Advanced Studies in Pure Mathematics 22, 1993
Progress in Differential Geometry
pp. 265–297

A Geometric Construction of Laguerre-Forsyth's Canonical Forms of Linear Ordinary Differential Equations

Yutaka Se-ashi

§0. Introduction

The purpose of this paper is to reformulate the fundamental results of E.J. Wilczynski's book [5] by applying E. Cartan's method of moving frames.

Let E be a vector bundle over a 1-dimensional manifold M. By taking a local coordinate system t in M and a moving frame $\{e_1, \cdots, e_r\}$ of E, we express each cross section s of E in the form:

$$s = \sum_{\alpha=1}^{r} y_\alpha e_\alpha, \tag{0.1}$$

where $y_1, \cdots, y_r$ are functions on M. Let $\mathcal{D}$ be a system of homogeneous linear ordinary differential equations on E of order n given in the form:

$$\left(\frac{d}{dt}\right)^n y_\alpha + \sum_{k=1}^{n}\sum_{\beta=1}^{r} a_{\alpha\beta}^{(k)}(t)\left(\frac{d}{dt}\right)^{n-k} y_\beta = 0, \quad \alpha = 1, \cdots, r. \tag{0.2}$$

Corresponding to (0.2), we define $r \times r$ matrices $A^{(1)}(t), \cdots, A^{(n)}(t)$ by

$$A^{(k)}(t) = (a_{\alpha\beta}^{(k)}(t)), \quad k = 1, \cdots, n. \tag{0.3}$$

For another local coordinate system in M and another moving frame, we also express $\mathcal{D}$ in a similar way as (0.2) and give $r \times r$ matrices as (0.3).

In modern terminology, Wilczynski showed the following facts:

(A) There exists a pair $(t, \{e_\alpha\})$ satisfying the following condition

$$A^{(1)}(t) = 0 \quad \text{and} \quad \mathrm{Tr}\ A^{(2)}(t) = 0. \tag{L.F}$$

Received March 25, 1991.
Revised April 12, 1991.

(B) If both $(t, \{e_\alpha\})$ and $(t', \{e'_\alpha\})$ satisfy the condition (L.F), then there exists $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ and $C = (c_{\alpha\beta}) \in GL(r, \mathbb{R})$ such that

$$t' = \frac{at+b}{ct+d},$$

$$e'_\beta = (ct+d)^{n-1} \sum_{\alpha=1}^{r} c_{\alpha\beta} e_\alpha, \quad \beta = 1, \cdots, r.$$

(C) Let $(t, \{e_\alpha\})$ be a pair satisfying the condition (L.F). For each integer k, $2 \leqq k \leqq n$, let θ_k be the cross section of the vector bundle $\overset{k}{\otimes} T(M)^* \otimes \mathrm{End}(E)$ which corresponds to the $r \times r$ matrix valued k covariant tensor field

$$\sum_{j=0}^{k-2} (-1)^j \frac{(2k-j-2)!(n-k+j)!}{j!(k-j-1)!} \left(\frac{d}{dt}\right)^j A^{(k-j)}(t)(dt)^k$$

under the trivialization with respect to $\{e_\alpha\}$. Then the definition of θ_k does not depend on the choice of $(t, \{e_\alpha\})$, and hence θ_k is an invariant of $\mathcal{D}$. Moreover $\theta_2, \cdots, \theta_n$ form a fundamental system of invariants of $\mathcal{D}$.

He studied mainly the case $r = 1$ and gave

$$\frac{k!(k-2)!}{n!(2k-3)!}\theta_k, \quad k = 3, \cdots, n,$$

as a fundamental system of invariants. (In this case, the invariant θ_2 automatically vanishes.) On the other hand, for the case $r \geqq 2$, he did not give a fundamental system of invariants. Following him, we will call the system (0.2) with the condition (L.F) a Laguerre-Forsyth's canonical form of $\mathcal{D}$.

Let us proceed to the description of the contents of this paper. Let $\{s_1, \cdots, s_{nr}\}$ be a fixed family of linearly independent solutions of $\mathcal{D}$. We define a map κ of M into the Grassmann manifold $\mathrm{Gr}(\mathbb{R}^{nr}, r)$ as follows: By using a pair $(t, \{e_\alpha\})$, we define an $nr \times r$ matrix $Y_1(x) = (y_{\alpha\beta}(x))$, $x \in M$, by

$$s_\alpha(x) = \sum_{\beta=1}^{r} y_{\alpha\beta}(x) e_{\beta x}, \quad \alpha = 1, \cdots, nr, \tag{0.4}$$

and an $nr \times nr$ matrix $Y(x)$, $x \in M$, by

$$
\begin{aligned}
&Y(x) = (Y_1(x), \cdots, Y_n(x)), \\
&Y_k(x) = \frac{1}{(k-1)!}\left(\frac{d}{dt}\right)^{k-1} Y_1(x), \quad k = 2, \cdots, n.
\end{aligned}
\tag{0.5}
$$

Then we define $\kappa(x)$ to be the point of $\mathrm{Gr}(\mathbb{R}^{nr}, r)$ corresponding to the r-dimensional subspace of $\mathbb{R}^{nr}$ spanned by the column vectors of $Y_1(x)$.

The general linear group $GL(nr, \mathbb{R})$ acts transitively on $\mathrm{Gr}(\mathbb{R}^{nr}, r)$. Let K be the isotropy subgroup of $GL(nr, \mathbb{R})$ at an origin of $\mathrm{Gr}(\mathbb{R}^{nr}, r)$. Then we regard $GL(nr, \mathbb{R})$ as a principal K-bundle over $\mathrm{Gr}(\mathbb{R}^{nr}, r)$ with projection π_0. Let P be the principal K-bundle over M defined by

$$P = \{(x, Z) \in M \times GL(nr, \mathbb{R}) | \kappa(x) = \pi_0(Z)\},$$

and let ω be the $gl(nr, \mathbb{R})$ valued 1-form on P induced by the Maurer-Cartan form of $GL(nr, \mathbb{R})$.

The main result is to show the unique existence of a normal reduction Q of P. Here the normal reduction Q is defined by the condition that the restriction of ω to Q is $\mathfrak{h} + \mathfrak{m}$ valued, where $\mathfrak{h}$ (resp. $\mathfrak{m}$) is the subalgebra (resp. the subspace) of $gl(nr, \mathbb{R})$ defined in §2. The restriction of ω to Q is decomposed into the two components $\chi_{\mathfrak{h}}$ and $\chi_{\mathfrak{m}}$. The 1-form $\chi_{\mathfrak{h}}$ is a flat Cartan connection in Q and the 1-form $\chi_{\mathfrak{m}}$ induces differential invariants in Q corresponding to $\theta_1, \cdots, \theta_n$. Using the absolute parallelism induced by the Cartan connection $\chi_{\mathfrak{h}}$, we give a vector field on Q whose integral curve corresponds to a Laguerre-Forsyth's canonical form of $\mathcal{D}$. We apply Cartan's reduction method developed in [1] to the construction of the normal reduction Q.

In Appendix, we construct Q as a reduction of the frame bundle $\mathcal{F}(J^{n-1}(E))$ of $J^{n-1}(E)$, where $J^{n-1}(E)$ is the $(n-1)$-th jet bundle of E. The connection form of the affine connection of $J^{n-1}(E)$ which is associated with $\mathcal{D}$ takes the place of the 1-form ω, in this case.

Preliminary remarks

1. Throughout this paper, we always assume the differentiability of class C^∞, though the argument goes through in complex analytic category with suitable modifications.

2. As we are mainly concerned with local properties of linear ordinary differential equations, base manifolds will be assumed to be simply connected unless otherwise stated.

3. We will frequently write any $nr \times nr$ matrix A in the form:

$$A = (A_1 \cdots A_n) \quad \text{or} \quad A = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \dots & A_{nn} \end{pmatrix},$$

where A_i, $1 \leqq i \leqq n$ and A_{ij}, $1 \leqq i, j \leqq n$ are $nr \times r$ matrices and $r \times r$ matrices respectively. We sometimes simply write $A = (A_i)$ or $A = (A_{ij})$.

4. As to Lie groups and principal bundles, we use the standard notations and terminology as in [2]. Especially let K be a Lie group and P a principal K-bundle over a base manifold M. For $A \in K$, R_A denotes the right translation induced by A. Let $\mathfrak{k}$ be the Lie algebra of K. For $X \in \mathfrak{k}$, X^* denotes the vertical vector field on P induced by the 1-parameter group of right translations $\{R_{\exp tX}\}$. The vector field X^* is called the fundamental vector field corresponding to X.

5. Cartan connections. Let H/H_0 be a homogeneous space of a Lie group H over its closed subgroup H_0. Let $\mathfrak{h}$ and $\mathfrak{h}_0$ be the Lie algebras of H and H_0 respectively. Let Q be a principal H_0-bundle over a manifold M, where dim M = dim H/H_0, and let ω be a $\mathfrak{h}$ valued 1-form on Q. Then we say that ω is a Cartan connection on Q of type H/H_0 if the following conditions are satisfied:

(C.1) $\omega(v) \neq 0$ for every non-zero tangent vector v of Q.

(C.2) $R_{A*}\omega = \mathrm{Ad}(A^{-1})\omega, \quad A \in H_0$.

(C.3) $\omega(X^*) = X$ for every $X \in \mathfrak{h}_0$.

Let Ω be the 2-form on Q defined by

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega].$$

The 2-form Ω is called the curvature form of the Cartan connection ω. If $\Omega = 0$, then the Cartan connection ω is said to be flat.

§1. Characteristic maps

1.1. Characteristic maps

Let E be a vector bundle over a 1-dimensional manifold M. As noted in Preliminary remarks, we assume that M is simply connected. Let $\mathcal{D}$ be a system of homogeneous linear ordinary differential equations on E as considered in Introduction.

Let Sol($\mathcal{D}$) denote the space of all solutions of $\mathcal{D}$. As is well known, Sol($\mathcal{D}$) is an nr-dimensional vector space. For a moment, let us fix a basis $\{s_1, \cdots, s_{nr}\}$ of Sol($\mathcal{D}$).

Taking a local coordinate system t in M and a moving frame $\{e_1, \cdots, e_r\}$ of E, we define, for each $x \in M$, the matrices $Y_1(x), \cdots, Y_n(x)$ and $Y(x)$ by (0.4) and (0.5). It is well known that $Y(x)$ is non-singular, and in particular the r column vectors of $Y_1(x)$ are linearly independent.

Taking another coordinate system t' in M and another moving frame $\{e'_1, \cdots, e'_r\}$ of E, we define matrices $Y'_1(x), \cdots, Y'_n(x)$ and $Y'(x)$ in the same way. We write $e'_1, \cdots, e'_n$ in the form:

$$e'_{\beta x} = \sum_{\alpha=1}^{r} c_{\alpha\beta}(x) e_{\alpha x}, \quad x \in M,$$

where $C(x) = (c_{\alpha\beta}(x)) \in GL(r, \mathbb{R})$. The proof of the next lemma is straightforward.

