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Lefschetz Principle in the Theory of Prehomogeneous Vector Spaces

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§0. Introduction

The theory of prehomogeneous vector spaces is originated by M. Sato in 1961 in order to give a testing ground for a general theory of linear differential equations [11]. A polynomial f, such as the quadratic form or the determinant, has an invariance with respect to a very large group and it is characterized up to constant by this invariance. This distinctive property inherits to a complex power of f and its Fourier transform. Using this fact, we can show that

$$\text{Fourier transform of } f^s = f^{-s} \times \text{ (some factors).}$$

Starting from this equality, we get linear differential equations whose fundamental solutions can be written explicitly, and also we get applications for the theory of zeta functions [13], [14].

This is an outline of the Sato's original theory of prehomogeneous vector spaces. Thus we can say that the Sato's original theory is based on the invariance with respect to a very large group. Invariance with respect to a Lie group is nothing but invariance with respect to its Lie algebra, which can also be expressed as a system of linear differential equations of first order. Thus we can also say that the Sato's original theory is based on this system of differential equations. But it seems that this system of differential equations is not good. For instance, it may not be holonomic unless the prehomogeneous vector space has a finite number of orbits. This fact often imposes the so-called finite orbit condition, which does not seem to be natural or necessary in many cases. Thus in order to get a more natural theory, it seems necessary to consider linear differential equations of general order together with these first order equations.

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The purpose of this note is to give a brief account of an attempt to get such a natural theory of prehomogeneous vector spaces, based on a system of differential equations which are not necessarily of first order.

One of the advantages of our theory is that it explains to some extent the following principle which we call the Lefschetz principle. *Whatever is true for prehomogeneous vector spaces over the real, complex, p-adic or finite field is also true for the other fields.* This principle seems to hold with a surprising accuracy. See §8. What we can explain at present is, without doubt, only a small portion of it.

§1. Invariant theory

Let G be a connected reductive group over the complex number field $\mathbb{C}$, $V = \mathbb{C}^n$ and $\rho : G \to GL(V)$ an algebraic homomorphism. Let $(G_{\mathbb{Z}}, \rho_{\mathbb{Z}}, V_{\mathbb{Z}})$ be a triple of a reductive group scheme $G_{\mathbb{Z}}$ over $\mathbb{Z}$, an affine space $V_{\mathbb{Z}}$ over $\mathbb{Z}$, and a homomorphism $\rho_{\mathbb{Z}} : G_{\mathbb{Z}} \to GL(V_{\mathbb{Z}})$. We call $(G_{\mathbb{Z}}, \rho_{\mathbb{Z}}, V_{\mathbb{Z}})$ a $\mathbb{Z}$-form of (G, ρ, V) if (G, ρ, V) is obtained from $(G_{\mathbb{Z}}, \rho_{\mathbb{Z}}, V_{\mathbb{Z}})$ by tensoring $\mathbb{C}$, that is,

$$V = V_{\mathbb{Z}} \times_{\mathrm{spec}(\mathbb{Z})} \mathrm{spec}(\mathbb{C}), \quad \text{etc.}$$

(We will denote the set of rational points by $V(\mathbb{Z})$ etc. Do not confuse $V_{\mathbb{Z}}$ with $V(\mathbb{Z})$.) We know that a $\mathbb{Z}$-form always exists. We fix a $\mathbb{Z}$-form.

In this note, we understand the objects of invariant theory as follows:

Geometric case. Let $\phi : G \to GL(M)$ be a rational representation. The first object of the invariant theory is an M-valued regular function on V such that $f(gv) = \phi(g)f(v)$ $(g \in G, v \in V)$, which is called a (vector valued) relative invariant. (Here, we do not go into the geometric study of the positive characteristic case.)

