $\nu(s)$) is the cut point distance to C along $\gamma_s|[0,\infty)$ (respectively $\gamma_s|(-\infty, 0]$). $z(\rho(s), s)$ is called a cut point to C along γ_s and $\gamma_s|[0, \rho(s)]$ is a maximal C-segment contained in $\gamma_s|[0,\infty)$. A cut point is a first focal point of a C-segment or the intersection of at least two distinct C-segments.

A cut point at C is called *normal* if it is the endpoint of exactly two distinct C-segments and is not a first focal point along either of them. A cut point to C which is not normal is called *anormal.* An anormal cut point $z(\rho(s), s)$ (or $z(\nu(s), s)$) is called *totally nondegenerate* iff $z(\rho(s), s)$ (or $z(\nu(s), s)$) is not a first focal point to C along any C-segment ending at $z(\rho(s), s)$ (or $z(\nu(s), s)$). An anormal cut point is called *degenerate* iff it is not totally nondegenerate. A number $t > 0$ is called *anormal* iff there exists a value $s \in \rho^{-1}(t)$ (or $s \in \nu^{-1}(-t)$) such that $z(t,s)$ (or $z(-t,s)$) is anormal. It $t > 0$ is not anormal, then t is called *normal.* Also $t > 0$ is called *exceptional* iff it is either anormal or normal but there exists an s such that $\rho(s) = t$ (or $\nu(s) = -t$) and $\rho' = 0$ (or $\nu' = 0$) at s. A positive number t is by definition *non-exceptional* iff it is not exceptional.

§2. The case where C bounds a domain

Throughout this section let $M \setminus C$ have two components and M_1 the component containing $\{z(\rho(s), s)\ ;\ \rho(s) < \infty\}$. Note that the sets $\{z(\rho(s), s)\ ;\ \rho(s) < \infty\}$ and $\{z(\nu(s), s)\ ;\ \nu(s) > -\infty\}$ have no common point. We only restrict to consider M_1 , since the same discussion holds for $M \setminus M_1$.

We begin with the discussion of degenerate cut points that was not discussed in [2]. It seems to the authors that the lack of degenerate cut points in [2] would cause unclearness in the proof of Theorem 6.2 in [2]. The following Lemma 2.1 is useful to prove our results.

Lemma 2.1. *The set $F = \{s \in [0, L_0]\ ;\ \rho(s) < P(s)$, but $z(\rho(s), s) \in M_1$ is a degenerate cut point along some C-segment$\}$ is of Lebesgue measure zero.*

Proof. It suffices for the proof to show that for any $s \in F$ there exists a positive δ such that $F \cap (s - \delta, s + \delta)$ is of Lebesgue measure zero. Let $s_0 \in F$ and set $p = z(\rho(s_0), s_0)$. Choose a small positive ϵ such that B_ϵ is an open normal convex ϵ-ball around p. For each $s \in [0, L_0]$ with $z(\rho(s), s) = p$ let s' denote the common point of ∂B_ϵ and $\gamma_s([0, \rho(s_0)])$. The circle ∂B_ϵ is naturally oriented. Define the oriented open subarc from s'_1 to s'_2 of ∂B_ϵ by $(s'_1,\ s'_2)$. For each $s \in [0, L_0] \setminus \{s_0\}$ with $z(\rho(s), s) = p$ let $D(s'_0, s')$ (respectively $D(s', s'_0)$) be the disk domain bounded by three arcs $\gamma_{s_0}|[\rho(s_0) - \epsilon, \rho(s_0)]$, $\gamma_s|[\rho(s_0) - \epsilon, \rho(s_0)]$ and (s'_0, s') (respectively $D(s', s'_0)$). Since $\rho(s_0) < P(s_0)$, there exist $s_+, s_- \in [0, L_0]$ such that $z(\rho(s_+), s_+) = z(\rho(s_-), s_-) = p$ and such that $D_+ := D(s'_0, s'_+)$ and $D_- := D(s'_-, s'_0)$ are disjoint and they do not contain any C-segment passing through p. Let (π, NC, M) be the normal bundle over C with projection π, total space NC and base space M. Since p is not a focal point to C along γ_{s_0},there exist a neighborhood V of $\rho(s_0) \cdot \dot\gamma_{s_0}(0)$ in NC and a neighborhood U of p in M such that the restriction $\exp_V$ of the normal exponential map to V is a diffeomorphism of V onto U. Since p is a degenerate cut point, there is a C-segment ending at p along which p is the first focal point to C. Suppose $P(s_+) = \rho(s_+)$. Choose a positive number ϵ_1 such that U contains $z(\rho(s), s)$ and $z(P(s), s)$ for all $s \in [s_+ - \epsilon_1, s_+ + \epsilon_1]$. From construction of D_+ we can choose a positive number $\delta_1 < \epsilon_1$ such that if $z(\rho(s_1), s_1) = z(\rho(s), s)$ for $s_1 \in [0, L_0]$, $s \in (s_0, s_0 + \delta_1)$, then $s = s_1$ or $s_1 \in (s_+ - \epsilon_1, s_+)$. Let $v : (s_+ - \epsilon_1, s_+) \to (s_1, s_0 + \delta_1)$ be defined as