Lemma 1.1. (1) $Y'_1(x) = Y_1(x)^t C(x)^{-1}$, $x \in M$.
(2) *For every* k, $2 \leqq k \leqq n$, $Y'_k(x)$ *can be written in the form:*

$$Y'_k(x) = \left(\frac{dt}{dt'}\right)^{k-1} Y_k(x)^t C(x)^{-1} + \sum_{j=1}^{k-1} Y_j(x) D_{jk}(x),$$

where $D_{1k}(x), \cdots, D_{k-1\ k}(x)$ *are* $r \times r$ *matrices.*

Let $\mathrm{Gr}(\mathbb{R}^{nr}, r)$ be the Grassmann manifold consisting of all r-dimensional subspaces of $\mathbb{R}^{nr}$. For each $x \in M$, let $\kappa(x)$ $(\in \mathrm{Gr}(\mathbb{R}^{nr}, r))$ be the subspace of $\mathbb{R}^{nr}$ spanned by the r column vectors of $Y_1(x)$. By (1) of Lemma 1.1, the definition of $\kappa(x)$ does not depend on the choice of $(t, \{e_\alpha\})$. It is not difficult to see that the assignment $x \to \kappa(x)$ gives an immersion of M into $\mathrm{Gr}(\mathbb{R}^{nr}, r)$. We call the map κ the characteristic map of $\mathcal{D}$ (corresponding to the basis $\{s_\alpha\}$ of $\mathrm{Sol}(\mathcal{D})$).

1.2. The induced principal bundles P

Consider the general linear group $GL(nr, \mathbb{R})$ acting on $\mathbb{R}^{nr}$ on the left. The group $GL(nr, \mathbb{R})$ acts on the Grassmann manifold $\mathrm{Gr}(\mathbb{R}^{nr}, r)$ in a natural way. Let o be the point of $\mathrm{Gr}(\mathbb{R}^{nr}, r)$ corresponding to the subspace spanned by the first r vectors of the natural basis of $\mathbb{R}^{nr}$. The isotropy subgroup K of $GL(nr, \mathbb{R})$ at o is given by:

$$K = \{A \in GL(nr, \mathbb{R}) | A_{21} = \cdots = A_{n1} = 0\}.$$

We denote by π_0 the natural projection of $GL(nr, \mathbb{R})$ onto $\mathrm{Gr}(\mathbb{R}^{nr}, r)$ under the identification $\mathrm{Gr}(\mathbb{R}^{nr}, r) \simeq GL(nr, \mathbb{R})/K$.

Using the characteristic map κ, we define a submanifold P of the direct product $M \times GL(nr, \mathbb{R})$ by

$$P = \{(x, Z) \in M \times GL(nr, \mathbb{R}) | \kappa(x) = \pi_0(Z)\}.$$

We denote by π_M (resp. π_G) the projection of P onto M (resp. the projection of P onto $GL(nr, \mathbb{R})$). Clearly P is a principal K-bundle over M with the projection π_M.

The $gl(nr, \mathbb{R})$-valued 1-form ω on P. Let ω be the pull back of the Maurer-Cartan form of $GL(nr, \mathbb{R})$ by the projection π_G. The 1-form ω possesses the following properties:

(ω.1) $\omega(v) \neq 0$ for every non-zero tangent vector v of P.

(ω.2) $R_{A*}\omega = \mathrm{Ad}(A^{-1})\omega$ for all $A \in K$.

(ω.3) $\omega(X^*) = X$ for all $X \in \mathfrak{k}$, where $\mathfrak{k}$ stands for the Lie algebra of K.

The standard cross sections. We fix a local coordinate system t in M and a moving frame $\{e_\alpha\}$ of E. We define $Y(x) \in GL(nr, \mathbb{R})$, $x \in M$, by (0.5). Let σ be the map of M into $M \times GL(nr, \mathbb{R})$ defined by

$$\sigma(x) = (x, Y(x)), \quad x \in M.$$

Obviously σ is a cross section of P, that is, $\sigma(x) \in P$ for all $x \in M$. The cross section σ will be called the standard cross section of P (corresponding to $(t, \{e_\alpha\})$).

1.3. The bundle homomorphism $\varepsilon : P \to \mathcal{F}(E)$

Let $\mathcal{F}(E)$ be the frame bundle of E. Let $\varepsilon : K \to GL(r, \mathbb{R})$ be the homomorphism defined by

$$\varepsilon(A) = {}^tA_{11}^{-1}, \quad A \in K.$$

Here, we will define a natural bundle homomorphism ε of P onto $\mathcal{F}(E)$ corresponding to the group homomorphism $\varepsilon : K \to GL(r, \mathbb{R})$.

For each $p \in P$, we write p in the form:

$$p = (x, Z), \quad x \in M, \quad Z \in GL(nr, \mathbb{R}).$$

We further write Z in the form: $Z = (Z_1, \cdots, Z_n)$.

Lemma 1.2. *There exists a unique basis $\{\varepsilon_1(p), \cdots, \varepsilon_r(p)\}$ of E_x such that*

$$s_\alpha(x) = \sum_{\beta=1}^{r} z_{\alpha\beta}\varepsilon_\beta(p), \quad \alpha = 1, \cdots, nr,$$

where $Z_1 = (z_{\alpha\beta})$.

Proof. We take a local coordinate system t in M and a moving frame $\{e_\alpha\}$ of E. Let $Y(x)$ be the $nr \times nr$-matrix defined by (0.5). Since $\pi_0(Z_1) = \pi_0(Y_1(x))$, we can write Z_1 in the form:

$$Z_1 = Y_1(x)C, \quad \text{where} \quad C \in GL(r, \mathbb{R}).$$

Let $\{\varepsilon_1(p), \cdots, \varepsilon_r(p)\}$ be the basis of E_x defined by

$$\varepsilon_\beta(p) = \sum_{\alpha=1}^{r} c'_{\alpha\beta} e_{\alpha x}, \quad 1 \leqq \beta \leqq r$$

where ${}^tC^{-1} = (c'_{\alpha\beta})$. It is easy to see that $\{\varepsilon_\alpha(p)\}$ is the desired basis of E_x. The uniqueness is obvious. Q.E.D.

Let $\varepsilon : P \to \mathcal{F}(E)$ be the map defined by

$$\varepsilon(p) = \{\varepsilon_\alpha(p)\}.$$

It is easily checked that $\varepsilon : P \to \mathcal{F}(E)$ is a bundle homomorphism corresponding to $\varepsilon : K \to GL(r, \mathbb{R})$.

Remark. Let σ be the standard cross section of P corresponding to $(t, \{e_\alpha\})$. Then,

$$\varepsilon(\sigma(x)) = \{e_{\alpha x}\} \quad \text{for all} \quad x \in M. \tag{1.1}$$

1.4. Expressions in local coordinate systems

We fix a local coordinate system t in M and a moving frame $\{e_\alpha\}$ of E. Here, we will give a local expression of the 1-form ω by using $(t, \{e_\alpha\})$. We express $\mathcal{D}$ as (0.2).

Proposition 1.3. *Let σ be the standard cross section of P corresponding to $(t, \{e_\alpha\})$. For each $x \in M$, we write the $nr \times nr$ matrix $(\sigma^*\omega)(d/dt)_x$ in the form:*

$$(\sigma^*\omega)\left(\frac{d}{dt}\right)_x = (X_{ij}(x)),$$

where $X_{ij}(x)$ are $r \times r$ matrices. Then,

$$X_{k+1\ k}(x) = kI, \ k = 1, \cdots, n-1. \tag{1.2}$$

$$X_{n-k+1\ n}(x) = -\frac{(n-k)!}{(n-1)!}\ {}^tA^{(k)}(t(x)), \ k = 1, \cdots, n. \tag{1.3}$$

$$X_{ij}(x) = 0 \text{ for the remaining pairs of indices } (i, j). \tag{1.4}$$

Proof. We define $Y(x) = (Y_k(x)) \in GL(nr, \mathbb{R})$, $x \in M$, by (0.5). Then we have the following equalities:

$$\frac{dY_k}{dt}(x) = kY_{k+1}(x), \quad k = 1, \cdots, n-1,$$
$$\frac{dY_n}{dt}(x) = -\sum_{k=1}^{n} \frac{(n-k)!}{(n-1)!} Y_{n-k+1}(x)\ {}^tA^{(k)}(t(x)).$$

On the other hand, since $Y(x)^{-1}dY/dt(x) = X(x)$, we have

$$\frac{dY_k}{dt}(x) = \sum_{j=1}^{n} Y_j(x) X_{jk}(x), \quad k = 1, \cdots, n.$$

The assertion follows from these equalities. Q.E.D.

Remark. It is easy to verify that if a cross section σ of P satisfies (1.1), (1.2) and (1.4), then σ is the standard cross section corresponding to $(t, \{e_\alpha\})$.

1.5. The compatibility relative to the choice of the basis of $\mathrm{Sol}(\mathcal{D})$

Let $\{s'_1, \cdots, s'_{nr}\}$ be a basis of $\mathrm{Sol}(\mathcal{D})$ such that

$$s'_\alpha = \sum_{\beta=1}^{nr} a_{\alpha\beta} s_\beta, \quad \alpha = 1, \cdots, nr,$$

where $A = (a_{\alpha\beta}) \in GL(nr, \mathbb{R})$. With respect to $\{s'_\alpha\}$, we define κ', P', ω' and ε' as before. We define a transformation Φ of $M \times GL(nr, \mathbb{R})$ by

$$\Phi(x, Z) = (x, AZ), \quad x \in M \quad \text{and} \quad Z \in GL(nr, \mathbb{R}).$$

The next proposition is obvious.

Proposition 1.4. (1) $\kappa' = A^\circ \kappa$.

(2) *Φ maps P onto P' and induces a bundle isomorphism Φ_P of P onto P' such that $\Phi_P * \omega' = \omega$ and $\varepsilon' \circ \Phi_P = \varepsilon$.*

In view of Proposition 1.4, the subsequent argument goes through without reference to the choice of $\{s_\alpha\}$.

§2. A reduction theorem for the principal bundle P

2.1. Notations

Let $S^{n-1}(\mathbb{R}^2)$ be the symmetric tensor product of $\mathbb{R}^2$ of degree $n-1$ and $S^{n-1}(\mathbb{R}^2)\otimes\mathbb{R}^r$ be the tensor product of $S^{n-1}(\mathbb{R}^2)$ and $\mathbb{R}^r$. Let $\{v_1, v_2\}$ be the natural basis of $\mathbb{R}^2$ and $\{u_1,\cdots,u_r\}$ be the natural basis of $\mathbb{R}^r$. Taking $\{\binom{n-1}{0}v_1^{n-1}\otimes u_1,\cdots,\binom{n-1}{0}v_1^{n-1}\otimes u_r,\ \binom{n-1}{1}v_1^{n-2}\otimes v_2\otimes u_1,\cdots,\binom{n-1}{1}v_1^{n-2}\otimes v_2\otimes u_r,\cdots,\binom{n-1}{n-1}v_2^{n-1}\otimes u_1,\cdots,\binom{n-1}{n-1}v_2^{n-1}\otimes u_r\}$ as a basis of $S^{n-1}(\mathbb{R}^2)\otimes\mathbb{R}^r$, we identify $S^{n-1}(\mathbb{R}^2)\otimes\mathbb{R}^r$ with $\mathbb{R}^{nr}$.

