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Compactification of Submanifolds in Euclidean Space by the Inversion

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Dedicated to Professor Tominosuke Otsuki

Introduction

Let $\bar{M}$ be an n-dimensional compact connected C^2 submanifold in the N-dimensional Euclidean space R^N. Let Ψ be the inversion of R^N, which is defined by $\Psi(x) = x/|x|^2$ for x in $R^N \cup \{\infty\}$. If the origin O is contained in $\bar{M}$, $\Psi(\bar{M})$ becomes a noncompact, complete, connected C^2 submanifold properly immersed into R^N. If we denote the second fundamental form of $\Psi(\bar{M})$ by B, $|x|^2|B|$ ($x \in \Psi(\bar{M})$) is bounded on $\Psi(\bar{M})$. In this paper we study the image by the inversion of a noncompact, complete, connected C^2 submanifold M of dimension $n \geq 2$ which is properly immersed into R^N. We are particularly interested in the smoothness of $\Psi(M)$ at the origin O. We say that M satisfies the condition $\mathrm{P}(\alpha)$ if $|x|^\alpha|B|$ ($x \in M$) is bounded on M. We prove that if M satisfies $\mathrm{P}(2+\varepsilon)$ for some positive constant ε, then the image of each end of M by Ψ is C^2 at O (Theorem 2). Boundedness of $|x|^2|B|$ (i.e., $\mathrm{P}(2)$) is not sufficient to assure that $\Psi(M)$ is C^2 at O, while $\Psi(M)$ is C^1 at O if $\mathrm{P}(1+\varepsilon)$ is satisfied for some $\varepsilon > 0$ (Theorem 1).

Noncompact submanifolds satisfying $\mathrm{P}(1+\varepsilon)$ are studied by Kasue and Sugahara ([4], [5]). They show that those submanifolds become totally geodesic under certain additional conditions on the mean curvature or the sectional curvature. We will make use of some of their results in our proof. As a direct consequence of our theorems, we see that if M satisfies $\mathrm{P}(1+\varepsilon)$, the Gauss map is continuous at infinity, and if M satisfies $\mathrm{P}(2+\varepsilon)$, then M is conformally equivalent to a compact C^2 Riemannian manifold punctured at a finite number of points. We also show that the total integral of the Lipschitz-Killing curvature over the unit normal bundle is an integer if M satisfies $\mathrm{P}(2+\varepsilon)$ (Theorem 3).

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These properties have been studied for submanifolds with $\int_M |B|^n < \infty$ in [8] when $\dim M = 2$, and in [1] when M is minimal. We note that if M satisfies $\mathrm{P}(1+\varepsilon)$, $\int_M |B|^n$ is finite (Proposition 4.1), and if M is minimal, $\dim M \geq 3$ and $\int_M |B|^n$ is finite, then M satisfies $\mathrm{P}(n)$ ([1]).

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§1. Asymptotic behavior of submanifolds

Let $\langle\ ,\ \rangle$ denote the standard inner product of R^N. We denote the covariant differentiation of R^N by D. For x in R^N let $|x| = \langle x, x\rangle^{1/2}$. Let $B(R) = \{x \in R^N : |x| < R\}$ and $S(R) = \{x \in R^N : |x| = R\}$.

Throughout this paper, M will denote a noncompact, complete, connected C^2 submanifold of dimension $n \geq 2$ properly immersed into R^N. For x in M let T_xM and $T_x^\perp M$ denote the tangent and the normal space of M at x respectively. The second fundamental form $B : T_xM \times T_xM \to T_x^\perp M$ is defined by $B(X,Y) = (D_XY)^\perp$, where $(D_XY)^\perp$ is the normal component of D_XY. We also define the shape operator $A_\xi : T_xM \to T_xM$ with respect to a unit normal vector ξ by $A_\xi X = -(D_X\xi)^\top$, where $(D_X\xi)^\top$ is the tangential component of $D_X\xi$. We denote by ∇ the covariant differentiation of M with respect to the induced metric. Let $r(x) = |x|$ for x in M.

Definition. We say that M satisfies the condition $\mathrm{P}(\alpha)$ if there exists a constant K such that

$$r^\alpha |B| \leq K$$

holds at every point of M.