Arithmetic case. Let k be any commutative ring, and $R : G(k) \to GL(M)$ a finite or infinite dimensional 'representation'. The second object of the invariant theory is an M-valued 'function' on $V(k)$ such that $f(gv) = R(g)f(v)$ $(g \in G(k), v \in V(k))$. Here we use the words 'representations' and 'functions' quite vaguely. We include hyperfunctions, D-modules, mixed Hodge modules, l-adic étale perverse sheaves, crystals and so on, in the word 'functions'.

In this note, we restrict ourselves to the case where M is one dimensional. First, we consider the geometric case, and next, we go into the arithmetic case where k is the complex, real or finite field.

§2. Prehomogeneous vector spaces (geometric case)

Let (G, ρ, V) be a triple as in §1. Such a triple is called a *prehomogeneous vector space,* or PV in short, if there is a Zariski dense G-orbit in V. Let ϕ be a character of G, and f a polynomial function on V such that $f(gv) = \phi(g)f(v)$ $(g \in G, v \in V)$. Let $V^\vee$ be the dual space of V, and $\rho^\vee$ the contragradient representation of ρ. Then $(G, \rho^\vee, V^\vee)$ is also a prehomogeneous vector space, and there exists a polynomial function $f^\vee$ on $V^\vee$ such that $f^\vee(gv^\vee) = \phi(g)^{-1}f^\vee(v^\vee)$ $(g \in G, v^\vee \in V^\vee)$. Let $\Omega = f^{-1}(\mathbb{C}^\times)$. Then there exists a unique G-orbit O_0 which is Zariski open in Ω, and a unique G-orbit O_1 which is Zariski closed in Ω. Define $\Omega^\vee$, $O_0^\vee$ and $O_1^\vee$ in the same way for the dual space.

Theorem 1 [3, 1.18]**.** (1) *Let* $F = \operatorname{grad}\log f$ *and* $F^\vee = \operatorname{grad}\log f^\vee$. *Then* $F(\Omega) = O_1^\vee$ *and* $F^\vee(\Omega^\vee) = O_1$.

(2) *The morphisms* F *and* $F^\vee$ *induce isomorphisms* $O_1 \to O_1^\vee$ *and* $O_1^\vee \to O_1$*, which are inverse of each other.*

(3) *Let* $(TO_1^\vee)^\perp$ *be the conormal bundle of* $O_1^\vee$*, i.e.,*

$$(TO_1^\vee)^\perp = \{(v, v^\vee) \in V \times V^\vee \mid v^\vee \in O_1^\vee, v \perp T_{v^\vee}O_1^\vee (\subset V^\vee)\}.$$

Then the following diagram is commutative.

$$\begin{tikzcd}
(TO_1^\vee)^\perp \arrow[rr, "\Phi"] \arrow[dr, "\text{proj.}"'] & & \Omega \arrow[dl, "F"] \\
& O_1^\vee &
\end{tikzcd}$$

where $\Phi(v, v^\vee) = v + F^\vee(v^\vee)$.

Remark. In the original theory of M. Sato, a prehomogeneous vector space is called *regular* if F is generically surjective. In the regular case, it is known that $\Omega = O_0 = O_1$ and $\Omega^\vee = O_0^\vee = O_1^\vee$. Hence $F(\Omega) = \Omega^\vee$, and Ω is a single G-orbit. These are the most important facts in the geometric part of the Sato's original theory. But even without the regularity condition, the above theorem says that Ω can be identified with a vector bundle on the single affine orbit $O_1^\vee$. Thus the above theorem enables us to do without the regularity condition.

§3. D-modules on $V(\mathbb{C})$

We use the following conventions;

$$n = \dim V = \dim V^\vee;$$
$(x_1, \ldots, x_n)$ be a coordinate of V;
$(y_1, \ldots, y_n)$ the dual coordinate of $V^\vee$;
$$D = D(V) = \mathbb{C}[x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}];$$
$$D = D(V^\vee) = \mathbb{C}[y_1, \ldots, y_n, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}].$$

The algebra isomorphism $\mathcal{F} : D(V) \to D(V^\vee)$ given by $x_i \to \sqrt{-1}\,\frac{\partial}{\partial y_i}$ and $\frac{\partial}{\partial x_i} \to \sqrt{-1}\, y_i$, formally defines the concept of *Fourier transformation* of D-modules.

