Green Function on Self-Similar Trees

Masami Okada

§1.   Introduction

There are typical examples of symmetric homogeneous spaces where the canonical Green functions associated with the Laplacians are explicitly calculated [29]. However, it is usually difficult to calculate them on models without nice symmetry or rich group structure.

In this note we shall study one dimensional models which have self-similar structure instead of symmetric one and shall derive a functional equation via a scaling argument which determines in principle the Green function. An asymptotic expansion of the Green function will also be discussed which gives the decay order of the heat kernel as time goes to infinity.

We are partly motivated by fractal geometry. In fact self-similar trees are typical fractal models [19] [13] [11] and the asymptotic decay order of heat kernels is in general closely related to the so called spectral dimension of fractal models [25]. Note also that the tree structure is omnipresent in the natural world [19] [18].

We hope that our approach also enrich the knowledge on the spectral geometry (or differential geometry) and on the brownian motion on various models.

§2.   Self-similar tree

Let X be the self-similar tree network depicted as in the following Figure 1.

Let the length of PP′, PQ and PR be respectively 1, r and s. Here the self-similarity means that the lengths of QS, QT, RU and RV are respectively r^2, rs, rs and s^2 and moreover other branches are defined in the same manner.

First we choose coordinate x such that the point O and P corresponds respectively to $x = 0$ and $x = 1/2$. To simplify the notation, Q
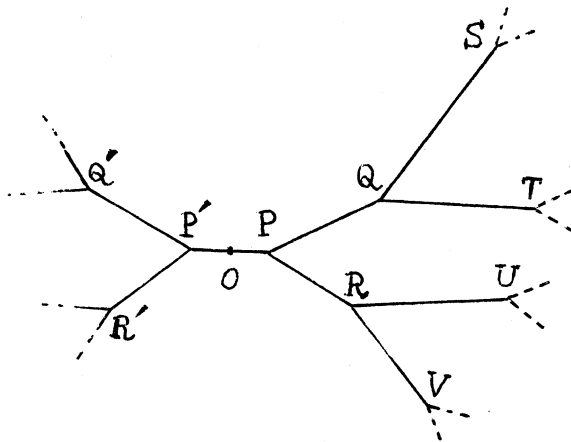

Fig. 1.

and R correspond respectively to $x = 1/2+r$ (on PQ) and $x = 1/2+s$ (on PR) without ambiguity. Then the canonical Green function $G(x, 0, \lambda)$ associated with the canonical Laplacian on X is formally defined for $\lambda \in \mathbf{C} - \mathbf{R}_-$ by

$$(1) \qquad (\lambda - \frac{d^2}{dx^2})G = \delta_0 \text{ and } G \to 0, \text{ as } \text{dist}(x, O) \to \infty.$$

Lemma. *Set* $u(x) = G(x, 0, \lambda)$ *modulo constant multiple. Then* $u(x)$ *can be written as follows.*

For $x \in OP$,

$$(2) \qquad u(x) = e^{\sqrt{\lambda}x} + \theta(\lambda)e^{-\sqrt{\lambda}x},$$

for $x \in PQ$,

$$(3) \qquad u(x) = c(\lambda)\{e^{\sqrt{\lambda}(x-r/2-1/2)} + \theta(r^2\lambda)e^{\sqrt{\lambda}(r/2+1/2-x)}\},$$

and for $x \in PR$,

$$(4) \qquad u(x) = d(\lambda)\{e^{\sqrt{\lambda}(x-s/2-1/2)} + \theta(s^2\lambda)e^{\sqrt{\lambda}(s/2+1/2-x)}\},$$

where c, d *and* θ *can be determined via a functional identity.*

Proof. First we give the outline of the proof. The Green function is harmonic on X–{O} which means that $u(x)$ is continuous and should

satisfy the conservation law of heat flux at the point P. See [9] for example. These three equations determine the three functions c, d, and θ. Then it suffices to show that $u(x)$ defined above can be extended in such a way that it satisfies the same conditions at other nodal points Q, R, etc.

