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Maslov Class of an Isotropic Map-Germ Arising from One-Dimensional Symplectic Reduction

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Dedicated to Professor Noboru Tanaka on his 60th birthday

§0. Introduction

Let (M^{2n}, ω) be a symplectic manifold of dimension $2n$ and $N^n \subset M^{2n}$ be a Lagrangian submanifold with singularities. For each regular point x of N, $T_x N$ is a Lagrangian subspace of the sympletic vector space $T_x M$.

To investigate the local structure of N near a singular point x_0 of N, it is natural to study the behavior of the distribution $\{T_x N \mid x \text{ is a regular point of } N\}$ near x_0. Then we can grasp an invariant of the singularity, which is called the Maslov class in this paper.

In studying the problem of Lagrangian immersion of surfaces to four dimensional symplectic manifolds, Givental' [G] introduced a Lagrangian variety, so called an open Whitney umbrella or an unfolded Whitney umbrella, and investigated some local and global problems. In particular, he calculated the "Maslov index" of an open Whitney umbrella. The main purpose of this paper is to generalize the result of Givental'.

Singular Lagrangian varieties appear typically in the process of symplectic reduction (see §5, [A2] and [I1]).

Note that singular Lagrangian varieties obtained by reduction are parametrized by isotropic mappings.

Originally, the notion of Maslov class (Keller-Maslov-Arnol'd class) stemed from the asymptotic method of linear partial differential equation, representation theory, and symplectic topology ([A1], [GS], [Gr], [Hö], [M], [V], [W]).

Maslov classes represent obstruction for transversality of two Lagrangian subbundles (see §1). Applying this understanding, we define

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the Maslov class of isotopic mapping as an obstruction for extendability of a partially defined Lagrangian subbundle. Further, we show several results on Maslov classes of isotropic mappings obtained by one-dimensional reduction process.

The first result is on vanishing of Maslov classes: The Maslov class of an isotropic map-germ obtained by one-dimensional reduction of a Lagrangian manifold is zero (Theorem 6.1).

Thus, for a singularity of one-dimensional symplectic reduction of an isotropic manifold, the Maslov class has a meaning of obstruction for representability as an intersection of a Lagrangian submanifold and a hypersurface.

In general, Maslov classes do not vanish. We give local models of singularities of isotropic mappings generically obtained by one-dimensional reduction of isotropic submanifolds, up to local symplectic diffeomorphisms of the reduced symplectic manifold (Theorem 7.1). These models are open Whitney umbrellas of arbitrary dimension and their suspensions, and their Maslov classes proves not to vanish (Theorem 8.3). Therefore, we see that a generic isotropic submanifold in a hypersurface of a symplectic manifold is not an intersection of a Lagrangian submanifold and the hypersurface, locally at each point where the characteristic direction is tangent to the hypersurface (Corollary 8.4).

In §1, we recall the notion of classical Maslov class. In the next section, we define the Maslov class of an isotropic mapping. In §3, the notion of symplectic equivalence of isotropic mappings is introduced. The Maslov class of an isotropic map-germ is defined in §4.

After a preliminary on the symplectic reduction in §5, Theorem 6.1 is stated in §6 and proved in §§9–10. Theorem 7.1 is stated in §7. The proof is given in [I2]. Theorem 8.3 is stated in §8 and proved in §§11–14.

Throughout this paper, all manifolds and maps are of class C^∞.

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§1. Classical Maslov class

Let (M^{2n},ω) be a symplectic manifold, $N^n \subset M$ a Lagrangian submanifold ($\omega|N = 0$), and $\pi : M \longrightarrow B$ a Lagrangian fibration. Then the symplectic vector bundle $\mathbf{E} = TM|N$ has two Lagrangian subbundles $\mathbf{L} = \operatorname{Ker}\pi_*|N$ and $\mathbf{L}' = TN$.

In general, for a symplectic vector bundle $\mathbf{E}$ of rank $2n$ over N and Lagrangian subbundles $\mathbf{L}$ and $\mathbf{L}'$ of $\mathbf{E}$, the Maslov class $m(\mathbf{E};\mathbf{L},\mathbf{L}') \in H^1(N,\mathbf{Z})$ is defined as follows.

Consider the bundle $\Lambda(\mathbf{E})$ over N of Lagrangian subspaces of fibers of $\mathbf{E}$. The Lagrangian subbundle $\mathbf{L}'$ defines a section $s(\mathbf{L}') : N \longrightarrow \Lambda(\mathbf{E})$ by $s(\mathbf{L}')(x) = \mathbf{L}'_x \in \Lambda(\mathbf{E}_x) \subset \Lambda(\mathbf{E}), x \in N$.

Let Ω denote the symplectic form of $\mathbf{E}$. Then there exist a complex structure J and a Hermitian form G on $\mathbf{E}$, unique up to homotopy, such that Ω is the imaginary part of G. Denote by g the real part of G.

Let $S = \{e_1, \dots, e_n\}$ be an orthonormal frame of $\mathbf{L}$ over an open subset U of N, with respect to g. Then S turns out to be a unitary frame of the Hermitian vector bundle $(\mathbf{E}; J, G)$ over U. Then we have an isomorphism $\mathbf{E}|U \cong \mathbf{C}^n \times U$ as Hermitian vector bundles, which maps $\mathbf{L}|U$ to $\mathbf{R}^n \times U$. Since $U(n)$ acts on the space $\Lambda(\mathbf{C}^n)$ of Lagrangian subspaces of $\mathbf{C}^n$ transitively, $\Lambda(\mathbf{C}^n)$ is identified with the homogeneous space $\Lambda(n) = U(n)/O(n)$ (see [A1]). Thus we have $\Lambda(\mathbf{E})|U \longrightarrow \Lambda(n)$, which are glued into a C^∞ mapping $\Phi(\mathbf{L}) : \Lambda(\mathbf{E}) \longrightarrow \Lambda(n)$ (cf.[F]).

