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Gauss Maps of Complete Minimal Surfaces

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§1. Introduction

In 1961, R. Osserman showed that the Gauss map of a complete nonflat minimal surface immersed in $\mathbf{R}^3$ cannot omit a set of positive logarithmic capacity ([16]). Afterwards, F. Xavier proved that the Gauss map of such a surface can omit at most six points ([25]). In 1988, the author has shown that the number of exceptional values of the Gauss map of such a surface is at most four ([6]). Here, the number four is best-possible. Indeed, there are many examples of nonflat complete minimal surfaces in $\mathbf{R}^3$ whose Gauss maps omit four values. Moreover, he revealed some relations between these results and the defect relation in Nevanlinna theory on value distribution of meromorphic functions, and gave some modified defect relation for the Gauss map of such a surface in [8]. Recently, as an improvement of these results, X. Mo and R. Osserman showed that, if the Gauss map of a nonflat complete minimal surface M immersed in $\mathbf{R}^3$ takes on five distinct values only a finite number of times, then M has finite total curvature ([14]).

The author gave also modified defect relations for the Gauss map G of a complete minimal surface immersed in $\mathbf{R}^m$ for the case where G is nondegenerate as a map into $P^{m-1}(\mathbf{C})$ and, as its application, he showed that G can omit at most $m(m+1)/2$ hyperplanes in general position ([9]). Here, the number $m(m+1)/2$ is best-possible for arbitrary odd numbers and some small even numbers ([7]). Recently, M. Ru showed that the "nondegenerate" assumption of the above result can be dropped ([20]). In [10], the author introduced a new definition of modified defect and proved a refined modified defect relation for the Gauss map of complete minimal surfaces possibly with branch points and gave some improvements of the above-mentioned results in [9], [14] and [20].

The purpose of this lecture is to survey the above-mentioned results more precisely and to give the outline of their proofs. We first give

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the definition of modified defect and some fundamental properties in §2. We next explain a modified defect relation for a holomorphic map of an open Riemann surface with a complete pseudo-metric into the complex projective space $P^n(\mathbf{C})$ and some consequences of it in §3. The outline of its proof is given in §4. After these expositions, we discuss the value distributions of the Gauss maps of complete minimal surfaces in $\mathbf{R}^m$ in the last two sections.

§2. Modified defect for a holomorphic curve in $P^n(\mathbf{C})$

Let M be an open Riemann surface. We consider a function u on M possibly with singularities in a discrete subset of M.

Definition 2.1. We call u to be a function with *mild singularities* on an open set D in M if u is a C^∞ function on D except a discrete set and around each point $a \in D$ we can write

$$|u| = |z-a|^\sigma |\log|z-a||^\tau u^* \tag{2.2}$$

with a holomorphic local coordinate z, a positive continuous function u^* and real numbers σ and τ.

For a function u with mild singularities on D, we define by

$$\nu_u(a) := \text{the number } \sigma \text{ in the expression (2.2) for some } \tau \text{ and } u^*$$

the divisor $\nu_u : D \to \mathbf{R}$. Here, a divisor on D means a map $\nu : D \to \mathbf{R}$ such that the support $|\nu| := \{z; \nu(z) \neq 0\}$ is discrete. For a nonzero meromorphic function ψ, $\nu_\psi(a)$ is nothing but the order of ψ at a.

Let ν be a divisor on M. We denote by $[\nu]$ the (1, 1)-current corresponding to ν, namely, the map $[\nu] : \mathcal{D} \to \mathbf{C}$ defined by

$$[\nu](\varphi) := \int_M \nu\varphi = \sum_{z\in M} \nu(z)\varphi(z) \qquad (\varphi \in \mathcal{D}),$$

where $\mathcal{D}$ denotes the space of all C^∞ differentiable functions on M with compact supports. In some cases, a $(1,1)$-form Ω on M is regarded as a current on M defined by $\Omega(\varphi) := \int_M \varphi\Omega$ for each $\varphi \in \mathcal{D}$.

For two $(1,1)$-currents Ω_1, Ω_2 and a positive constant c, by $\Omega_1 \prec_c \Omega_2$ we mean that there are a divisor ν and a bounded continuous nonnegative function k with mild singularities such that $\nu \geq c$ on $|\nu|$ and

$$\Omega_1 + [\nu] = \Omega_2 + dd^c \log k^2,$$

where $d^c = \dfrac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$. We write $\Omega_1 \prec \Omega_2$ if $\Omega_1 \prec_c \Omega_2$ for some $c > 0$.

Let $f : M \to P^n(\mathbf{C})$ be a holomorphic map which is nondegenerate, namely, whose image is not included in any hyperplane, and let

$$H : a_0 w_0 + \cdots + a_n w_n = 0$$

be a hyperplane in $P^n(\mathbf{C})$. Take a representation $f = (f_0 : \cdots : f_n)$ on M which is reduced, namely, whose components f_i are holomorphic functions without common zero. Set $F(H) := a_0 f_0 + \cdots + a_n f_n$ and define $\nu(f, H) := \nu_{F(H)}$. The n-truncated pull-back $f^*(H)^{[n]}$ of H as divisor is defined by $f^*(H)^{[n]} := [\min(\nu(f, H), n)]$. We see easily $f^*(H)^{[n]} \prec \Omega_f$, where Ω_f denotes the pull-back of the Fubini-Study metric on $P^n(\mathbf{C})$ by f, namely, $\Omega_f = dd^c \log \|f\|^2$ for $\|f\| := (\sum_{i=0}^n |f_i|^2)^{1/2}$.