The homomorphism $\rho: SL(2,\mathbb{R})\to GL(nr,\mathbb{R})$. Let ρ_1 be the representation of $SL(2,\mathbb{R})$ on $S^{n-1}(\mathbb{R}^2)$ defined by

$$\rho_1(A)w_1\cdots w_r = Aw_1\cdots Aw_r,\quad A\in SL(2,\mathbb{R}),\quad w_1,\cdots,w_r\in\mathbb{R}^2,$$

and let ρ_2 be the trivial representation of $SL(2,\mathbb{R})$, i.e.,

$$\rho_2(A)w = w,\quad A\in SL(2,\mathbb{R}),\quad w\in\mathbb{R}^r.$$

Then we define ρ to be the homomorphism corresponding to the representation $\rho_1\otimes\rho_2$ of $SL(2,\mathbb{R})$ on $S^{n-1}(\mathbb{R}^2)\otimes\mathbb{R}^r$ under the identification of $S^{n-1}(\mathbb{R}^2)\otimes\mathbb{R}^r$ with $\mathbb{R}^{nr}$. Let ρ_* denote the homomorphism of $sl(2,\mathbb{R})$ onto $gl(nr,\mathbb{R})$ induced by ρ. Let $\{s_{-1}, s_0, s_1\}$ be the basis of $sl(2,\mathbb{R})$ defined by

$$s_{-1}=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad s_0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad s_1=\begin{pmatrix}0&1\\0&0\end{pmatrix}.$$

Then ρ_* maps s_{-1}, s_0 and s_1 respectively to

$$U_{-1}=\begin{pmatrix}0&&&&&&\\ I&0&&&&&0\\ &2I&0&&&&\\ &&\ddots&\ddots&&&\\ &&&&0&&\\ &0&&&(n-2)I&0&\\ &&&&&(n-1)I&0\end{pmatrix},$$

$$U_0=\begin{pmatrix}(n-1)I&&&&0\\ &(n-3)I&&&\\ &&\ddots&&\\ &0&&-(n-3)I&\\ &&&&-(n-1)I\end{pmatrix},$$

$$U_1 = \begin{pmatrix} 0 & (n-1)I & & & & \\ & 0 & (n-2)I & 0 & & \\ & & 0 & \ddots & & \\ & & & 2I & & \\ & 0 & & 0 & I & \\ & & & & 0 \end{pmatrix},$$

where I stands for the $r \times r$ unit matrix.

The homomorphism $\lambda : GL(r, \mathbb{R}) \to GL(nr, \mathbb{R})$. Let λ be the homomorphism of $GL(r, \mathbb{R})$ onto $GL(nr, \mathbb{R})$ defined by

$$\lambda(C) = \begin{pmatrix} {}^tC^{-1} & & & \\ & {}^tC^{-1} & & 0 \\ 0 & & \ddots & \\ & & & {}^tC^{-1} \end{pmatrix}, \quad C \in GL(r, \mathbb{R}).$$

The subgroups H and H_0 of $GL(nr, \mathbb{R})$. Let H be the subgroup of $GL(nr, \mathbb{R})$ defined by

$$H = \rho(SL(2, \mathbb{R})) \cdot \lambda(GL(r, \mathbb{R})),$$

and let H_0 be the subgroup of K defined by

$$H_0 = H \cap K.$$

Let $SL(2, \mathbb{R})_0$ be the set of all upper triangular matrices of $SL(2, \mathbb{R})$. Then it should be noted that

$$H_0 = \rho(SL(2, \mathbb{R})_0) \cdot \lambda(GL(r, \mathbb{R})),$$

and hence

$$H/H_0 \simeq SL(2, \mathbb{R})/SL(2, \mathbb{R})_0 \simeq P(\mathbb{R}^2),$$

where $P(\mathbb{R}^2)$ is the real projective line. We denote by $\mathfrak{h}$ and $\mathfrak{h}_0$ the Lie algebras of H and H_0 respectively.

The subspace $\mathfrak{m}$ of $gl(nr, \mathbb{R})$. Let $\mathfrak{m}$ be the subspace of $gl(nr, \mathbb{R})$ consisting of all elements $X = (X_{ij})$ of $gl(nr, \mathbb{R})$ satisfying

$$\begin{aligned} &X_{ij} = 0 \quad \text{for} \quad j \neq n, \\ &X_{nn} = 0, \\ &\mathrm{Tr}\ X_{n-1\ n} = 0. \end{aligned}$$

One should note that $\mathfrak{m}$ is $\mathrm{Ad}(H_0)$ invariant.

2.2. The normal reduction of P to H_0

Let P be the principal K-bundle over M defined as in 1.2. A reduction Q of P to H_0 is said to be normal if the restriction of ω to Q takes values in $\mathfrak{h} + \mathfrak{m}$.

Proposition 2.1. *Let Q be a normal reduction of P to H_0, and χ be the restriction of ω to Q. Let $\chi_{\mathfrak{h}}$ and $\chi_{\mathfrak{m}}$ be the $\mathfrak{h}$-component of χ and the $\mathfrak{m}$-component of χ respectively. Then,*

(1) $\chi_{\mathfrak{h}}$ *is a flat Cartan connection of type H/H_0 in Q.*

(2) $\chi_{\mathfrak{m}}$ *is a tensorial form, that is, the following equalities are satisfied:*

$$R_{A*}\chi_{\mathfrak{m}} = \mathrm{Ad}(A^{-1})\chi_{\mathfrak{m}} \qquad \text{for all} \quad A \in H_0.$$

$$\chi_{\mathfrak{m}}\ (X^*) = 0 \qquad \text{for all} \quad X \in \mathfrak{h}_0.$$

Proof. Since both $\mathfrak{h}$ and $\mathfrak{m}$ are $\mathrm{Ad}(H_0)$ invariant, we have

$$R_{A*}\chi_{\mathfrak{h}} = \mathrm{Ad}(A^{-1})\chi_{\mathfrak{h}} \quad \text{and} \quad R_{A*}\chi_{\mathfrak{m}} = \mathrm{Ad}(A^{-1})\chi_{\mathfrak{m}}$$

for any $A \in H_0$. By definition, we have

$$\chi_{\mathfrak{h}}\ (X^*) = X \quad \text{and} \quad \chi_{\mathfrak{m}}\ (X^*) = 0 \quad \text{for any} \quad X \in \mathfrak{h}_0.$$

Clearly we have $\chi_{\mathfrak{h}}\ (v) \neq 0$ for every non-zero tangent vector v of Q. Therefore $\chi_{\mathfrak{h}}$ is a Cartan connection of type H/H_0 in Q. Since M is 1-dimensional, $\chi_{\mathfrak{h}}$ is flat. Q.E.D.

We are now in a position to state the main theorem.

Theorem 2.2. *There exists a unique normal reduction of P to H_0.*

This theorem will be proved in the next section. The uniqueness of the normal reduction yields the following

Proposition 2.3. *Let Q be the normal reduction of P. Let σ be a cross section of P and $\sigma^*\omega$ be the pull back of ω by σ. If $\sigma^*\omega$ takes values in $\mathfrak{h} + \mathfrak{m}$, then $\sigma(x) \in Q$ for all $x \in M$. In other words, σ is a cross section of Q.*

Proof. Let Q' be the reduction of P to H_0 which contains the subset $\sigma(M)$ of P. We want to show that the reduction Q' is normal. Let χ' be the restriction of ω to Q'. We first show that $\chi'(v) \in \mathfrak{h} + \mathfrak{m}$, for any $v \in T(Q)_{\sigma(x)}$, $x \in M$. It is sufficient to consider the following two cases:

Case 1. v is vertical. In this case v can be written in the form $v = X^*$, where $X \in \mathfrak{h}_0$. Hence we have

$$\chi'(v) = X \in \mathfrak{h}_0 \subset \mathfrak{h} + \mathfrak{m}.$$

Case 2. $v = \sigma_*(d/dt)$, where t is a local coordinate system in M. We have

$$\chi'(v) = (\sigma^*\omega)\left(\frac{d}{dt}\right) \in \mathfrak{h} + \mathfrak{m}.$$

We can write any point q of Q' in the form:

$$q = \sigma(x)A, \quad x \in M, \quad A \in H_0.$$

From the equality $R_{A*}\chi'_q = \mathrm{Ad}(A^{-1})\chi'_{\sigma(x)}$ and the fact that $\mathfrak{h} + \mathfrak{m}$ is $\mathrm{Ad}(H_0)$ invariant, it follows that χ'_q takes values in $\mathfrak{h} + \mathfrak{m}$. Therefore Q' is a normal reduction. Q.E.D.

§3. Proof of Theorem 2.2

In this section, we will prove Theorem 2.2.

3.1. Algebraic preliminaries

For each integer k, $-n+1 \leqq k \leqq n-1$, let $\mathfrak{g}_k$ (resp. $\mathfrak{g}^{(k)}$) be the subspace of $gl(nr, \mathbb{R})$ consisting of all elements $X = (X_{ij})$ such that

$$X_{ij} = 0 \quad \text{for} \quad j \neq i + k$$
$$(\text{resp. } X_{ij} = 0 \quad \text{for} \quad j < j + k).$$

It is easy to see that $gl(nr, \mathbb{R})$ becomes a graded Lie algebra with the direct sum decomposition $gl(nr, \mathbb{R}) = \sum_k \mathfrak{g}_k$, that is, $[\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{j+k}$, for any $-n+1 \leqq j,\ k \leqq n-1$. One should note that, for any $k \geqq 0$, $\mathfrak{g}^{(k)}$ becomes a graded subalgebra of $gl(nr, \mathbb{R})$ with the direct sum decomposition $\mathfrak{g}^{(k)} = \sum_{j \geq k} \mathfrak{g}_j$.

For $k = -1, 0, 1$, let ℓ_k be the subspace of $sl(2, \mathbb{R})$ defined by $\ell_k = \mathbb{R}s_k$. It is also easy to see that $sl(2, \mathbb{R})$ becomes a graded Lie algebra with the direct sum decomposition $sl(2, \mathbb{R}) = \sum_k \ell_k$, and that $\rho_*(\ell_k) \subset \mathfrak{g}_k$ for $k = -1, 0, 1$.

The operators ∂_k. For each integer k, $-n+2 \leqq k \leqq n$, we put $C^{k,0} = \mathfrak{g}_{k-1}$ and $C^{k,1} = \mathfrak{g}_{k-1} \otimes \ell^*_{-1}$. On the analogy of the so-called Spencer complex, we define an operator $\partial_k : C^{k,0} \to C^{k-1,1}$ by

$$(\partial_k X)(s) = [\rho_*(s), X], \quad X \in C^{k,0} \quad \text{and} \quad s \in \ell_{-1}.$$

A simple calculation shows the following

Lemma 3.1. (1) Ker $\partial_k = 0$, $2 \leqq k \leqq n$.

(2) $C^{k,1} = \mathrm{Im}\ \partial_{k+1} + (\mathfrak{h} + \mathfrak{m})_{k-1} \otimes \ell^*_{-1}$, *for any* k, *where* $(\mathfrak{h} + \mathfrak{m})_{k-1} = \mathfrak{g}_{k-1} \cap (\mathfrak{h} + \mathfrak{m})$.

The subgroup $H^{(0)}$ of K. Let G_0 and $G^{(0)}$ be the subgroups of $GL(nr, \mathbb{R})$ defined respectively by

$$G_0 = \{A = (A_{ij}) |\ A_{ij} = 0 \quad \text{for} \quad i \neq j\},$$
$$G^{(0)} = \{A = (A_{ij}) |\ A_{ij} = 0 \quad \text{for} \quad i > j\}.$$

Clearly, $G_0 \subset G^{(0)} \subset K$. It is easy to see that the Lie algebras of G_0 and $G^{(0)}$ agree with $\mathfrak{g}_0$ and $\mathfrak{g}^{(0)}$ respectively. The next lemmas are obvious.

Lemma 3.2. (1) $\mathfrak{g}_k$ *is* $\mathrm{Ad}(G_0)$*-invariant for every* k.

(2) $\mathfrak{g}^{(k)}$ *is* $\mathrm{Ad}(G^{(0)})$*-invariant for every* $k \geqq 0$.

Lemma 3.3. (1) *Every element* A *of* $G^{(0)}$ *can be written uniquely in the form:*

$$A = A_0 \exp X_1 \cdots \exp X_{n-1},$$

where $A_0 \in G_0$ *and* $X_k \in \mathfrak{g}_k$. *Moreover the assignment* $A \longrightarrow A_0$ *gives a homomorphism of* $G^{(0)}$ *onto* G_0.

