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A Note on Lie Contact Manifolds

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Dedicated to Professor T. Otsuki on his 75th birthday

§1. Introduction

The classical projective and conformal connections of H. Weyl fit into harmonic theory in the Spencer cohomology of graded Lie algebras in the sense that the curvature forms of such connections are harmonic. These structures are treated systematically by N. Tanaka [T1] as special cases of a structure associated with an $\mathfrak{l}$-system, called later a graded Lie algebra of the first kind. A lucid explanation of this theory is given by T. Ochiai [O] where he rebuilds Tanaka theory using semisimple flat homogeneous spaces as model spaces.

To deal with more general structures such as CR-structure, Tanaka developed the theory to simple graded Lie algebras of contact type [T2] and then to the full class of simple graded Lie algebras [T3]. The argument essentially depends on the generalized prolongation scheme and on the harmonic theory in the refined Spencer cohomology of Lie algebras. The vanishing of certain cohomology group guarantees the existence and uniqueness of *the normal Cartan connection* (= *Tanaka connection*, for short), attached to the equivalence class of the structure. Though the curvature form of Tanaka connection is no more harmonic in general, its harmonic part gives a fundamental system of invariants of the structure.

Going back to the starting point, we know that the study of projective and conformal structures on a manifold has a background of the classical projective and conformal geometry. This reminds us of another classical geometry, Lie's sphere geometry. Then what kind of structure corresponds to this geometry? Why has this object not yet been investigated? H. Sato [S, SY] is probably the first to consider this problem and finds a Lie contact structure, which is a structure on a contact manifold with model space T_1S^n of which transformation group is the

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Lie transformation group $PO(n+1,2)$. Noting that the Lie algebra of $PO(n+1,2)$ is a simple graded Lie algebra of contact type, Sato claims that the developed Tanaka theory plays a main role on this structure. A typical and important example of Lie contact structure is found on the unit tangent bundle T_1M of a riemannian manifold M. Now, how can we express Tanaka connection on this structure? The author answered this question in [M2] and obtained a close relation between the Lie contact structure on T_1M and the conformal structure on M.

The purpose of this note is to give a survey of these results, and add some explanations. In particular, Theorem 1 in §4, which clarifies the relation between Tanaka connection and the normal conformal connection, is due to Sato, who suggested it to the author, observing the result in [M2]. Recently, this relation is also investigated by K. Yamaguchi in a different way.

For these valuable suggestions as well as criticisms, the author would like to express her hearty thanks to Professors H. Sato and K. Yamaguchi.

§2. $\tilde{G}$-structure and Cartan connection

Let M be an n-dimensional differentiable manifold and let $F(M)$ be the linear frame bundle over M. For a Lie subgroup $\tilde{G}$ of $GL(n,\mathbf{R})$, a $\tilde{G}$-reduction $\tilde{P}$ of $F(M)$ is called a $\tilde{G}$-structure on M. When $\pi\colon \tilde{P} \to M$ is a principal $\tilde{G}$-bundle over M, an $\mathbf{R}^n$-valued 1-form $\tilde{\theta}$ defined by

$$\tilde{\theta}(X) = u^{-1}(\pi_* X), \qquad u \in P, \quad X \in T_u\tilde{P}$$

is called the basic form, which satisfies

(1) $\tilde{\theta}(X) = 0$ if and only if X is a vertical vector.

(2) $R_a^*\tilde{\theta} = a^{-1}\tilde{\theta}, \quad a \in \tilde{G}$.

Sometimes, we define a $\tilde{G}$-structure by a pair $(\tilde{P}, \tilde{\theta})$, satisfying (1) and (2).

Let G/G' be a homogeneous space of dimension n and let $\mathfrak{g}$ and $\mathfrak{g}'$ be the Lie algebra of G and G', respectively.

Definition. *A Cartan connection (P,ω) of type G/G'* is by definition

C1 P is a principal fiber bundle over M with the structure group G'.

C2 ω is a $\mathfrak{g}$-valued 1-form on P satisfying

(a) $\omega(X) = 0$ implies $X = 0, \quad X \in TP$.

(b) $R_a^*\omega = \mathrm{Ad}(a^{-1})\omega, \quad a \in G'$.
(c) $\omega(A^*) = A, \quad A \in \mathfrak{g}'$.

When (P, ω) is a Cartan connection of type G/G', let θ be the $\mathfrak{m}$-component of ω, where $\mathfrak{m} = T_0(G/G')$. Putting $\tilde{P} = P/\mathrm{Ker}\,\rho$ where ρ is the isotropy representation, we denote the projection $P \to \tilde{P}$ by $\tilde{\rho}$. Then $(\tilde{P}, \tilde{\theta})$, where $\theta = \tilde{\rho}^*\tilde{\theta}$, is a $\tilde{G}$-structure on $M, \tilde{G} = \rho(G')$. To give a standard choice of connection which induces a given $\tilde{G}$-structure, Tanaka defined normal Cartan connections as follows.

Let $\mathfrak{g} = \sum_{p \in \mathbf{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra of non-compact type with subalgebra $\mathfrak{m} = \sum_{p<0} \mathfrak{g}_p$. A cochain complex $(C^q(\mathfrak{m}, \mathfrak{g}), \partial)$ is given where $C^q(\mathfrak{m}, \mathfrak{g}) = \mathfrak{g} \otimes \wedge^q(\mathfrak{m}^*)$, and $\partial : C^q \to C^{q+1}$ is the coboundary operator [T3, K]. Let $\partial^* : C^{q+1} \to C^q$ be the adjoint operator with respect to the metric $(X, Y) = -B(X, \sigma Y)$ defined by the Killing form B and the involution σ of $\mathfrak{g}$. Explicitly, they are given by