We set $M(R) = M \setminus B(R)$. Since M is properly immersed, $M(R)$ is a union of a finite number of submanifolds $M_1(R), \ldots, M_q(R)$ and $\partial M_\lambda(R) = M_\lambda(R) \cap S(R)$ is compact for each $\lambda = 1, \ldots, q$. The following lemma is due to Kasue ([4, Lemma 2]).

Lemma 1.1. *Suppose M satisfies $P(1+\varepsilon)$ for some positive constant ε. Then there exist positive constants C_1 and R_1 such that*

(1) $\quad |\nabla r| \geq C_1^{-1}$ *for all x in M with $r \geq R_1$,*

(2) $\quad M_\lambda(R_1)$ *is diffeomorphic to $\partial M_\lambda(R_1) \times [R_1, \infty)$ for each $\lambda = 1, \ldots, q$.*

$M_\lambda(R)$ $(R \geq R_1)$ is called an end of M. In the following argument, we assume that the position vector of a point x in R^N is denoted by the same letter x. For x in M, regarding the vector x as a tangent vector to R^N at the point x, we denote by $x^\top$ (resp. $x^\perp$) the image of x by the orthogonal projection from R^N onto the tangent space T_xM (resp. the normal space $T_x^\perp M$) of M at x.

Lemma 1.2. $x^\top = r\nabla r$.

Proof. The gradient vector of $\langle x, x\rangle^{1/2}$ as a function on R^N is given by $r^{-1}x$. For x in M we take its tangential component to see that $\nabla r = r^{-1}x^\top$.

We will use several results from [5] to prove our theorems.

Lemma 1.3. *Suppose M satisfies $P(1+\varepsilon)$ with $\varepsilon > 0$. Then:*

(1) *For any constant δ satisfying $\delta < \min\{\varepsilon, 1\}$, $r^{-1+\delta}|x^\perp|$ tends to zero as $r \to \infty$.* ([5, Lemma 5 (ii)])

(2) *For every $t \geq R_1$ any two points on $\partial M_\lambda(t)$ can be joined by a curve on $\partial M_\lambda(t)$ whose length is less than $C_2 t$, where C_2 is a constant which does not depend on t.* ([5, Lemma 6])

(3) *The second fundamental form of $t^{-1}\partial M_\lambda(t)$ as a submanifold of $S(1)$ tends to zero as $t \to \infty$.* ([5, Lemma 7])

For a submanifold satisfying $\mathrm{P}(2+\varepsilon)$ $(\varepsilon > 0)$ we have the following lemma.

Lemma 1.4. *Suppose M satisfies $P(2+\varepsilon)$ with $\varepsilon > 0$. Then $x^\perp$ is continuous at infinity on each end $M_\lambda(R_1)$.*

Proof. Let x be a point in $M(R_1)$. We first observe that, for X, Y in T_xM and N in $T_x^\perp M$,

$$\begin{aligned} \langle D_X x^\perp, Y\rangle &= -\langle D_X Y, x^\perp\rangle \\ &= -\langle B(X,Y), x^\perp\rangle \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} \langle D_X x^\perp, N\rangle &= \langle D_X(x - x^\top), N\rangle \\ &= \langle X - D_X x^\top, N\rangle \\ &= -\langle B(X, x^\top), N\rangle . \end{aligned} \tag{1.2}$$

Hence there exists a constant C_3 such that

$$\begin{aligned}|D_X x^\perp| &\le C_3 |B| \ |X| \ |x| \\ &\le K C_3 r^{-1-\varepsilon} |X|\,.\end{aligned} \tag{1.3}$$

Now we fix an end $M_\lambda(R_1)$. Let y be a point in $\partial M_\lambda(R_1)$ and let γ_y be the integral curve of $|\nabla r|^{-2}\nabla r$ on $M_\lambda(R_1)$ which starts at y. γ_y is parametrized by r. Set $N(r) = (\gamma_y(r))^\perp$. $N(r)$ is the restriction of the vector field $x^\perp$ to γ_y. By (1.3), we have

$$\begin{aligned}\left|\frac{dN}{dr}\right| &\le K C_3 r^{-1-\varepsilon} |\nabla r|^{-1} \\ &\le K C_1 C_3 r^{-1-\varepsilon}\,.\end{aligned} \tag{1.4}$$

(1.4) implies that $N(r)$ converges to a constant vector N_y as $r \to \infty$.