$$v(s) := z_0^{-1} \circ \pi \circ (\exp_V^{-1})(z(\rho(s), s))$$

If $s \in (s_+ - \epsilon, s_+)$ satisfies $P'(s) = 0$ and $P(s) = \rho(s)$, then $v'(s) = 0$, and hence s is a critical point of v. Let $K \subset (s_+ - \epsilon_1, s_+)$ be the set of all critical points of v. If $s \in (s_0, s_0 + \delta_1)$ is an element of F, then there exists an $s_1 \in [0, L_0]$ such that $z(\rho(s), s) = z(\rho(s_1), s_1)$, $P(s_1) = \rho(s_1)$. It follows from the choice of δ_1 and Proposition 2.1 in [5] that $P'(s_1) = 0$ and $s_1 \in (s_+ - \epsilon_1, s_+)$. Therefore we find an $s_1 \in K$ such that $z(\rho(s), s) = z(\rho(s_1), s_1) = z(P(s_1), s_1)$. This fact means that $(s_0, s_0 + \delta_1) \cap F$ is contained entirely in $v(K)$. The Sard Theorem implies that $v(K)$ is of Lebesgue measure zero. If $\rho(s_+) < P(s_+)$, then there exists a positive number δ such that $(s_0, s_0 + \delta) \cap F = \emptyset$. Summing up these discussion we observe that there exists a positive number δ_1 such that $(s_0, s_0 + \delta_1) \cap F$ is of measure zero.

An analogous discussion applies to D_- to prove that $(s_0 - \delta_1', s_0) \cap F$ is of measure zero for some positive number δ_1'. This completes the proof of Lemma 2.1.

Let $D := \{(t, s)\ ;\ 0 \le t < \rho(s), 0 \le s \le L_0\}$ and $\chi(t, s)$ the characteristic function of D such that $\chi(s, t) = 1$ or 0 according as $(t, s) \in D$ or not. For any $t \ge 0$ set

$$L(t) := \int_0^{L_0} \chi(t, s) f(t, s)\, ds$$

This $L(t)$ is the length of $S(t) = \{x \in M_1 | d(x, C) = t\}$ if t is a non-exceptional value. We define for $t \ge 0$ the set $Q(t)$ as follows.

$$Q(t) := \{s \in \rho^{-1}(t)\ ;\ z(s, t) \text{ is normal and } \rho'(s) = 0\}.$$

$Q(t)$ has the property that elements in it are pairwise disjoint, and hence it is of Lebesgue measure zero except for an at most countable set of $[0, \infty)$. We define for $t \ge 0$ the function

$$J(t) := \sum_{0 \le u \le t} \int_{Q(u)} f(u, s)\, ds.$$

Note that L and also J is discontinuous at $t = t_0$ iff the Lebesgue measure of $Q(t_0)$ is positive.

In order to prove Theorem 2.2 we shall need some basic tools from measure theory which is referred to [6]. Let h be a continuous function of bounded variation defined on a closed interval $[a, b]$. Then the function h defines a *Lebesgue-Stieltjes measure* Λ_h such that $\Lambda_h((x, y])$ for each subinterval $(x, y]$ of $[a, b]$ equals the total variation of h on $[x, y]$. It is known that any Borel set B in $[a, b]$ is Λ_h-measurable. For each Lebesgue measurable set $S \subset \mathbf{R}$, $|S|$ denotes its Lebesgue measure.

Theorem 2.2. *The function $H(t) = L(t) + J(t)$ is absolutely continuous on any compact subinterval of* $[0, \infty)$.