Let U be a simply connected open subset of $f^{-1}(\mathbb{C}^\times)$ (with respect to the classical topology) and $f(x)^s$ a single valued branch on $\{(s,x) \in \mathbb{C} \times U\}$. Then

$$f^\vee(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) f(x)^{s+1} = b(s) f(x)^s$$

with a polynomial $b(s) \in \mathbb{C}[s]$, which is called the *b-function*. Let f^α be the natural generator of the D-module $N(\alpha) = D[s]f^s/(s-\alpha)D[s]f^s$ for a fixed complex number α. Then $N(\alpha)$ is generated by f^α; $N(\alpha) = Df^\alpha$. Note that f^α is an element of an abstract D-module.

Theorem 2 [3, 3.11]. *Let*

$$A_+ = \{\alpha \in \mathbb{C} \mid b(\alpha + j) \neq 0 \text{ for } j = 0, 1, 2, \ldots\} \quad \text{and}$$
$$A_- = \{\alpha \in \mathbb{C} \mid b(\alpha - j) \neq 0 \text{ for } j = 1, 2, \ldots\}.$$

(1) *The Fourier transform of Df^α is given by*

$$\mathcal{F}(Df^\alpha) = \begin{cases} \mathcal{F}(Df^\alpha)[f^{\vee -1}], & \text{if } \alpha \in A_+, \\ (\mathcal{F}(Df^{-\alpha})[f^{\vee -1}])^*, & \text{if } \alpha \in A_-. \end{cases}$$

Here $*$ denotes the dual D-module; $(-)^* = \mathrm{Ext}^n_D(-, D) \otimes_{\mathbb{C}} \mathbb{C}(dx_1 \wedge \cdots \wedge dx_n)$.

(2) *Let Du_α be the $D(V^\vee)$-module defined by*

$$\text{(A)} \quad \left(\sum_{i,j=1}^{n} (-a_{ji} y_j \frac{\partial}{\partial y_i}) - (\alpha\phi + \phi_0)(A) \right) u_\alpha = 0 \quad \text{for } A \in \mathrm{Lie}(G)$$

and

$$\text{(B)} \qquad a(y)u_\alpha = 0 \quad \textit{for any polynomial } a(y) \textit{ vanishing identically on } O_1^\vee.$$

Then

$$\mathcal{F}(Df^\alpha)[f^{\vee -1}] = (Du_\alpha)[f^{\vee -1}].$$

Here $\rho(A) = (a_{ij})$, $\phi_0(A) = \mathrm{trace}(\rho(A))$, *and we have written the character of* $\mathrm{Lie}(G)$ *corresponding to the character* ϕ *of* G *by the same letter.*

Remark. The relative invariance of f^α with respect to the group G can be expressed in terms of the corresponding Lie algebra. If we use a coordinate system, this invariance can be expressed as a system of linear differential equations of first order

$$\text{(C)} \qquad \left(\sum_{i,j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_i} - \alpha\phi(A) \right) v_\alpha = 0 \quad \text{for } A \in \mathrm{Lie}(G).$$

It is easy to see that (A) is the Fourier transform of (C). In the original theory of M. Sato, this fact was the keystone. But, as is explained in the introduction, the differential equation (C) is not enough, and it becomes necessary to consider equations of general order together with (C). In other words, instead of the D-module defined by (C), we need to consider its quotient. This quotient is the D-module Df^α, whose Fourier transform is determined by the above theorem to some extent.