Next, let us see the details.

a) continuity at $x = 1/2$:

By setting $x = 1/2$ in (2), (3) and (4), we have

$$
(5) \quad \begin{aligned} u(x) = e^{\sqrt{\lambda}/2} + \theta(\lambda)e^{-\sqrt{\lambda}/2} &= c(\lambda)\{e^{-r\sqrt{\lambda}/2} + \theta(r^2\lambda)e^{r\sqrt{\lambda}/2}\}, \\ &= d(\lambda)\{e^{-s\sqrt{\lambda}/2} + \theta(s^2\lambda)e^{s\sqrt{\lambda}/2}\}. \end{aligned}
$$

b) conservation of heat flux:

At $x = 1/2$ the sum of first derivatives is equal to zero, i.e.,

$$
(6) \quad \begin{aligned} e^{\sqrt{\lambda}/2} - \theta(\lambda)e^{-\sqrt{\lambda}/2} &= c(\lambda)\{e^{-r\sqrt{\lambda}/2} - \theta(r^2\lambda)e^{r\sqrt{\lambda}/2}\} \\ &\quad + d(\lambda)\{e^{-s\sqrt{\lambda}/2} - \theta(s^2\lambda)e^{s\sqrt{\lambda}/2}\}. \end{aligned}
$$

In this way c, d and θ are determined by (5) and (6) and we shall get a functional equation for θ.

Now let us proceed to show that if we define u appropriately on QS, QT, RU and RV, then the above conditions a) and b) are also satisfied on the points Q and R. In fact it suffices to define u for $x \in$ QS by

$$
(7) \quad u(x) = c(\lambda)c(r^2\lambda)\{e^{\sqrt{\lambda}(x - r^2/2 - r - 1/2)} + \theta(r^4\lambda)e^{\sqrt{\lambda}(r^2/2 + r + 1/2 - x)}\},
$$

and for $x \in$ QT by

$$
(8) \quad u(x) = c(\lambda)d(r^2\lambda)\{e^{\sqrt{\lambda}(x - rs/2 - r - 1/2)} + \theta(r^2s^2\lambda)e^{\sqrt{\lambda}(rs/2 + r + 1/2 - x)}\}.
$$

Then note that the conditions at Q are also satisfied for this u since by the definition $c(r^2\lambda)$, $d(r^2\lambda)$ and $\theta(r^2\lambda)$ in (7) and (8) correspond to $c(\lambda)$, $d(\lambda)$ and $\theta(\lambda)$ in (3) and (4) respectively. The same scaling argument can also be applied to other nodal points. Q.E.D.

§3. A functional equation

Let $w \equiv w(\lambda) = e^{\sqrt{\lambda}/2}$. Then (5) gives,

$$
c(\lambda) = \frac{w + \theta w^{-1}}{w^{-r} + \theta(r^2\lambda)w^r} \quad \text{and} \quad d(\lambda) = \frac{w + \theta w^{-1}}{w^{-s} + \theta(s^2\lambda)w^s},
$$

and hence (6) implies

$$(9) \qquad \frac{w^2 - \theta(\lambda)}{w^2 + \theta(\lambda)} = \frac{w^{-2r} - \theta(r^2\lambda)}{w^{-2r} + \theta(r^2\lambda)} + \frac{w^{-2s} - \theta(s^2\lambda)}{w^{-2s} + \theta(s^2\lambda)} \, .$$

Theorem 1.   *Let $H(\lambda) \equiv 2\sqrt{\lambda}\, G(0,0,\lambda)$ and $\phi(\lambda) = \tanh\sqrt{\lambda}/2$. Then, H satisfies the following functional equation:*

$$(10) \qquad \frac{1 - \phi(\lambda)H(\lambda)}{H(\lambda) - \phi(\lambda)} = \frac{1 + \phi(r^2\lambda)H(r^2\lambda)}{H(r^2\lambda) + \phi(r^2\lambda)} + \frac{1 + \phi(s^2\lambda)H(s^2\lambda)}{H(s^2\lambda) + \phi(s^2\lambda)} \, .$$

Proof.   By the definition of G

$$G(0,0,\lambda) = \frac{\theta + 1}{2\sqrt{\lambda}\,(\theta - 1)} \, ,$$

which implies $\theta = \frac{H+1}{H-1}$. Therefore from (9) follows the theorem.

$$\text{Q.E.D.}$$

This may be one of few situations where a functional equation appears naturally in geometry in an explicit manner. Further, this functional equation may itself be interesting. In the next section, however, we shall investigate only the asymptotic behavior of the solution H near $\lambda = 0$, $\lambda \in \mathbf{C} - \mathbf{R}_-$.

§4.   Asymptotic expansion I (conjecture)

Let us study how to determine the first terms of the asymptotic expansion of H, as λ tends to zero in $\mathbf{C} - \mathbf{R}_-$. To this end the Tauberian theorem is useful since the heat kernel P_t which is the inverse Laplace transform of G may provide information on G in some cases. We shall state our result as a conjecture since mathematically rigorous proof has not yet been completed.