Set $\Phi = \Phi(\mathbf{L}) \circ s(\mathbf{L}') : N \longrightarrow \Lambda(n) = U(n)/O(n)$. Then the homotopy type of Φ is independent of the choice of (J, G).

If $S' = \{e'_1, \dots, e'_n\}$ is an orthonormal frame of $\mathbf{L}'$ over U, then, at $x \in U$, $e'_j = \sum_{i=1}^n a_{ij} e_i$ for some $A = (a_{ij}) \in U(n)$. Then $\Phi(x) = [A] \in \Lambda(n)$. Remark that $G(e'_j, e_i) = a_{ij}$.

Define the Maslov class $m(\mathbf{E};\mathbf{L},\mathbf{L}')$ of triplet $(\mathbf{E};\mathbf{L},\mathbf{L}')$ to be the image of the generator of $H^1(S^1,\mathbf{Z}) \cong \mathbf{Z}$, relatively to counterclockwise orientation, under

$$(\det{}^2 \circ \Phi)^* : H^1(S^1,\mathbf{Z}) \longrightarrow H^1(N,\mathbf{Z}),$$

where $\det^2 : U(n)/O(n) \longrightarrow S^1$ is defined by $\det^2([A]) = (\det A)^2$, $A \in U(n)$.

The following is well known (cf.[V]) or easily proved.

Lemma 1.1. *The Maslov class satisfies following properties:*

(0) $m(\mathbf{E};\mathbf{L},\mathbf{L}) = 0$.

(1) $m(\mathbf{E};\mathbf{L},\mathbf{L}') + m(\mathbf{E};\mathbf{L}',\mathbf{L}'') = m(\mathbf{E};\mathbf{L},\mathbf{L}'')$.

(2) $\mathbf{L}$ *and* $\mathbf{L}'$ *are transverse in* $\mathbf{E}$, *then* $m(\mathbf{E};\mathbf{L},\mathbf{L}') = 0$.

(3) *If there is an isomorphism between symplectic vector bundles* $\mathbf{E}_1$ *and* $\mathbf{E}_2$ *mapping* $\mathbf{L}_1, \mathbf{L}'_1$ *to* $\mathbf{L}_2, \mathbf{L}'_2$, *then* $m(\mathbf{E}_1;\mathbf{L}_1,\mathbf{L}'_1) = m(\mathbf{E}_2;\mathbf{L}_2,\mathbf{L}'_2)$.

(4) $m(h^*\mathbf{E}; h^*\mathbf{L}, h^*\mathbf{L}') = h^*m(\mathbf{E};\mathbf{L},\mathbf{L}') \in H^1(P,\mathbf{Z})$.

Here $\mathbf{L}, \mathbf{L}', \mathbf{L}''$ *are Lagrangian subbundles of* $\mathbf{E}$ *and* h *is a mapping from a manifold* P *to* N.

Remark 1.2. Vaisman [V] defines the Maslov classes $\mu_h(\mathbf{E};\mathbf{L},\mathbf{L}') \in H^{4h-3}(N,\mathbf{R}), h = 1,2,3,\dots,$ for two Lagrangian subbundles $\mathbf{L},\mathbf{L}'$ of a symplectic vector bundle $\mathbf{E}$ over N, such that

(i) $\mu_1(\mathbf{E};\mathbf{L},\mathbf{L}') = (1/2)m(\mathbf{E};\mathbf{L},\mathbf{L}') \in H^1(N,\mathbf{R})$;

(ii) μ_h satisfies the properties of Lemma 1.1.

Returning to the situation we begun with, we define the Maslov class $m(N)$ by $m(N) = m(TM|N;\ \mathrm{Ker}\ \pi_*|N, TN) \in H^1(N,\mathbf{Z})$.

§2. Maslov class of an isotropic mapping

Let (M^{2n},ω) be a symplectic manifold of dimension $2n$, and N^n be a C^∞ manifold of dimension n.

A C^∞ mapping $f : N \longrightarrow M$ is called an isotropic mapping if, for each $x \in N$, the image of $T_xf : T_xN \longrightarrow T_xM$ is an isotropic subspace of the symplectic vector space T_xM, that is, if $f^*\omega = 0$.

For an isotropic mapping f, set

$$\Sigma = \Sigma(f) = \{x \in N \mid T_xf \text{ is not injective}\}.$$

Then the restriction $f|N - \Sigma : N - \Sigma \longrightarrow M$ is a Lagrangian immersion.

Set $\Lambda(M) = \Lambda(TM)$, and denote by $\pi : \Lambda(M) \longrightarrow M$ the canonical projection. In the symplectic vector bundle π^*TM over $\Lambda(M)$, define the tautological bundle $\mathcal{L}$ by $\mathcal{L}_{(y,\lambda)} = \lambda \subset T_yM$, $(y,\lambda) \in \Lambda(M)$.

Associated to f, $\varphi(f) : N - \Sigma \longrightarrow \Lambda(M)$ is defined by $\varphi(f)(x) = (f(x),\ \mathrm{Im}(T_xf))$. Then $\pi \circ \varphi(f) = f$.

Set $\mathbf{L}_f = \varphi(f)^*\mathcal{L}$. Then $\mathbf{L}_f$ is a Lagrangian subbundle of $f^*TM = \varphi(f)^*\pi^*TM$ over $N - \Sigma$.

Definition 2.1. Assume f^*TM has a Lagrangian subbundle $\mathbf{L}$ (over N). Then define the Maslov class of f by

$$m(f) = \delta(m(f^*TM;\mathbf{L},\mathbf{L}_f)) \in H^2(N, N-\Sigma;\mathbf{Z}),$$

where $\delta : H^1(N-\Sigma;\mathbf{Z}) \longrightarrow H^2(N,N-\Sigma;\mathbf{Z})$ is the coboundary map.