To prove that N_y does not depend on y we will show that for any y_1 and y_2 in $\partial M_\lambda(R_1)$ and any positive number η we have $|N_{y_1} - N_{y_2}| < \eta$. We first take $R_2 \ge R_1$ such that

$$|(\gamma_{y_j}(R_2))^\perp - N_{y_j}| < \frac{\eta}{3} \tag{1.5}$$

for $j = 1, 2$. By Lemma 1.3 (2), there exists a curve σ on $\partial M_\lambda(R_2)$ which joins $\gamma_{y_1}(R_2)$ and $\gamma_{y_2}(R_2)$ and has length less than $C_2 R_2$, where C_2 is a constant independent of R_2. We parametrize σ by its arclength s. Let $N(s) = (\sigma(s))^\perp$. By (1.3), we have

$$\left|\frac{dN}{ds}\right| \le K C_3 R_2^{-1-\varepsilon}\,.$$

Hence

$$\begin{aligned}|(\gamma_{y_2}(R_2))^\perp - (\gamma_{y_1}(R_2))^\perp| &= \left|\int_\sigma \frac{dN}{ds} ds\right| \\ &\le \int_\sigma \left|\frac{dN}{ds}\right| ds \\ &\le K C_2 C_3 R_2^{-\varepsilon}\,.\end{aligned}$$

If we take R_2 sufficiently large, it is possible to have

$$|(\gamma_{y_2}(R_2))^\perp - (\gamma_{y_1}(R_2))^\perp| < \frac{\eta}{3}\,. \tag{1.6}$$

It follows from (1.5) and (1.6) that

$$
\begin{aligned}
|N_{y_2} - N_{y_1}| &\le |N_{y_2} - (\gamma_{y_2}(R_2))^\perp| + |(\gamma_{y_2}(R_2))^\perp - (\gamma_{y_1}(R_2))^\perp| \\
&\quad + |(\gamma_{y_1}(R_2))^\perp - N_{y_1}| \\
&< \eta\,.
\end{aligned}
$$

This completes the proof of Lemma 1.4.

§2. C^1 compactification by the inversion

Let $R^N \cup \{\infty\}$ be the union of R^N and the point of infinity. The inversion Ψ is a map from $R^N \cup \{\infty\}$ onto $R^N \cup \{\infty\}$ which is defined by $\Psi(x) = \langle x, x\rangle^{-1} x$ for all x in $R^N \setminus \{O\}$, $\Psi(O) = \infty$ and $\Psi(\infty) = O$. If X and Y are tangent vectors of R^N at x, then

$$d\Psi(X) = \langle x, x\rangle^{-1} X - 2\langle x, x\rangle^{-2} \langle x, X\rangle\, x$$

and we have

$$\langle d\Psi(X), d\Psi(Y)\rangle = \langle x, x\rangle^{-2} \langle X, Y\rangle\ .$$

Let $\bar{M} = \Psi(M)$. We denote the second fundamental form of $\bar{M}$ by $\bar{B}$. Let $\bar{x} = \Psi(x)$ and $\bar{r} = \langle \bar{x}, \bar{x}\rangle^{1/2}$. We have $\bar{r} = r^{-1}$, where $r = \langle x, x\rangle^{1/2}$. For a unit tangent vector X and a unit normal vector ξ of M at x we set $\bar{X} = r^2 d\Psi(X)$ and $\bar{\xi} = r^2 d\Psi(\xi)$. $\bar{X}$ (resp. $\bar{\xi}$) is a unit tangent (resp. normal) vector of $\bar{M}$ at $\bar{x}$.