Conjecture.   *For $g(\lambda) = G(0,0,\lambda)$, we have the following expansion as $\lambda \to 0$ in $\mathbf{C} - \mathbf{R}_-$:*

Case 1. If $1/r + 1/s < 1$ then $g(\lambda) \sim a_\alpha \lambda^\alpha$, where α satisfies

$$(11) \qquad r^{-2\alpha-1} + s^{-2\alpha-1} = 1 \qquad (-1/2 < \alpha < 0).$$

Case 2. If $1/r + 1/s = 1$ then $g(\lambda) \sim a_0 \log \lambda, \quad (a_0 \neq 0)$.

Case 3. *If* $1/r + 1/s > 1$ *and* $\max(r, s) > 1$, *then there are two subcases. Let* β *be defined by*

$$(12) \qquad r^{2\beta-1} + s^{2\beta-1} = \left(1/r + 1/s\right)^2 \quad (0 < \beta).$$

(i) *Then if* β *is not an integer, i.e.,* $n < \beta < n + 1$,

$$g(\lambda) \sim a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1} + a_\beta\lambda^\beta \quad (a_\beta \neq 0),$$

and (ii) *if* β *is an integer, i.e.,* $\beta = n$,

$$g(\lambda) \sim a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1} + a_n\lambda^n \log \lambda \quad (a_n \neq 0).$$

where $a_0 = (r + s + rs)/\{4(r + s - rs)\}$.

Let us briefly mention other cases.

If $\max(r, s) \leq 1$ then $g(\lambda)$ may be an analytic function in a neighborhood of $\lambda = 0$. In particular, if $r = s = 1$ this is shown in [8].

The case where $r + s < 1$ is exceptional since then the solution of the functional equation (10) is not unique. In fact as is easily seen, the following function g_1 also satisfies (10):

$$(13) \qquad g_1(\lambda) = \frac{a_{-1}}{\lambda} + a_0 + \cdots, \text{ where } a_{-1} = \frac{1 - r - s}{1 + r + s}.$$

To understand this apparently peculiar situation it suffices to observe that in this case the total length of X is finite ($= \frac{1+r+s}{1-r-s}$). It turns out that g and g_1 correspond respectively to the Dirichlet and Neumann boundary conditions. Note that under both conditions our computation which uses the scaling property of the model X are justified. Further, note that these two boundary conditions make no difference provided $r + s \geq 1$. This phenomenon is explained by means of probability theory. See Proposition in the section 6.

§5. Asymptotic expansion II (computation)

We would like to present the idea of the "proof", although as we have already mentioned, some parts of the following argument have not yet been rigorously verified. The following is actually difficult for us to prove directly from (10).

$$(14) \qquad \frac{\log H(\lambda)}{\log \lambda} \longrightarrow c, \text{ as } \lambda \to 0, \lambda \in \mathbf{C} - \mathbf{R}_-,$$

with a nonzero constant c, or equivalently via the Tauberian theorem

$$(15) \qquad \frac{\log P_t(0,0)}{\log t} \longrightarrow -c - 1/2 \, , \text{ as } t \to \infty.$$

In the sequel we shall assume (14).

Remark 1. Since the left hand side of (15) is bounded, we need only to show

$$\frac{\log P_t(0,0)}{\log t} \text{ is a monotone function of } t \text{ for large } t.$$

However this kind of property seems to be unknown in general.

"Proof" of Conjecture. Case 1. $1/r + 1/s < 1$. First we know $0 \le c \le 1/2$ since it is known that the Brownian motion on X is recurrent if and only if $1/r + 1/s \le 1$. See [6] [17] [22] for example. Next, as $\phi(\lambda) = \sqrt{\lambda}/2 + \cdots$, (10) implies

$$(16) \qquad \frac{1}{H(\lambda) - \frac{\sqrt{\lambda}}{2}} = \frac{1}{H(r^2\lambda) + \frac{r\sqrt{\lambda}}{2}} + \frac{1}{H(s^2\lambda) + \frac{s\sqrt{\lambda}}{2}} \, ,$$

up to an error $O(\sqrt{\lambda})$ and hence provided $c < 1/2$,

$$(17) \qquad \qquad 1 = \frac{H(\lambda)}{H(r^2\lambda)} + \frac{H(\lambda)}{H(s^2\lambda)} \, ,$$

up to the same error.