Remark that $m(f^*TM;\mathbf{L},\mathbf{L}_f) \in H^1(N-\Sigma;\mathbf{Z})$. For another Lagrangian subbundle $\mathbf{L}'$ of f^*TM over N, we have

$$m(f^*TM;\mathbf{L}',\mathbf{L}_f) = m(f^*TM;\mathbf{L}',\mathbf{L}) + m(f^*TM;\mathbf{L},\mathbf{L}_f),$$

in $H^1(N-\Sigma;\mathbf{Z})$, by Lemma 1.1.(2).

Since $m(f^*TM; \mathbf{L}', \mathbf{L})$ comes from an element of $H^1(N; \mathbf{Z})$, we have

$$\delta(m(f^*TM; \mathbf{L}', \mathbf{L}_f)) = \delta(m(f^*TM; \mathbf{L}, \mathbf{L}_f)).$$

Therefore $m(f)$ is independent of the choise of $\mathbf{L}$.

Remark 2.2. By Remark 1.2 and the same argument as above, we can define a class $\mu_h(f) \in H^{4h-2}(N, N-\Sigma; \mathbf{R})$ for an isotropic mapping $f : N \longrightarrow M$ by

$$\mu_h(f) = \delta(\mu_h(f^*TM; \mathbf{L}, \mathbf{L}_f)),$$

$h = 1, 2, \ldots$, provided f^*TM has a Lagrangian subbundle $\mathbf{L}$.

§3. Symplectic equivalence

Let $f : N^n \longrightarrow (M^{2n}, \omega)$ and $f' : N'^n \longrightarrow (M'^{2n}, \omega')$ be isotropic mappings.

Definition 3.1. A pair (σ, τ) of a diffeomorphism $\sigma : N \longrightarrow N'$ and a symplectic diffeomorphism $\tau : M \longrightarrow M', \tau^*\omega' = \omega$, is called a symplectic equivalance between f and f' if $\tau \circ f = f' \circ \sigma$. Then we call f is symplectically equivalent to f, and write $f \sim f'$.

If (σ, τ) is a symplectic equivalence between f and f', then σ induces an isomorphism $\sigma^* : H^2(N', N' - \Sigma'; \mathbf{Z}) \longrightarrow H^2(N, N - \Sigma; \mathbf{Z})$, where $\Sigma' = \Sigma(f')$, and $\sigma^* m(f') = m(f)$, if f^*TM has a Lagrangian subbundle $\mathbf{L}$. In fact, τ induces isomorphisms

$$\tau' : \tau^*TM' \longrightarrow TM \quad \text{and} \quad \tau'' : \sigma^* f'^* TM' = f^* \tau^* TM' \longrightarrow f^*TM$$

of symplectic vector bundles over M and over N respectively, and τ'' maps $\sigma^*\mathbf{L}_{f'}$ to $\mathbf{L}_f$. Thus

$$\begin{aligned} \sigma^* m(f'^*TM'; \tau''^{-1}\mathbf{L}, \mathbf{L}_{f'}) &= m(\sigma^* f'^* TM'; \sigma^* \tau''^{-1}\mathbf{L}, \sigma^*\mathbf{L}_{f'}) \\ &= m(f^*TM; \mathbf{L}_1, \mathbf{L}_f). \end{aligned}$$

for some Lagrangian subbundle $\mathbf{L}_1$ of f^*TM, by Lemma 1.1.(3) and (4). Therefore

$$\begin{aligned} \sigma^* m(f') &= \sigma^* \delta m(f'^*TM'; \tau''^{-1}\mathbf{L}, \mathbf{L}_{f'}) \\ &= \delta(m(f^*TM; \mathbf{L}_1, \mathbf{L}_f)) \\ &= m(f) \end{aligned}$$

§4. Maslov class of an isotropic map-germ

Let $f : N^n, x \longrightarrow (M^{2n}, \omega)$ be a germ of an isotropic mapping. For each representative (f, U) such that $f : U \longrightarrow M$ is isotropic and U is a contractible neighborhood of x, we have $m(f, U) \in H^2(U, U - \Sigma; \mathbf{Z})$, since f^*TM is trivial over U. If V is a contractible neighborhood of x, with $V \subset U$ and $\iota^* : H^2(U, U - \Sigma; \mathbf{Z}) \longrightarrow H^2(V, V - \Sigma; \mathbf{Z})$ is the restriction, then $\iota^*(m(f, U)) = m(f, V)$ by Lemma 1.1.(4). Set

$$H^2(N, N - \Sigma; \mathbf{Z})_x = \varinjlim H^2(U, U - \Sigma; \mathbf{Z}),$$

where U runs over contractible neighborhoods of x. Then we have an element

$$m(f) \in H^2(N, N - \Sigma; \mathbf{Z})_x.$$

We call it the Maslov class of the isotropic map-germ f.

We can define the notion of symplectic equivalence between two isotropic map-germs in a similar manner to that in §3.

If (σ, τ) is a symplectic equivalance between f and $f' : N', x' \longrightarrow (M', \omega')$, then $\sigma^* : H^2(N', N' - \Sigma'; \mathbf{Z}) \longrightarrow H^2(N, N - \Sigma; \mathbf{Z})$ maps $m(f')$ to $m(f)$.

§5. Symplectic reduction

Let $(M^{2(n+k)}, \omega)$ be a symplectic manifold of dimension $2(n + k)$, and $K^{2n+k} \subset M$ be a coisotropic submanifold of codimension k. We denote by $(TK)^{\perp}$ the skew-orthogonal complement to TK in $TM|K$.

Remark that the rank of $(TK)^{\perp}$ is equal to k. Since K is coisotropic, $(TK)^{\perp} \subset TK$, and hence $(TK)^{\perp}$ is integrable ([AM]). We call $(TK)^{\perp}$ (resp. induced foliation on K) the characteristic distribution (resp. foliation) relative to K.

Let $x \in K$. Then, in an open neighborhood U of x in K, a submersion $\pi : U \longrightarrow {M'}^{2n}$ is induced, where M' is the leaf space. Then M' has a unique symplectic structure ω', up to symplectic diffeomorphisms of M', such that $\pi^*\omega' = \omega|K$ ([AM]). M' is called the reduction of M by K.