$$(\partial c)(X_1 \wedge \cdots \wedge X_{q+1}) = \sum_i (-1)^{i+1}[X_i, c(X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_{q+1})]$$
$$+ \sum_{i<j} (-1)^{i+j} c([X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_{q+1}),$$
$$(\partial^* c)(X_1 \wedge \ldots \wedge X_{q-1}) = \sum_j [e_j^*, c(e_j \wedge X_1 \wedge \cdots \wedge X_{q-1})]$$
$$+ \frac{1}{2} \sum_{i,j} (-1)^{i+1} c([e_j^*, X_i]_- \wedge e_j \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_{q-1}),$$

where $c \in C^q, X_1, \ldots, X_{q+1} \in \mathfrak{m}$ and $[e_j^*, X_i]_-$ denotes the $\mathfrak{m}$-component of $[e_j^*, X_i]$ with respect to the decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}'$, $\{e_j^*\}$ is the base of $\mathfrak{m}^* = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ dual to a base $\{e_j\}$ of $\mathfrak{m}$ with respect to B. Define

$$\wedge_i^q = \sum \mathfrak{g}_{r_1}^* \wedge \cdots \wedge \mathfrak{g}_{r_q}^*,$$

where the summation is taken over $r_1, \ldots, r_q < 0, \sum_{k=1}^q r_k = i$. Then we have $\wedge^q(\mathfrak{m}^*) = \sum_i \wedge_i^q$. Put

$$C^{p,q} = \sum_j \mathfrak{g}_j \otimes \wedge_{j-p-q+1}^q.$$

When (P, ω) is a Cartan connection of type G/G' where the Lie subalgebra of G (G', resp.) is $\mathfrak{g}$ ($\mathfrak{g}'$, resp.), the coefficient $K(z)$ of the curvature form $\Omega = \frac{1}{2} K\theta \wedge \theta$ (P, ω) belongs to $C^2(\mathfrak{m}, g), z \in P$. Let K^p be the $C^{p,2}$-component of K.

Definition. (P,ω) is *a normal Cartan connection of type G/G'* if the curvature form $\Omega = \frac{1}{2}K\theta \wedge \theta$ satisfies

N1 $K^p = 0 \quad (p < 0)$
N2 $\partial^* K^p = 0 \quad (p \geq 0)$.

Definition. A simple graded Lie algebra $\mathfrak{g}$ is called *of the μ-th kind* if $\mathfrak{g}_p = 0, (p < -\mu)$ and $\mathfrak{g}_{-\mu} \neq 0$. When $\mathfrak{g}$ is of the second kind and $\dim \mathfrak{g}_{-2} = 1$, $\mathfrak{g}$ is called *of contact type*.

Remark 1. When $\mathfrak{g}$ is of the first kind, **N1** means ω is torsion free while **N2** means that the curvature form is harmonic. When $\mathfrak{g}$ is of contact type, **N1** is satisfied if and only if the associated $\tilde{G}$-structure is of type $\mathfrak{m}$ (see [T3] for definition. Here, for simplicity, we adopt **N1** as a definition of *$\tilde{G}$-structure of type* $\mathfrak{m}$, when the $\tilde{G}$-structure is induced from (P,ω)). As we see later, this is the case for conformal contact and Lie contact structures. In the following, to avoid more definitions, let $\mathfrak{g}$ be of the first kind or of contact type. The following is important:

Fact 1 [T3]. *When $H^{q,1}(\mathfrak{m}, g) = 0$ for $q \geq 1$, there exists a unique normal Cartan connection of type G/G' attached to the isomorphism class of $\tilde{G}$-structures of type* $\mathfrak{m}$.

§3. Definitions and basic facts of Lie contact structures

Let $\mathbf{R}^N_k$ be the N-dimensional real vector space equipped with the scalar product $\langle\ ,\ \rangle_k$ of signature $(+,\dots,+,-,\dots,-)$, where $-$ appear k-times, $0 \leq k \leq N$. The projective space associated with $\mathbf{R}^N_k$ is denoted by $P^{N-1}_k\mathbf{R}$. We identify $S^n = \{x \in \mathbf{R}^{n+1}_0 \mid \langle x, x\rangle_0 = 1\}$ with $Q^n = \{[y] \in P^{n+1}_1\mathbf{R} \mid \langle y, y\rangle_1 = 0\}$, by the correspondence

$$S^n \ni x \mapsto y = (x,1) \in \mathbf{R}^{n+2}_1.$$

Then the projective transformation group of $P^{n+1}_1\mathbf{R}$ fixing Q^n is $L = PO(n+1,1)$, *the Möbius group*.

Fact 2. *L acts on S^n transitively and $S^n = L/L'$, for an isotropy subgroup L'.*

Let Σ be the set of all oriented hyperspheres in S^n (including point spheres). An element of Σ is given by (m,θ), where $m \in S^n$ is the center of the hypersphere and $0 \leq \theta < \pi$ is the oriented radius. Identify Σ with $Q^{n+1} = \{[k] \in P^{n+2}_2\mathbf{R} \mid \langle k, k\rangle_2 = 0\}$ by

$$\Sigma \ni (m,\theta) \mapsto k = (m, \cos\theta, \sin\theta) \in \mathbf{R}^{n+3}_2.$$

The projective transformation group of $P_2^{n+2}\mathbf{R}$ fixing Q^{n+1} is $G = PO(n+1,2)$, the so called *Lie transformation group*. $[k_1] \in Q^{n+1}$ is in oriented contact with $[k_2] \in Q^{n+1}$ if and only if $\langle k_1, k_2\rangle_2 = 0$. A pair (k_1, k_2) in Q^{n+1} satisfying $\langle k_1, k_2\rangle_2 = 0$, defines a line l in Q^{n+1}, which consists of points $[ak_1 + bk_2] \in Q^{n+1}$, $a, b \in \mathbf{R}$. Let Λ^{2n-1} be the set of all lines in Q^{n+1}:

$$\Lambda^{2n-1} = \{\, (k_1, k_2) \mid \langle k_i, k_j\rangle_2 = 0, i, j = 1, 2\,\}.$$

G acts on Λ^{2n-1} since G preserves $\langle\, ,\, \rangle_2$. We identify $T_1S^n = \{\, (u,v) \in S^n \times S^n \mid \langle u, v\rangle_0 = 0\,\}$ with Λ^{2n-1} by

$$T_1S^n \ni (u,v) \mapsto (k_1, k_2) \in \Lambda^{2n-1}$$

where $k_1 = (u, 1, 0)$ and $k_2 = (v, 0, 1)$. It is now clear that the line (k_1, k_2) is identified with the family of oriented hyperspheres through u normal to v.