Definition 2.3. We define the *modified H-defect* of H for f by

$$D_f(H) := 1 - \inf\{\eta; f^*(H)^{[n]} \prec \eta\,\Omega_f \text{ on } M - K \text{ for a compact set } K\}.$$

For a not necessarily nondegenerate holomorphic map f of M into $P^n(\mathbf{C})$, if $f(M) \subseteq H$, we set $D_f(H) = 0$, and otherwise we define H-defect for f by H-defect for the map f considered as a map into the smallest projective linear subspace of $P^n(\mathbf{C})$ including $f(M)$.

The modified H-defect has the following properties.

Proposition 2.4. (i) $0 \le D_f(H) \le 1$.

(ii) *If there exists a bounded nonzero holomorphic function g on $M - K$ for a compact set K such that $\nu_g \ge \min(\nu(f, H), n)$ on $M - K$, or particularly, if $\# f^{-1}(H) < \infty$, then $D_f(H) = 1$.*

(iii) *If $\nu(f, H) \ge m$ at every $a \in f^{-1}(H) - K$ for some compact set K, then $D_f(H) \ge 1 - n/m$.*

Proof. The assertion (i) is trivial and (ii) is also obvious because

$$f^*(H)^{[n]} + [\nu_g - \min(\nu(f, H), n)] = [\nu_g] = dd^c \log |g|^2$$

on $M - K$ by Poincaré-Lelong formula. Moreover, (iii) is true because

$$f^*(H)^{[n]} + \left[\frac{n}{m}\nu(f, H) - \min(\nu(f, H), n)\right] = \frac{n}{m} dd^c \log \|f\|^2 + dd^c \log k^2$$

on $M - K$ for the bounded function $k := (|F(H)|/\|f\|)^{n/m}$.

We recall the classical defect for a nondegenerate holomorphic map of (an open neighborhood) of $\Delta_{R,\infty} := \{z; R \le |z| < +\infty\}$ into $P^n(\mathbf{C})$.

The order function of f and the counting function (truncated by n) of a hyperplane H for f are defined by

$$T_f(r) = \int_R^r \frac{dt}{t} \int_{R\leq|z|<t} \Omega_f \qquad (R < r < +\infty),$$

$$N_f(r)^{[n]} = \int_R^r \frac{dt}{t} \int_{R\leq|z|<t} f^*(H)^{[n]} \qquad (R < r < +\infty),$$

respectively. The classical defect (truncated by n) is defined by

$$\delta_f(H)^{[n]} = 1 - \limsup_{r\to\infty} \frac{N_f(r)^{[n]}}{T_f(r)}.$$

We can prove the following relation.

Proposition 2.5. *Let f be a nondegenerate holomorphic map of an open Riemann surface M into $P^n(\mathbf{C})$. Assume that there is a biholomorphic map Φ of an open neighborhood of $\Delta_{R,\infty}$ onto an open set in M such that $\tilde{f} := f \cdot \Phi$ has an essential singularity at ∞. Then, for every hyperplane H*

$$0 \leq D_f(H) \leq \delta_{\tilde{f}}(H)^{[n]} \leq 1.$$

Proof. We take a nonnegative constant η such that

$$f^*(H)^{[n]} + [\nu] = \eta\Omega_f + dd^c \log k^2$$

on $M - K$ for a compact set K, a bounded continuous function k with mild singularities and a divisor ν satisfying the condition that $\nu \geq c$ on $|\nu|$ for some $c > 0$. Then, by the monotonicity of integral, we see

$$N_{\tilde{f}}(r)^{[n]} \leq \eta T_{\tilde{f}}(r) + O(\log r) \qquad (R < r < +\infty)$$

and so $1 - \eta \leq 1 - N_{\tilde{f}}(r)^{[n]}/T_{\tilde{f}}(r) + O(\log\, r)/T_{\tilde{f}}(r)$. This concludes the desired inequality.

§3. Modified defect relation

Let $N \geq n$ and $q > 2N - n + 1$ and consider q hyperplanes $H_1, \ldots, H_q$ in $P^n(\mathbf{C})$. After W. Chen ([2]), we give the following :

Definition 3.1. We say that $H_1, \dots, H_q$ are in *N-subgeneral position* if, for every $1 \le j_0 < \cdots < j_N \le q$,

$$H_{j_0} \cap \ldots \cap H_{j_N} = \emptyset.$$

In [15], E. I. Nochka has given the following theorem :

Theorem 3.2. *For given hyperplanes $H_1, \dots, H_q$ in N-subgeneral position, there are some constants $\omega(1), \dots, \omega(q)$ and θ such that*

(i) $0 < \omega(j) \le \theta \ (1 \le j \le q)$ *and* $\dfrac{n+1}{2N-n+1} \le \theta \le \dfrac{n+1}{N+1}$,

(ii) $\sum_{j=1}^{q} \omega(j) = n + 1 + \theta(q - 2N + n - 1)$,

(iii) *if* $R \subset Q$ *and* $0 < \#R \le N+1$, *then* $\sum_{j \in R} \omega(j) \le d(R)$.

For the proof, see [2].

Definition 3.3. We call constants $\omega(j)$ and θ with the properties (i) $\sim$ (iii) *Nochka weights* and *a Nochka constant* for H_j's respectively.

By definition, $H_j (1 \le j \le q)$ are in general position if and only if they are in n-subgeneral position. If $H_1, \dots, H_q$ are in general position, then we have necessarily $\omega(1) = \cdots = \omega(q) = \theta = 1$.

We give here the classical defect relation improved by E. I. Nochka.