(1) By this reduction procedure, Lagrangian submanifolds of M also reduces to "Lagrangian varieties".

Now, let $L^{n+k} \subset M$ be a Lagrangian submanifold and $x \in L$. If $N = L \cap K$ is an n-dimensional submanifold of K in a neighborhood of x, then $f = \pi|N : N, x \longrightarrow M'$ is an isotropic map-germ. In fact, $f^*\omega' = \pi^*\omega'|N = \omega \mid N = 0$.

Remark that f is an immersion at x if and only if $T_xL \cap (T_xK)^\perp = 0$.

In particular, if L is transverse to K, then we have an immersed Lagrangian submanifold in the reduced symplectic manifold M'.

In fact, in this case, $T_xL \cap (T_xK)^\perp = T_xL \cap (T_xL + T_xK)^\perp = T_xL \cap (T_xM)^\perp = T_xL \cap \{0\} = \{0\}$.

In general, f is not an immersion and has a singularity.

Definition 5.1. Let f be the same as in the above. Then f is called an isotropic map-germ arising from a k-dimensional reduction of a Lagrangian manifold.

(2) Moreover, let N^n be an isotropic submanifold of $M^{2(n+k)}$ contained in K^{2n+k} and containing x. Then $f = \pi|N : N, x \longrightarrow M'$ is isotropic, and f is immersive if and only if $T_xN \cap (T_xK)^\perp = 0$.

Definition 5.2. Such a germ f is simply called an isotropic map-germ arising from k-dimensional reduction.

§6. Reduction of a Lagrangian manifold and Maslov class

In §4, we have defined the Maslov class $m(f) \in H^2(N, N - \Sigma; \mathbf{Z})_x$ for an isotropic map germ $f : N^n, x \longrightarrow M^{2n}$, where Σ is the singular set of f.

Theorem 6.1. *Let $f : N, x \longrightarrow M$ be an isotropic map-germ. If f is symplectically equivalent to an isotropic map-germ arising from a one-dimensional reduction of a Lagrangian manifold, then $m(f) = 0 \in H^2(N, N - \Sigma; \mathbf{Z})_x$.*

Precisely, for any open neighborhood U of x, and for any respersentative $f : U \longrightarrow M$ of f, there exist a contractible open neighborhood V such that $x \in V \subset U$ and $m(f|V) = 0$ in $H^2(V, V - \Sigma; \mathbf{Z})$.

§7. Local models for generic one-dimensional reductions

We consider, by symplectic equivalences, a generic local classification of isotropic mappings arising from symplectic reduction relative to a hypersurface (i.e., one-dimensional reduction) (see Definition 5.2).

Let (M^{2n+2}, ω) be a symplectic manifold, K^{2n+1} be a hypersurface of M, and N^n be an n-dimensional manifold.

Denote by $\mathcal{I}$ the set of isotropic embeddings $i : N \longrightarrow M$ with $i(N) \subset K$, endowed with the Whitney C^∞ topology.

Next we introduce special isotropic map-germs $f_{n,k}$ as local models for singularities of isotropic mappings. Consider the cotangent bundle $T^*\mathbf{R}^n$ with canonical coordinates $q_1, \dots, q_n; p_1 \dots, p_n$ and with the symplectic form $\omega = \sum_i dp_i \wedge dq_i$. Besides, consider the space $\mathbf{R}^n$ with coordinates $x_1, \dots, x_n$. Then

$$f_{n,k} : \mathbf{R}^n, 0 \longrightarrow T^*\mathbf{R}^n, \ 0 \le k \le \left[\frac{n}{2}\right],$$

is defined by

$$q_i \circ f_{n,k} = x_i, \ 1 \le i \le n-1,$$

$$u = q_n \circ f_{n,k} = \frac{x_n^{k+1}}{(k+1)!} + \sum_{i=1}^{k-1} x_i \frac{x_n^{k-i}}{(k-i)!},$$

$$v = p_n \circ f_{n,k} = \sum_{i=0}^{k-1} x_{k+i} \frac{x_n^{k-i}}{(k-i)!},$$

and

$$p_j \circ f_{n,k} = \int_0^{x_n} \left(\frac{\partial v}{\partial x_j}\frac{\partial u}{\partial x_n} - \frac{\partial v}{\partial x_n}\frac{\partial u}{\partial x_j} \right) dx_n, \ 1 \le j \le n-1.$$

Remark that each $f_{n,k}$ is a polynomial mapping of kernel rank one and of very simple form.

Theorem 7.1. *There exists an open dense subset $\mathcal{G}$ in $\mathcal{I}$ such that, for each $i \in \mathcal{G}$ and for each $x \in N$, the isotropic map-germ $f : N^n, x \longrightarrow M'^{2n}$ arising from one-dimensional reduction relative to K is symplectically equivalent to some $f_{n,k}, 0 \le k \le [n/2]$.*

§8. Maslov class of an open Whitney umbrella

Let us study properties of local models $f_{n,k} : \mathbf{R}^n, o \longrightarrow T^*\mathbf{R}^n, 0 \le k \le [n/2]$.

For $k = 0$, $f_{n,0}$ is just the zero-section $\zeta_n : \mathbf{R}^n, 0 \longrightarrow T^*\mathbf{R}^n$ and is an immersion.

For $k \neq 0$, we easily verify that

$$\Sigma = \Sigma(f_{n,k}) = \{\partial u/\partial x_n = \partial v/\partial x_n = 0\}$$

is a submanifold of codimension 2 in $\mathbf{R}^n$. Thus we have

$$(*) : H^2(\mathbf{R}^n, \mathbf{R}^n - \Sigma; \mathbf{Z})_0 \cong H^1(\mathbf{R}^n - \Sigma; \mathbf{Z})_0 \cong \mathbf{Z}.$$

By defnition, we can write $f_{n,k} = f_{2k,k} \times \zeta_{n-2k}$. Then $f_{n,k}$ is a "suspension" of $f_{2k,k}$.