Fact 3. *G acts on T_1S^n transitively and $T_1S^n = G/G'$, for an isotropy subgroup G'.*

Lemma 1. *An element $f \in L$ is lifted to Lie transformations $f^{\pm} \in G$ by*

$$f^{\pm}(v) = \pm f_* v/\|f_* v\|, \qquad v \in T_1S^n.$$

Proof. Let $e_0, \ldots, e_{n+2}$ be the standard base of $\mathbf{R}_2^{n+3}$, i.e. such that

$$(\langle e_\alpha, e_\beta\rangle_2)_{0\le\alpha,\beta\le n+2} = \begin{pmatrix} I_{n+1} & 0 \\ 0 & -I_2 \end{pmatrix}.$$

Then L is embedded in G by

$$L \ni f \overset{\iota}{\mapsto} \begin{pmatrix} \pm f & 0 \\ 0 & 1 \end{pmatrix} \in G.$$

Now, recall the meaning of $S\colon T_1S^n \to T_1S^n$, $S \in G$. If we identify $(u,v) \in T_1S^n$, $\langle u, v\rangle_0 = 0$, with a family of hyperspheres through u normal to v, $S(u,v)$ corresponds to a family of hyperspheres through some point $\tilde{u}$ normal to some fixed vector $\tilde{v}$ at $\tilde{u}$. When $S = \iota f \in G$, we have in particular $\tilde{u} = f(u)$ (here we identify $S^n \cong Q^n$) and $\tilde{v}$ is normal to the image of every hypersphere through u normal to v, under f. But since f is conformal, $f_* v$ is also normal to these image hyperspheres, and hence $\tilde{v}$ is parallel with $f_* v$. Q.E.D.

Let G_M be the image of L in G via the map $\iota(f) = f^+$.

Fact 4. *G_M acts on T_1S^n transitively and $T_1S^n = G_M/G'_M$ for an isotropy subgroup G'_M.*

Let $\mathfrak{l}, \mathfrak{l}', \mathfrak{g}, \mathfrak{g}', \mathfrak{g}_M$ and $\mathfrak{g}'_M$ be the Lie algebras of L, L', G, G', G_M and G'_M, respectively. The following expression of these Lie algebras in certain bases is significant and is used in the last section. A base of $\mathbf{R}^{n+2}_1$ is given by $e_0, \dots, e_{n+1}$, and we change it by

$$\begin{cases} \tilde{e}_0 = \dfrac{-e_0 + e_{n+1}}{2}, \\ \tilde{e}_i = e_i, \qquad 1 \leq i \leq n, \\ \tilde{e}_{n+1} = e_0 + e_{n+1}. \end{cases}$$

With respect to this base, we have

$$\varepsilon = ((\tilde{e}_\alpha, \tilde{e}_\beta)_1)_{0 \leq \alpha, \beta \leq n+1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$L = \{A \in GL(n+2, \mathbf{R}) \mid {}^tA\varepsilon A = \varepsilon\},$$
$$\mathfrak{l} = \{X \in \mathfrak{gl}(n+2, \mathbf{R}) \mid {}^tX\varepsilon + \varepsilon X = 0\}.$$

Fact 5. *The Lie algebra $\mathfrak{l}$ associated with the homogeneous space $S^n = L/L'$ is a simple graded Lie algebra of the first kind, i.e.*

$$\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1, \quad [\mathfrak{l}_p, \mathfrak{l}_q] = \mathfrak{l}_{p+q},$$

where $\mathfrak{l}_{-1} = T_0(L/L')$, $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$,

$$\mathfrak{l}_0 = \left\{ \begin{pmatrix} r & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -r \end{pmatrix} \mid r \in \mathbf{R}, a \in \mathfrak{o}(n) \right\} \simeq \mathfrak{co}(n),$$

$$\mathfrak{l}_{-1} = {}^t\mathfrak{l}_1 = \left\{ \begin{pmatrix} 0 & {}^tb & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \mid b \in \mathbf{R}^n \right\} \simeq \mathbf{R}^n.$$

Next, change $e_0, \dots, e_{n+2}$ by

$$\begin{cases} f_0 = \dfrac{-e_0 + e_{n+1}}{2}, \qquad f_{n+1} = e_0 + e_{n+1}, \\ f_1 = \dfrac{-e_1 + e_{n+2}}{2}, \qquad f_{n+2} = e_1 + e_{n+2}, \\ f_i = e_i, \quad 2 \leq i \leq n \end{cases}$$

Then we have

$$\varepsilon' = (\langle f_\alpha, f_\beta\rangle_2)_{0\le\alpha,\beta\le n+2} = \begin{pmatrix} 0 & 0 & -I_2 \\ 0 & I_{n-1} & 0 \\ -I_2 & 0 & 0 \end{pmatrix}$$
$$G = \{S \in GL(n+3, \mathbf{R}) \mid {}^tS\varepsilon' S = \varepsilon'\}$$
$$\mathfrak{g} = \{X \in \mathfrak{gl}(n+3, \mathbf{R}) \mid {}^tX\varepsilon' + \varepsilon' X = 0\}.$$

