Advanced Studies in Pure Mathematics 21, 1992
Zeta Functions in Geometry
pp. 71–86

The Relation between the η-Invariant and the Spin Representation in Terms of the Selberg Zeta Function

Masato Wakayama

§1. Introduction

Roughly speaking, we will show that the "ratio"

$$\frac{\det(\text{Cayley transform of } \sqrt{\Delta})}{\text{SZF attached to the difference of spin representations}}$$

is essentially equal to the exponential of η-invariant. Of course, we mean "det" the functional determinant. So we need the discussion on the regularization of these kind of determinants. This invariant was introduced by Atiyah-Patodi-Singer and indices the spectral asymmetry of (an odd dimensional) Riemannian manifold. Namely, at least formally (or symbolically),

$$\begin{aligned}
&\eta\text{-invariant}\\
&=\text{the internal index (signature) of an infinite quadratic form}\\
&\quad \overline{{}^t d\mathbb{X}}\sqrt{\Delta}d\mathbb{X} \text{ on } d\Omega^{(\text{dimension } -1)/2}\\
&=\sharp\{\text{positive eigenvalues of } \sqrt{\Delta}\} - \sharp\{\text{negative eigenvalues of } \sqrt{\Delta}\}.
\end{aligned}$$

By the way, as to the result on the SZF related to the spin representations on the compact quotient of the hyperbolic space of dimension $4n-1$, there is a work by Milson. For computing the η-invariant in terms of the SZF, he found the intermediate formula (Selberg trace formula for "odd" type). Our result mentioned above is the one for the Milson's type SZF. (But we also consider the objects which are associated with the finite dimensional unitary representations of the fundamental group.) However this type of zeta function is actually defined by the "difference" of the two spin representations that decompose the action of $SO(2\rho)$ on

Received December 9, 1990.

$\wedge^p\mathbb{C}^{2p}$. On the other hand, we can naturally define the SZF as usual, (in our situation, this means that the "sum" of two spin representations). Using these kind of SZF, we can separate the spin representations at the spectrum level and hence pick up and define a new SZF exclusively attached to only one spin representation. That is to say, we are able to show that

$$\text{the "Characteristic polynomial" of } \sqrt{\Delta} \equiv \text{a new SZF.}$$

§2. The root of the Hodge Laplacian and the η-invariant

Let $X = \Gamma\backslash H = \Gamma\backslash G/K = \Gamma\backslash SO_0(2n-1,1)/SO(2n-1)$ be a compact oriented hyperbolic manifold of odd dimension $\ell = 2n-1$, Γ its fundamental group. Let $G = KA_{\mathfrak{p}}N$ be the Iwasawa decomposition of G. The centralizer M of $A_{\mathfrak{p}}$ in K is isomorphic to $SO(\ell-1)$. Let χ be an unitary representation of Γ on $\mathbb{C}^m$. Let $\Omega^q = \Omega^q_\chi$ be the space of all χ-twisted q-forms on H (that is, $\gamma^*\omega = \chi(\gamma)\omega$, values in $\mathbb{C}^m$). In other words, we consider the flat unitary bundle associated with χ. Consider the operator A^e (resp. A^o) on (χ-twisted) even (resp. odd) forms on H, defined on Ω^{2p} (resp. Ω^{2p-1}) by the formula

$$A^e = i^n(-1)^{p+1}(*d - d*) \curvearrowright \Omega^{\text{even}} = \oplus_{p=0}^{n-1}\Omega^{2p}, \tag{2-1}$$

$$A^o = i^n(-1)^{np}((-1)^p * d + d*) \curvearrowright \Omega^{\text{odd}} = \oplus_{p=1}^{n}\Omega^{2p-1}, \tag{2-2}$$

where d is the exterior differential and $*$ is the Hodge duality operator defined by the metric.