Definition 8.1. $f_{2n,n}$ is called a $2n$-dimensional open Whitney umbrella.

Remark 8.2. $f_{2,1}$ is just the (2-dimensional) open Whitney umbrella introduced by Givental' [G].

For Maslov classes, we have

Theorem 8.3. *Under the identification* $(*)$,

$$m(f_{n,k}) = \begin{cases} 0 & \textit{for} \quad k = 0, \\ \pm 2 & \textit{for} \quad 0 < k \leq \left[\dfrac{n}{2}\right]. \end{cases}$$

Corollary 8.4. *For a generic, that is, belonging to $\mathcal{G}$ in Theorem 7.1, isotropic submanifold $i : N^n \longrightarrow K^{2n+1} \subset M^{2n+2}$, if T_xN contains the characteristic direction of K at a point $x \in N$, then N is never representable as an intersection of any Lagrangian submanifold and K, as germ at x*

Proof. If N were an intersection of a Lagrangian submanifold and K, then the Maslov class of isotropic map-germ arising from the reduction relative to K would vanish by Theorem 6.1.

By Theorem 7.1, that map-germ is symplectically equivalent to some $f_{n,k}, k \neq 0$. By Theorem 8.3, $m(f_{n,k}), k \neq 0$, does not vanish. Combined with the argument in §4, this leads to a contradiction. Q.E.D.

§9. Reduction of symplectic vector bundles

(1) Let $\mathbf{E}$ be a symplectic vector bundle over a manifold X, and $\mathbf{K}$ be a coisotropic subbundle. Then the bundle $\mathbf{K}/\mathbf{K}^\perp$ has the induced symplectic structure, where $\mathbf{K}^\perp$ is the skew-orthogonal complement of $\mathbf{K}$ in $\mathbf{E}$ (see [AM],[W]).

Let $\mathbf{L}$ be a Lagrangian subbundle of $\mathbf{E}$. If $\mathbf{L} \subset \mathbf{K}$, then $\mathbf{K}^\perp \subset \mathbf{L}^\perp = \mathbf{L} \subset \mathbf{K}$, and $\mathbf{L}/\mathbf{K}^\perp \subset \mathbf{K}/\mathbf{K}^\perp$ is a Lagrangian subbundle.

Lemma 9.1. *Let* $\mathbf{E}$ *be a symplectic vector bundle over* X, $\mathbf{K}$ *a coisotropic subbundle of* $\mathbf{E}$, *and* $\mathbf{L}$ *(resp.* $\mathbf{L}'$*) a Lagrangian subbundle contained in* $\mathbf{K}$. *Then* $m(\mathbf{K}/\mathbf{K}^{\perp};\mathbf{L}/\mathbf{K}^{\perp},\mathbf{L}'/\mathbf{K}^{\perp}) = m(\mathbf{E};\mathbf{L},\mathbf{L}')$ *in* $H^1(X;\mathbf{Z})$ *(cf.§1).*

Proof. Set $\operatorname{rank}\mathbf{E} = 2(n+k)$ and $\operatorname{rank}\mathbf{K} = 2n+k$. Then $\operatorname{rank}\mathbf{K}^{\perp} = k$. Compare $\Phi_1 = \Phi(\mathbf{L})\circ s(\mathbf{L}') : X \longrightarrow \Lambda(n+k)$ and $\Phi_2 = \Phi(\mathbf{L}/\mathbf{K}^{\perp})\circ s(\mathbf{L}'/\mathbf{K}^{\perp}) : X \longrightarrow \Lambda(n)$.

Set $\Lambda(n+k,k) = \{\lambda \in \Lambda(\mathbf{C}^{n+k}) \mid \lambda \subset \mathbf{C}^n \times \mathbf{R}^k\}$. Then we can choose a Hermitian structure on $\mathbf{E}$ such that $\Phi_1(X) \subset \Lambda(n+k,k)$ and $\tilde{\pi}\circ\Phi_1 = \Phi_2$, where $\pi : \mathbf{C}^n \times \mathbf{C}^k \longrightarrow \mathbf{C}^n$ is the projection and $\tilde{\pi} : \Lambda(n+k,k) \longrightarrow \Lambda(n)$ is defined by $\tilde{\pi}(\lambda) = \pi(\lambda) \subset \mathbf{C}^n$ $(\lambda \in \Lambda(n+k,k))$. Remark that $\det^2\circ\tilde{\pi} = \det^2 : \Lambda(n+k,k) \longrightarrow S^1$. Then $\det^2\circ\Phi_2 = \det^2\circ\tilde{\pi}\circ\Phi_1 = \det^2\circ\Phi_1$. Thus we have the required result.

(2) We apply Lemma 9.1 to the situation of §5,(1).

Shrinking K around x if necessary, we assume that the characteristic foliation of K comes from a submersion $\pi : K \longrightarrow M'$, $N = L\cap K$ is an n-dimensional submanifold in K, and that K and N are contractible.

Set $\mathbf{E} = TM|N-\Sigma, \mathbf{K} = TK|N-\Sigma, \mathbf{K}' = (TN)^{\perp}|N-\Sigma, \mathbf{L} = TL|N-\Sigma$, and $\mathbf{L}' = TN + (TK)^{\perp}|N-\Sigma$.

Note that $TN+(TK)^{\perp}$ is a direct sum in TM over $N-\Sigma$. Therefore, $\mathbf{L}'$ is a subbundle of $\mathbf{E}$ of rank $n+k$. Furthermore, $\mathbf{L}'^{\perp} = \mathbf{K}'\cap\mathbf{K} \supset \mathbf{L}'$. Hence $\mathbf{L}'$ is Lagrangian.

Thus we have a symplectic vector bundle $\mathbf{E}$, coisotropic subbundles $\mathbf{K}$ and $\mathbf{K}'$ of rank $2n+k$ and $n+2k$, respectively, and Lagrangian subbundles $\mathbf{L}$ and $\mathbf{L}'$ with $\mathbf{L}\subset\mathbf{K}', \mathbf{L}'\subset\mathbf{K}$, and $\mathbf{L}'\subset\mathbf{K}'$.

