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On a Theorem of Edmonds

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§1. Introduction

For an action of a cyclic group of odd order $m \geqq 3$ on a manifold, the normal bundle of the fixed point set is orientable. It is false for actions of the cyclic group of order 2. Edmonds showed the following

Theorem (Edmonds [E]). *If $\mathbb{Z}_2$ acts smoothly on an n dimensional spin manifold M, preserving its orientation and spin structure, then the fixed point set $F = M^{\mathbb{Z}_2}$ is orientable.*

Bott and Taubes gave another proof in [B-T]. The purpose of this short note is to give an elementary proof of this theorem and consider the spin^c case.

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§2. Review on Clifford algebras and spin structures

Let V be a vector space with an positive definite inner product. The Clifford algebra $\mathrm{Cl}(V)$ associated to V is defined as the quotient algebra of the tensor algebra over V by the ideal generated by $v \otimes v + |v|^2$ where $v \in V$. $\mathrm{Cl}(V)$ is not an algebra with $\mathbb{Z}$-grading, but there is a filtration as follows:

$$\mathrm{Cl}(V)^k = \textit{linear span of } \{v_1 \cdots v_j \in \mathrm{Cl}(V); v_i \in V, j \leqq k\}.$$

It is easy to see that the associated graded module of filtered module $\mathrm{Cl}(V)$ is the exterior algebra $\Lambda(V)$. $\mathrm{Cl}(V)$ contains the spin group $\mathrm{Spin}(V)$ which is the double covering group of $SO(V)$. More precisely $\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathrm{Cl}(V)_0$, where $\mathrm{Pin}(V)$ is the multiplicative group generated by unit vectors in V, and $\mathrm{Cl}(V)_0$ is the even part of $\mathrm{Cl}(V)$

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with respect to the $\mathbb{Z}_2$-grading. The $SO(V)$ action on V extends to the action on $\mathrm{Cl}(V)$ as algebra automorphisms.

Let M be a Riemannian manifold. $\mathrm{Cl}(M)$ denotes the Clifford algebra bundle on M associated to the tangent bundle TM. A spin structure on an n-dimensional oriented manifold M is a principal $\mathrm{Spin}(n)$-bundle $\widetilde{P}$ on M, which is a double covering of the principal $SO(n)$-bundle P associated to TM. Remark that $\mathrm{Cl}(M)$ contains $\widetilde{P} \times_{\mathrm{Ad}} \mathrm{Spin}(n)$.

§3. Proof

Let τ be an involution on a spin manifold M, preserving orientation and a spin structure $\widetilde{P}$. Then we have $\tau^*\widetilde{P} \cong \widetilde{P}$, which implies that τ can be lifted to $\tilde{\tau} : \widetilde{P} \to \widetilde{P}$. As τ is an involution, $\tilde{\tau}^2$ is a covering transformation of $\widetilde{P} \to P$ and an bundle automorphism of $\widetilde{P} \to M$. Restricted to the fixed point set $F = M^\tau$, $\tilde{\tau}_F : \widetilde{P}|_F \to \widetilde{P}|_F$ is a bundle automorphism acting trivially on the base space F, i.e. a gauge transformation. It is well known that it can be seen as a section of the adjoint bundle. In our case, $\tilde{\tau}_F$ defines a section of $\mathrm{Cl}(M)|_F$, where we choose a $\mathbb{Z}_2$-invariant Riemannian metric on M. Let $TM|_F = TF \oplus N$ be the decomposition into the tangent bundle TF of F and the normal bundle N. Here we recall the following fact.

Fact. *Let $U = V \oplus W$ be a direct sum of vector spaces with positive definite inner products. Only elements in $\mathrm{Cl}(V \oplus W)$ which act on V by -1 and on W by 1 are $\pm e_1 \cdot e_2 \cdot \dots \cdot e_l$, where $\{e_1, e_2, \cdots, e_l\}$ is an orthonormal basis of V.*

Let $p \in F$. Then τ acts on N_p by -1 and T_pF by 1. Since τ acts on M preserving orientation, the rank of N is even and $\pm v_1 \cdot v_2 \cdot \dots \cdot v_l$ is an element of $\mathrm{Spin}(N_p)$, where $\{v_1, v_2, \cdots, v_l\}$ is an orthonormal basis of N_p. Thus $\tilde{\tau}_F(p)$ corresponds to $\pm v_1 \cdot v_2 \cdot \dots \cdot v_l$. As we have seen in §2, the graded algebra bundle associated to the filtered algebra bundle $\mathrm{Cl}(M)$ is the exterior algebra bundle of TM. $\tilde{\tau}_F$ belongs to $\mathrm{Cl}(M)^l$, and corresponds to $\pm v_1 \wedge v_2 \wedge \cdots \wedge v_l$, therefore it determines the orientation of the normal bundle N, which implies the orientablity of the fixed point set F.

§4. Spin^c case

Let $\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} S^1$. A spin^c structure on an n-dimensional oriented manifold is a principal $\mathrm{Spin}^c(n)$-bundle Q such that $Q \times_\rho SO(n)$ is the principal $SO(n)$-bundle associated to TM, and ρ

is the natural homomorphism from $\mathrm{Spin}^c(n)$ to $SO(n)$. We can show the following

Proposition. *Let M be a spin^c manifold and Q a spin^c structure on M. If an smooth involution τ on M can be lifted to Q as a periodic mapping, then the fixed point set F is orientable.*

Proof. As in §2, N denotes the normal bundle of F, and $\tilde{\tau}_F : Q|_F \to Q|_F$ denotes the lifting of τ restricted to F with period s. $\tilde{\tau}_F$ defines a section of $\mathrm{Cl}(M)|_F \otimes \mathbb{C}$, and a section of $\bigwedge(TM)|_F \otimes \mathbb{C}$. For $p \in F$, $\tilde{\tau}_F$ corresponds to $\exp(\frac{2\pi i m}{s}) \cdot v_1 \wedge v_2 \wedge \cdots \wedge v_l$, where $\{v_1, v_2, \cdots, v_l\}$ is an orthonormal vasis of N_p. Consider $\mathrm{Re}(\tilde{\tau}_F)$ or $\mathrm{Im}(\tilde{\tau}_F)$, it defines an orientation of N, which implies the conclusion.

References

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