(2) *Every element* A *of* H_0 *can be written uniquely in the form:*

$$A = A_0 \exp X_1, \quad A_0 \in G_0 \cap H, \quad X_1 \in \mathfrak{g}_1 \cap \mathfrak{h}.$$

Let $H^{(0)}$ *be the set of all elements* A *of* $G^{(0)}$ *of the form:*

$$A = A_0 \exp X_1 \cdots \exp X_{n-1}, \quad A_0 \in G_0 \cap H, \quad X_k \in \mathfrak{g}_k. \tag{3.1}$$

Clearly $H_0 \subset H^{(0)} \subset K$. *By (1) of Lemma* 3.3, $H^{(0)}$ *is a subgroup of* $G^{(0)}$ *whose Lie algebra* $\mathfrak{h}^{(0)}$ *coincides with the subspace* $\mathfrak{h}_0 + \mathfrak{g}^{(1)}$ *of* $gl(nr, \mathbb{R})$.

Lemma 3.4. (1) $\mathbb{R}U_{-1} + \mathfrak{g}^{(0)}$ *is* $\mathrm{Ad}(H^{(0)})$*-invariant.*

(2) $\mathfrak{m}$ *is* $\mathrm{Ad}(H^{(0)})$*-invariant.*

Proof. We will prove only (1). The proof of (2) is much easier. Let A be any element of $H^{(0)}$. By definition A can be written in the form

$$A = A_0 \exp X_1 \cdots \exp X_{n-1}, \quad A_0 \in G_0 \cap H, \quad X_k \in \mathfrak{g}_k.$$

Since $A \in G^{(0)}$, we have $\mathrm{Ad}(A)\mathfrak{g}^{(0)} = \mathfrak{g}^{(0)}$. Hence to prove the assertion, it suffices to show that $\mathrm{Ad}(A)U_{-1} \in \mathbb{R}U_{-1} + \mathfrak{g}^{(0)}$. Clearly we have, for any $k \geqq 1$,

$$\mathrm{Ad}(\exp X_k)(U_{-1}) \equiv U_{-1} \pmod{\mathfrak{g}^{(0)}},$$

and hence

$$\mathrm{Ad}(A)(U_{-1}) \equiv \mathrm{Ad}(A_0)(U_{-1}) \pmod{\mathfrak{g}^{(0)}}.$$

Since $A_0 \in H$, we have

$$\mathrm{Ad}(A_0)(U_{-1}) \in \mathfrak{h} \subset \mathbb{R}U_{-1} + \mathfrak{g}^{(0)}.$$

Therefore we have the assertion. Q.E.D.

The subgroups $H^{(1)}, \cdots, H^{(n-1)}$ of K. Let $H^{(1)}, \cdots, H^{(n-1)}$ be the subgroups of K defined inductively by

$$H^{(k)} = \{A \in H^{(k-1)} \mid \mathrm{Ad}(A) \text{ preserves } \mathbb{R}U_{-1} + \mathfrak{h}^{(k-1)} + \mathfrak{m}\},$$

where $\mathfrak{h}^{(k-1)}$ stands for the Lie algebra of $H^{(k-1)}$.

Lemma 3.5. (1) *For any integer $k \geqq 1$, $H^{(k)}$ consists of the elements A of $G^{(0)}$ of the form:*

$$A = A_0 \exp X_1 \exp X_{k+1} \cdots \exp X_{n-1},$$

where $A_0 \in G_0 \cap H$, $X_1 \in \mathfrak{g}_1 \cap \mathfrak{h}$ and $X_j \in \mathfrak{g}_j$ for $j \geqq k+1$. In particular $H^{(n-1)} = H_0$.

(2) *For every $A \in H^{(k-1)} \backslash H^{(k)}$,*

$$\mathrm{Ad}(A)(\mathbb{R}U_{-1} + \mathfrak{h}^{(k-1)} + \mathfrak{m}) \cap (\mathbb{R}U_{-1} + \mathfrak{h}^{(k-1)} + \mathfrak{m}) = \mathfrak{h}^{(k-1)} + \mathfrak{m}.$$

Proof. (1) We first remark that

$$H^{(k)} = \{A \in H^{(k-1)} \mid \mathrm{Ad}(A)U_{-1} \in \mathbb{R}U_{-1} + \mathfrak{h}^{(k-1)} + \mathfrak{m}\}.$$

In fact $\mathfrak{h}^{(k-1)}$ is $\mathrm{Ad}(H^{(k-1)})$-invariant, and by Lemma 3.4, $\mathfrak{m}$ is also $\mathrm{Ad}(H^{(k-1)})$-invariant.

The proof is by induction on k. We first consider the case where $k = 1$. By definition, any element A of $H^{(0)}$ can be written as (3.1). Since $\mathrm{Ad}(A_0)U_{-1} \in \mathfrak{h} \subset \mathbb{R}U_{-1} + \mathfrak{h}^{(0)}$, we have $A_0 \in H^{(1)}$. Hence $A \in H^{(1)}$, if and only if $A_0^{-1}A \in H^{(1)}$. In a similar way as in the proof of Lemma 3.4, we can show that

$$\mathrm{Ad}(A_0^{-1}A)(U_{-1}) \equiv [X_1, U_{-1}] \pmod{\mathbb{R}U_{-1} + \mathfrak{h}^{(0)}}.$$

Therefore $A \in H^{(1)}$ if and only if $[X_1, U_{-1}] \in \mathfrak{g}_0 \cap \mathfrak{h}$. A simple calculation shows that this is equivalent to $X_1 \in \mathfrak{g}_1 \cap \mathfrak{h}$.

Assume that the assertion is true for $k-1$. Then, any element A of $H^{(k-1)}$ can be written in the form: $A = A_0 \exp X_1 \exp X_k \cdots \exp X_{n-1}$, where $A_0 \in G_0 \cap H$, $X_1 \in \mathfrak{g}_1 \cap \mathfrak{h}$ and $X_j \in \mathfrak{g}_j$ for $j \geqq k$. Hence $\mathfrak{h}^{(k-1)} = \mathfrak{h}_0 + \mathfrak{g}^{(k)}$. As above, we can show that $A_0 \exp X_1 \in H^{(k)}$. Accordingly, without loss of generality, we may assume that A is of the form

$$A = \exp X_k \cdots \exp X_{n-1}, \quad X_j \in \mathfrak{g}_j \quad j = k, \cdots, n-1.$$

Then we have

$$\mathrm{Ad}(A)(U_{-1}) \equiv [X_k, U_{-1}] \pmod{\mathbb{R}U_{-1} + \mathfrak{h}^{(k-1)} + \mathfrak{m}}.$$

Therefore $A \in H^{(k)}$ if and only if

$$[X_k, U_{-1}] \in \mathfrak{g}_{k-1} \cap (\mathfrak{h} + \mathfrak{m}).$$

This is equivalent to $X_k = 0$. (2) follows from the above arguments.

Q.E.D.

3.2. The normal reduction of P to $H^{(0)}$

A reduction $Q^{(0)}$ of P to $H^{(0)}$ is said to be normal if the restriction of ω to $Q^{(0)}$ takes values in $\mathbb{R}U_{-1} + \mathfrak{g}^{(0)}$.

Proposition 3.6. *There exists a unique normal reduction of P to $H^{(0)}$.*

Proof. We need the next lemma, which follows immediately from the definition of the group $H^{(0)}$.

Lemma 3.7. *Let A, $A' \in GL(nr, \mathbb{R})$. The following conditions are mutually equivalent:*

(i) *There exists $B \in H^{(0)}$ such that $A' = AB$.*

(ii) *There exist $a \in \mathbb{R}\backslash\{0\}$, $C \in GL(r, \mathbb{R})$ and a family of $r \times r$ matrices D_{jk}, $j < k$, such that*

$$A'_k = a^{k-1} A_k C + \sum_{j=1}^{k-1} A_j D_{jk} \quad k = 1, \cdots, n.$$

We first show the existence. We take a local coordinate system t in M and a moving frame $\{e_\alpha\}$ of E. Let σ be the standard cross section of P corresponding to $(t, \{e_\alpha\})$. Let $Q^{(0)}$ be the unique reduction of P which contains the submanifold $\sigma(M)$. By Lemmas 1.1 and 3.7, it follows that the definition of $Q^{(0)}$ does not depend on the choice of $(t, \{e_\alpha\})$.

We want to show that $Q^{(0)}$ is a normal reduction of P to $H^{(0)}$. Let $\omega^{(0)}$ be the restriction of ω to $Q^{(0)}$. We assume that, for any $x \in M$, $\omega^{(0)}$ takes values in $\mathbb{R}U_{-1} + \mathfrak{g}^{(0)}$ at $\sigma(x)$. It suffices to consider the following two cases:

Case 1. v is vertical. In this case, v can be written in the form $v = X^*$, where $X \in \mathfrak{h}^{(0)}$. Hence we have

$$\omega^{(0)}(v) = X \in \mathfrak{h}^{(0)} \subset \mathbb{R}U_{-1} + \mathfrak{g}^{(0)}.$$

Case 2. $v = \sigma_*(d/dt)$. The assertion is a consequence of Proposition 1.3.

For a general point q of $Q^{(0)}$, we can write q in the form

$$q = \sigma(x)A, \quad x \in M \quad \text{and} \quad A \in H^{(0)}.$$

Now the assertion follows from Lemma 3.4 and the equality

$$R_{A*}\omega^{(0)} = \mathrm{Ad}(A^{-1})\omega^{(0)}.$$

We next show the uniqueness. Let $Q^{(0)'}$, be a normal reduction of P to $H^{(0)}$. We must show that $Q^{(0)'}$, coincides with $Q^{(0)}$. Let σ' be any cross section of $Q^{(0)'}$. We write σ' in the form

$$\sigma'(x) = (x, Y'(x)), \quad x \in M, \quad Y'(x) \in GL(nr, \mathbb{R}).$$

By assumption, $\sigma'^*\omega$ takes values in $\mathbb{R}U_{-1} + \mathfrak{g}^{(0)}$. We can take a coordinate system t in M in such a way that $\mathbb{R}U_{-1}$-component of the function $(\sigma'^*\omega)(d/dt)$ is equal to U_{-1}. Then we have, for every $x \in M$,

$$\begin{aligned}
&Y'(x)^{-1}\frac{dY'}{dt}(x) \\
&= (\sigma'^*\omega)\left(\frac{d}{dt}\right)(x) \\
&= \begin{pmatrix}
X_{11}(x) & \cdot & \cdot & \cdot & \cdot\ \cdot & X_{1n}(x) \\
I & X_{22}(x) & & & & \cdot \\
 & 2I & \ddots & & & \cdot \\
 & \ddots & & & & \cdot \\
0 & & (n-2)I & X_{n-1\ n-1}(x) & & \cdot \\
 & & & (n-1)I & & X_{nn}(x)
\end{pmatrix},
\end{aligned}$$

where $X_{jk}(x)$, $j \leqq k$, are $r \times r$ matrices. From this we obtain the following equalities

$$(3.2) \qquad Y'_{k+1}(x) = \frac{1}{k} \frac{dY'}{dt} k(x) - \frac{1}{k} \sum_{j \leq k} Y'_j(x) X_{jk}(x), \quad k = 1, \cdots, n.$$

We take a moving frame $\{e_\alpha\}$ of E in such a way that $e_\alpha(x) + \varepsilon_\alpha(\sigma'(x))$, $\alpha = 1, \cdots, r$, for all $x \in M$, where $\varepsilon = \{\varepsilon_\alpha\}$ denotes the bundle homomorphism of P to $\mathcal{F}(E)$ defined in the paragraph 1.3. We write the standard cross section σ in the form

$$\sigma(x) = (x, Y(x)), \quad x \in M, \quad Y(x) \in GL(nr, \mathbb{R}).$$

Then, for every $x \in M$, we have the following equalities

$$(3.3) \qquad Y_{k+1}(x) = \frac{1}{k} \frac{d}{dt} Y_k(x), \quad x \in M, \quad k = 1, \cdots, n-1.$$

Furthermore we have

$$(3.4) \qquad Y'_1(x) = Y_1(x) \quad x \in M.$$

Combining (3.2), (3.3) and (3.4), we can inductively show that there exists a family of $r \times r$ matrices $D_{jk}(x)$, $j < k$, such that

$$Y'_k(x) = Y_k(x) + \sum_{j=1}^{k-1} Y_j(x) D_{jk}(x),$$

for $k = 1, \cdots, n$. From Lemma 3.7, we see that there exists $B(x) \in H^{(0)}$ such that $\sigma'(x) = \sigma(x)B(x)$. This means that $\sigma'(x) \in Q^{(0)}$. Q.E.D.