Theorem 3.4. *Let $f : \Delta_{R,\infty} \to P^n(\mathbf{C})$ be a nondegenerate holomorphic map with an essential singularity at ∞. Then, for arbitrary hyperplanes $H_j (1 \le j \le q)$ in N-subgeneral position with Nochka constants $\omega(j)$, it holds that*

$$\sum_{j=1}^{q} \omega(j) \delta_f(H_j)^{[n]} \le n + 1.$$

For the proof, see [15] or [2].

Definition 3.5. We call ds^2 a *pseudo-metric* on M if, for each holomorphic local coordinate z, it is written as $ds^2 = \lambda_z^2 |dz|^2$ with a nonnegative function λ_z which has mild singularities. A continuous pseudo-metric ds^2 means a pseudo-metric such that λ_z is continuous.

We define the divisor of a pseudo-metric $ds^2 = \lambda_z^2 |dz|^2$ by $\nu_{ds} := \nu_{\lambda_z}$.

For a pseudo-metric $ds^2 = \lambda_z^2 |dz|^2$ the Ricci form is defined by

$$\mathrm{Ric}_{ds^2} := -dd^c \log \lambda_z^2$$

as a current, and the Gaussian curvature of ds^2 is given by $K_{ds^2} = \Delta \log \lambda_z / \lambda_z^2$ only on the set $M_1 := \{ds^2 \neq 0\}$, which is called to be strictly negative if $K_{ds^2} \leq -Cds^2$ on M_1 for some $C > 0$. A Riemann surface M whose universal covering is biholomorphic with the unit disc has the unique complete conformal metric with constant curvature -1, which we call Poincaré metric of M and denote by $d\sigma_M^2$ in the following.

To state the modified defect relation, we give two more definitions.

Definition 3.6. We define the *H-order* of f by

$$\rho_f := \inf\{\rho; -\mathrm{Ric}_{ds^2} \prec \rho\Omega_f \text{ on } M - K \text{ for some compact set } K\}.$$

Definition 3.7. Let M be an open Riemann surface of finite type, namely, M is biholomorphic with a compact Riemann surface $\bar{M}$ with finitely many points removed. A holomorphic map f of M into $P^n(\mathbf{C})$ is said to be *transcendental* if f has no holomorphic extension to $\bar{M}$.

The modified defect relation is stated as follows :

Theorem 3.8. *Let M be an open Riemann surface with a complete continuous pseudo-metric ds^2 and f a nondegenerate holomorphic map of M into $P^n(\mathbf{C})$. Take hyperplanes $H_1, \ldots, H_q$ in N-subgeneral position with Nochka constants $\omega(j)$. If M is not of finite type or else f is transcendental, then*

$$\sum_{j=1}^{q} \omega(j) D_f(H_j) \leq n + 1 + \frac{\rho_f n(n+1)}{2}. \tag{3.9}$$

The outline of the proof of Theorem 3.8 will be given in §4. We give here the following corollary to this theorem.

Corollary 3.10. *Let M be an open Riemann surface with a complete pseudo-metric and $f : M \to P^n(\mathbf{C})$ a nondegenerate holomorphic map. If M is not of finite type, then for arbitrary hyperplanes $H_1, \ldots, H_q$ in N-subgeneral position,*

$$\sum_{j=1}^{q} D_f(H_j) \leq (2N - n + 1) + \frac{\rho_f n(2N - n + 1)}{2}. \tag{3.11}$$

Proof. Let A denote the right hand side of (3.11). According to Theorems 3.2 and 3.8, we have

$$A \geq 2N - n + 1 + \frac{\rho_f \sigma_n}{\theta} \geq q + \frac{1}{\theta}\left(n + 1 + \rho_f \sigma_n - \sum_{j=1}^{q} \omega(j)\right)$$
$$\geq q + \sum_{j=1}^{q} \frac{\omega(j)}{\theta}(D_f(H_j) - 1) \geq q + \sum_{j=1}^{q}(D_f(H_j) - 1) = \sum_{j=1}^{q} D_f(H_j),$$

where $\sigma_n := n(n+1)/2$. This gives Corollary 3.10.

§4. The proof of the modified defect relation

In this section, we shall give the outline of the proof of Theorem 3.8. For this purpose, we first give the following theorem.

Theorem 4.1. *Let M be an open Riemann surface with a complete continuous pseudo-metric ds^2 and let $d\tau^2$ be a continuous pseudo-metric on $M - K$ with strictly negative curvature for some compact set K. Assume that there exists a constant p with $0 < p < 1$*

$$- \mathrm{Ric}_{ds^2} \prec_{1-p} p\,(- \mathrm{Ric}_{d\tau^2}) \tag{4.2}$$

on $M - K$. Then M is of finite type.

Proof. Taking a nowhere zero holomorphic 1-form ω, we write $ds^2 = \lambda^2|\omega|^2$ and $d\tau^2 = \eta^2|\omega|^2$. By assumption, we can take a divisor ν and a continuous nonnegative bounded function k on $M - K$ with mild singularities such that $\nu(z) \geq 1 - p$ for every $z \in |\nu|$ and

$$dd^c \log \lambda^2 + [\nu] = p\, dd^c \log \eta^2 + dd^c \log k^2.$$

Here, we may assume that ν and k are defined on M and $0 \leq k \leq 1$. Set $u := k\eta^p/\lambda$. Then, $\log u$ is harmonic outside $K \cup |\nu|$, $\nu_u \geq 1 - p$ on $|\nu| - K$ and $\lambda = k\eta^p/u \leq \eta^p/u$. Define a new pseudo-metric

$$d\rho^2 := u^{-2/(1-p)}|\omega|^2$$

on M and set $M_1 := \{a \in M; \nu_{d\rho}(a) = 0\}$. Then, $d\rho^2$ is a flat metric on $M_1 - K$ and, since $\nu_u \geq 1-p$ on $|\nu_u| - K$, $\nu_{d\rho} \leq -1$ on $M - (K \cup M_1)$.