Since M' is a symplectic reduction of M relative to K, we have an isomorphism $\alpha : TK/(TK)^{\perp} \longrightarrow \pi^*TM'$, which induces an isomorphism $\beta : TK/(TK)^{\perp}|N \longrightarrow f^*TM'$.

For each $y \in N-\Sigma$, $\beta(\mathbf{L}'_y/\mathbf{K}^{\perp}_y) = T_yf(T_yN) = (\mathbf{L}_f)_y$ in the fiber $(f^*TM')_y$ over y. By restriction, β induces an isomorphism

$$\gamma : \mathbf{K}/\mathbf{K}^{\perp} \longrightarrow f^*TM'|N-\Sigma$$

such that $\gamma(\mathbf{L}'/\mathbf{K}^{\perp}) = \mathbf{L}_f$.

Therefore, for a Lagrangian subbundle $\mathbf{L}_1$ of f^*TM' over N,

$$m(f^*TM';\mathbf{L}_1,\mathbf{L}_f) = m(\mathbf{K}/\mathbf{K}^{\perp};\beta^{-1}(\mathbf{L}_1),\mathbf{L}'/\mathbf{K}^{\perp})$$

in $H^1(N-\Sigma;\mathbf{Z})$, by Lemma 1.1.(3).

Take the Lagrangian subbundle $\mathbf{L}_2$ of $TM|N$ contained in $TK|N$ which projects to $\beta^{-1}(\mathbf{L}_1) \subset (TK/(TK)^{\perp})|N$. Then, by Lemma 9.1,

$$m(\mathbf{K}/\mathbf{K}^{\perp}; \beta^{-1}(\mathbf{L}_1), \mathbf{L}'/\mathbf{K}^{\perp}) = m(\mathbf{E}; \mathbf{L}_2, \mathbf{L}').$$

By Lemma 1.1.(1),

$$m(\mathbf{E}; \mathbf{L}_2, \mathbf{L}') = m(\mathbf{E}; \mathbf{L}_2, \mathbf{L}) + m(\mathbf{E}; \mathbf{L}, \mathbf{L}').$$

Since $\mathbf{L} = TL|N - \Sigma$ is a restriction of the Lagrangian subbundle $TL|N$ over N, $m(\mathbf{E}; \mathbf{L}_2, \mathbf{L})$ is the restriction of an element in $H^1(N; \mathbf{Z})$.

Since these arguments are valid over any contractible neighborhood V of x in N, we have Theorem 6.1 if $m(\mathbf{E}; \mathbf{L}, \mathbf{L}') = 0$ in $H^1(N - \Sigma, \mathbf{Z})_x$.

Furthermore, using Lemma 9.1 again, we see

$$m(\mathbf{E}; \mathbf{L}, \mathbf{L}') = m(\mathbf{K}'/\mathbf{K}'^{\perp}; \mathbf{L}/\mathbf{K}'^{\perp}, \mathbf{L}'/\mathbf{K}'^{\perp}).$$

In the next section, we will show that the right hand side is equal to zero in $H^1(N - \Sigma, \mathbf{Z})_x$, at least if $k = 1$.

§10. Proof of Theorem 6.1

It is sufficient to show $m(\mathbf{K}'/\mathbf{K}'^{\perp}; \mathbf{L}/\mathbf{K}'^{\perp}, \mathbf{L}'/\mathbf{K}'^{\perp}) = 0$ in $H^1(V - \Sigma; \mathbf{Z})$ for any sufficiently small contractible neighborhood V of x, in the notation of §9,(2).

Let $h = 0$ be a local equation of K in M, where $h \in C^{\infty}(M)$. By the sign of h, $L - N$ is devided into two: $L - N = L_+ \cup L_-, L_{\pm} = \{y \in N \mid \pm h(y) > 0\}$.

Take a vector field v tangent to L toward L_+ at $x \in N$. Then $dh(v) \geq 0$.

Let w be the Hamiltonian vector field with Hamiltonian h. Then the imaginary part of $G(w, v)$ is equal to $\Omega(w, v) = (w \rfloor \Omega)(v) = -dh(v) \leq 0$.

Remark that normalized v (resp. w) turns into an orthonormal frame of $\mathbf{L}/\mathbf{K}'^{\perp}$ (resp. $\mathbf{L}'/\mathbf{K}'^{\perp}$). Therefore, for $\Phi : V - \Sigma \longrightarrow \Lambda(1) = U(1)/O(1)$ in the definition of Maslov class, we see that $\det \circ \Phi : V - \Sigma \longrightarrow S^1 \subset \mathbf{C}$ has non-positive imaginary part. Therefore, $\det \circ \Phi$ is homotopically zero, and so is $\det^2 \circ \Phi$. Thus

$$m(\mathbf{K}'/\mathbf{K}'^{\perp}; \mathbf{L}/\mathbf{K}'^{\perp}, \mathbf{L}'/\mathbf{K}'^{\perp}) = (\det{}^2 \circ \Phi)^* 1 = 0$$

in $H^1(V - \Sigma; \mathbf{Z})$ for any sufficiently small contractible neighborhood V of x in N. Q.E.D.

§11. Variety of singular isotropic jets

Let N be a manifold of dimension n and M be a symplectic manifold of dimension $2n$. In the k-jet bundle $J^k(N,M)$, we set

$$J^k_I(N,M) = \{j^k f(x) \in J^k(N,M) \mid f : N, x \longrightarrow M \text{ is isotropic}\},$$

and

$$\widetilde{\Sigma} = \{j^1 f(x) \in J^1_I(N,M) \mid f : N, x \longrightarrow M \text{ is not an immersion}\}.$$

Further, set

$$\widetilde{\Sigma}^i = \{j^1 f(x) \in J^1_I(N,M) \mid \text{ dim Ker } T_x f = i\}.$$

Then we have

$$\widetilde{\Sigma} = \bigcup_{i=1}^{n} \widetilde{\Sigma}^i.$$

Set

$$\bar{\Sigma}^j = \bigcup_{i=j}^{n} \widetilde{\Sigma}^i.$$

Proposition 11.1. *The set $J^1_I(N,M) - \bar{\Sigma}^2$ of isotropic 1-jets with kernel dimension ≤ 1 is a submanifold of $J^1(N,M)$. Further, $\widetilde{\Sigma}^1$ is a submanifold of $J^1_I(N,M) - \bar{\Sigma}^2$ of codimension 2.*

Set $V = \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{C}^n) \cong M_n(\mathbf{C})$.