Lemma 2.1. *For any tangent vectors X and Y of M at x, we have*

$$\bar{B}(\bar{X}, \bar{Y}) = r^4 d\Psi(B(X, Y)) + 2r^2 \langle X, Y\rangle\, d\Psi(x^\perp)\,.$$

Proof. We have

$$
\begin{aligned}
D_{\bar{X}}\bar{Y} &= r^2 D_X(Y - 2r^{-2}\langle x, Y\rangle\, x) \\
&= r^2 D_X Y + (4r^{-2}\langle x, X\rangle \langle x, Y\rangle - 2\langle X, Y\rangle \\
&\quad - 2\langle x, D_X Y\rangle) x - 2\langle x, Y\rangle X \\
&= r^4 d\Psi(D_X Y) - 2r^2 \langle x, Y\rangle\, d\Psi(X) + 2r^2 \langle X, Y\rangle\, d\Psi(x)\,.
\end{aligned}
\tag{2.1}
$$

In the last equality, we note that $d\Psi(x) = -r^{-2}x$. Since $d\Psi$ maps tangent spaces and normal spaces of M onto tangent spaces and normal spaces of $\bar{M}$ respectively, the lemma follows from (2.1).

Lemma 2.2. *Suppose M satisfies $P(1+\varepsilon)$ with $\varepsilon > 0$. Then there exists a positive constant δ such that $\bar{r}^{1-\delta}|\bar{B}|$ is bounded in a neighborhood of O in $\bar{M}$.*

Proof. For any unit tangent vectors X and Y of M at x it follows from Lemma 2.1 that

$$
\begin{aligned}
|\bar{B}(\bar{X},\bar{Y})| &= |r^2 B(X,Y) + 2\langle X,Y\rangle x^{\perp}| \\
&\leq r^2|B| + 2|x^{\perp}|\,.
\end{aligned}
\tag{2.2}
$$

Let δ be any constant such that $0 < \delta < \min\{\varepsilon, 1\}$. Then, by Lemma 1.3 (1) and the condition $\mathrm{P}(1+\varepsilon)$, there exists a constant C_4 such that

$$
|\bar{B}| \leq C_4 r^{1-\delta}\,. \tag{2.3}
$$

Now we have the lemma since $\bar{r} = r^{-1}$.

We write $M_\lambda = M_\lambda(R_1)$ $(\lambda = 1, \ldots, q)$ and $\bar{M}_\lambda = \Psi(M_\lambda)$.

Lemma 2.3. *Suppose M satisfies $P(1+\varepsilon)$ with $\varepsilon > 0$. Let $R \geq R_1$. Then any two points $\bar{x}_1$, $\bar{x}_2$ in $B(1/R) \cap \bar{M}_\lambda$ can be joined by a curve on $\bar{M}$ whose length is less than C_5/R, where C_5 is a constant which does not depend on R.*

Proof. Let $\bar{\gamma}_i$ be the integral curve of $|\nabla\bar{r}|^{-2}\nabla\bar{r}$ on $\bar{M}_\lambda$ which passes through $\bar{x}_i$ $(i = 1, 2)$. $\bar{\gamma}_i$ is parametrized by $\bar{r}$. Let $\bar{y}_i = \bar{\gamma}_i(1/R)$. Since $\nabla\bar{r} = -r^2 d\Psi(\nabla r)$, it follows from Lemma 1.1 that

$$
|\nabla\bar{r}| = r^2|d\Psi(\nabla r)| = |\nabla r| \geq C_1^{-1}
$$

for all $\bar{x}$ in $\bar{M}$ with $\bar{r} \leq 1/R_1$. Hence the length of $\bar{\gamma}_i$ between $\bar{x}_i$ and $\bar{y}_i$ is less than C_1/R. By Lemma 1.3 (2), there exists a curve σ on $\partial M_\lambda(R)$ which joins $\Psi(y_1)$ and $\Psi(y_2)$ and has length less than $C_2 R$, where C_2 is a constant which does not depend on R. Let $\bar{\sigma} = \Psi(\sigma)$. Then $\bar{\sigma}$ joins $\bar{y}_1$ and $\bar{y}_2$ and has length less than C_2/R. Connecting $\bar{\gamma}_1$, $\bar{\sigma}$ and $\bar{\gamma}_2$, we obtain a curve in $\bar{M}_\lambda$ which joins $\bar{x}_1$ and $\bar{x}_2$ and has length less than $(2C_1 + C_2)/R$.