We recall here the following theorem of A. Huber ([13]).

Theorem 4.3. *For an open Riemann surface M, if there is a complete metric $d\rho^2$ on M such that*

$$\int_M \max(-K_{d\rho^2}, 0)\Omega_{d\rho^2} < +\infty,$$

then M is of finite type, where $\Omega_{d\rho^2}$ denotes the area form associated with $d\rho^2$.

To prove Theorem 4.1, it suffices to show the following fact.

(4.4). *The surface M_1 is complete with respect to the metric $d\rho^2$.*

In fact, by the aid of Theorem 4.3 we can conclude from (4.4) that M_1, and so M, are of finite type because

$$\int_{M_1} \max(-K_{d\rho^2}, 0)\Omega_{d\rho^2} = \int_K \max(-K_{d\rho^2}, 0)\Omega_{d\rho^2} < \infty.$$

Assume that M_1 is not complete, and so $d_0 := \mathrm{dist}_{d\rho}(K, \partial M_1)$ is finite, where $\mathrm{dist}_{d\rho}(K, \partial M_1)$ denotes the distance between K and ∂M_1. Then we can find a continuous curve $\gamma_0(t)\,(0 \leq t < 1)$ such that $\gamma_0(0) \in K$, $\gamma_0(t)$ tends to ∂M_1 as $t \to 1$ and the length $L_{d\rho}(\gamma_0)$ of γ_0 is smaller than $2d_0$ and take a point $p_0 := \gamma_0(t_0)\,(0 \leq t_0 < 1)$ such that $\mathrm{dist}_{d\rho}(K, p_0) > d_0/2$ and $L_{d\rho}(\gamma_0|[t_0, 1)) < d_0/2$, where $\gamma|[\alpha, \beta)$ denotes the part of γ from $t = \alpha$ to $t = \beta$.

Since $d\rho^2$ is flat on $M_1 - K$, there is an isometry Φ of a disc $\Delta_R := \{w \in \mathbf{C}; |w| < R\}$ with the standard metric onto an open neighborhood of p_0 in $M_1 - K$ with the metric $d\rho^2$ such that $\Phi(0) = p_0$. Take the largest $R(\leq +\infty)$ such that there is a local isometry Φ of Δ_R onto some open set in $M_1 - K$ with $\Phi(0) = p_0$. Then, $R \leq L_{d\rho}(\gamma_0|[t_0, 1)) < d_0/2$ and there is a line segment Γ joining the origin and a point in $\partial\Delta_R$ such that $\gamma := \Phi(\Gamma)$ tends to the boundary of $M_1 - K$. Then, if γ tends to K or to the set $M - (K \cup M_1)$, then we have an absurd conclusion $R \geq \mathrm{dist}_{d\rho}(K, p_0) > d_0/2$ or $R = L_{d\rho}(\gamma) = +\infty$ respectively, because $\nu_{d\rho} \leq -1$ on $M - (K \cup M_1)$. Therefore, γ tends to the boundary of M.

Now, we shall estimate the length $L_{ds}(\gamma)$ of γ. To this end, we set $\tilde{\eta} := d\tau/d\rho$. Then, we have $\eta = \tilde{\eta}u^{-1/(1-p)}$. So,

$$\begin{aligned} L_{ds}(\gamma) &\leq \int_\gamma u^{-1}\eta^p|\omega| \leq \int_\gamma u^{-1}\tilde{\eta}^p u^{-\frac{p}{1-p}}|\omega| \\ &\leq \int_\gamma \tilde{\eta}^p d\rho = \int_\Gamma (\tilde{\eta}\cdot\Phi)^p \Phi^*(d\rho) = \int_\Gamma (\tilde{\eta}\cdot\Phi)^p |dw|. \end{aligned}$$

On the other hand, the curvature of $\Phi^*(d\tau)$ is strictly negative on Δ_R by assumption. By the generalized Schwarz lemma we obtain

$$\Phi^*(d\tau) = (\tilde{\eta}\cdot\Phi)|dw| \leq C_0\, d\sigma_{\Delta_R} \leq C_1 \frac{R}{R^2 - |w|^2}|dw|$$

for some constants C_0 and C_1. Therefore, we have

$$L_{ds}(\gamma) \leq C_2 \int_\Gamma \left(\frac{R}{R^2 - |w|^2} \right)^p |dw| \leq \frac{C_3}{1-p} R^{1-p} < \infty$$

for some constants C_2 and C_3. This contradicts the completeness of M with respect to ds^2. Thus, we conclude (4.4) and so Theorem 4.1.

Now, we start to prove Theorem 3.8. To this end, we may assume $\rho_f < +\infty$ and M is not of finite type because Theorem 3.8 is obvious from Proposition 2.5 and Theorem 3.4 for the other cases.

Take arbitrary constants $\rho > 0$ and η_j $(1 \leq j \leq q)$ such that

$$(4.5) \qquad -\mathrm{Ric}_{ds^2} \prec \rho\, \Omega_f, \qquad f^*(H_j)^{[n]} \prec \eta_j\, \Omega_f$$

on $M' := M - K$ for a compact set K. By definition, there are divisors ν_j and bounded continuous nonnegative functions k_j with mild singularities such that $\nu_j \geq c_j$ on $|\nu_j|$ for some $c_j > 0$ and

$$f^*(H_j)^{[n]} + [\nu_j] = \eta_j\, \Omega_f + dd^c \log k_j^2$$

on M'. Set $h_j := k_j \|f\|^{\eta_j}$. Then, $\log h_j$ are harmonic on $M' - |\nu_{h_j}|$.