Consequently, $c = -2\alpha - 1$ with α defined in (11).

Case 2. $1/r + 1/s = 1$. This case corresponds to $c = 1/2$ in previous case. We start from (16). Let us set $h(\lambda) = H(\lambda)/\sqrt{\lambda}$.

Then we can show that $h(\lambda)$ tends slowly to infinity as λ tends to 0 in view of (16) and thus

$$(18) \qquad \begin{aligned} & \frac{1}{h(\lambda)}\left(1 + \frac{1}{2h(\lambda)}\right) \\ = & \frac{1}{rh(r^2\lambda)}\left(1 - \frac{1}{2h(r^2\lambda)}\right) + \frac{1}{sh(s^2\lambda)}\left(1 - \frac{1}{2h(s^2\lambda)}\right), \end{aligned}$$

up to an error $o(1)$, i.e.,

$$\frac{h(\lambda) - h(r^2\lambda)}{r\,h(r^2\lambda)} + \frac{h(\lambda) - h(s^2\lambda)}{s\,h(s^2\lambda)} = \frac{1}{2h(\lambda)} + \frac{h(\lambda)}{2rh(r^2\lambda)} + \frac{h(\lambda)}{2sh(s^2\lambda)}.$$

Therefore, as $1/r + 1/s = 1$ we may expect that

$$(19) \qquad h(\lambda) - h(r^2\lambda) \sim \text{ constant} \neq 0, \text{ as } \lambda \to 0,$$

from which follows $h(\lambda) \sim c \log \lambda$, with nonzero c.

Case 3. $1/r + 1/s > 1$ and $\max(r, s) > 1$. In this case we take the Taylor expansion of ϕ up to the order $[\beta]$ and the coefficients a_k, $k = 0, 1, \cdots, [\beta]$ can be computed as far as

$$(20) \qquad (1/r + 1/s)^2 \neq r^{2k-1} + s^{2k-1}.$$

This fact is easily shown inductively by the same method of comparing terms of the same order. Since the the rest of the proof is analogous to previous cases, it is omitted. Q.E.D.

As a consequence, it turns out that if $r+s \geq 1$ then the solution H is unique. Further, we derive the following from the asymptotic expansion by the Tauberian theorem: *Provided our cojecture is correct.*

$$(21) \qquad \begin{aligned} P_t(0,0) &= O(t^{-\alpha-1}) \text{ if } 1/r + 1/s < 1, \\ &= O(t^{-\beta-1}) \text{ if } 1/r + 1/s \geq 1, \end{aligned}$$

where α and β are determined by (11) *and* (12) *respectively. Also note that P_t is rapidly decreasing if* $\max(r, s) \leq 1$.

Remark 2. If $r = s$ the above argument is made rigorous since there exists only one parameter r and the functional equation (10) becomes simpler.

§6. The remaining case : $0 < r < s = \infty$

In the previous sections r and s were finite. Let us treat the case where $s = \infty$. The method is similar and simpler.

Theorem 2. *Denote $e^{-\sqrt{\lambda}}$ by $\rho(\lambda)$. Then for H defined as in Theorem 1 , we get*

$$(22) \qquad H(r^2\lambda) = \frac{\psi(\lambda)H(\lambda) - \xi(\lambda)}{\eta(\lambda)H(\lambda) - \zeta(\lambda)},$$

where

$$\psi = \rho^{r-1} - \rho^r - 3\rho^{-1} - 1,$$
$$\xi = \rho^{r-1} - \rho^r - 3\rho^{-1} + 1,$$
$$\eta = \rho^{r-1} - \rho^r + 3\rho^{-1} + 1,$$
$$and \quad \zeta = \rho^{r-1} + \rho^r + 3\rho^{-1} - 1.$$

Proof. It suffices to use again the scaling argument due to the self-similarity of X and the details of computation are omitted.   Q.E.D.

As before, from Theorem 2. it follows then

Corollary.   *As* $\lambda \to 0$, $\lambda \in \mathbf{C} - \mathbf{R}_-$,

$$(23) \qquad G(0,0,\lambda) = a_0 - \frac{r^2}{(1-r)^3(1+r)}\sqrt{\lambda} + \cdots \quad \text{if } r < 1,$$

$$= \lambda^{-1/4} \text{ if } r = 1 \text{ and } = O(\frac{1}{\sqrt{\lambda}\log\lambda}) \quad \text{if } r > 1.$$

Therefore, as $t \to \infty$, $P_t(0,0) = O(t^{-3/2})$, *if* $r < 1$, $= O(t^{-3/4})$, *if* $r = 1$, *and* $= O\left(\dfrac{1}{\sqrt{t}\log t}\right)$, *if* $r > 1$.