§4. Perverse sheaves on $V(\mathbb{C})$

There is a canonical two fold covering $\pi^\vee : \tilde{O}_1^\vee \to O_1^\vee$, which is possibly disconnected. The direct image sheaf $\pi_*^\vee \mathbb{C}$ is a direct sum of the constant sheaf $\mathbb{C}$ and a certain locally constant sheaf $H^\vee$. Let $\mathcal{O}^{an}$ be the sheaf of holomorphic functions and let $\mathrm{Sol}(-) = R\underline{Hom}_D(-, \mathcal{O}^{an})$.

Theorem 3 [3, 3.23]. *Let* $j : \Omega \to V$, $j^\vee : \Omega^\vee \to V^\vee$, $i : O_1 \to \Omega$ *and* $i^\vee : O_1^\vee \to \Omega^\vee$ *be the inclusion mappings,* $n = \dim V = \dim V^\vee$ *and* $m = \dim O_1 = \dim O_1^\vee$.
(1) *The sheaf of holomorphic solutions of* Df^α *is given by*

$$\mathrm{Sol}(Df^\alpha) = \begin{cases} Rj_*(\mathbb{C}f^\alpha), & \textit{if } \alpha \in A_+ \\ j_!(\mathbb{C}f^\alpha), & \textit{if } \alpha \in A_-. \end{cases}$$

(2) *The sheaf of holomorphic solutions of* $\mathcal{F}(Df^\alpha)$ *is given by*

$$\mathrm{Sol}(\mathcal{F}(Df^\alpha)) = \begin{cases} j_!^\vee i_*^\vee(\mathbb{C}f^{\vee-\alpha}\otimes H^\vee)[m-n], & \textit{if } \alpha\in A_+, \\ Rj_*^\vee i_*^\vee(\mathbb{C}f^{\vee-\alpha}\otimes H^\vee)[m-n], & \textit{if } \alpha\in A_-. \end{cases}$$

Theorem 4. *Let* $\mathcal{F}_{\mathrm{geom}}$ *be the Fourier-Sato transformation (= the geometric Fourier transformation of Brylinski-Malgrange-Verdier)* [1], [4], [6]. *Then*

$$\mathcal{F}_{\mathrm{geom}}(j_!(\mathbb{C}f^\alpha)) = Rj_*^\vee i_*^\vee(\mathbb{C}f^{\vee-\alpha}\otimes H^\vee)[m-n], \textit{ and}$$
$$\mathcal{F}_{\mathrm{geom}}(Rj_*(\mathbb{C}f^\alpha)) = j_!^\vee i_*^\vee(\mathbb{C}f^{\vee-\alpha}\otimes H^\vee)[m-n].$$

§5. Hyperfunctions on $V(\mathbb{R})$

Since

$$R\Gamma_{V(\mathbb{R})}R\underline{Hom}_D(-,\mathcal{O}^{an}) = R\underline{Hom}_D(-,R\Gamma_{V(\mathbb{R})}(\mathcal{O}^{an})),$$

by applying $R\Gamma_{V(\mathbb{R})}$ to the first part of Theorem 3, we get the following result.

Theorem 5. *If* $\alpha\in A_+$, *every hyperfunction solution of* Df^α *on* $\Omega(\mathbb{R})$ *can be uniquely extended to a solution on* $V(\mathbb{R})$.

Since the hyperfunction solutions of Df^α on $\Omega(\mathbb{R})$ can be easily determined, the above theorem determines the hyperfunction solutions of Df^α on the whole space $V(\mathbb{R})$. In the same way, applying $R\Gamma_{V^\vee(\mathbb{R})}$ to the second part of Theorem 3, we can also determine the hyperfunction solutions of $\mathcal{F}(Df^\alpha)$ on $V^\vee(\mathbb{R})$ if $\alpha\in A_-$. Comparing these two results, we can calculate the Fourier transforms of the hyperfunction solutions of Df^α. See [3, §4] for the details.

§6. l-Adic étale perverse sheaves on $V(\overline{\mathbb{F}_q})$

In this section, we denote by $\mathbb{F}_q$ the finite field with q elements and assume that the characteristic p of the finite field $\mathbb{F}_q$ is sufficiently large. In order to consider 'reduction modulo p', we assume that $f\in\mathbb{Z}[V]$ and $f^\vee\in\mathbb{Z}[V^\vee]$.