For our purpose, we have only to show that

$$(4.6) \qquad \gamma := \theta(q - 2N + n - 1) - \sum_{j=1}^{q} \omega(j)\eta_j \leq \rho\sigma_n.$$

In fact, if this is true, then we easily obtain (3.9) from the definitions of the modified defect and ρ_f because (4.6) can be rewritten

$$\sum_{j=1}^{q} \omega(j)(1 - \eta_j) \leq n + 1 + \rho\sigma_n$$

by the use of Theorem 3.2, (ii).

Assume that $\gamma > \rho\sigma_n$. We shall show that there exists a pseudo-metric on M' with strictly negative curvature which satisfies (4.2) for a suitable constant p with $0 < p < 1$, which leads to a contradiction by Theorem 4.1 and concludes (4.6). To this end, we represent each H_j as

$$H_j : a_{j0}w_0 + \cdots + a_{jn}w_n = 0 \qquad (1 \leq j \leq q).$$

Take a reduced representation $f = (f_0 : \cdots : f_n)$ and set $F(H_j) := a_{j0}f_0 + \cdots + a_{jn}f_n$. Moreover, for an arbitrarily fixed holomorphic local coordinate z we consider the functions

$$|F_k| := \left(\sum_{0 \le j_0 < \cdots < j_k \le n} |W(f_{j_0}, \ldots, f_{j_k})|^2 \right)^{1/2},$$

$$|F_k(H_j)| := \left(\sum_{0 \le i_1 < \cdots < i_k \le n} \left| \sum_{j \ne i_1, \ldots, i_k} a_j W(f_j, f_{i_1}, \ldots, f_{i_k}) \right|^2 \right)^{1/2},$$

$$\varphi_k(H_j) := \frac{|F_k(H_j)|^2}{|F_k|^2},$$

where $W(g_0, \ldots, g_k)$ denotes the Wronskian of holomorphic functions $g_0, \ldots, g_k$. Now, choosing some ε with $\gamma > \varepsilon\sigma_{n+1}$, we set

$$\eta_z := \left(\frac{\|f\|^{\gamma - \varepsilon\sigma_{n+1}} |F_n| \prod_{j=1}^q |h_j|^{\omega(j)} \prod_{k=0}^n |F_k|^\epsilon}{\prod_{j=1}^q (|F(H_j)| \prod_{k=0}^{n-1} \log(a/\varphi_k(H_j)))^{\omega(j)}} \right)^{\frac{1}{\sigma_n + \epsilon\tau_n}}$$

and define the pseudo-metric $d\tau^2 := \eta_z^2 |dz|^2$, which is well-defined on $M - K$. Set $\varphi := |F_n| / \prod_{j=1}^q |F(H_j)|^{\omega(j)}$. Then we can prove that $\nu_\varphi + \sum_{j=1}^q \omega(j) \min(\nu(f, H_j), n) \ge 0$ (cf., [10, §2]). This implies that $\nu_0 \ge c'$ on $|\nu_0|$ for some $c' > 0$ and $d\tau^2$ is a continuous pseudo-metric on M'. Moreover, we can prove that $d\tau^2$ has strictly negative curvature on M'(cf., [10, §5]).

For some holomorphic local coordinate z and each pair of indices j, k, we choose indices $i_1, \ldots, i_k$ with $1 \le i_1 < \cdots < i_k \le q$ such that

$$\psi_{jk}^z := \sum_{\ell \ne i_1, \ldots, i_k} a_{j\ell} W(f_\ell, f_{i_1}, \ldots, f_{i_k}) \not\equiv 0.$$

For convenience' sake, we set $\psi_{j0}^z = F(H_j)$. By the theorem of identity, $\psi_{jk}^z \not\equiv 0$ for every holomorphic local coordinate z. We now define

$$k := \left(\frac{\prod_{1 \le j \le q, 0 \le k \le n-1} |\psi_{jk}^z|^{\epsilon/q} \log^{\omega(j)}(a/\varphi_k(H_j))}{\prod_{0 \le k \le n-1} |F_k|^\epsilon} \right)^{1/(\sigma_n + \epsilon\tau_n)}.$$

Then, k is bounded because

$$\frac{|\psi_{jk}^z|^{\epsilon/q} \log^{\omega(j)}(a/\varphi_k(H_j))}{|F_k|^{\epsilon/q}} \le \left(\frac{|F_k(H_j)|}{|F_k|} \right)^{\epsilon/q} \log^{\omega(j)}(a/\varphi_k(H_j))$$
$$\le \sup_{0 < x \le 1} x^{\epsilon/q} \log^{\omega(j)} \left(\frac{a}{x} \right) < +\infty.$$

Set $v := |\varphi| \prod_{j=1}^{q} |h_j|^{\omega(j)} \prod_{1 \leq j \leq q, 0 \leq k \leq n-1} |\psi_{jk}^{z}|^{\epsilon/q}$. Then $\log v$ is harmonic outside $|\nu_v|$. If we choose a constant ε which is smaller than and sufficiently near to the number $(\gamma - \rho\sigma_n)/(\sigma_{n+1} + \rho\tau_n)$ and set $p := \rho(\sigma_n + \varepsilon\tau_n)/(\gamma - \varepsilon\sigma_{n+1})$, it holds that $-\mathrm{Ric}_{ds^2} \prec_{1-p} \rho\,\Omega_f$, $0 < p < 1$ and $\nu_v \geq (\gamma - \varepsilon\sigma_{n+1})(1-p)/\rho$. on $|\nu_v|$. Moreover, we see easily

$$\rho\,\Omega_f + \left[\frac{\rho}{\gamma - \varepsilon\sigma_{n+1}}\nu_v\right] = p\,dd^c \log \eta_z^2 + dd^c \log k^{2p}.$$

This shows that (4.2) holds on M'. Thus, we have proved Theorem 3.8.