Proof. Let $[a, b]$ be a compact subinterval of $[0, \infty)$. In order to prove the theorem we shall show that for any positive ϵ there exists a positive $\eta = \eta(\epsilon, a, b)$ such that if $\delta_1, \delta_2, \ldots, \delta_k$ are non-overlapping subintervals of $[a, b]$, then

$$\text{(2.1)} \qquad \sum_{i=1}^{k} |\delta_i H| < (L_0 + 2)\epsilon \text{ whenever } \sum_{i=1}^{k} |\delta_i| < \eta$$

where $\delta_i H = H(\tau) - H(\sigma)$, $|\delta_i| = \tau - \sigma$ if $\delta_i = (\sigma, \tau]$. Let $\epsilon > 0$ be fixed. It follows from Proposition 3.1 in [5] that the set $T_b := \{s \in [0, L_0]\ ;\ \rho(s) \leq b, z(\rho(s), s)$ is a totally nondegenerate anormal point$\}$ is finite. Let $c = c(b)$ be a constant satisfying

$$|f(t,s)| \leq c,\ |f_t(t,s)| \leq c,\ (t,s) \in [0,b] \times [0, L_0]$$

By Lemma 2.1 the set F^ϵ defined by

$$F^\epsilon = \{s \in [0, L_0]\ ;\ \rho(s) \leq b,\ s \in F,\ f(\rho(s), s) \geq \epsilon/2\}$$

is compact and of Lebesgue measure zero. Here there exists a set V^ϵ with $|V^\epsilon| < \epsilon/c$ consisting of a finite number of open subintervals of $[0, L_0]$ such that $V^\epsilon \supset T_b \cap F^\epsilon$. Let Q^ϵ be the set

$$Q^\epsilon := \{s \in [0, L_0]\ ;\ \rho(s) \leq b, f(\rho(s), s) \leq \epsilon/2\}.$$

Since Q^ϵ is compact, Q^ϵ can be covered by a set S^ϵ consisting of a finite number of open subintervals of $[0, L_0]$ on which $f(\rho(s), s) < 3\epsilon/4$. Then the set $R^\epsilon = [0, L_0] - (S^\epsilon \cup V^\epsilon)$ consists of a finite number of closed subintervals $I_1, \ldots, I_p$ of $[0, L_0]$. It follows from construction of R^ϵ and from Proposition 2.2 in [5] that ρ is smooth at each point $s \in R^\epsilon$ if $\rho(s) \leq b$. Hence the function $\rho_b :=$**Max** $\{\rho, b\}$ is Lipschitz continuous on each closed intervals $I_j, j = 1, \ldots, p$. In particular the restriction ρ_j of ρ_b to I_j is of bounded variation. If Λ_j denotes the Lebesgue-Stieltjes measure defined by ρ_j, then we observe from Corollary 3.1 in [2] that

$$\text{(2.2)} \qquad \sum_{j=1}^{k} \Lambda_j(\rho_j^{-1}(\delta_i)) = \int_\sigma^\tau n(r)\, dr$$

where $n(r)$ is the Lebesgue summable function defined by the number of the elements of the set $\{s \in R^\epsilon\ ;\ \rho(s) = r\}$. Let $O(i)$ be an open set containing $R(i) = \cup_{\sigma<t\le\tau} Q(t)$ such that $|O(i) - R(i)| < |\delta_i|$. Setting $S(i) = \rho^{-1}(\delta_i)$, we define

$$\begin{aligned} S_1 &= (S(i) - R(i)) \cap O(i) \\ S_2 &= (S(i) - R(i)) \cap [\{s\ ;\ f(\rho(s), s) < \epsilon\} \cup V^\epsilon] \\ S_3 &= (S(i) - R(i)) - (S_1 \cup S_2). \end{aligned}$$

Making use of the inequality (6.20) in [2], we obtain

$$\tag{2.3} \begin{aligned} |\delta_i H| &\le \sum_{j=1}^{3} \int_{S_j} f(\rho(s), s)\, ds + 2cL_0|\delta_i| \\ &\le c|\delta_i| + \epsilon|S(i)| + c|V^\epsilon \cap S(i)| + c|S_3| + 2cL_0|\delta_i| \end{aligned}$$

Since $S_3 \subset R^\epsilon$ and $S_3 \cap O(i) = \emptyset$, ρ is smooth at each point of S_3 and $|\rho'| \ge c_1$ on S_3 holds for some positive constant $c_1 = c_1(\epsilon, a, b)$. From the property of the Lebesgue-Stieltjes measure Λ_j we obtain

$$\sum_{j=1}^{p} \Lambda_j(I_j \cap S_3) \ge c_1 \sum_{j=1}^{p} |I_j \cap S_3| = c_1|R^\epsilon \cap S_3| = c_1|S_3|.$$