Remark 3. The case $r = 1$ was already established in [9] and the case $r > 1$ can be easily shown by an alternative probabilistic argument.

Now let us investigate the cases where $r < 1$ and either $s = \infty$ or $1 \leq s < \infty$ from a probabilistic point of view. The following proposition explains why the boundary condition doesn't affect the heat kernel [Green function].

Proposition.   *If* $r < 1$ *and* $r + s \geq 1$, *then the Brownian particle never returns to the origin after hitting the boundary.*

Proof. We consider the case $s < \infty$. First we recall that $v(x) \equiv G(x,0,0)/G(0,0,0)$ is nothing but the probability that a particle starting from the point at x ever hits the origin (see [3] for example) and this v has already been treated in the preceding sections. In fact, $v(x) = u(x)/u(0)$ by the unicity of the Green function, where u is given by (2) and

$$\theta(\lambda) = \frac{2\sqrt{\lambda}G(\lambda)+1}{2\sqrt{\lambda}G(\lambda)-1} \sim -1 - 4a_0\sqrt{\lambda},$$

as $\lambda \to 0$. Therefore $c(0) = s/(r+s)$ and $d(0) = r/(r+s)$ in view of the identities in the section 4. Consequently if x is at the middle point of the interval whose length is $r^k s^l$, then

$$v(x) = O(\frac{r^k s^l}{(r+s)^{k+l}}\frac{1 + \theta(r^{2k}s^{2l}\lambda)}{1+\theta(\lambda)}) = O((\frac{rs}{r+s})^{k+l}).$$

It follows then that $v(x) \to 0$ as x tends to a boundary point. The case where $s = \infty$ is similar. In this case $c(0) = 1$ but the function θ yields the same conclusion.   Q.E.D.

The above phenomenon is a consequence of the dangling effect of H. Kesten [14]. It may be interesting to note that the decay order of the heat kernel does not depend on the particular value of $r < 1$.

§7.   Generalizations and questions

Apart from making arguments rigorous in some parts of the section 5, there remains questions and possibility of generalizations.

(1) Trees without strict self-similarity. If trees are not exactly self-similar, our method of explicit computation can not be applied in general. Nevertheless, if the kth branches have lengths not 2^k but $l_k = k^\gamma 2^k$, $-\infty < \gamma < \infty$ for example, small correction will be sufficient for the relevant term in the expansion. In fact, if $r = s = k^\gamma 2^k$ then it is likely that

$$(24) \qquad\qquad P_t(0,0) = O(\frac{(\log t)^\gamma}{t}).$$

However this question has to be investigated more systematically. Moreover, when the number of branches are not fixed at each nodal point, another difficulty is caused, although self-similar trees with certain periodicity may be treated by the same method as ours.

(2) Models of higher dimension.

(3) On point spectrum. I am indebted to Prof. K. Aomoto for this problem. What can we reduce from (10) on the point spectrum? We have only a partial answer to this question: If λ_0 is a point spectrum the Laplacian, then $G(0,0,\lambda_0) = \infty$, therefore putting $H(\lambda_0) = \infty$ in (10) we get formally

$$(25) \qquad\qquad -\phi(\lambda_0) = \phi(r^2\lambda_0) + \phi(s^2\lambda_0).$$

Besides this, very few seem to be known on this question. We add only two facts:

It may be interesting to compare our models with the lattice models where the second order differential operator (Laplacian) is replaced by the adjacent second order difference operator when $r = 1$. See [2] [3] [5]

[6] [10] etc. Point spectrums exist in the former model and not in the latter [2].

Another curious thing occurs when $r + s < 1$. It seems that every $\lambda \in \mathbf{R}_-$ is actually a point spectrum for the Neumann problem (i.e., with reflecting boundary condition in terms of probability theory) with respect to the usual line measure on X.

References

[1] S. Alexander and R. Orbach, Density of states on fractals, J. Phys. Lett., **43** (1982), 625–631.

[2] K. Aomoto, Point spectrum on a quasi homogeneous tree, to appear, Pacific J. of Math..