§5. Gauss maps of complete minimal surfaces in $\mathbf{R}^m$

Let $x : M \to \mathbf{R}^m$ be a (possibly branched) minimal surface. By defintion, M is an open Riemann surface, $x = (x_1, \ldots, x_m)$ is a nonconstant map whose components x_i are harmonic and satisfy the condition

$$f_1^2 + \cdots + f_m^2 = 0$$

for holomorphic functions $f_i := \partial x_i/\partial z$ locally defined with a holomorphic local coordinate z on M. The pseudo-metric ds^2 on M induced from $\mathbf{R}^m$ are locally written as $ds^2 = 2\|f\|^2|dz|^2$ (cf., [18]). The set S of all branch points coincides with the set of common zeros of the functions $f_i (1 \leq i \leq m)$ and we have $\nu_{ds} = \min\{\nu_{f_i}; 1 \leq i \leq m\}$.

As is well-known, the set of all oriented 2-planes in $\mathbf{R}^m$ may be identified with the quadric $Q_{m-2}(\mathbf{C})$ in $P^{m-1}(\mathbf{C})$. By definition, the Gauss map G of M maps each $p \in M - S$ to the point in $Q_{m-2}(\mathbf{C})$ corresponding to the oriented tangent plane of M at p and it is locally given by $G = (f_0 : \cdots : f_n)$ on $M - S$. Take a nonzero holomorphic function h on M with $\nu_h = \nu_{ds}$. If we set $g_i := f_i/h (1 \leq i \leq m)$, we have $G = (g_1 : g_2 : \cdots : g_m)$ outside S, which is holomorphically extended across S. So, we may consider the Gauss map G as a holomorphic map of M into $P^{m-1}(\mathbf{C})$.

A surface with a pseudo-metric is called to be flat if the Gaussian curvature identically vanishes. It is easily seen that a minimal surface is flat if and only if the Gauss map is a constant.

Definition 5.1. The *total curvature* of M is defined by $-\int_M \mathrm{Ric}_{ds^2}$.

Proposition 5.2. *A complete minimal surface $x : M \to \mathbf{R}^m$ has finite total curvature if and only if M is of finite type and the Gauss map of M is not transcendental.*

For the proof, see [3].

Definition 5.3. We define the *branching H-order* of M by

$$\rho_{ds} := \inf\{\rho; [\nu_{ds}] \prec \rho\,\Omega_G \text{ on } M - K \text{ for some compact set } K\}.$$

Obviously, if $x : M \to \mathbf{R}^m$ is an immersion, then $\rho_{ds} = 0$.

Theorem 5.4. *Let $x : M \to \mathbf{R}^m$ be a complete nonflat minimal surface with infinite total curvature and $G : M \to P^N(\mathbf{C})$ the Gauss map of M, where $N = m - 1$. Consider the smallest linear subspace $P^n(\mathbf{C})$ of $P^N(\mathbf{C})$ which includes $G(M)$. Then, for arbitrary hyperplanes $H_1, \ldots, H_q (1 \le j \le q)$ in $P^N(\mathbf{C})$ located in general position,*

$$\sum_{j=1}^{q} D_G(H_j) \le 2N - n + 1 + \frac{(1+\rho_{ds})n(2N-n+1)}{2}.$$

Proof. By assumption, the Gauss map G is nondegenerate as the map into $P^n(\mathbf{C})$. On the other hand, the metric of M is given by $ds^2 = 2\|f\|^2|dz|^2 = 2|h|^2\|g\|^2|dz|^2$ for a holomorphic function h and a reduced representation $g = (g_1 : \cdots : g_m)$. For each $\rho \ge 0$ such that $[\nu_{ds}] \prec \rho\,\Omega_G$ on M outside a compact set K, we have $-\operatorname{Ric}_{ds^2} \prec (\rho + 1)\Omega_G$ on $K - M$. Taking the infimum of the right hand side for various ρ, we obtain $\rho_G \le \rho_{ds} + 1$. Since $H_1, \ldots, H_q$ considered as hyperplanes in $P^n(\mathbf{C})$ are located in N-subgeneral position, Theorem 5.4 is now an immediate consequence of Corollary 3.10.

Theorem 5.5. *Let G be the Gauss map of a nonflat complete minimal surface immersed in $\mathbf{R}^m$ with infinite total curvature. Then, for arbitrary hyperplanes $H_1, \ldots, H_q$ in $P^{m-1}(\mathbf{C})$ located in general position,*

$$\sum_{j=1}^{q} D_G(H_j) \le \frac{m(m+1)}{2}.$$

Proof. By assumption, Theorem 5.4 is valid for some n with $1 \le n \le N$. Therefore, we have

$$\sum_{j=1}^{q} D_G(H_j) \le 2N - n + 1 + \frac{n(2N-n+1)}{2}$$
$$= \frac{(N+1)(N+2) - (N-n)(N-n-1)}{2} \le \frac{m(m+1)}{2}.$$

This gives Theorem 5.5.

In view of Proposition 2.4, Theorem 5.5 yields the following :

Corollary 5.6. *Let M be a nonflat complete minimal surface immersed in $\mathbf{R}^m$ with infinite total curvature, and let G be the Gauss map of M. If $G^{-1}(H_j)$ are finite for q hyperplanes $H_1, \dots, H_q$ in $P^{m-1}(\mathbf{C})$ located in general position, then $q \leq m(m+1)/2$.*

We have also the following result by Ru ([20]).