Fact 6. *The Lie algebra $\mathfrak{g}$ associated with the homogeneous space $T_1S^n = G/G'$ is a simple graded Lie algebra of the second kind, or more precisely, of contact type* [S], *i.e.*

$$\mathfrak{g} = \sum_{p=-2}^{2} \mathfrak{g}_p, \quad [\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}, \quad \dim \mathfrak{g}_{-2} = 1,$$

$$\mathfrak{g}_{-2} = {}^t\mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid c = \begin{pmatrix} 0 & p \\ -p & 0 \end{pmatrix} \right\},$$

$$\mathfrak{g}_{-1} = {}^t\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & {}^tb & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \mid b = (b_1, b_2), b_i \in \mathbf{R}^{n-1} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -{}^ta \end{pmatrix} \mid a \in \mathfrak{gl}(2, \mathbf{R}), e \in \mathfrak{o}(n-1) \right\}.$$

Putting $\mathfrak{m} = T_0(G/G')$, we have $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, and $\mathfrak{g}' = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

Since G_M preserves $\mathbf{R}_1^{n+2} = \{\sum_{\alpha=0}^{n+2} x_\alpha f_\alpha \mid x_1 + 2x_{n+2} = 0\}$, we have easily

Fact 7. *As a Lie algebra associated with $T_1S^n = G_M/G'_M$, $\mathfrak{g}_M$ is given by*

(3.1)

$$\mathfrak{g}_M = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \{\mathfrak{g}_M \cap (\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2)\}$$

$$= \left\{ \begin{pmatrix} r & -\frac{p}{2} & {}^tb_1 & 0 & p \\ -q & 0 & -{}^tb_2 & -p & 0 \\ d_1 & \frac{b_2}{2} & e & b_1 & -b_2 \\ 0 & -\frac{q}{2} & {}^td_1 & -r & q \\ \frac{q}{2} & 0 & \frac{{}^tb_2}{2} & \frac{p}{2} & 0 \end{pmatrix} \;\middle|\; \begin{matrix} p, q, r \in \mathbf{R}, \\ b_1, b_2, d_1 \in \mathbf{R}^{n-1}, \\ e \in \mathfrak{o}(n-1) \end{matrix} \right\}.$$

In particular,

(3.2)

$$\begin{aligned} \mathfrak{g}'_M &= \mathfrak{g}_M \cap \mathfrak{g}' \\ &= \left\{ \begin{pmatrix} r & 0 & 0 & 0 & 0 \\ -q & 0 & 0 & 0 & 0 \\ d_1 & 0 & e & 0 & 0 \\ 0 & -\frac{q}{2} & {}^td_1 & -r & q \\ \frac{q}{2} & 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; \begin{matrix} p, q, r \in \mathbf{R}, \\ d_1 \in \mathbf{R}^{n-1}, \\ e \in \mathfrak{o}(n-1) \end{matrix} \right\} \\ &\simeq \mathbf{R}^n \oplus \mathfrak{co}(n-1). \end{aligned}$$

In fact, with respect to the base $\tilde{e}_0, \ldots, \tilde{e}_{n+2}$ of $\mathbf{R}_2^{n+3}$, where $\tilde{e}_{n+2} = e_{n+2}$, $\iota_*\mathfrak{l}$ is given by

$$\iota_*\mathfrak{l} = \{X = \begin{pmatrix} r & {}^tb & 0 & 0 \\ d & a & b & 0 \\ 0 & {}^td & -r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid r \in \mathbf{R}, b, d \in \mathbf{R}^n, a \in \mathfrak{o}(n)\},$$

and if we put

$$b = \begin{pmatrix} p \\ b_1 \end{pmatrix}, \quad d = \begin{pmatrix} q \\ d_1 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & {}^tb_2 \\ -b_2 & e \end{pmatrix}, \tag{3.3}$$

where $p, q \in \mathbf{R}$, $b_1, b_2, d_1 \in \mathbf{R}^{n-1}$ and $e \in \mathfrak{o}(n-1)$, the expression of X in the base $\{f_0, \ldots, f_{n+2}\}$ is exactly as in (3.1).

We denote by ρ the isotropy representation of each homogeneous space above. Let M and $F(M)$ be as in §2 and put $\tilde{L} = \rho(L') \subset GL(n, \mathbf{R})$. It is easy to see that

$$(3.4)\quad \rho(A) = \alpha k, \qquad A = \begin{pmatrix} \alpha & 0 & 0 \\ * & k & 0 \\ * & * & \alpha^{-1} \end{pmatrix} \in L', \quad \alpha \neq 0, \quad k \in O(n),$$

and $\tilde{L} = CO(n)$.

Definition. An $\tilde{L}$-reduction of $F(M)$ is called *a conformal structure on M.*

Let $F(N)$ be the frame bundle of a $(2n-1)$-dimensional contact manifold N. It is well known that $F(N)$ is reduced to a $G_0^\sharp$-bundle $L^\sharp(N)$, where

$$G_0^\sharp = \left\{ \begin{pmatrix} \alpha & 0 \\ * & CSp(n-1) \end{pmatrix} \mid \alpha \neq 0 \right\}.$$

Fact 8 [SY,M1]. *$\tilde{G}_M = \rho(G'_M)$ and $\tilde{G} = \rho(G')$ are subgroups of $G_0^\sharp$. In fact, we have*

$$\tilde{G} = \left\{ \begin{pmatrix} \det A & 0 & 0 \\ * & & \\ & h \otimes A & \\ * & & \end{pmatrix} \mid h \in O(n-1),\ A \in GL(2, \mathbf{R}) \right\}$$

$$\tilde{G}_M = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha h & 0 \\ * & \gamma h & h \end{pmatrix} \mid \alpha \neq 0,\ h \in O(n-1) \right\}.$$

Denoting by $\tilde{O}(n-1)$ the subgroup of $\tilde{G}$ given by $A = I_2$, we obtain

$$(3.5)\qquad \tilde{O}(n-1) \subset \tilde{G}_M \subset \tilde{G}.$$

Definition. A $\tilde{G}_M$-reduction of $L^\sharp(N)$ is called *a conformal contact structure on N.*

Definition. A $\tilde{G}$-reduction of $L^\sharp(N)$ is called *a Lie contact structure on N.*