[3] P. Cartier, Fonctions harmoniques sur un arbre, Symposia Math., **9** (1972), 203–270.

[4] E.B. Davies, "Heat kernels and spectral theory", Cambridge Univ. Press, 1989.

[5] J. Dodziuk, Difference equations, isoperimetric inequality and transience of certain random walks, Trans. Amer. Math. Soc., **284** (1984), 787–794.

[6] P.G. Doyle and J.L. Snell, "Random Walks and Electric Networks", Math. Asso. America, Washington D.C., 1984.

[7] A. Figà-Talamanca and T. Steger, Harmonic analysis on trees, Symposia. Math., **29** (1987), 163–182.

[8] M. Fukushima and T. Shima, On a spectral analysis for the Sierpinski gasket, to appear.

[9] B. Gaveau, M. Okada and T. Okada, Explicit heat kernels on graphs and spectral analysis,, Several Complex Variables (J.E. Fφrnaess, ed.) Math. Notes, **38** (1991), 360–384, Princeton Univ. Press.

[10] P. Gerl, Continued fraction methods for random walks on N and on trees, Lecture Notes in Math. Springer, **1064** (1984), 131–146.

[11] M. Hata, On the structure of self-similar sets, Japan J. Appl. Math., **2** (1985), 381–414.

[12] H. Heyer, "Convergence of Markov chains on a homogeneous tree", Seminar on Probability (Japan), 1988.

[13] J.E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J., **30** (1981), 713–747.

[14] H. Kesten, Subdiffusive behavior of random walk on a random cluster, Ann. Inst. H.Poincaré, **22** (1986), 425–487.

[15] J. Kigami, Harmonic calculus on P.C.F. self-similar sets, Trans. Amer. Math. Soc., to appear.

[16] A. Koranyi and M.A. Picardello, Boundary behaviour of eigenfunctions of the Laplace operator on trees, Ann. Sc. Norm. Sup. Pisa, **13** (1986), 389–399.

[17] T. Lyons, A simple criterion for transience of a reversible Markov chain, The Annals of Probability, **11** (1983), 403–413.

[18] N. MacDonald, "Trees and Networks in Biological Models", John Wiley and Sons, 1983.

[19] B.B. Mandelbrot, "The Fractal Geometry of Nature", W.H. Freeman and Co., San Fransisco, 1983.

[20] B. Mohar and W. Woess, A survey on spectra of infinite graphs, Bull. London Math. Soc., **21** (1989), 209–234.

[21] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. J. Math., **80** (1958), 931-954.

[22] C.St.J.A. Nash-Williams, Random walks and electric currents in networks, Proc. Cam. Phil. Soc., **55** (1959), 181–194.

[23] M. Okada, T. Sekiguchi and Y. Shiota, Heat kernels on infinite graph networks and deformed Sierpinski gaskets, Japan J. Appl. Math., **7** (1990), 527–543.

[24] T. Okada, Asymptotic behaviour of skew conditional heat kernels on graph network , to appear.

[25] R. Rammal and G. Toulouse, Random walks on fractal structures and percolation clusters, J. Phys. Lett., **44** (1983), 13–22.

[26] J.P. Roth, Le spectre du laplacien sur un graphe, Lecture Notes in Math, Springer, **1096** (1984), 521–539.

[27] S. Sawyer, Isotropic random walks in a tree, Z. Wahrsch. Verw. Gebierte, **42** (1978), 278–292.

[28] J.P. Serre, "Astérisque; Arbres, Amalgams,SL_2", Soc. Math. de France, 1977.

[29] T. Sunada, "Fundamental Groups and Laplacians   (in japanese)", Kinokuniya, Tokyo, 1988.

[30] A. Telcs, Random walks on graphs, electric networks and fractals, Probab. Th. Rel. Fields, **82** (1989), 435–449.

[31] N.Th. Varopoulos, Long range estimates for Markov chains, Bull. Sci. Math.(2), **109** (1985), 225–252.

[32] M. Yamaguti and J. Kigami, Some remarks on Dirichlet problem of Poisson equation., "Analyse Mathématique et Applications", Paris, Gauthier-Villars, 1988, pp. 465–471.

[33] H. Yokoi, Denumerable Markov chains and nested fractals, Master Thesis, Tôhoku Univ. (1991).

Department of Mathematics
College of General Education
Tôhoku University
Kawauchi Sendai 980
Japan
Fax 022-213-2870
E-mail OKMAT@JPNTUVMO.Bitnet