3.3. The normal reduction of P to $H^{(k)}$

For each integer $k \geqq 1$, we say that a reduction $Q^{(k)}$ of P to $H^{(k)}$ is normal if the restriction of ω to $Q^{(k)}$ takes values in $\mathbb{R}U_{-1} + \mathfrak{h}^{(k-1)} + \mathfrak{m}$. Since $H^{(n-1)} = H_0$ and $\mathbb{R}U_{-1} + \mathfrak{h}^{(n-2)} + \mathfrak{m} = \mathfrak{h} + \mathfrak{m}$, a normal reduction of P to $H^{(n-1)}$ is nothing but a normal reduction of P to H_0 defined in §2. Hence Theorem 2.2 follows from the following

Proposition 3.8. *For every $k \geqq 1$, there exists a unique normal reduction of P to $H^{(k)}$.*

Proof. The proof is by induction on k. Let us consider the case where $k = 1$. We first show the existence. Let $Q^{(0)}$ be the normal reduction of P to $H^{(0)}$ and $\omega^{(0)}$ be the restriction of ω. We define $Q^{(1)}$ to be the set of all points q of $Q^{(0)}$ such that $\omega_q^{(0)}$ takes values in $\mathbb{R}U_{-1} + \mathfrak{h}^{(0)} + \mathfrak{m}$.

Now we want to show that, for every $x \in M$, the fiber $Q_x^{(1)}$ of $Q^{(1)}$ over x is non-empty. For this purpose, we fix an arbitrary point q of $Q_x^{(0)}$ and an arbitrary subspace V_q of $T(Q^{(0)})_q$ such that the projection $\pi_{M^*} : V_q \to T(M)_x$ is an isomorphism. For each j, $-n+1 \leqq j \leqq n-1$, let $\omega_j^{(0)}$ denote the $\mathfrak{g}_j$-component of $\omega^{(0)}$. Since $Q^{(0)}$ is a normal reduction, $(\omega_{-1}^{(0)})_q$ takes values in $\mathbb{R}U_{-1}$ $(= \rho_*(\ell_{-1}))$ and gives an isomorphism of V_q onto $\mathbb{R}U_{-1}$. Hence, for every $s \in \ell_{-1}$, there exists a unique element $v(s)$ of V_q such that $(\omega_{-1}^{(0)})(v(s)) = s$.

This being prepared, for each j, $-n+2 \leqq j \leqq n$, we define $u_j \in C^{j,1}$ $(= \mathfrak{g}_{j-1} \otimes \ell_{-1}^*)$ by

$$u_j(s) = (\omega_{j-1}^{(0)})(v(s)), \quad s \in \ell_{-1}. \tag{3.5}$$

Since $Q^{(0)}$ is a normal reduction, we have

$$\begin{aligned} u_j &= 0 \quad \text{for} \quad j \leqq -1, \\ u_0 &= \rho_*. \end{aligned}$$

Taking any $X \in \mathfrak{g}_1$, we put $A = \exp X$ and $q' = qA$. For each j, $-n+2 \leqq j \leqq n$, we also define $u'_j \in C^{j,1}$ by

$$u'_j(s) = (\omega_{j-1}^{(0)})(R_{A*}v(s)), \quad s \in \ell_{-1}. \tag{3.6}$$

Since $R_{A*}\omega_j^{(0)} = \omega_j^{(0)}$ for every $j \leqq -1$ and $R_{A*}\omega_0^{(0)} = \omega_0^{(0)} - [X, \omega_{-1}^{(0)}]$, we have

$$\begin{aligned} u'_j &= u_j = 0 \quad \text{for all} \quad j \leqq -1 \\ u'_0 &= u_0 = \rho_* \\ u'_{-1} &= u_1 + \partial_2 X. \end{aligned} \tag{3.7}$$

By Lemma 3.1, we can choose X in such a way that

$$u'_1 \in (\mathfrak{h} + \mathfrak{m})_0 \otimes \ell_{-1}^*. \tag{3.8}$$

Then, by (3.7) and (3.8), we have

$$\omega^{(0)}(R_{A*}v(s)) \in \mathbb{R}U_{-1} + \mathfrak{h}^{(0)} + \mathfrak{m}$$

for all $s \in \ell_{-1}$. On the other hand, we have

$$\omega^{(0)}(v) \in \mathfrak{h}^{(0)}$$

for any vertical tangent vector v of $Q^{(0)}$ at q'. Therefore we see that $\omega_{q'}^{(0)}$ takes values in $\mathbb{R}U_{-1} + \mathfrak{h}^{(0)} + \mathfrak{m}$ and hence $q' \in Q_x^{(1)}$.

From (2) of Lemma 3.5 and the definitions of $H^{(1)}$ and $Q^{(1)}$, it is clear that $Q^{(1)}$ is a normal reduction of P to $H^{(1)}$.

Next we show the uniqueness. Let $Q^{(1)'}$ be a normal reduction of P to $H^{(1)}$. Let $Q^{(0)'}$ be the reduction of P to $H^{(0)}$ which contains $Q^{(1)'}$. It is not difficult to see that $Q^{(0)'}$ is a normal reduction of P to $H^{(0)}$. Hence, $Q^{(1)'} \subset Q^{(0)'} = Q^{(0)}$. Take any point q' of $Q^{(1)'}$. From the assumption that $Q^{(1)'}$ is normal, we easily see that $\omega_{q'}^{(0)}$ takes values in $\mathbb{R}U_{-1} + \mathfrak{h}^{(0)} + \mathfrak{m}$. Hence we have $q' \in Q^{(1)}$. We have thus proved the case $k = 1$.

Assume that the assertion is true for $k-1$. Let $Q^{(k-1)}$ be the normal reduction of P to $H^{(k-1)}$ and $\omega^{(k-1)}$ be the restriction of ω to $Q^{(k-1)}$. In a similar way as in the case $k = 1$, let $Q^{(k)}$ be the set of all points q of $Q^{(k-1)}$ such that $\omega_q^{(k-1)}$ takes values in $\mathbb{R}U_{-1} + \mathfrak{h}^{(k-1)} + \mathfrak{m}$. We claim that, for every $x \in M$, the fiber $Q_x^{(k)}$ of $Q^{(k)}$ at x is non-empty. We fix an arbitrary point q of $Q_x^{(k-1)}$ and an arbitrary subspace V_q of $T(Q^{(k-1)})_q$ such that $\pi_{M*} : V_q \to T(M)_x$ is an isomorphism. Taking any $X \in \mathfrak{g}_k$, we define u_j, $u'_j \in C^{j,1}$ respectively as (3.5) and (3.6). Then we have

$$\begin{aligned} &u'_j = u_j = 0 && \text{for} \quad j \leqq -1, \\ &u'_0 = u_0 = \rho_*, && \\ &u'_j = u_j \in (\mathfrak{h} + \mathfrak{m})_{j-1} \otimes \ell^*_{-1} && \text{for} \quad 1 \leqq j \leqq k-1, \\ &u'_k = u_k + \partial_{k+1} X. && \end{aligned} \tag{3.9}$$

By Lemma 3.1, we can take $X \in \mathfrak{g}_k$ in such a way that

$$u'_k \in (\mathfrak{h} + \mathfrak{m})_{k-1} \otimes \ell^*_{-1}. \tag{3.10}$$

Then, from (3.9) and (3.10), we see that $q \exp X \in Q^{(k)}$, proving the assertion. It is clear that $Q^{(k)}$ is a normal reduction of P to $H^{(k)}$.

The uniqueness can be shown in quite similar manner as in the case $k = 1$. Q.E.D.

§4. Canonical forms and differential invariants

Let Q be the normal reduction of P to H_0. As before, let χ be the restriction of ω to Q and $\chi_{\mathfrak{h}}$ (resp. $\chi_{\mathfrak{m}}$) be the $\mathfrak{h}$-component of χ (resp. the $\mathfrak{m}$-component of χ).

4.1. The vector field U^*_{-1} and the functions $A^{(1)}, \cdots, A^{(n)}$

Since $\chi_{\mathfrak{h}}$ is a Cartan connection in Q, $\chi_{\mathfrak{h}}$ gives a linear isomorphism of $T(Q)_q$ onto $\mathfrak{h}$ at each point q of Q. For any $X \in \mathfrak{h}$, let X^* be the vector field on Q defined by

$$\chi_{\mathfrak{h}}(X^*) = X. \tag{4.1}$$

For $X \in \mathfrak{h}_0$, the above notation X^* is compatible with the standard notation of the fundamental vector field corresponding to X (see Preliminary remarks).

Lemma 4.1. (1) *For any* $T = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in SL(2,\mathbb{R})_0$ *and any* $C \in GL(r,\mathbb{R})$, *the following equalities are satisfied:*

$$R_{\rho(T)*}(U^*_{-1}) = a^2 U^*_{-1} - abU^*_0 - b^2 U^*_1,$$
$$R_{\lambda(C)*}(U^*_{-1}) = U^*_{-1}.$$

(2) $[U^*_0, U^*_{-1}] = -2U^*_{-1}$ *and* $[U^*_1, U^*_{-1}] = U^*_0$.

Proof. These assertions follow immediately from the facts that

$$R_{A*}\chi_{\mathfrak{h}} = \mathrm{Ad}(A^{-1})\chi_{\mathfrak{h}} \qquad \text{and} \qquad d\chi_{\mathfrak{h}} + \frac{1}{2}[\chi_{\mathfrak{h}} \wedge \chi_{\mathfrak{h}}] = 0,$$

where $A \in H_0$. Q.E.D.

We write the $nr \times nr$ matrix $\chi_{\mathfrak{m}}(U^*_{-1})_q$, $q \in Q$, in the form:

$$\chi_{\mathfrak{m}}(U^*_{-1})q = (X_{ij}(q)),$$

where $X_{ij}(q)$ are $r \times r$ matrices. Since $\chi_{\mathfrak{m}}(U^*_{-1})_q \in \mathfrak{m}$, we have

$$X_{ij}(q) = 0 \quad \text{for} \quad j \neq n,$$
$$X_{nn}(q) = 0,$$
$$\mathrm{Tr}\ X_{n-1\, n}(q) = 0.$$

For each integer k, $2 \leqq k \leqq n$, let $A^{(k)}(q) = (a^{(k)}_{\alpha\beta}(q))$, $q \in Q$, be the $r \times r$-matrices defined by the equation:

$$A^{(k)}(q) = -\frac{(n-1)!}{(n-k)!}\,{}^t X_{n-k+1\ n}(q).$$

Note that $\mathrm{Tr} A^{(2)}(q) = 0$.