Let (M, g) be an n-dimensional riemannian manifold and let

$$(3.6)\qquad O(n) \to Q_g \xrightarrow{\pi_g} M$$

be the associated principal $O(n)$-bundle over M. According to [KN, Proposition 5.5 in Chapter I], we define *the lifted riemannian structure on* T_1M by

$$O(n-1) \to Q_g \overset{\tilde{\pi}_g}{\to} Q_g/O(n-1) = T_1M, \tag{3.7}$$

where

$$\tilde{\pi}_g^{-1}(z_1) = \{e(z) \mid z = (z_1, \dots z_n) \in Q_g, \\ e(z) = (z_i^h, z_j^v), 1 \le i \le n, 2 \le j \le n\},$$

using the horizontal (resp. vertical) lift z_i^h (resp. z_i^v) of $z_i \in T_1M$ with respect to the Levi-Civita connection on M. We distinguish the total space of (3.6) and (3.7) by Q_g and P_g, respectively, where Q_g is diffeomorphic to P_g via the map

$$\psi : Q_g \ni z \mapsto e(z) \in P_g.$$

Since the $O(n-1)$-action on P_g is given by $e(z)h = e(zh'), h' = \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}$ and easily seen to coincide with $\tilde{O}(n-1)$, we obtain

Lemma 2 [SY,M1]. *Let M be an n-dimensional riemannian manifold. Then on the unit tangent bundle T_1M of M exist a conformal contact structure and a Lie contact structure.*

Proof. They are given by

$$\tilde{P}_M = P_g \times_{\tilde{O}(n-1)} \tilde{G}_M, \qquad \tilde{P} = P_g \times_{\tilde{O}(n-1)} \tilde{G},$$

respectively, by virtue of (3.5). Q.E.D.

Remark 2. P_g is an $O(n-1)$-reduction of the principal $O(2n-1)$-bundle over the riemannian manifold (T_1M, s_g) where s_g is the metric induced from the Sasakian metric on TM.

§4. Geometry of unit tangent bundles

4.1. Riemannian case

Let (M, g) be an n-dimensional riemannian manifold and let Q_g, P_g and ψ be as in the last section. Let $A = \left\{ \begin{pmatrix} k & \xi \\ 0 & 1 \end{pmatrix} \mid k \in O(n), \xi \in \mathbf{R}^n \right\}$.

When (Q_g, χ) is a Cartan connection of type $A/O(n)$, define a 1-form $\tilde{\chi}$ on P_g by

$$\tilde{\chi}(X) = \chi(\psi_*^{-1}X), \qquad X \in TP_g.$$

Putting $B = \left\{ \begin{pmatrix} h' & 0 & \eta_1 \\ 0 & h & \eta_2 \\ 0 & 0 & 1 \end{pmatrix} \mid h \in O(n-1), \eta_1 \in \mathbf{R}^n, \eta_2 \in \mathbf{R}^{n-1} \right\}$, we show that $(P_g, \tilde{\chi})$ is a Cartan connection of type $B/O(n-1)$. In fact, the Lie algebra $\mathfrak{a}$ of A and $\mathfrak{b}$ of B are isomorphic (as a vector space) by

$$\text{(4.1)} \qquad \mathfrak{a} \ni \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & {}^tb_2 & p \\ -b_2 & e & b_1 \\ 0 & 0 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} \tilde{e} & 0 & b \\ 0 & e & -b_2 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{b},$$

where $a = \begin{pmatrix} 0 & {}^tb_2 \\ -b_2 & e \end{pmatrix} \in \mathfrak{o}(n)$, $b = \begin{pmatrix} p \\ b_1 \end{pmatrix} \in \mathbf{R}^n$, $e \in \mathfrak{o}(n-1)$ and $\tilde{e} = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}$. Moreover, **C2** follows from the commutative diagram

$$\begin{array}{ccc} P_g & \xrightarrow{\psi^{-1}} & Q_g \\ {\scriptstyle R_h}\downarrow & & \downarrow{\scriptstyle R_{h'}} \\ P_g & \xrightarrow{\psi^{-1}} & Q_g \end{array} \qquad h \in O(n-1),$$

and from

$$\tilde{\chi}(E^*) = \chi(\psi_*^{-1}E^*) = \chi(E^*) = E, \qquad E \in \mathfrak{o}(n-1),$$

where E^* denotes the fundamantal vector field on P_g and Q_g. The decomposition $\chi = \theta + \chi_0 + \chi_1$ with respect to $\mathfrak{a} \ni \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \leftrightarrow (b, e, -b_2) \in \mathbf{R}^n \oplus \mathfrak{o}(n-1) \oplus \mathbf{R}^{n-1}$ determines the basic form θ of χ. On the other hand, (4.1) implies that the basic form $\tilde{\theta}$ of $\tilde{\chi}$ is given by

$$\tilde{\theta}(X) = (\theta + \chi_1)(\psi_*^{-1}X).$$

Remark 3. When χ is the Levi-Civita connection on M, $\tilde{\chi}$ is *not* in general the Levi-Civita connection on (T_1M, s_g), since the torsion appears whenever (M, g) is not riemannian flat.

Definition. When χ is the Levi-Civita connection on M, $\tilde{\chi}$ is called *the lifted riemannian connection on* T_1M.