From (2.2) and the above inequality, we get

$$\tag{2.4} |S_3| \le c_1^{-1} \sum_{j=1}^{p} \Lambda_j(I_j \cap S_3) \le c_1^{-1} \sum_{j=1}^{p} \Lambda_j(I_j \cap \rho^{-1}(\delta_i)) = c_1^{-1} \int_\sigma^\tau n(r)\, dr.$$

From inequalities (2.3) and (2.4) we have

$$\tag{2.5} \sum_{i=1}^{k} |\delta_i H| \le c(1 + 2L_0) \sum_{i=1}^{k} |\delta_i| + (L_0 + 1)\epsilon + cc_1^{-1} \sum_{i=1}^{k} \int_{\delta_i} n(r)\, dr.$$

The inequality (2.5) implies that we can find a positive $\eta = \eta(\epsilon, a, b)$ satisfying (2.1). Note that the function $n(r)$ is Lebesgue summable.

§3. The case where C bounds no domain

We deal with the case where a closed curve C does not bound any domain of M. Our situation means that there exists a cut point $p \in M$

to C such that $p = z(\rho(s_1), s_1) = z(\nu(s_2), s_2)$ for some $s_1, s_2 \in [0, L_0]$. Three types of cut points to C appear. A cut point p to C is by definition of ρ-type (respectively ν-type) iff all C-segments ending at p are tangent to N (respectively to $-N$) at their starting points. A cut point p to C is of *mixed type* iff $p = z(\rho(s_1), s_1) = z(\nu(s_2), s_2)$ for some $s_1, s_2 \in [0, L_0]$. For a mixed type cut point to C the normality, anormality, degeneracy and all other properties are well defined by the same manner as before. These properties are defined for t-value where $S(t)$ contains a mixed type cut point having the corresponding properties. Let F_+, F_- be the sets

$$F_+ := \{s \in [0, L_0] \ ; \ \rho(s) < P(s), \text{ but } z(\rho(s), s) \text{ is a degenerate cut point}\}$$

$$F_- := \{s \in [0, L_0] \ ; \ \nu(s) > Q(s), \text{ but } z(\nu(s), s) \text{ is a degenerate cut point}\}.$$

Since the proof of Lemma 2.1 is done by a local discussion in a small convex ball around a cut point, we obtain the following lemma by a similar discussion.

Lemma 3.1. *The set $F := F_+ \cup F_-$ is of Lebesgue measure zero.*

Let $D_+ := \{(t, s) \ ; \ 0 \le t < \rho(s), s \in [0, L_0]\}$ and $D_- := \{(t, s) \ ; \ \nu(s) < t \le 0, s \in [0, L_0]\}$. We then define two functions L_+ and L_- on $[0, \infty)$ by

$$L_+(t) := \int_0^{L_0} \chi_+(t, s) f(t, s)\, ds$$

$$L_-(t) := \int_0^{L_0} \chi_-(t, s) f(-t, s)\, ds$$

where $\chi_+(t, s)$ and $\chi_-(t, s)$ are the characteristic functions of D_+ and D_- respectively. If $t > 0$ is non-exceptional, then the function

$$L(t) := L_+(t) + L_-(t)$$

is nothing but the length of $S(t) = \{x \in M \ ; \ d(x, C) = t\}$.

Note that if $t_0 > 0$ is a normal exceptional value, then $S(t_0)$ consists of a set of piecewise smooth curves. However the length of $S(t_0)$ is not necessarily equal to $L(t_0)$ but equal to

$$L(t_0) + \frac{1}{2}\{\int_{Q_+(t_0)} f(t_0, s)\, ds + \int_{Q_-(t_0)} f(-t_0, s)\, ds\}.$$

Here we set

$$Q_+(t) := \{s \in \rho^{-1}(t)\ ;\ z(t,s) \text{ is normal and } \rho'(s) = 0\},$$
$$Q_-(t) := \{s \in \nu^{-1}(-t)\ ;\ z(-t,s) \text{ is normal and } \nu'(s) = 0\}.$$

In order to define $J(t)$ in this case we need to set

$$J_+(t) := \sum_{0 \le u \le t} \int_{Q_+(t)} f(u, s)\, ds,$$
$$J_-(t) := \sum_{0 \le u \le t} \int_{Q_-(t)} f(-u, s)\, ds.$$

We then define $J(t)$ as follows.

$$J(t) := J_+(t) + J_-(t).$$

By a similar discussion as in the proof of Theorem 2.2 we obtain the following

Theorem 3.2. *The function $H(t) = L(t) + J(t)$ is absolutely continuous on any compact subinterval of $[0, \infty)$.*

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