Corollary 5.7. *The Gauss map of a nonflat complete minimal surface immersed in $\mathbf{R}^m$ can omit at most $m(m+1)/2$ hyperplanes in general position.*

Proof. If M has infinite total curvature, then this is a direct result of Corollary 5.6. Otherwise, take the smallest projective linear subspace $P^n(\mathbf{C})$ of $P^{m-1}(\mathbf{C})$. If given hyperplanes are in general position in $P^n(\mathbf{C})$, then by the result of Chern and Osserman ([3]) G can omit at most $n(n+3)/2 (< m(m+1)/2)$ hyperplanes in general position. By the use of Theorem 3.2, the arguments in [3] is available for the case where given hyperplanes are in general position in $P^{m-1}(\mathbf{C})$(cf., [21]).

Here, the number $m(m+1)/2$ is best-possible for an arbitrary odd numbers and some small even numbers m. In fact, we can construct some complete minimal surfaces in $\mathbf{R}^m$ whose Gauss maps are non-degenerate and omit $m(m+1)/2$ hyperplanes in general position for such numbers. For the details, see [7].

Now, we consider a holomorphic curve in $\mathbf{C}^m$ given by a nonconstant holomorphic map $w = (w_1, w_2, \dots, w_m) : M \to \mathbf{C}^m$. The space $\mathbf{C}^m$ is identified with $\mathbf{R}^{2m}$ by associating $(x_1+\sqrt{-1}y_1, \dots, x_m+\sqrt{-1}y_m) \in \mathbf{C}^m$ with $(x_1, y_1, \dots, x_m, y_m)$. The curve $w : M \to \mathbf{C}^m$ is considered as a minimal surface $w = (x_1, y_1, \dots, x_m, y_m) : M \to \mathbf{R}^{2m}$. By Cauchy-Riemann's equations, $f_i := \partial x_i/\partial z = \sqrt{-1}\partial y_i/\partial z (1 \leq i \leq m)$. So, the Gauss map of M is given by $G = (f_1 : -\sqrt{-1}f_1 : \cdots : f_m : -\sqrt{-1}f_m)$, and therefore the image $G(M)$ of G is included in the projective subspace

$$P^{m-1}(\mathbf{C}) := \{(u_1 : v_1 : \cdots : u_m : v_m); u_i = -\sqrt{-1}v_i (1 \leq i \leq m)\}$$

of $P^{2m-1}(\mathbf{C})$. As a consequence of Theorem 3.8, we have the following :

Corollary 5.8. *Let $w : M \to \mathbf{C}^m$ be a holomorphic curve in $\mathbf{C}^m$ which is complete and not included in any affine hyperplane, and let G be* *the Gauss map of M considered as a map of M into the above-mentioned space $P^{m-1}(\mathbf{C})$. If M is not of finite type, then*

$$\sum_{j=1}^{q} D_G(H_j) \leq m + \frac{(\rho_{ds}+1)m(m-1)}{2}$$

for arbitrary hyperplanes $H_1, \ldots, H_q$ in $P^{m-1}(\mathbf{C})$ in general position.

For the proof, see [10, §6].

§6. The Gauss maps of minimal surfaces in $\mathbf{R}^3$ or $\mathbf{R}^4$

We next consider a minimal surface $x = (x_1, x_2, x_3) : M \to \mathbf{R}^3$. In this case, the quadric $Q_1(\mathbf{C})$ is canonically biholomorphic with $P^1(\mathbf{C})$. Instead of the Gauss map $G : M \to Q_1(\mathbf{C})$ we may study the classical Gauss map $g : M \to P^1(\mathbf{C})$ defined by $g = (f_3 : f_1 - \sqrt{-1}f_2)$, where $f_i := \partial x_i/\partial z (i = 1, 2, 3)$. Then, the metric of M is given by $ds^2 = |h|^2(|g_0|^2 + |g_1|^2)^2|dz|^2$ for a reduced representation $g = (g_0 : g_1)$ and a nonzero holomorphic function h with $\nu_h = \min(\nu_{f_1}, \nu_{f_2}, \nu_{f_3})$. Since $\nu_{ds} = \nu_h$, we have $-\operatorname{Ric}_{ds} \leq \rho + 2$ whenever $[\nu_{ds}] \prec \rho\Omega_g$. This yields $\rho_g \leq \rho_{ds} + 2$. From Theorem 3.8, we can easily conclude the following :

Theorem 6.1. *Let $x : M \to \mathbf{R}^3$ be a nonflat complete minimal surface with infinite total curvature and let g be the classical Gauss map. Then, for arbitrary distinct points $\alpha_1, \ldots, \alpha_q$ in $P^1(\mathbf{C})$,*

$$\sum_{j=1}^{q} D_g(\alpha_j) \leq 4 + \rho_{ds}.$$

Here, we can construct an example of a nonflat complete minimal surface in $\mathbf{R}^3$ with $\rho_{ds} = 2$ whose Gauss map omit six distinct points in $P^1(\mathbf{C})$ (cf., [10, §6]).

Relating to Theorem 6.1, we can prove the following theorem for noncomplete minimal surfaces in $\mathbf{R}^3$.