Lemma 4.2. (1) *For any* $T = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in SL(2,\mathbb{R})_0$ *and any* $C \in GL(r,\mathbb{R})$, *the following equalities are satisfied:*

$$A^{(k)}(q\rho(T)) = \sum_{j=0}^{k-2} j! \binom{k-1}{j}\binom{n-k+j}{j} a^{-2k+j}(-b)^j A^{(k-j)}(q),$$
$$A^{(k)}(q\lambda(C)) = C^{-1}A(q)C.$$

(2) $U_0^* A^{(k)} = -2kA^{(k)}$,

$$U_1^* A^{(k)} = -(k-1)(n-k+1)A^{(k-1)}.$$

Proof. (1) We will show only the first equality. The second equality is also proved in a similar manner. The proof is based on (2) of Proposition 2.1 and on (1) of Lemma 4.1. We have

$$\begin{aligned}(4.2)\qquad \chi_{\mathfrak{m}}\ (U^*_{-1})_{q\rho(T)} &= a^{-2}\chi_{\mathfrak{m}}\ (R_{\rho(T)*}(U^*_{-1})_q) \\ &\quad + a^{-1}b\chi_{\mathfrak{m}}\ (U_0^*)_{q\rho(T)} + a^{-2}b^2\chi_{\mathfrak{m}}\ (U_1^*)_{g\rho(T)} \\ &= a^{-2}\,\mathrm{Ad}(\rho(T)^{-1})\chi_{\mathfrak{m}}\ (U^*_{-1})_q.\end{aligned}$$

We write $\rho(T)$ and $\rho(T)^{-1}$ in the form $\rho(T) = (T_{ij})$ and $\rho(T)^{-1} = (T'_{ij})$, where T_{ij}, T'_{ij}, $1 \leqq i,\ j \leqq n$ are $r \times r$ matrices. Then it is easy to see that

$$\begin{aligned}(4.3)\qquad & T_{ij} = T'_{ij} = 0 \quad \text{for} \quad i > j, && \text{for} \quad i \leqq j, \\ & T_{ij} = \binom{n-i}{j-i} a^{n+1-i-j} b^{j-i} I && \text{for} \quad i \leqq j, \\ & T'_{ij} = \binom{n-i}{j-i} a^{-n-1+i+j}(-b)^{j-i} I && \text{for} \quad i \leqq j.\end{aligned}$$

From (4.2) and (4.3) it follows that

$$\begin{aligned} X_{n-k+1\ n}(q\rho(T)) &= a^{-2}\sum_{j=0}^{k-1} T'_{n-k+1\ n-k+j+1} X_{n-k+j+1\ n}(q) T_{nn} \\ &= \sum_{j=0}^{k-1} \binom{k-1}{j} a^{-2k+j}(-b)^j X_{n-k+j+1\ n}. \end{aligned}$$

The assertion follows from this equality.

(2) Substituting $\exp ts_0$ for T in the first equality of (1), we have

$$A^{(k)}(q\ \exp tU_0) = \exp(-2kt)A^{(k)}(q).$$

By differentiating both sides of this equality with respect to the parameter t, we obtain the first equality of (2). The second equality is proved quite similarly. Q.E.D.

4.2. Laguerre-Forsyth's canonical forms

Here, we will show that there exists one to one correspondence between the Laguerre-Forsyth's canonical forms of $\mathcal{D}$ and the integral curves of the vector field U^*_{-1}.

Let $\gamma(t)$ be an integral curve of U^*_{-1}. We write $\gamma(t)$ in the form

$$\gamma(t) = (x(t), Y(t)), \quad x(t) \in M, \quad Y(t) \in GL(nr, \mathbb{R}).$$

Clearly the map $t \to x(t)$ is an immersion and hence the parameter t can be regarded as a local coordinate system in M. Let σ be the cross section of Q defined by the assignment $\sigma : x(t) \to \gamma(t)$. Then we have

$$(\sigma^*\omega)(\frac{d}{dt})_{x(t)} = \chi(U^*_{-1})_{\gamma(t)} = U_{-1} + \chi_{\mathfrak{m}}\ (U^*_{-1})_{\gamma(t)}.$$

Hence equations (1.2) and (1.4) are satisfied. From the remark following Proposition 1.3, we see that σ is the standard cross section of P with respect to $(t, \{e_\alpha\})$, where $\{e_\alpha\}$ is defined by

$$e_\alpha(x) = \varepsilon_\alpha(\sigma(x)), \quad x \in M, \quad \alpha = 1, \cdots, r. \tag{4.4}$$

By Proposition 1.3, $\mathcal{D}$ is written in the form

$$\left(\frac{d}{dt}\right)^n y_\alpha + \sum_{k=1}^{n}\sum_{\beta=1}^{r} a^{(k)}_{\alpha\beta}(\gamma(t)) \left(\frac{d}{dt}\right)^{n-k} y_\beta = 0, \quad \alpha = 1, \cdots, r. \tag{4.5}$$

This is nothing but one of the Laguerre Forsyth's canonical forms of $\mathcal{D}$.

Theorem 4.3. *For any integral curve $\gamma(t)$ of U^*_{-1}, (4.5) gives a Laguerre-Forsyth's canonical form of $\mathcal{D}$. Conversely every Laguerre-Forsyth's canonical form can be thus obtained.*

Proof. It is sufficient to show the converse. We take any Laguerre-Forsyth's canonical form of $\mathcal{D}$. Let t be the corresponding coordinate system in M and $\{e_\alpha\}$ be the corresponding moving frame of E, and let σ be the standard cross section of P corresponding to $(t, \{e_\alpha\})$. By Proposition 1.3, the 1-form $\sigma^*\omega$ takes values in $\mathfrak{h}+\mathfrak{m}$, and by Proposition 2.3, σ is a cross section of Q. Let $\gamma(t)$ be the curve in Q satisfying $\gamma(t(x)) = \sigma(x)$. Clearly $\gamma(t)$ is an integral curve of U^*_{-1}. It is obvious that the Laguerre-Forsyth's canonical form corresponds to this integral curve $\gamma(t)$. Q.E.D.

4.3. Transformations between canonical forms

Here, we will interpret the fact (B) in terms of the relation between the integral curves of U^*_{-1}. Let $\gamma(t)$ and $\gamma'(t)$ be integral curves of U^*_{-1}. Regarding t and t' as local coordinate systems in M, we write t' in the form: $t' = t'(t)$.

Theorem 4.4. *There exist $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ and $C = (c_{\alpha\beta}) \in GL(r,\mathbb{R})$ such that the following equalities are satisfied.*

$$\begin{aligned} t'(t) &= \frac{at+b}{ct+d}, \\ \gamma'(t'(t)) &= \gamma(t)\rho(T(t))\lambda(C), \end{aligned} \tag{4.6}$$

where $T(t)$ is the element of $SL(2,\mathbb{R})_0$ defined by

$$T(t) = \begin{pmatrix} (ct+d)^{-1} & -c \\ 0 & ct+d \end{pmatrix}.$$

Proof. For any $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ and any $C \in GL(r,\mathbb{R})$, we define a curve $\delta(s)$ by the right-hand side of (4.6), i.e.,

$$\begin{aligned} s = s(t) &= \frac{at+b}{ct+d}, \\ \delta(s(t)) &= \gamma(t)\rho(T(t))\lambda(C). \end{aligned}$$

We claim that $\delta(s)$ is an integral curve of U_{-1}^*. For this purpose, we write $\gamma(t)$ and $\delta(s)$ in the form:

$$\begin{aligned}\gamma(t) &= (x(t), Y(t)), \quad x(t) \in M, \quad Y(t) \in GL(r, \mathbb{R}),\\ \delta(s) &= (z(s), W(s)), \quad z(s) \in M, \quad W(s) \in GL(r, \mathbb{R}).\end{aligned}$$

We have $z(s(t)) = x(t)$ and $W(s(t)) = Y(t)\rho(T(t))\lambda(C)$. Hence, it follows that

$$\begin{aligned}\chi_{\mathfrak{h}}\left(\delta_*\left(\frac{d}{ds}\right)\right) &= [W(s)^{-1}\frac{dW}{ds}(s)]_{\mathfrak{h}}\\ &= \left(\frac{dt}{ds}\right)\lambda(C)^{-1}\rho(T(t))^{-1}[Y(t)^{-1}\frac{dY}{dt}(t)]_{\mathfrak{h}}\ \ \rho(T(t))\lambda(C)\\ &\quad + \left(\frac{dt}{ds}\right)\lambda(C)^{-1}\rho(T(t))^{-1}\frac{d}{dt}\rho(T(t))\lambda(C),\end{aligned}$$

where $[Y(t)^{-1}dY/dt(t)]_{\mathfrak{h}}$ and $[W(s)^{-1}dW/ds(s)]_{\mathfrak{h}}$ denote the $\mathfrak{h}$ components of $Y(t)^{-1}dY/dt(t)$ and $W(s)^{-1}dW/ds(s)$ respectively. Using the equality $[Y(t)^{-1}dY/dt(t)]_{\mathfrak{h}} = U_{-1} = \rho_*(s_{-1})$ and the fact that ρ is a homomorphism of $SL(2, \mathbb{R})$ onto $GL(nr, \mathbb{R})$, we have

$$\begin{aligned}&\rho(T(t)^{-1})[Y(t)^{-1}\frac{dY}{dt}(t)]_{\mathfrak{h}}\ \ \rho(T(t)) = \rho_*(\mathrm{Ad}(T(t)^{-1}s_{-1})),\\ &\rho(T(t)^{-1})\frac{d}{dt}\rho(T(t)) = \rho_*(T(t)^{-1}\frac{dT}{dt}(t)).\end{aligned}$$

A direct calculation shows that

$$\mathrm{Ad}(T(t)^{-1})s_{-1} + T(t)^{-1}\frac{dT}{dt}(t) = \frac{1}{(ct+d)^2}s_{-1}.$$

From these equalities and $dt/ds = (ct+d)^2$, we conclude that

$$\chi_{\mathfrak{h}}\left(\delta_*\left(\frac{d}{ds}\right)\right) = U_{-1}.$$

Therefore $\delta(s)$ is an integral curve of U_{-1}^*.

We can choose $T \in SL(2, \mathbb{R})$ and $C \in GL(r, \mathbb{R})$ in such a way that $\gamma'(t_0) = \delta(t_0)$ for some $t_0 \in \mathbb{R}$. Since both $\gamma'(t')$ and $\delta(s)$ are integral curves of U_{-1}^*, $\gamma'(t')$ coincides with $\delta(s)$, i.e., $t' = s$ and $\gamma'(t') = \delta(s)$.

Q.E.D.

Corollary 4.5. *Under the same notations as in Theorem 4.4, the following equalities are satisfied:*

$$\varepsilon_\beta(\gamma'(t')) = (ct+d)^{n-1}\sum_{\alpha=1}^{r} c_{\alpha\beta}\varepsilon_\alpha(\gamma(t)), \quad \beta = 1,\cdots,r,$$

$$A^{(k)}(\gamma'(t')) = \sum_{j=0}^{k-2} j!\binom{k-1}{j}\binom{n-k+j}{j}\cdot (ct+d)^{2k-j}c^j C^{-1}A^{(k-j)}(\gamma(t))C.$$

Proof. The first assertion follows from the fact that

$$\varepsilon\circ\rho(T(t)) = (ct+d)^{n-1}I \quad \text{and} \quad \varepsilon\circ\lambda(C) = C.$$

The second assertion follows from (1) of Lemma 4.2. Q.E.D.