4.2. Conformal case

In the following we assume that $\dim M \geq 3$. Let M be a manifold with conformal structure. Recall that a conformal structure corresponds uniquely to the normal Cartan connection (Q_L, ω) of type L/L', called the normal conformal connection, where

$$L' \to Q_L \xrightarrow{\pi} M \tag{4.2}$$

is the associated principal L'-bundle [OG]. By Fact 5, we have

$$\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1 = \{ \begin{pmatrix} r & 0 & 0 \\ d & a & 0 \\ 0 & {}^t d & -r \end{pmatrix} \mid r \in \mathbf{R}, a \in \mathfrak{o}(n), d \in \mathbf{R}^n \} = \mathfrak{co}(n) \oplus \mathbf{R}^n.$$

Define a subalgebra $\mathfrak{h}'$ of $\mathfrak{l}'$ by

$$\mathfrak{h}' = \{ \begin{pmatrix} r & 0 & 0 \\ d & a & 0 \\ 0 & {}^t d & -r \end{pmatrix} \in \mathfrak{l}' \mid a = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \in \mathfrak{o}(n-1) \} = \mathfrak{co}(n-1) \oplus \mathbf{R}^n,$$

and let H' be the corresponding connected Lie subgroup. Then we obtain a principal H'-bundle

$$H' \to Q_L \xrightarrow{\tilde{\pi}} Q_L/H', \tag{4.3}$$

the total space of which is denoted by P_L to distinguish from (4.2). We call P_L *the lifted conformal structure.* Let g be a riemannian metric on M belonging to the conformal class. It is easy to see that Q_L/H' is identified with $T_1 M$. Noting that for any $l' \in L'$, there exists $k \in O(n)$ such that $k^{-1} l' \in H'$, and $H' \overset{\iota}{\cong} G'_M$, define $\tilde{\psi}: Q_L = Q_g \times_{O(n)} L' \to P_g \times_{O(n-1)} G'_M$ by

$$Q_L \ni (z, l') \mapsto (e(zk), \iota(k^{-1} l')) \in P_g \times_{O(n-1)} G'_M, \quad z \in Q_g,\ l' \in L',$$

which is well-defined since if $k_1, k_2 \in O(n)$ are such that $k_1^{-1} l'$, $k_2^{-1} l' \in H'$, it follows $k_1^{-1} l' (k_2^{-1} l')^{-1} = k_1^{-1} k_2 \in H' \cap O(n) = O(n-1)$, and we get

$$\begin{aligned}(e(zk_1), \iota(k_1^{-1} l')) &= (e(zk_1) k_1^{-1} k_2, (k_1^{-1} k_2)^{-1} \iota(k_1^{-1} l')) \\ &= (e(zk_2), \iota(k_2^{-1} l')).\end{aligned}$$

Moreover, it is easy to see that $\tilde{\psi}$ is a diffeomorphism. Thus identifying P_L with $P_g \times_{O(n-1)} G'_M$ and using $Q_L \overset{\tilde{\psi}}{\cong} P_L$, we can define a 1-form $\tilde{\omega}$

on P_L by

$$\tilde{\omega}(X) = \omega(\tilde{\psi}_*^{-1} X), \qquad X \in TP_L.$$

As before, $\tilde{\omega}$ is a Cartan connection of type G_M/G'_M on P_L, since $\mathfrak{l}$ is isomorphic to $\mathfrak{g}_M$ by

(4.4)

$$\mathfrak{l} \ni \begin{pmatrix} r & {}^t b & 0 \\ d & a & b \\ 0 & {}^t d & -r \end{pmatrix} \overset{\iota_*}{\mapsto} \begin{pmatrix} r & -\frac{p}{2} & {}^t b_1 & 0 & p \\ -q & 0 & -{}^t b_2 & -p & 0 \\ d_1 & \frac{b_2}{2} & e & b_1 & -b_2 \\ 0 & -\frac{q}{2} & {}^t d_1 & -r & q \\ \frac{q}{2} & 0 & \frac{{}^t b_2}{2} & \frac{p}{2} & 0 \end{pmatrix} \in \mathfrak{g}_M,$$

where we use (3.3), and since the following diagram is commutative.

$$\begin{array}{ccc} P_L & \xrightarrow{\tilde{\psi}^{-1}} & Q_L \\ R_{\iota(s)} \downarrow & & \downarrow R_s \\ P_L & \xrightarrow{\tilde{\psi}^{-1}} & Q_L \end{array} \qquad s \in H'.$$

A decomposition $\omega = \theta + \omega_c + \omega'$ with respect to $\mathfrak{l} \ni \begin{pmatrix} r & {}^t b & 0 \\ d & a & b \\ 0 & {}^t d & -r \end{pmatrix} \leftrightarrow$ $(b, (r, a), d) \in \mathbf{R}^n \oplus \mathfrak{co}(n) \oplus \mathbf{R}^n$, and $\omega_c = \omega_c^1 + \omega_c^2$ with respect to $\mathfrak{co}(n) \ni (r, a) \leftrightarrow (-b_2, (r, e)) \in \mathbf{R}^{n-1} \oplus \mathfrak{co}(n-1)$ determines the basic form $\tilde{\theta}$ of $\tilde{\omega}$ by $\tilde{\theta} = \theta + \omega_c^1$.

Definition. When ω is the normal conformal connection on M, we call $\tilde{\omega}$ *the lifted conformal connection on* $T_1 M$.

Remark 4. Note the difference between the conformal connection and the lifted conformal connection on $(T_1 M, s_g)$. The former is a Cartan connection of type $PO(2n, 1)/K = S^{2n-1}$ (K is an isotropy subgroup), while the latter is of type $G_M/G'_M = T_1 S^n$.