Theorem 6.2. *Let $x : M \to \mathbf{R}^3$ be a nonflat minimal surface and $g : M \to P^1(\mathbf{C})$ the classical Gauss map. If there exist distinct points $\alpha_1, \ldots, \alpha_q \in P^1(\mathbf{C})$ and positive integers $m_1, \ldots, m_q$ satisfying the condition that each $g - \alpha_j$ has no zeros with multiplicity $< m_j$ and*

$$\sum_{j=1}^{q} \left(1 - \frac{1}{m_j}\right) > 4,$$

then there is a constant $C > 0$ depending only on α_j and m_j such that

$$|K(p)| \leq \frac{C}{\mathrm{dist}_{ds^2}(p, \partial M)^2} \qquad (p \in M).$$

For the proof, see [6] and [19].

We next consider a complete minimal surface $x : M \to \mathbf{R}^4$. In this case, the Gauss map G of M is a map into $Q_2(\mathbf{C})$, which is canonically identified with $P^1(\mathbf{C}) \times P^1(\mathbf{C})$. Instead of the Gauss map $G : M \to Q_2(\mathbf{C})(\subset P^3(\mathbf{C}))$ we consider the map $g : M \to P^1(\mathbf{C}) \times P^1(\mathbf{C})$, which we call the classical Gauss map of M.

We can prove the following defect relation.

Theorem 6.3. *Let $x : M \to \mathbf{R}^4$ be a complete minimal surface not of finite type and $g = (g_1, g_2) : M \to P^1(\mathbf{C}) \times P^1(\mathbf{C})$ the classical Gauss map of M. Take two systems of distinct points $\{\alpha_1, ..., \alpha_{q_1}\}$ and $\{\beta_1, ..., \beta_{q_2}\}$.*

(i) *If g_1 and g_2 are nonconstant and, moreover, $\sum_{i=1}^{q_1} D_{g_1}(\alpha_i) > 2$ and $\sum_{j=1}^{q_2} D_{g_2}(\beta_j) > 2$, then*

$$\frac{1}{\sum_{i=1}^{q_1} D_{g_1}^H(\alpha_i) - 2} + \frac{1}{\sum_{j=1}^{q_2} D_{g_2}^H(\beta_j) - 2} \geq 1.$$

(ii) *If g_1 is nonconstant and g_2 is a constant, then*

$$\sum_{j=1}^{q_1} D_{g_1}(\alpha_j) \leq 3.$$

The proof is omitted. For the details, see [8].

References

[1] L. V. Ahlfors, "Conformal invariants, Topics in Geometric Function Theory", McGraw-Hill Book Company, New York, 1973.

[2] W. Chen, Defect relations for degenerate meromorphic maps, Trans. Amer. Math. Soc., **319** (1990), 499–515.

[3] S. S. Chern and R. Osserman, Complete minimal surfaces in euclidean n-space, J. Analyse Math., **19** (1967), 15–34.

[4] M. J. Cowen and P.A. Griffiths, Holomorphic curves and metrics of negative curvature, J. Analyse Math., **29** (1976), 93–153.

[5] R. S. Earp and H. Rosenberg, On values of the Gauss map of complete minimal surfaces in $\mathbf{R}^3$, Comment. Math. Helv., **63** (1988), 579–586.
[6] H. Fujimoto, On the number of exceptional values of the Gauss map of minimal surfaces, J. Math. Soc. Japan, **40** (1988), 235–247.
[7] ———, Examples of complete minimal surfaces in $\mathbf{R}^m$ whose Gauss maps omit $m(m+1)/2$ hyperplanes in general position, Sci. Rep. Kanazawa Univ., **33** (1988), 37–43.
[8] ———, Modified defect relations for the Gauss map of minimal surfaces, J. Differential Geometry, **29** (1989), 245–262.
[9] ———, Modified defect relations for the Gauss map of minimal surfaces II, J. Differential Geometry, **31** (1990), 365–385.
[10] ———, Modified defect relations for the Gauss map of minimal surfaces III, preprint.
[11] W. K. Hayman, "Meromorphic functions", Oxford Math Monographs, Clarendon Press, Oxford, 1964.
[12] D. A. Hoffman and R. Osserman, The geometry of the generalized Gauss map, Mem. Amer. Math. Soc., No. 236 (1986).
[13] A. Huber, On subharmonic functions and differential geometry in the large, Comment. Math. Helv., **32** (1957), 13–72.
[14] X. Mo and R. Osserman, On the Gauss map and total curvature of complete minimal surfaces and an extension of Fujimoto's theorem, J. Differential Geometry, **31** (1990), 343 – 355.
[15] E. I. Nochka, On the theory of meromorphic functions, Soviet Math. Dokl., **27** (2) (1983), 377 – 381.
[16] R. Osserman, Minimal surfaces in the large, Comm. Math. Helv., **35** (1961), 65–76.
[17] ———, Global properties of minimal surfaces in $\mathbf{E}^3$ and $\mathbf{E}^n$, Ann. of Math., **80** (1964), 340–364.
[18] ———, "A survey of minimal surfaces", Dover, New York, 1986.
[19] A. Ros, The Gauss map of minimal surfaces, preprint.
[20] M. Ru, On the Gauss map of minimal surfaces immersed in $\mathbf{R}^n$, preprint.
[21] ———, Complete minimal surfaces in $\mathbf{R}^n$ of finite total curvature, preprint.
[22] B. V. Shabat, "Distribution of values of holomorphic mappings", Transl. Math. Monographs, AMS, 1985.
[23] M. Tsuji, "Potential theory in modern function theory", Maruzen, Tokyo, 1959.
[24] B. White, Complete surfaces of finite total curvature, J. Differential Geometry, **26** (1987), 315–326.
[25] F. Xavier, The Gauss map of a complete non-flat minimal surface cannot omit 7 points of the sphere, Ann. of Math., **113** (1981), 211–214.

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