4.4. Differential invariants $\Omega_2,\cdots,\Omega_n$

We say that an $r\times r$ matrix valued function $\Omega = (\Omega_{\alpha\beta})$ on Q is said to be an invariant of weight k if it satisfies the following two conditions:

$$\text{(I.1)} \quad \Omega(q\rho(T)) = a^{-2k}\Omega(q) \quad \text{for all} \quad T = \begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} \in SL(2,\mathbb{R})_0 \quad \text{and all } q\in Q.$$

$$\text{(I.2)} \quad \Omega(q\lambda(C)) = C^{-1}\Omega(q)C \quad \text{for all} \quad C\in GL(r,\mathbb{R}) \quad \text{and all } q\in Q.$$

We may replace condition (I.1) by the following condition:

$$\text{(I.1)}' \quad U_0^*\Omega = -2k\Omega \quad \text{and} \quad U_1^*\Omega = 0.$$

Let χ_{-1} denote the $\mathbb{R}U_{-1}$ component of χ with respect to the direct sum decomposition $\mathfrak{h} = \mathbb{R}U_{-1} + \mathfrak{h}_0$. The next lemma is obvious.

Lemma 4.6. (1) $\chi_{-1}(v) = 0$ *for any vertical tangent vector* v *of* Q.

(2) $R_{\rho(T)*}\chi_{-1} = a^2\chi_{-1}$ for all $T = \begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} \in SL(2,\mathbb{R})_0$, $R^*_{\lambda(C)}\chi_{-1} = \chi_{-1}$ for all $C\in GL(r,\mathbb{R})$.

For any invariant $\Omega = (\Omega_{\alpha\beta})$ of weight k on Q, we define an $r \times r$ matrix valued covariant tensor field $\Omega' = (\Omega'_{\alpha\beta})$ of degree k on Q by

$$\Omega'_q = \overbrace{\chi_{-1} \otimes \cdots \otimes \chi_{-1}}^{k \text{ times}} \otimes \Omega(q).$$

Proposition 4.7. (1) $\Omega'(v_1, \cdots, v_k) = 0$ *whenever at least one of the tangent vectors* v_i *of* Q *is vertical.*

(2) $R_{\rho(T)*}\Omega' = \Omega'$ *for any* $T \in SL(2, \mathbb{R})_0$.
$R_{\lambda(C)*}\Omega' = C^{-1}\Omega' C$ *for any* $C \in GL(r, \mathbb{R})$.

(3) *There exists a unique cross section* θ *of the vector bundle* $\overset{k}{\otimes} T(M)^* \otimes \operatorname{End}(E)$ *such that, for every* $q \in Q$ *and every* β, $1 \leqq \beta \leqq r$,

$$\theta_q(\pi_* v_1, \cdots, \pi_* v_k)\varepsilon_\beta(q) = \sum_{\alpha=1}^{r} \Omega'_{\alpha\beta}(v_1, \cdots, v_k)\varepsilon_\alpha(q),$$

where $v_1, \cdots, v_k \in T(Q)_q$ *and* π *denotes the projection of* Q *onto* M.

Proof. (1) and (2) follow from the definition of the invariants and Lemma 4.6.

(3) For each $x \in M$, we take any $q \in Q$ such that $\pi(q) = x$. From (1), we see that there exists a unique element θ'_q of $\overset{k}{\otimes} T(M)^*_x \otimes \operatorname{End}(E_x)$ such that

$$\theta'_q(\pi_* v_1, \cdots, \pi_* v_k)\varepsilon_\beta(q) = \sum_{\alpha=1}^{r} \Omega'_{\alpha\beta}(v_1, \cdots, v_k)\varepsilon_\alpha(q),$$

for all $v_1, \cdots, v_k \in T(Q)_q$. We put $\theta_x = \theta'_q$. To see that the definition of θ_x does not depend on the choice of q, it suffices to remark the following equalities:

$$\varepsilon_\alpha(q\rho(T)) = a^{-n+1}\varepsilon_\alpha(q), \ 1 \leqq \alpha \leqq r, \text{for all} \quad T \in SL(2, \mathbb{R})_0$$
$$\varepsilon_\beta(q\lambda(C)) = \sum_{\alpha=1}^{r} c_{\alpha\beta}\varepsilon_\beta(q), \ 1 \leqq \beta \leqq r, \text{for all} \quad C \in GL(r, \mathbb{R}).$$

Q.E.D.

For each integer k, $2 \leqq k \leqq n$, let θ_k be the $r \times r$-matrix-valued function on Q defined by

$$\theta_k(x) = \sum_{j=0}^{k-2} (-1)^j \frac{(2k-j-2)!(n-k+j)!}{j!(k-j-1)!} (U^*_{-1})^j A^{(k-j)}(x), \quad x \in M.$$

Remark. In the case where $r = 1$, the invariant Ω_2 automatically vanishes.

Theorem 4.8. *Ω_k is an invariant of weight k.*

Proof. For the sake of simplicity, we put

$$A^{(k,j)} = (U_{-1}^*)^j A^{(k)}.$$

The assertion is an immediate consequence of the following

Lemma 4.9. (1) $U_0^* A^{(k,j)} = -2(k+j)A^{(k,j)}$.

(2) $U_1^* A^{(k,j)} = -(2k+j-1)j\ A^{(k,j-1)} - (k-1)(n-k+1)A^{(k-1,j)}$.

(3) $A^{(k,j)}(q\lambda(C)) = C^{-1}A^{(k,j)}(q)C$ *for all* $q \in Q$ *and* $C \in GL(r, \mathbb{R})$.

Proof. By (2) of Lemma 4.1 and by (2) of Lemma 4.2, we have

$$\begin{aligned}
U_0^* A^{(k,j)} &= U_0^*(U_{-1}^*)^j A^{(k)} \\
&= \sum_{i=1}^{j} (U_{-1}^*)^{j-i}[U_0^*, U_{-1}^*](U_{-1}^*)^{i-1}A^{(k)} \\
&\qquad + (U_{-1}^*)^j U_0^* A^{(k)} \\
&= -2j(U_{-1}^*)^j A^{(k)} - 2k(U_{-1}^*)^j A^{(k)} \\
&= -2(k+j)A^{(k,j)},
\end{aligned}$$

proving (1). Similarly we have

$$\begin{aligned}
U_1^* A^{(k,j)} &= U_1^*(U_{-1}^*)^j A^{(k)} \\
&= \sum_{i=1}^{j} (U_{-1}^*)^{j-i}[U_1^*, U_{-1}^*](U_{-1}^*)^{i-1}A^{(k)} \\
&\qquad + (U_{-1}^*)^j U_1^* A^{(k)} \\
&= \sum_{i=1}^{j} (U_{-1}^*)^{j-i} U_0^* A^{(k,i-1)} \\
&\qquad - (k-1)(n-k+1)A^{(k-1,j)}.
\end{aligned}$$

From (1), it follows that

$$\begin{aligned} U_1^* A^{(k,j)} &= -\sum_{i=1}^{j} 2(k+i-1)A^{(k,j-1)} \\ &\qquad - (k-1)(n-k+1)A^{(k-1,j)} \\ &= -(2k+j-1)j\ A^{(k,j-1)} - (k-1)(n-k+1)A^{(k-1,j)}, \end{aligned}$$

proving (2). (3) follows directly from (1) of Lemma 4.1 and (1) of Lemma 4.2. Q.E.D.

Let θ_k be the cross section of $\overset{k}{\otimes} T(M)^* \otimes \mathrm{End}(E)$ corresponding to Ω_k. Let $\gamma(t)$ be an integral curve of U_{-1}^*. As in 4.2, we take t as a local coordinate system in M and give the moving frame $\{e_\alpha\}$ of E defined by (4.4). In terms of $(t, \{e_\alpha\})$, θ_k is expressed by the $r \times r$-matrix-valued k-covariant tensor field

$$\sum_{j=0}^{k-2} (-1)^j \frac{(2k-j-2)!(n-k+j)!}{j!(k-j-1)!} \left(\frac{d}{dt}\right)^j A^{(k-j)}(\gamma(t))(dt)^k.$$

Note that $A^{(j)}(\gamma(t))$, $j = 2, \cdots, n$, are coefficients of (4.5).

§5. Normal reductions and isomorphisms

In this section, we will show the compatibility of the normal reductions of Theorem 2.2 with isomorphisms of linear ordinary differential equations.

5.1. Isomorphisms of linear ordinary differential equations

Let E' be a vector bundle over a 1-dimensional manifold M' of the same rank as E. As in §1, we consider a system $\mathcal{D}'$ of linear ordinary differential equations on E'.

Let ϕ be a bundle isomorphism of E onto E'. For each cross section s' of E', we define a cross section $\phi^* s'$ of E by

$$(\phi^* s')(x) = \phi_x^{-1}(s'(\phi'(x)),$$

where $x \in M$ and ϕ' denotes the diffeomorphism of M onto M' induced by ϕ.

We say that ϕ is an isomorphism of $\mathcal{D}$ onto $\mathcal{D}'$ if $\mathcal{D}'$ corresponds to $\mathcal{D}$ under ϕ. In this case, we have $\phi^* s' \in \mathrm{Sol}(\mathcal{D}')$ for any $s' \in \mathrm{Sol}(\mathcal{D}')$.

5.2 Normal reductions and isomorphisms

We fix a basis $\{s'_1, \cdots, s'_{nr}\}$ of $\mathrm{Sol}(\mathcal{D}')$. For any isomorphism ϕ of $\mathcal{D}$ onto $\mathcal{D}'$, we define an $nr \times nr$-matrix $A(\phi) = (a_{\alpha\beta}(\phi))$ as follows: As we have remarked above, $\phi^* s'_\alpha \in \mathrm{Sol}(\mathcal{D})$ for every α. Hence $\phi^* s'_\alpha$ can be written as a linear combination of $s_1, \cdots, s_{nr}$. We define $A(\phi)$ by

$$\phi^* s'_\alpha = \sum_{\beta=1}^{r} a_{\alpha\beta}(\phi) s_\beta, \quad \alpha = 1, \cdots, nr.$$

Using the matrix $A(\phi)$, we define a bundle isomorphism Φ of $M \times GL(nr, \mathbb{R})$ onto $M' \times GL(nr, \mathbb{R})$ as follows

$$\Phi(x, Z) = (\phi'(x), A(\phi)Z),$$

where $x \in M$ and $Z \in GL(nr, \mathbb{R})$.

Associated with $\mathcal{D}'$, we define a principal K-bundle P' over M' and a $gl(nr, \mathbb{R})$-valued 1-form ω' on P' as before. Let Q' be the normal reduction of P' to H_0, and let χ' be the restriction of ω' to Q'.

Theorem 5.1. (1) *For every isomorphism ϕ of $\mathcal{D}$ onto $\mathcal{D}'$, Φ maps Q onto Q' and induces a bundle isomorphism Φ_Q of Q onto Q' satisfying $\Phi_Q^* \chi' = \chi$.*

(2) *For every bundle isomorphism Ψ of Q onto Q' satisfying $\Psi^* \omega' = \omega$, there exists a unique isomorphism ϕ of $\mathcal{D}$ onto $\mathcal{D}'$ such that $\Phi_Q = \Psi$.*

Proof. (1): We first show that Φ maps P onto P'. We choose a moving frame $\{e_\alpha\}$ of E and a moving frame $\{e'_\alpha\}$ of E' in such a way that $\phi^* e'_\alpha = e_\alpha$, $\alpha = 1, \cdots, n$, and then define Y_1 and Y'_1 by (0.4). It is easy to see that

$$Y'_1(\phi'(x)) = A(\phi) Y_1(x).$$

From this we easily see that Φ maps P onto P' and induces a bundle isomorphism Φ_P of P onto P'. Moreover we have $\Phi_P^* = \omega$. From the uniqueness of the normal reduction of P to H_0, we conclude that Φ_P maps Q onto Q', that is, Φ maps Q onto Q'. Hence Φ induces a bundle isomorphism Φ_Q of Q onto Q'. It is clear that $\Phi_Q^* \chi' = \chi$.