Let $\langle \cdot, \cdot \rangle$ denote the standard Hermitian structure on $\mathbf{C}^n$. Define the symplectic structure $[\cdot,\cdot]$ on $\mathbf{C}^n$ by $[u,v] = \mathrm{Im}\,\langle u,v\rangle, u,v \in \mathbf{C}^n$. Let $X \subset V$ be the set of isotropic linear maps $\mathbf{R}^n \longrightarrow \mathbf{C}^n$, and $\Sigma^i \subset X$ be the set of isotropic linear maps $\mathbf{R}^n \longrightarrow \mathbf{C}^n$ with kernel dimension i. Set $S^j = \bigcup_{i=j}^n \Sigma^i$.

To prove Proposition 11.1, it is sufficient to show

Lemma 11.2. *X is a real algebraic variety in V, with $\mathrm{Sing}(X) \subset S^2$. Further, Σ^i is a submanifold of V of dimension $(1/2)\{n(3n+1) - i(3i+1)\}$. In particular, Σ^1 is a submanifold of codimension 2 in $X - S^2$.*

Proof. Denote by $\mathrm{Alt}(n)$ the set of skewsymmetric bilinear forms $a : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}$ on $\mathbf{R}^n$ and by $Sp(n)$ the group of symplectic linear isomorphisms on $(\mathbf{C}^n, [\,,\,])$.

Set $G = GL(n,\mathbf{R}) \times Sp(n)$. Define G-actions on $V = \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{C}^n)$ and $\mathrm{Alt}(n)$ by

$$(\sigma, \tau)\ell = \tau \circ \ell \circ \sigma^{-1},$$

$$(\sigma, \tau)a = a \circ (\sigma^{-1} \times \sigma^{-1})$$

for $(\sigma, \tau) \in G, \ell \in V, a \in \mathrm{Alt}(n)$, respectively.

Consider the map $\rho : V \longrightarrow \mathrm{Alt}(n)$ defined by $\rho(\ell)(u, v) = [\ell u, \ell v]$, $\ell \in V, u, v \in \mathbf{R}^n$. Then ρ is a G-equivariant polynomial map and $X = \rho^{-1}(O)$. In particular, X is a real algebraic variety.

Let $\ell \in X$. Then $\mathrm{rank}\,(\ell) = i, 0 \leq i \leq n$ if and only if there exists $g \in G$ such that

$$g \cdot \ell = \begin{pmatrix} E_i & O \\ O & O \end{pmatrix}.$$

In fact, if $\mathrm{rank}\,(\ell) = i$, then there exists $\tau \in U(n) \subset Sp(n)$ such that $\tau(\mathrm{image}\,\ell) = \mathbf{R}^i \times 0 \subset \mathbf{C}^n$. Thus, for some $\sigma \in GL(n, \mathbf{R})$, $\tau \circ \ell \circ \sigma^{-1}$: $\mathbf{R}^n \longrightarrow \mathbf{C}^n$ is the projection to $\mathbf{R}^i \times 0 \subset \mathbf{C}^n$. The converse is clear.

Remark that the matrix representation of ρ is

$$A + \sqrt{-1}B \mapsto {}^tBA - {}^tAB \in \mathrm{Alt}(n), \quad A, B \in M_n(\mathbf{R}).$$

Let ℓ be isotropic. Then ρ is a submersion at ℓ if and only if $\mathrm{rank}\,(\ell) \geq n - 1$. To see this, we may assume

$$\ell = \begin{pmatrix} E_i & O \\ O & O \end{pmatrix}$$

without loss of generality. The tangent map of ρ at ℓ,

$$T_\ell(\rho) : T_\ell V \longrightarrow T_{\rho(\ell)}\,\mathrm{Alt}(n),$$

is described by

$$A' + \sqrt{-1}B' \mapsto {}^tB'\ell - {}^t\ell B' = \begin{pmatrix} {}^tB_{11} - B_{11} & -B_{12} \\ {}^tB_{12} & O \end{pmatrix},$$

where

$$B' = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

B_{11}, B_{12}, B_{21} and B_{22} are real matrices of type $(i, i), (i, n-i), (n-i, i)$ and $(n-i, n-i)$, respectively. Therefore, $T_\ell(\rho)$ is surjective if and only if $i = n$ or $i = n - 1$. Thus $\mathrm{Sing}(X) \subset S^2$.

Define a subset Y of $X \times \Lambda(n)$ to be the totality of pairs (ℓ, λ) such that the image of ℓ is contained in λ. The projection $Y \to \Lambda(n)$ is a fibration with fiber diffeomorphic to $M_n(\mathbf{R}) = \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}^n)$. Set $Y^i = Y \cap (\Sigma^i \times \Lambda(n))$. Then Y^i is a submanifold of Y of codimension i^2. On the other hand, the projection $Y^i \to X \subset V$ is a fibration with fiber

$\Lambda(i)$, that is, of constant kernel dimension $(1/2)i(i+1)$. Therefore, Σ^i is a submanifold of V and $\dim \Sigma^i = \dim Y^i - (1/2)i(i+1) = \dim Y - (1/2)i(3i+1) = (1/2)n(3n+1) - (1/2)i(3i+1)$. Q.E.D.

§12. Universal Maslov class

The calculation of Maslov classes of isotropic map-germs can be reduced to that in jet spaces.