4.3. Lie contact case

Let $P = P_L \times_{G'_M} G'$ and extend the lifted conformal connection

$(P_L, \tilde{\omega})$ flattly to $(P, \tilde{\omega})$, i.e.

$$\begin{aligned}\tilde{\omega}(X) &= \mathrm{Ad}(a^{-1})\tilde{\omega}(Y) + A \\ &= \mathrm{Ad}(a^{-1})\omega(\psi_*^{-1}Y) + A,\end{aligned}$$

where $X \in T_uP, a \in G', ua \in P_L, Y \in T_{ua}P_L, A \in \mathfrak{g}'$ and

$$X = R_{a_*}Y + A^*.$$

Now, compare this connection with Tanaka connection τ obtained in [M2]. The latter is given, after a long calculation, as follows : Let $p \in M$ and let (x^i) be a geodesic normal coordinate in a neighborhood U of p, such that

$$p = (0, \ldots, 0), \qquad g_{ij}(0) = \delta_{ij}, \qquad \{{}^{\,i}_{jk}\}(0) = 0,$$

where g_{ij} and $\{{}^{\,i}_{jk}\}$ are, respectively, the coefficients of the riemannian metric g of M and the Christoffel's symbols of its riemannian connection. Take a local coordinate (x^i, z^i_j) of Q_g so that

$$z_i = z_i^j \frac{\partial}{\partial x^j}, \qquad g_{ij} z^i_k z^j_l = \delta_{kl}.$$

Let $(s^\alpha_\beta)_{0 \le \alpha, \beta \le n+2} \in G'$. It is shown in [M2, §2] that $(x^i, z^i_j, s^a_b, s_i, s_{\bar{i}}, s_1)$ is a local coordinate of P around $e(z(0))$ where $0 \le a, b \le 1, s_i = s_i^{n+1}, s_{\bar{i}} = s_i^{n+2}, 2 \le i \le n$ and $s_1 = s_0^{n+2}$.

Fact 9 [M2]. *Let $(x^i, z^i_j, s^a_b, s_i, s_{\bar{i}}, s_1)$ be the local coordinate chosen as above. Then at $e = e(z) = (x^i, z^i_j, \delta^a_b, \mathbf{0}, \mathbf{0}, 0) \in P_g$, Tanaka connection (P, τ) is given by*

$$\tau = \begin{pmatrix} \tau^0_0 & \tau^0_1 & \theta^i & 0 & \theta^1 \\ \tau^1_0 & \tau^1_1 & \theta^{\bar{i}} & -\theta^1 & 0 \\ & & & & \\ \tau_i & \tau_{\bar{i}} & \tau^i_j & \theta^i & \theta^{\bar{i}} \\ & & & & \\ 0 & -\tau_1 & \tau_i & -\tau^0_0 & -\tau^1_0 \\ \tau_1 & 0 & \tau_{\bar{i}} & -\tau^0_1 & -\tau^1_1 \end{pmatrix},$$

$$
(4.5)\qquad
\begin{aligned}
&\theta^i = g_{jk} z_i^k dx^j, \qquad 1 \le i \le n,\\
&\theta^{\bar{i}} = g_{jk} z_i^k (dz_1^j + \{{}^{j}_{st}\} z_1^s dx^t), \qquad 2 \le i \le n,\\
&\tau_j^i = g_{uv} z_i^v (dz_j^u + \{{}^{u}_{st}\} z_j^s dx^t), \qquad 2 \le i, j \le n,\\
&\tau_0^0 = ds_0^0, \qquad \tau_1^0 = ds_1^0 + A_{11}^0 \theta^1,\\
&\tau_0^1 = ds_0^1 + \sum_{j=1}^{n} A_{0j}^1 \theta^j, \qquad \tau_1^1 = ds_1^1,\\
&\tau_i = ds_i + \sum_{j=1}^{n} A_{ij} \theta^j, \qquad 2 \le i \le n,\\
&\tau_{\bar{i}} = ds_{\bar{i}} + A_{\bar{i}\bar{i}} \theta^{\bar{i}}, \qquad 2 \le i \le n,\\
&\tau_1 = ds_1 + \sum_{j=1}^{n} A_{1j} \theta^j,
\end{aligned}
$$

where

(4.6)

$$
\begin{aligned}
&A_{11}^0 = A_{\bar{i}\bar{i}} = -\frac{1}{2},\\
&A_{0j}^1 = \frac{1}{n-2} R_{1j}, \quad A_{01}^1 = \frac{1}{n-2} R_{11} - \frac{R}{2(n-1)(n-2)},\\
&A_{ij} = -\frac{1}{n-2} R_{ij} + \frac{R}{2(n-1)(n-2)} \delta_{ij}, \quad A_{i1} = -\frac{1}{n-2} R_{1i},\\
&A_{1i} = -\frac{1}{2(n-2)} R_{1i}, \quad A_{11} = -\frac{1}{2(n-2)} R_{11} + \frac{R}{4(n-1)(n-2)},
\end{aligned}
$$

using the Ricci curvature R_{ij} and the scalar curvature R of M at (x^i). Denoting the component of the curvature by $K^i, K^{\bar{i}}, K_j^i, K_b^a, K_i, K_{\bar{i}}, K_1$, respectively, we obtain

$$
\begin{aligned}
&K_{1j}^{\bar{i}} = C_{11j}^i,\ K_{jk}^{\bar{i}} = C_{1jk}^i,\ K_{01i}^1 = C_{11i},\ K_{0ij}^1 = C_{1ij},\\
&K_{jk1}^i = C_{jk1}^i,\ K_{jkl}^i = C_{jkl}^i,\ K_{i1j} = -C_{i1j},\ K_{ijk} = -C_{ijk},\\
&K_{\bar{i}1j} = -\frac{1}{2} C_{11j}^i,\ K_{\bar{i}jk} = -\frac{1}{2} C_{1jk}^i,\ K_{11j} = -\frac{1}{2} C_{11j},\ K_{1jk} = -\frac{1}{2} C_{1jk}
\end{aligned}
$$

for $2 \le i, j, k, l \le n$, and all other components vanish, where C_{jkl}^i and C_{ijk} are the coefficients of Weyl's confomal curvature.

Theorem 1. *Let M be an n-dimensional manifold with conformal* *structure*

$$L' \to Q_L \to M,$$

and let (Q_L, ω) *be the normal conformal connection. Moreover, let* P_L *be the lifted conformal structure*

$$H' \to Q_L \to Q_L/H',$$

and let $\tilde{\omega}$ *be the lifted conformal connection. Then on the Lie contact structure* $P = P_L \times_{G'_M} G'$, *Tanaka connection* (P, τ) *coincides with the connection* $(P, \tilde{\omega})$ *which is flatly extended from* $(P_L, \tilde{\omega})$.