Theorem 1. *Let M be a noncompact, complete, connected C^2 submanifold of dimension $n \geq 2$ properly immersed into R^N. Suppose M satisfies the condition $P(1+\varepsilon)$ for some positive constant ε. Then the image of each end M_λ by the inversion is C^1 at the origin O.*

Proof. We will use the generalized Gauss map $\bar{G}$ which maps each point $\bar{x}$ of $\bar{M}$ to the n-dimensional linear subspace parallel to the tangent space of $\bar{M}$ at $\bar{x}$. $\bar{G}$ defines a map from $\bar{M}$ into the Grassmannian manifold $G_n(R^N)$ which consists of n-dimensional (unoriented) linear subspaces of R^N. $G_n(R^N)$ has the standard invariant metric g as a symmetric space. If $\bar{Y}$ and $\bar{Z}$ are tangent vectors of $\bar{M}$ at $\bar{x}$, we have

$$(2.4) \qquad g\,(\,d\bar{G}(\bar{Y}), d\bar{G}(\bar{Z})\,) = \sum_{i=1}^{n} \langle \bar{B}(\bar{Y}, \bar{X}_i), \bar{B}(\bar{Z}, \bar{X}_i) \rangle \;,$$

where $\{\bar{X}_1, \ldots, \bar{X}_n\}$ is an orthonormal base of $T_{\bar{x}}\bar{M}$ ([6]). By Lemma 2.3, any two points $\bar{x}_1$, $\bar{x}_2$ in $B(1/R) \cap \bar{M}_\lambda$ ($R \geq R_1$) is joined by a curve $\bar{\tau}$ in $\bar{M}_\lambda$ whose length is less than C_5/R, where C_5 is a constant which does not depend on R. Now we use (2.4) to see that the length of $\bar{G}(\bar{\tau})$ is less than $C_5\sqrt{n}|\bar{B}|R^{-1}$. Hence, by Lemma 2.2, the length of $\bar{G}(\bar{\tau})$ is less than $C_6 R^{-\delta}$ for some positive constants δ and C_6. This implies that, for an open subset $\bar{U}$ of $\bar{M}_\lambda$ containing O, $\bar{G}(\bar{U})$ converges to a certain point in $G_n(R^N)$ when $\bar{U}$ shrinks to the point O. Hence the tangent space $T_{\bar{x}}\bar{M}_\lambda$ converges to an n-dimensional linear subspace P as $\bar{x}$ in $\bar{M}_\lambda$ approaches O, which means that $\bar{M}_\lambda$ is C^1 at O.

Corollary. *Let M be as in Theorem 1. Then the Gauss map $G\colon M \to G_n(R^N)$ is continuous at infinity on each end.*

Remarks. (1) Theorem 1 does not hold if $n = 1$.

(2) The condition P(1) is not sufficient for $\bar{M}$ to be C^1 at O. Such an example is given in [8]; If M is the graph of a smooth function $z = u(x, y)$ which away from the origin is given by $u(x, y) = x\sin(\log(\log\rho))$ ($\rho = \sqrt{x^2 + y^2}$), then M satisfies P(1) but the Gauss map is not continuous at infinity.

§3. C^2 Compactification by the inversion

In this section we study the image by the inversion of a submanifold which satisfies $P(2+\varepsilon)$. As in §2, let $M_\lambda = M_\lambda(R_1)$ and $\bar{M}_\lambda = \Psi(M_\lambda)$ for $\lambda = 1, \ldots, q$.

Lemma 3.1. *Suppose M satisfies $P(2+\varepsilon)$ with $\varepsilon > 0$. Then for each $\lambda = 1, \ldots, q$ there exist a constant a_λ and a constant unit vector A_λ in R^N such that $\bar{B}(\cdot,\cdot)$ converges to $2a_\lambda \langle \cdot,\cdot \rangle A_\lambda$ as $\bar{x}$ in $\bar{M}_\lambda$ approaches O.*

Proof. By Lemma 1.4, there exist a constant a_λ and a constant unit vector A_λ for each λ such that $x^\perp$ converges to $a_\lambda A_\lambda$ when x lies in M_λ and $|x|$ tends to ∞. For any tangent vectors X and Y of M it follows from Lemma 2.1 that

$$\begin{aligned}\bar{B}(\bar{X},\bar{Y}) &= r^4 d\Psi(B(X,Y)) + 2\langle X,Y\rangle x^\perp \\ &\quad - 4r^{-2}\langle X,Y\rangle \langle x,x^\perp\rangle x\,.\end{aligned} \tag{3.1}$$