Let l be a prime number different from p, $\chi\in\mathrm{Hom}(\mathbb{F}_q^\times,\overline{\mathbb{Q}_l}^\times)$, and $\overline{\mathbb{Q}_l}$ denotes an algebraic closure of the l-adic number field. Let $\chi(0)=0$ by convention. Let $L(\chi)$ be the corresponding Lang torsor [2,p.171] on $\overline{\mathbb{F}_q}^\times$, where $\overline{\mathbb{F}_q}$ is an algebraic closure of $\mathbb{F}_q$. Note that $L(\chi)$ is also a Kummer torsor, and hence is obtained from the corresponding Kummer torsor over $\mathbb{C}^\times$ by 'reduction modulo p'.

Theorem 6. *Let $\psi \in \mathrm{Hom}(\mathbb{F}_q, \overline{\mathbb{Q}_l}^{\times}) - \{1\}$ and $\mathcal{F}_\psi$ be the geometric Fourier transformation of Deligne* [7]. *Then*

$$\mathcal{F}_\psi(j_! f^* L(\chi)) = Rj^\vee_* i^\vee_* (f^{\vee *} L(\chi^{-1}) \otimes H^\vee)[m-n], \quad \textit{and}$$
$$\mathcal{F}_\psi(Rj_* f^* L(\chi)) = j^\vee_! i^\vee_* (f^{\vee *} L(\chi^{-1}) \otimes H^\vee)[m-n].$$

Sketch of the proof. In principle, we want to obtain Theorem 6 from Theorem 4 by 'reduction modulo p'. But, since the definition of $\mathcal{F}_{\mathrm{geom}}$ involves the half-space, we can not consider its 'reduction modulo p'. Also, since the definition of $\mathcal{F}_\psi$ involves the Artin-Schreier sheaf, we can not obtain it as a result of 'reduction modulo p'. We avoid these difficulties as follows. It is known that f is a homogeneous polynomial. Let $d = \deg f$. Then

$$\begin{aligned}
&\Big(\sum_{x \in \mathbb{F}_q^\times} \chi(x^{-d})\psi(-x)\Big)\Big(\sum_{v \in V(\mathbb{F}_q)} \chi(f(v))\psi(\langle v^\vee, v\rangle)\Big)\\
&= \sum_{x \in \mathbb{F}_q^\times, v} \chi(f(x^{-1}v))\psi(\langle v^\vee, v\rangle - x)\\
&= \sum_{x \in \mathbb{F}_q^\times, v} \chi(f(v))\psi(x(\langle v^\vee, v\rangle - 1))\\
&= \sum_{x \in \mathbb{F}_q, v} \chi(f(v))\psi(x(\langle v^\vee, v\rangle - 1)) - \sum_v \chi(f(v))\\
&= q \sum_{v, \langle v^\vee, v\rangle = 1} \chi(f(v)) - \sum_v \chi(f(v)).
\end{aligned}$$

Thus we can eliminate the additive character ψ from the Fourier transform of $\chi(f(v))$ by multiplying the classical Gauss sum, whose property is well understood. Imitating the above calculation, we can eliminate the half space and the Artin-Schreier sheaf from $\mathcal{F}_{\mathrm{geom}}(j_!(\mathbb{C}f^\alpha))$ and $\mathcal{F}_\psi(j_! f^* L(\chi))$, respectively. Hence we can move from $\mathbb{C}$ to $\overline{\mathbb{F}_q}$ by 'reduction modulo p'.

§7. $\mathbb{C}$-valued functions on $V(\mathbb{F}_q)$

7.1. Applying the trace formula of Grothendieck [2] to Theorem 6, we get the corresponding result concerning $\mathbb{C}$-valued functions on $V(\mathbb{F}_q)$.