(2): We first remark that there exists a unique bundle isomorphism ϕ of E onto E' which makes the following diagram commutative:

$$\begin{array}{ccc} Q & \xrightarrow{\Psi} & Q' \\ \varepsilon\downarrow & & \varepsilon'\downarrow \\ \mathcal{F}(E) & \xrightarrow{\phi_{\mathcal{F}}} & \mathcal{F}(E') \end{array}$$

where $\phi_{\mathcal{F}}$ is the natural bundle isomorphism of $\mathcal{F}(E)$ onto $\mathcal{F}(E')$ induced by ϕ.

It suffices to show that ϕ is an isomorphism of $\mathcal{D}$ onto $\mathcal{D}'$. For this purpose, we take an integral curve $\gamma(t)$ of the vector field U^*_{-1} on Q. Let $\gamma'(t)$ be the curve in Q' defined by $\gamma'(t) = \Psi(\gamma(t))$. From the equality $\Psi^*\chi' = \chi$, we see that $\gamma'(t)$ is an integral curve of $U^{*\prime}_{-1}$, where U^*_{-1} is the vector field on Q' defined in the same way as U^*_{-1}. It is obvious that the canonical form of $\mathcal{D}'$ corresponding to $\gamma'(t)$ coincides with that of $\mathcal{D}$ corresponding to $\gamma(t)$. Q.E.D.

§6. Appendix

6.1. Jet bundles

Let E be a vector bundle over a 1-dimensional manifold M. We denote by $\Gamma(E)$ the space of all cross sections of E on M. In this section, we will not necessarily assume that M is simply connected, unless otherwise stated. For each $k \geqq 0$, let $J^k(E)$ be the k-th jet bundle of E, and for any $s \in \Gamma(E)$ and any $x \in M$, $j^k_x(s)$ denotes the k-th jet of s at x. For each integer k, $0 \leqq k \leqq n-1$, π^{n-1}_k denotes the natural projection of $J^{n-1}(E)$ onto $J^k(E)$.

We put $F^k(E) = \text{Ker } \pi^{n-1}_K$ for $0 \leqq k \leqq n-1$, and $F^{-1}(E) = J^{n-1}(E)$. Then we have a natural filtration

$$0 = F^{n-1}(E) \subset F^{n-2}(E) \subset \cdots \subset F^0 \subset F^{-1}(E) = J^{n-1}(E).$$

Moreover there is a natural bundle isomorphism of the quotient bundle $F^{k-1}(E)/F^k(E)$ onto the vector bundle $\overset{k}{\otimes} T(M)^* \otimes W$. This is defined as follows: We take a local coordinate system t in M and a moving frame $\{e_\alpha\}$ of E. Fixing a point x of M, to each $s \in \Gamma(E)$ with $j^{k-1}_x(s) = 0$, we assign

$$\frac{1}{k!}\sum_{\alpha=1}^{r}(\frac{d}{dt})^k y_\alpha(x)(dt)^k \otimes e_{\alpha x} \quad (\in \overset{k}{\otimes} T(M)^*_x \otimes E_x), \tag{6.1}$$

where y_α, $1 \leqq \alpha \leqq r$, are functions defined by (0.1). It is easy to see that the definition of (6.1) does not depend on the choice of $(t, \{e_\alpha\})$. It is also easy to see that (6.1) is equal to zero for every $s \in \Gamma(E)$ with $j^k_x(s) = 0$. Thus the assignment of s to the expression (6.1) induces the isomorphism $F^{k-1}(E)/F^k(E) \simeq \overset{k}{\otimes} T(M)^* \otimes E$.

6.2. The typical fiber of $J^{n-1}(E)$

Let V be an r-dimensional vector space with a fixed bases $\{v_1, \cdots, v_r\}$ and W a 1-dimensional vector space with a fixed bases $\{w\}$. Let F be the vecotr space defined by

$$F = V + W \otimes V + \cdots + \overset{n-1}{\otimes} W \otimes V \quad \text{(direct sum)}$$

Taking $\{v_1, \cdots, v_r, w \otimes v_1, \cdots, w \otimes v_r, \cdots, w^{n-1} \otimes v_1, \cdots, w^{n-1} \otimes v_r\}$ as a basis of F, we identify F with $\mathbb{R}^{nr}$. We put

$$F^k = \sum_{j>k} \overset{j}{\otimes} W \otimes V.$$

Then we have a natural filtration:

$$0 = F^{n-1} \subset F^{n-2} \subset \cdots \subset F^0 \subset F^{-1} = F$$

Note that F^{k-1}/F^k is isomorphic to $\overset{k}{\otimes} W \otimes V$.

6.3. The principal $H^{(0)}$-bundle $R^{(0)}$

Let $\mathcal{F}(J^{n-1}(E))$ be the frame bundle of $J^{n-1}(E)$. For every $x \in M$, we regard $\mathcal{F}(J^{n-1}(E))_x$ as the set of all linear isomorphisms of F onto $J^{n-1}(E)_x$.

Let $R_x^{(0)}$ be the set of all elements p of $\mathcal{F}(J^{n-1}(E))_x$ satisfying the following two conditions.

(R.1) $\qquad p(F^k) = F^k(E) \quad \text{for every} \quad 0 \leqq k \leqq n-1.$

By the first condition (R.1), every element p of $R_x^{(0)}$ induces an isomorphism p^k of F^{k-1}/F^k onto $F^{k-1}(E)_x/F^k(E)_x$ for each k, $0 \leqq k \leqq n-1$. Furthermore the induced isomorphism p^0 can be regarded as an isomorphism of V onto E_x. The second condition is

(R.2) There exists a linear isomorphism A of W onto $T(M)^*_x$ which makes the following diagram commutative:

$$\begin{array}{ccc}
F^{k-1}/F^k & \xrightarrow{p^k} & F^{k-1}(E)_x/F^k(E)_x \\
\downarrow & & \downarrow \\
\overset{k}{\otimes} W \otimes V & \xrightarrow{A^k \otimes p^0} & \overset{k}{\otimes} T(M)^*_x \otimes E_x
\end{array}$$

where $A^k : \overset{k}{\otimes} W \to \overset{k}{\otimes} T(M)^*_x$ denotes the isomorphism defined by

$$A^k(w_1 \otimes \cdots \otimes w_k) = Aw_1 \otimes \cdots \otimes Aw_k, \quad w_1, \cdots, w_k \in W.$$

We put $R^{(0)} = \bigcup_{x \in M} R_x^{(0)}$. It is not difficult to see that $R^{(0)}$ is a principal $H^{(0)}$-bundle over M under the identification of F with $\mathbb{R}^{nr}$.

Let t be a local coordinate system in M and $\{e_\alpha\}$ a moving frame of E. For each $x \in M$, let σ_x be the linear isomorphism of F onto $J^{n-1}(E)_x$ defined by

$$\sum_{k=0}^{n-1} \sum_{\alpha=1}^{r} \frac{1}{k!}\left(\frac{d}{dt}\right)^k y_\alpha(x) w^k \otimes v_\alpha \to j_x^{n-1}(s),$$

where $s = \sum y_\alpha e_\alpha \in \Gamma(E)$. It can be easily verified that $\sigma_x \in R_x^{(0)}$. Thus we have a cross section σ of $R^{(0)}$, which will be called the standard cross section of $R^{(0)}$ corresponding $(t, \{e_\alpha\})$.

6.4. The associated connection of $J^{n-1}(E)$

As before, let $\mathcal{D}$ be a system of linear ordinary differential equations on E of order n. Let us consider the system of first-order differential equations associated with $\mathcal{D}$. This can be regarded as a system of first-order equations on the $(n-1)$-th jet bundle $J^{n-1}(E)$ of E as follows; Let σ be the standard cross section of $R^{(0)}$ corresponding to a pair $(t, \{e_\alpha\})$. For every $\zeta \in \Gamma(J^{n-1}(E))$, we define a family of functions $f_{k\alpha}$, $1 \leqq k \leqq n$, $1 \leqq \alpha \leqq r$ on M by

$$\sigma_x^{-1}\zeta(x) = \sum_{k=1}^{n} f_{k\alpha}(x) w^{k-1} \otimes v_\alpha, \quad x \in M.$$

Then the system of first-order equations is given by

$$\frac{df_{k\alpha}}{dt} - k f_{k+1\ \alpha} = 0, \quad 1 \leqq k \leqq n-1,$$
$$\frac{df_{n\alpha}}{dt} + \sum_{k=1}^{n} \sum_{\beta=1}^{r} \frac{(n-k)!}{(n-1)!}\, a_{\alpha\beta}^{(k)} f_{k\beta} = 0.$$

Now we consider the affine connection of $J^{n-1}(E)$ associated with this system. Let ω be its connection form in $\mathcal{F}(J^{n-1}(E))$ and let $\omega^{(0)}$ be the restriction of ω to $R^{(0)}$. It is easy to see that a similar assertion as in Proposition 1.3 holds for the pair $(R^{(0)}, {}^t\omega^{(0)})$. Therefore we have the following

Proposition 6.1. ${}^t\omega^{(0)}$ *takes values in* $\mathbb{R}U_{-1} + \mathfrak{g}^{(0)}$.

6.5. A reduction theorem for $R^{(0)}$

A reduction R of $R^{(0)}$ to H_0 is said to be normal if the restriction of ${}^t\omega^{(0)}$ to R takes values in $\mathfrak{h}+\mathfrak{m}$. In view of Proposition 6.1, it is obvious that the proof of Proposition 3.8 is still valid, if we replace $Q^{(0)}$ by $R^{(0)}$. Therefore we obtain the following

Theorem 6.2. *There exists a unique normal reduction of $R^{(0)}$ to H_0.*

Remark. Assume that M is simply connected. We fix a basis $\{s_1, \cdots, s_{nr}\}$ of $\mathrm{Sol}(\mathcal{D})$. Then the system $\{j^{n-1}(s_1), \cdots, j^{n-1}(s_{nr})\}$ gives a trivialization: $J^{n-1}(E) \simeq M \times \mathbb{R}^{nr}$. Hence we can naturally identify $\mathcal{F}(J^{n-1}(E))$ with $M \times GL(nr, \mathbb{R})$. It is obvious that, under this identification, the correspondence $(x, y) \in R^{(0)} \to (x, \ {}^tY^{-1}) \in P$ maps the normal reduction of $R^{(0)}$ to that of P constructed in §3.

References

[1] E. Cartan, La théorie des groupes finis et continus et la géométrie différentielle, Gauthiers-Villars, Paris (1937).

[2] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Vol. 1, Interscience, New York (1963).

[3] H. Morikawa, Invariant theory (in Japanese), Kinokuniya (1977).

[4] ———, Some analytic and geometric applications of the invariant theoretic method, Nagoya Math. J., **80** (1980), 1-47.

[5] E.J. Wilczynski, Projective differential geometry of curves and ruled surfaces, B.G.Teubner. Leipzig (1906); reprint, Chelsea (1961)

Editor's Note. The present paper was written by the author before he died by an accident in 1989, when he was an associate professor of Department of Mathematics, Kitami Institute of Technology, Kitami 090, Japan. Correspondence should be sent to Professor K. Yamaguchi, Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060, Japan.