When $\bar{x}$ approaches O, we have $r \to \infty$ and hence

$$|r^4 d\Psi(B(X,Y))| \leq r^2|B| \leq Kr^{-\varepsilon} \quad \to 0$$

and

$$|r^{-2}\langle x,x^\perp\rangle x| = r^{-1}|x^\perp|^2 \quad \to 0.$$

Thus $\bar{B}(\bar{X},\bar{Y})$ converges to $2\langle X,Y\rangle a_\lambda A_\lambda = 2\langle \bar{X},\bar{Y}\rangle a_\lambda A_\lambda$ when $\bar{x}$ lies in $\bar{M}_\lambda$ approaches O.

Theorem 2. *Let M be a noncompact, complete, connected C^2 submanifold of dimension $n \geq 2$ properly immersed into R^N. Suppose M satisfies the condition $P(2+\varepsilon)$ for some positive constant ε. Then the image of each end M_λ by the inversion is C^2 at the origin O.*

Proof. Since $\bar{M}_\lambda$ is C^1 at O by Theorem 1, we may express a neighborhood $\bar{U}$ of O in $\bar{M}_\lambda$ as a graph

$$(x_1,\ldots,x_n) \mapsto (x_1,\ldots,x_n,f_{n+1}(x_1,\ldots,x_n),\ldots,f_N(x_1,\ldots,x_n))$$

with $f_\alpha(0,\ldots,0) = 0$ and $\dfrac{\partial f_\alpha}{\partial x_i}(0,\ldots,0) = 0$ for all $i = 1,\ldots,n$ and $\alpha = n+1,\ldots,N$. The normal space of $\bar{M}_\lambda$ at O is spanned by $\{E_\alpha : \alpha = n+1,\ldots,N\}$, where $E_\alpha = (\xi_1,\ldots,\xi_N)$ with $\xi_\alpha = 1$ and $\xi_s = 0$ for $s \neq \alpha$. Then we have

$$\lim_{\bar{x}\to O} \frac{\partial^2 f_\alpha}{\partial x_i \partial x_j} = \left\langle \lim_{\bar{x}\to O} \bar{B}(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}),E_\alpha\right\rangle . \tag{3.2}$$

Since $\bar{B}$ is continuous at O by Lemma 3.1, (3.2) shows that all f_α's have continuous second derivatives at O. Hence $\bar{M}_\lambda$ is C^2 at O.

Corollary. *Let M be as in Theorem 2. Then M is conformally equivalent to a compact C^2 Riemannian manifold $\bar{M}$ punctured at a finite number of points.*

Remarks. (1) The origin O is an umbilic point on each $\bar{M}_\lambda$. (cf. Lemma 3.1)

(2) If $\bar{N}$ is a compact C^2 submanifold of R^N containing O, $\Psi(\bar{N})$ satisfies P(2). But if we replace the condition P$(2+\varepsilon)$ in Theorem 2 by P(2), $\Psi(M)$ is not necessarily C^2 at O.

§4. Total curvatures

We mean by a total curvature the total integral of a geometric quantity defined through the second fundamental form of the submanifold. We will define two types of total curvatures. To do this let $\nu(M)$ denote the unit normal bundle of an n-dimensional submanifold M in R^N. Let $G(\xi)$ be the Lipshitz-Killing curvature of M with respect to a unit normal vector ξ, i.e., $G(\xi) = \det A_\xi$. We denote the volume of the k-dimensional unit sphere by c_k. We define $\sigma(M)$ and $\kappa(M)$ by

$$\sigma(M) = \int_M |B|^n$$

$$\kappa(M) = \frac{1}{c_{N-1}} \int_\nu G\,.$$

Proposition 4.1. *Let M be as in Theorem 1. Then $\sigma(M) < \infty$.*

Proof. Let K_t and β_t denote the sectional curvature of and the second fundamental form of $\partial M_\lambda(t)$ as a submanifold of $S(t)$, respectively. Since, by Lemma 1.3 (3), there exists a positive continuous function $\eta(t)$ which satisfies $t|\beta_t| \leq \eta(t)$ and $\eta(t) \to 0$ as $t \to \infty$, it follows from the Gauss equation that there exists a positive constant C_7 independent of t such that $K_t \geq C_7 t^{-2}$. By the standard comparison argument, we see that $\mathrm{Vol}(\partial M(t)) \leq C_8 t^{n-1}$ for some constant C_8 independent of t. If R is sufficiently large, we have