7.2. Although we have started the arithmetic study from D-modules, it is also possible to start it from mixed Hodge modules of M. Saito [9], [10]. We can consider the weight filtrations (one half of the mixed Hodge structure) in each step (except §5), and at this final step, we get a deeper information, i.e., the so-called Weil estimation.

Before giving further explanation of these results, let us present a conjecture.

7.3. We know that the b-function is of the form

$$b(s) = b_0 \prod_{j=1}^{d} (s + \alpha_j)$$

with positive rational numbers α_j. See §3 for $b(s)$. Consider the polynomial

$$b^{\exp}(t) = \prod_{j=1}^{d} (t - \exp(2\pi\sqrt{-1}\,\alpha_j)).$$

We can show that this polynomial is the minimal polynomial of the monodromy of the vanishing cycle sheaf $R\psi_f(\mathbb{C}) = R\psi_f(\mathbb{Q}) \otimes \mathbb{C}$. Hence $b^{\exp}(t) \in \mathbb{Q}[t]$. On the other hand, $\exp(2\pi\sqrt{-1}\,\alpha_j)$ are roots of unity. Hence $b^{\exp}(t)$ is a product of cyclotomic polynomials, and is written as

$$b^{\exp}(t) = \prod_{j \geq 1} (t^j - 1)^{e(j)}$$

with some integers $e(j)$, which are possibly negative. In order to state our conjecture, we need more notations.

$$r = \operatorname{card}\{j \mid \alpha_j \in \mathbb{Z}\} = \sum_{j \geq 1} e(j).$$

$r(-) =$ rank $=$ dimension of a maximal torus.

$s(-) =$ split rank $=$ dimension of a maximal split torus.

$Z_G(v^\vee) =$ isotropy group at $v^\vee \in V^\vee(\mathbb{F}_q)$.

$r(v^\vee) = r(G) - r(Z_G(v^\vee))$.

$s(v^\vee) = s(G) - s(Z_G(v^\vee))$.

$\eta(v^\vee) = (-1)^{r(v^\vee) - s(v^\vee)}$.

$$G(\chi, \psi) = \sum_{x \in \mathbb{F}_q^\times} \chi(x)\psi(x).$$

Conjecture A. *If the characteristic of* $\mathbb{F}_q$ *is sufficiently large,*

$$(*)\qquad \begin{aligned} &q^{-\frac{1}{2}\dim V} \sum_{v\in V(\mathbb{F}_q)} \chi(f(v))\psi(\langle v^\vee, v\rangle) \\ &= q^{\frac{1}{2}(\dim V^\vee - \dim O_1^\vee)} \prod_{j\geq 1}\left(\frac{G(\chi^j,\psi)}{\sqrt{q}}\right)^{e(j)} \cdot \\ &\cdot \chi\left(\frac{b_0 f^\vee(v^\vee)^{-1}}{\prod_{j\geq 1}(j^j)^{e(j)}}\right)\cdot \eta(v^\vee), \end{aligned}$$

for $v^\vee \in O_1^\vee(\mathbb{F}_q)$.

Remark 7.4. Our first result (cf. 7.1) says that

$$\text{left side of } (*) = \text{right side of } (*) \times C$$

with some constant C. At the same time, we can also show that the left side of (*) vanishes if $v^\vee \notin (\overline{O_1^\vee})(\mathbb{F}_q)$. The Weil estimation (cf. 7.2) says that the absolute value of this constant C at every archimedean place is equal to one.

7.5. Assume that the characteristic of $\mathbb{F}_q$ is not 2, and let $\chi_{1/2}$ be the unique non-trivial character of $\mathbb{F}_q^\times$ of order 2, i.e., the Legendre symbol. For $v^\vee \in V^\vee(\mathbb{F}_q)$, let $h^\vee(v^\vee)$ be the discriminant of the quadratic form Q determined by $\left(\frac{\partial^2 \log f^\vee}{\partial y_i \partial y_j}(v^\vee)\right)$, i.e., the discriminant of the quadratic form on $V^\vee(\mathbb{F}_q)/(\text{radical of } Q)$ induced by Q. Since $h^\vee(v^\vee)$ is an element of $\mathbb{F}_q^\times/\mathbb{F}_q^{\times 2}$, $\chi_{1/2}(h^\vee(v^\vee))$ is well-defined.