Proof. First, let $j : P_L \to P$ be the natural inclusion. Noting $P_L \cong P_g \times_{O(n-1)} G'_M$ and (3.2), we have

$$j^* ds_1^0 = 0, \qquad j^* ds_0^1 = -2j^* ds_1, \qquad j^* ds_1^1 = 0, \qquad j^* ds_{\bar{i}} = 0.$$

Thus we obtain

$$\begin{aligned}
&j^*\theta^i = \theta^i, \qquad 1 \le i \le n,\\
&j^*\theta^{\bar{i}} = \theta^{\bar{i}}, \qquad 2 \le i \le n,\\
&j^*\tau_j^i = \tau_j^i, \qquad 2 \le i, j \le n,\\
&j^*\tau_0^0 = ds_0^0,\\
&j^*\tau_1^0 = j^* ds_1^0 + A_{11}^0 j^*\theta^1 = -\frac{\theta^1}{2},\\
&j^*\tau_0^1 = \tau_0^1,\\
&j^*\tau_1^1 = j^* ds_1^1 = 0,\\
&j^*\tau_i = \tau_i, \qquad 2 \le i \le n,\\
&j^*\tau_{\bar{i}} = j^* ds_{\bar{i}} + A_{\bar{i}\bar{i}} j^*\theta^{\bar{i}} = -\frac{\theta^{\bar{i}}}{2}, \qquad 2 \le i \le n,\\
&j^*\tau_1 = j^* ds_1 + \sum_{j=1}^n A_{1j} j^*\theta^j = -\frac{1}{2} j^* ds_0^1 - \frac{1}{2(n-1)} R_{1j}\theta^j = -\frac{1}{2}\tau_0^1,
\end{aligned}$$

and hence $j^*\tau$ is a $\mathfrak{g}_M$-valued 1-form on P_L satisfying **C2**. Now, by $Q_L \overset{\tilde{\psi}}{\cong} P_L$, and $\mathfrak{l} \cong \mathfrak{g}_M$ using (3.3), we may consider $\nu = \tilde{\psi}^* j^* \tau$ as a Cartan connection of type L/L' on Q_L. In fact, putting

$$\nu = \begin{pmatrix} \nu_0^0 & \nu^i & 0 \\ \nu_i & \nu_j^i & \nu^i \\ 0 & \nu_i & -\nu_0^0 \end{pmatrix} \in \mathfrak{l}, \qquad 1 \le i, j \le n,$$

and noting (4.4), we have,

$$
\begin{aligned}
&\nu^1 = \theta^1, \\
&\nu^i = \theta^i, \qquad 2 \le i \le n, \\
&\nu^1_i = -\theta^{\bar{i}} = -\nu^i_1, \qquad 2 \le i \le n, \\
&\nu^0_0 = ds^0_0, \\
&\nu^i_j = \tau^i_j, \qquad 2 \le i, j \le n, \\
&\nu_1 = -\tau^1_0 \\
&\nu_i = \tau_i, \qquad 2 \le i \le n.
\end{aligned} \tag{4.7}
$$

When $\Phi = \frac{1}{2} N\theta \wedge \theta$ is its curvature form, we obtain

$$
\begin{aligned}
N^i &= 0, && 1 \le i \le n, \\
N^0_0 &= 0, \\
N^i_{jkl} &= C^i_{jkl}, && 1 \le i, j, k, l \le n, \\
N_{ijk} &= -C_{ijk}, && 1 \le i, j, k \le n,
\end{aligned}
$$

since the calculation is carried out in parallel with the calculation of K, because the structure equations of $\mathfrak{l}, \mathfrak{g}_M$ and $\mathfrak{g}$ correspond each other in the relation of (3.1) and (3.3). Thus N satisfies the normality condition of a Cartan connection (Q_L, ν) of type L/L', and by the uniqueness of such connection, we conclude $\nu = \omega$, and hence $\tau = \tilde{\omega}$. Q.E.D.

Remark 5. A local expression of the normal conformal connection ω is given, for instance, in [OG,§11]. Noting that ω^i_j there corresponds to $\nu^i_j - \nu^0_0 \delta^i_j$, and that the sign of C_{ijk} is opposite, we can prove $\nu = \omega$ directly by (4.5) $\sim$ (4.7).

Finally, by the argument in [M2, §4], we obtain

Theorem 2. *$(P, \tilde{\omega})$ is the normal Cartan connection of type G/G', which induces the Lie contact structure $\tilde{P}$ on T_1M of an n-dimensional riemannian manifold M, if $n \ge 3$. The fundamental system of invariants of the structure is given by the torsion part $K^0 = (K^{\bar{i}}_{jk})$ of the curvature form of $\tilde{\omega}$, when $n \ge 4$, and by $K^1 = (K^1_{0ij})$, when $n = 3$. In both cases, they are written down in terms of all the coefficients of Weyl's conformal curvature tensor of M.*

Corollary. *T_1M is Lie flat if and only if M is conformally flat.*

Remark 6. We may view a conformal structure $Q_L \to M$ as an enlarged bundle structure

$$Q_L = Q_g \times_{O(n)} L'$$

or

$$\tilde{Q}_L = Q_g \times_{O(n)} CO(n).$$

In this case, riemannnian flatness and conformal flatness are not, of course, equivalent. The Lie contact structure $\tilde{P}$, is also regarded as an enlarged bundle structure

$$P = P_L \times_{G'_M} G'$$

or

$$\tilde{P} = \tilde{P}_L \times_{\tilde{G}_M} \tilde{G}.$$

Thus, Corollary is a non-trivial fact, indeed, though it may be trivial that the conformally flatness is equivalent with the flatness of P_L.

Remark 7. When $n = 3$, we have $K^0 \equiv 0$, which means that the Lie contact structure is integrable. This structure is shown [SY] to be equivalent with a CR-structure with indefinite Levi form, discovered independently by H. Sato and LeBrun [LB].

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