Define $\Psi : J^1_I(N, M) - \widetilde{\Sigma} \longrightarrow \Lambda(M)$ by

$$\Psi(j^1 f(x)) = T_x f(T_x N) \subset T_{f(x)} M$$

for $j^1 f(x) \in J^1_I(N, M) - \widetilde{\Sigma}$.

Remark that $\Psi \circ j^1 f = \varphi(f)$ (see §2).

Definition 12.1. The universal Maslov class of an isotropic 1-jet $z = j^1 f(x)$ is defined by

$$m(z) = \delta(m(\Psi^* \pi^* TM; \mathbf{L}, \Psi^* \mathcal{L})) \in H^2(J^1_I(N, M), J^1_I(N, M) - \widetilde{\Sigma}; \mathbf{Z})_z,$$

where $\mathbf{L}$ is a Lagrangian subbundle of $\Psi^* \pi^* TM|U$ over a contractible neighborhood of z in $J^1_I(N, M)$ and $\mathcal{L} \subset \pi^* TM$ is the tautological Lagrangian subbundle over $\Lambda(M)$.

Lemma 12.2. *Let $f : N, x \longrightarrow M$ be an isotropic map-germ. Then $j^1 f : N, x \longrightarrow J^1_I(N, M)$ induces*

$$(j^1 f)^* : H^2(J^1_I(N, M), J^1_I(N, M) - \widetilde{\Sigma}; \mathbf{Z})_{j^1 f(x)} \longrightarrow H^2(N, N - \Sigma; \mathbf{Z})_x,$$

which maps $m(j^1 f(x))$ to $m(f)$.

Proof. We have

$$\begin{aligned} (j^1 f)^* m(j^1 f(x)) &= (j^1 f)^* \delta m(\mathbf{L}, \Psi^* \mathcal{L}) \\ &= \delta m((j^1 f)^* \mathbf{L}, (\Psi \circ j^1 f)^* \mathcal{L}) \\ &= \delta m((j^1 f)^* \mathbf{L}, (\varphi(f))^* \mathcal{L}) \\ &= \delta m((j^1 f)^* \mathbf{L}, \mathbf{L}_f) \\ &= m(f). \end{aligned}$$

§13. Calculation of an universal Maslov class

Proposition 13.1. *Let $z \in \widetilde{\Sigma}^1$. Then*

$$H^2(J^1_I(N,M), J^1_I(N,M) - \widetilde{\Sigma}; \mathbf{Z})_z \cong \mathbf{Z},$$

and $m(z) = \pm 2$.

Proof. The first half is clear from Proposition 11.1.

To see the second half, without loss of generality, we may assume that $z = j^1 f(0) \in J^1_I(\mathbf{R}^n, T^*\mathbf{R}^n)$ with $q_i \circ f = x_i, 1 \leq i \leq n-1, q_n \circ f = 0, p_i \circ f = 0, 1 \leq i \leq n$. Define $c : \mathbf{R}^2 \longrightarrow J^1_I(\mathbf{R}^n, T^*\mathbf{R}^n)$ by

$$c(t,s) = j^1(x_1, \dots, x_{n-1}, tx_n; 0, \dots, 0, sx_n)(0)$$

for $(t,s) \in \mathbf{R}^2$. Then $c(0) = z$ and c is transverse to $\widetilde{\Sigma}^1$.

Take a small loop $\ell_\varepsilon : S^1 \longrightarrow J^1_I(\mathbf{R}^n, T^*\mathbf{R}^n)$, where $\ell_\varepsilon(e^{i\theta}) = c(\varepsilon\cos\theta, \varepsilon\sin\theta)$. Then ℓ_ε is a generator of $H_1(J^1_I(\mathbf{R}^n, T^*\mathbf{R}^n) - \widetilde{\Sigma}; \mathbf{Z})_z$. Thus $|m(z)|$ is determined by the evaluation to ℓ_ε. Remark that $\Psi \circ \ell_\varepsilon$ is represented by

$$e^{i\theta} \mapsto \begin{pmatrix} E_{n-1} & O \\ O & e^{i\theta} \end{pmatrix} \in U(n).$$

Thus $\det^2 \circ \Psi \circ \ell_\varepsilon : S^1 \longrightarrow S^1$ is of degree 2. Therefore, $|m(z)| = 2$.

§14. Proof of Theorem 8.3

Lemma 14.1. *Let $f : N, x \longrightarrow M$ be isotropic. If $j^1 f(x) \in \widetilde{\Sigma}^1$ and $j^1 f$ is transverse to $\widetilde{\Sigma}^1$ in $J^1_I(N,M)$. Then*

$$H^2(N, N - \Sigma; \mathbf{Z})_x \cong \mathbf{Z},$$

and $m(f) = \pm 2$, where $\Sigma = \Sigma(f) = (j^1 f)^{-1}(\widetilde{\Sigma})$.

Proof. Since $j^1 f$ is transverse to $\widetilde{\Sigma}^1$, and $\Sigma = (j^1 f)^{-1}(\widetilde{\Sigma}^1)$, we see that Σ is a submanifold of codimension 2 in N near x, and

$$H^2(N, N - \Sigma; \mathbf{Z})_x \cong H^2(J^1_I(N,M), J^1_I(N,M) - \widetilde{\Sigma}; \mathbf{Z})_{j^1 f(x)} \cong \mathbf{Z}.$$

Since $m(j^1 f(x)) = \pm 2$ by Proposition 13.1, we have

$$m(f) = (j^1 f)^* m(j^1 f(x)) = \pm 2,$$

relatively to the above isomorphism.

By checking the 2-jets of $f_{n,k}$ at 0, we easily verify the following

Lemma 14.2. *The* 1*-jet extension* $j^1 f_{n,k}, k \neq 0$*, is transverse to* $\widetilde{\Sigma}^1$ *in* $J^1_I(\mathbf{R}^n, T^*\mathbf{R}^n)$ *at* $j^1 f_{n,k}(0)$.

Now, Theorem 8.3 follows from Lemmata 14.1 and 2.

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