Lemma 7.6.

$$m \equiv r \mod 2,$$

where $m = \dim O_1 = \dim O_1^\vee$ *Especially,* $\chi_{1/2}((-1)^{\frac{1}{2}(m+r)})$ *is well-defined.*

Proof. Applying the calculation in the 'proof' of Theorem 6 to the case where χ is the trivial character χ_0 of $\mathbb{F}_q^\times$, we get

$$\sum_{v\in V(\mathbb{F}_q)} \chi_0(f(v))\psi(\langle v^\vee, v\rangle) \in \mathbb{Z}.$$

On the other hand,

$$\left| q^{-\frac{n}{2}} \sum_{v\in V(\mathbb{F}_q)} \chi_0(f(v))\psi(\langle v^\vee, v\rangle)\right| = q^{\frac{1}{2}(n-m)-\frac{r}{2}}$$

(cf. 7.4). Hence $q^{\frac{1}{2}(m-r)} \in \mathbb{Z}$ and $m \equiv r \mod 2$.

Conjecture B. *If the characteristic of $\mathbb{F}_q$ is sufficiently large,*

$$(**) \qquad \eta(v^\vee) = \chi_{1/2}((-1)^{\frac{1}{2}(m+r)} \prod_{j \geq 1} j^{e(j)} \cdot h^\vee(v^\vee))$$

for $v^\vee \in O_1^\vee(\mathbb{F}_q)$.

Remark 7.7. We can show that

$$\text{left side of } (**) = \text{right side of } (**) \times \epsilon$$

with $\epsilon = \pm 1$, independent of $v^\vee$.

Remark 7.8. The above conjectures are
(1) compatible with the castling transformations.
Actually, the author has come to these conjectures by studying quantities which are invariant under the castling transformation. (See [12] for the definition of the castling transformation.) The author hopes to discuss such quantities elsewhere. The above conjectures are also
(2) compatible with the transformation of the form $f \to cf^k$, $f^\vee \to c^\vee f^{\vee k}$ with $c, c^\vee \in \mathbb{Z}$ and a natural number k,
(3) supported by the theory of generalized Gelfand-Graev representations of finite reductive groups due to N. Kawanaka [8], and
(4) supported by many examples.

§8. Dictionary

Assume the regularity condition for the prehomogeneous vector space (G, ρ, V) (cf. remark in §2). From the conjectures A and B, it follows that

$$q^{-\frac{n}{2}} \sum_{x \in V(\mathbb{F}_q)} \chi(f(x))\psi(\langle x, y\rangle)$$

$$(8.1) \qquad = \left(\prod_{j \geq 1} \left(\frac{G(\chi^j)}{\sqrt{q}} \right)^{e(j)} \right) \cdot \chi \left(\frac{b_0 f^\vee(y)^{-1}}{\prod_{j \geq 1} (j^j)^{e(j)}} \right) \cdot$$

$$\cdot \chi_{1/2} \left((-1)^{\frac{1}{2}(n+r)} \prod_{j \geq 1} j^{e(j)} \cdot h^\vee(y) \right).$$

In this section, we shall give a heuristic argument which explains this equality as an analogy of the following theorem.