$$\begin{aligned}
\int_{M_\lambda(R)} |B|^n &= \int_R^\infty \left(\int_{\partial M_\lambda(t)} |B|^n \right) dt \\
&\leq \int_R^\infty (K^n t^{-n(1+\varepsilon)} \mathrm{Vol}(\partial M_\lambda(t)))\, dt \\
&\leq \int_R^\infty K^n C_8 t^{-1-n\varepsilon}\, dt \\
&< \infty\,.
\end{aligned}$$

This yields $\sigma(M) < \infty$.

Remark. If M is a complete, connected, minimal submanifold of dimension $n \geq 3$ in R^N with $\sigma(M) < \infty$, then M satisfies $\mathrm{P}(n)$ (and hence $\mathrm{P}(2+\varepsilon)$). ([1]. See also [4].)

In order to apply results in [7], we imbed R^N into R^{N+1} as an N-dimensional linear subspace. Let p be a unit normal vector of R^N in

R^{N+1}. Let S^N be the unit sphere in R^{N+1}. The stereographic projection $\pi_p\colon S^N \setminus \{p\} \to R^N$ is given by

$$\pi_p(z) = p + \frac{1}{1-\langle z, p\rangle}(z-p)$$

for z in $S^N \setminus \{p\}$. The stereographic projections are related to the inversion by $\Psi = \pi_{-p} \circ \pi_p^{-1}$. If M is a submanifold of R^N as in Theorem 2, there exists a compact C^2 manifold $\widetilde{M}$ in S^N such that $\pi_p(\widetilde{M}) = M$ and $\pi_{-p}(\widetilde{M}) = \bar{M}$.

Lemma 4.1 ([7]). *If $n = \dim M$ is even, then*

$$\kappa(M) = \chi(\widetilde{M}) - 2q,$$

where $\chi(\widetilde{M})$ is the Euler characteristic of $\widetilde{M}$ and q is the number of the ends of M. If n is odd, then $\kappa(M) = 0$.

Since $\chi(\widetilde{M}) = \chi(\bar{M}) = \chi(M) + q$, we obtain the following theorem.

Theorem 3. *Let M be as in Theorem 2. If $n = \dim M$ is even, then*

$$\kappa(M) = \chi(M) - q,$$

where $\chi(M)$ is the Euler characteristic of M and q is the number of the ends of M. If n is odd, then $\kappa(M) = 0$.

Corollary. *Let M be as in Theorem 2 and $\dim M = 2$. Then*

$$\int_M K = 2\pi(\chi(M) - q),$$

where K denotes the Gaussian curvature of M.

Proof. Let $\nu(M,x)$ be the unit normal space of M at x. Then we have

$$\int_{\nu(M,x)} G = \frac{c_{N-3}}{N-2} K(x).$$

Hence

$$\begin{aligned}\kappa(M) &= \frac{1}{c_{N-1}} \int_M \left(\int_{\nu(M,x)} G \right) \\ &= \frac{1}{2\pi} \int_M K.\end{aligned}$$

Remark. White ([8]) proved that if $\sigma(M)$ is finite for an oriented surface M in R^N, then $\int_M K = 2\pi m$ for some integer m.

Remark. Another popular total curvature is the total mean curvature, which is defined by $\mu(M) = \int_M |H|^n$. Here H denotes the mean curvature vector of M. When $n = 2$, it is known that the total mean curvature is invariant under the inversion if both M and $\Psi(M)$ are compact ([3]). For a surface M satisfying the conditions in Theorem 2, one can show that $\mu(M) = \mu(\bar{M}) - 4\pi q$, where $\bar{M} = \Psi(M)$. If M is minimal and $q = 1$, one has $\mu(\bar{M}) = 4\pi$. Then a theorem by B.Y. Chen ([2]) implies that $\bar{M}$ is a round sphere. Therefore M must be totally geodesic. This is a special case of a theorem in [4], which says that if a complete minimal submanifold M properly immersed into R^N has one end and satisfies P(2) (or P$(1+\varepsilon)$ if $n \geq 3$), then M must be totally geodesic.

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