Theorem (M. Sato-T. Shintani [11], J. Igusa [5]). *Assume the regularity condition. Then*

$$(8.2)\qquad \begin{aligned}(2\pi)^{-\frac{n}{2}}\int_{V(\mathbb{C})} f(x)^s e^{i\langle x,y\rangle}dx_1\dots dx_n &= \left(\prod_{j=1}^{d}\frac{\Gamma(s+\alpha_j)}{\sqrt{2\pi}}\right)\cdot\\ &\cdot\left(b_0 f^\vee\left(\frac{y}{i}\right)^{-1}\right)^s i^{\frac{1}{2}(n-r)}\left((-1)^{\frac{1}{2}(n+r)}h^\vee\left(\frac{y}{i}\right)\right)^{\frac{1}{2}}.\end{aligned}$$

A part of this expression is justified by (7.6). For the full justification, see [11].

If we rely on the Lefschetz principle, it would be able to translate this theorem into a statement in the finite field case. Let us give a dictionary.

$\mathbb{C}$	$\mathbb{F}_q$
$t\to e^{-t}$	$\psi\in\mathrm{Hom}(\mathbb{F}_q,\mathbb{C}^\times)-\{1\}$
$s:t\to t^c$	$\chi\in\mathrm{Hom}(\mathbb{F}_q^\times,\mathbb{C}^\times)\simeq\left(\frac{1}{q-1}\mathbb{Z}\right)/\mathbb{Z}$
js	χ^j
$\frac{1}{2}$	$\chi_{1/2}$ Legendre symbol
$\Gamma(s)=\int_0^\infty t^s e^{-t}\frac{dt}{t}$	$G(\chi)=\sum_{t\in\mathbb{F}_q^\times}\chi(t)\psi(t)$
$2\pi i$	q

Since the multiplicative characters of $\mathbb{F}_q^\times$ are parametrized by a subset of $\mathbb{Q}$ modulo $\mathbb{Z}$, s and $s+1$ should correspond to the same character of $\mathbb{F}_q^\times$. Hence we shall consider s and $s+1$ equivalent and denote $s\sim s+1$.

Now let us start with the translation of the first factor of the right hand side of (8.2). Since

$$\prod_{j=1}^{d}(t-e^{2\pi i\alpha_j})=\prod_{j\geq 1}\left(\prod_{k=0}^{j-1}(t-e^{2\pi i\frac{k}{j}})\right)^{e(j)},$$

and $s \sim s+1$, we have

$$\prod_{j=1}^{d} \frac{\Gamma(s+\alpha_j)}{\sqrt{2\pi}}$$

$$\sim \prod_{j\geq 1} \left(\prod_{k=0}^{j-1} \frac{\Gamma(s+\frac{k}{j})}{\sqrt{2\pi}} \right)^{e(j)}$$

$$= \prod_{j\geq 1} \left(\frac{\Gamma(js)}{\sqrt{2\pi i}} \cdot i^{\frac{1}{2}} \cdot \frac{1}{(j^j)^s} \cdot j^{\frac{1}{2}} \right)^{e(j)}$$

$$= \left(\prod_{j\geq 1} \left(\frac{\Gamma(js)}{\sqrt{2\pi i}} \right)^{e(j)} \right) i^{\frac{r}{2}} \left(\frac{1}{\prod_j (j^j)^{e(j)}} \right)^s \left(\prod_j j^{e(j)} \right)^{\frac{1}{2}}.$$

Hence by (8.2),

$$(2\pi i)^{-\frac{n}{2}} \int_{V(\mathbb{C})} f(x)^s e^{-\langle x,y\rangle} dx_1 \dots dx_n$$

$$\sim \left(\prod_{j\geq 1} \left(\frac{\Gamma(js)}{\sqrt{2\pi i}} \right)^{e(j)} \right) \cdot$$

$$\cdot \left(\frac{b_0 f^\vee(y)^{-1}}{\prod_{j\geq 1} (j^j)^{e(j)}} \right)^s \cdot \left((-1)^{\frac{1}{2}(n+r)} \prod_{j\geq 1} j^{e(j)} \cdot h^\vee(y) \right)^{\frac{1}{2}}.$$

Thus, using the above dictionary, we can translate (8.2) into the identity (8.1).

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