Let A denote A^e (resp. A^o). Anyway, it is easy to see that the operator A is formally self-adjoint, elliptic and square A^2 is the Hodge Laplacian $\Delta = d\delta + \delta d$, where δ stands for the formal adjoint of d. Hence A is diagonalizable with real eigenvalues and the eigenvalues of A are square roots of those of Δ. It follows that they can be either positive or negative. By the way, Atiyah-Patodi-Singer introduced the so-called "eta-function"

$$\eta(\chi : s) = \eta_A(\chi : s) = \sum_{\lambda \neq 0} (\text{sign}\,\lambda)|\lambda|^{-s}, \tag{2-3}$$

where the summation is taken over all eigenvalues of A counting with multiplicity. It is easy to verify that

$$\begin{aligned} \eta(\chi : s) &= \mathrm{Tr}(A(\Delta)^{-(s+1)/2}) \\ &= \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s+1)/2}\, \mathrm{Tr}(Ae^{-t\Delta}) \frac{dt}{t} \end{aligned} \tag{2-4}$$

for $\mathrm{Re}\, s \gg 0$. By means of this formula, they proved that $\eta(\chi : s)$ has a meromorphic continuation to the entire complex plane and does not have a pole at zero. From this it follows that $\eta(\chi) = \eta(\chi : 0)$ is well-defined and is called the η-invariant.

§3. Formal definition of infinite determinants

For simplicity, we assume that $\ell = \dim X = 4n-1$ (that is, $G = SO_0(4n-1, 1)$) and put $\rho = \frac{\ell-1}{2} = 2n-1$. In this case, note that

$$*d : \Omega^{2n-1} \to \Omega^{2n-1}, \qquad d* : \Omega^{2n} \to \Omega^{2n}.$$

Define two self-adjoint square roots of Δ as follows :

$$\begin{aligned} B^o &= A^o|_{\text{the space of coclosed forms in } \Omega^{2n-1}} \\ &= *d|_{\text{the space of coclosed forms in } \Omega^{2n-1}}, \\ B^e &= A^e|_{\text{the space of closed forms in } \Omega^{2n}} \\ &= d*\,|_{\text{the space of closed forms in } \Omega^{2n}}. \end{aligned} \tag{3-1}$$

Since the Hodge operator $*$ gives the isomorphism

$$* : \Omega^{2n-1} \xrightarrow{\sim} \Omega^{2n},$$

it is easy to see that Spec B^o = Spec B^e. So we concentrate on the operator B^o. We prepare several notations. Let σ_q be the standard action of $M = SO(\ell-1) = SO(4n-2)$ on $\wedge^q \mathbb{C}^{\ell-1}$. Then σ_q is irreducible except when $q = \rho$ in which case it decomposes as the sum of the two spin representations σ^+, σ^-. Let $\cdots < -\lambda_{j+1} < -\lambda_j < \cdots < -\lambda_1 < \lambda_0 = 0 < \lambda_1 < \cdots < \lambda_k < \lambda_{k+1} < \cdots$ be the (possible) eigenvalues of B^o. We denote by $N(\sigma^+, \lambda_j)$ (resp. $N(\sigma^-, \lambda_j)$) the multiplicity with which the principal series representation $\pi_{\sigma^+, \lambda_j}$ (resp. $\pi_{\sigma^-, \lambda_j}$) induced from the representation $\sigma^+ \otimes \lambda_j \otimes 1$ (resp. $\sigma^- \otimes \lambda_j \otimes 1$) of the minimal parabolic subgroup $MA_{\mathfrak{p}}N$ occurs in $L^2(\Gamma \backslash G, \chi)$, the L^2-sections of the vector bundle associated with χ on $\Gamma\backslash G$. Note that $N(\sigma^{\pm}, \lambda_j)$ may be equal to 0. We put $m_j^{\pm} = N(\sigma^+, \lambda_j) \pm N(\sigma^-, \lambda_j)$.

Put $\lambda_{-j} = -\lambda_j$. Since $\pi_{\sigma^+, \lambda_j} \simeq \pi_{\sigma^-, -\lambda_j}$ (equivalent) as a unitary representation of G, it is clear that $N(\sigma^+, \pm\lambda_j) = N(\sigma^-, \mp\lambda_j)$. By $N(\chi : \pm\lambda_j)$ we denote this common value. Then the number $N(\chi : \lambda_j)\,(j \in \mathbb{Z})$ is nothing but the multiplicity of the eigenvalue λ_j of B^o. Furthermore, let b_q denotes the q-th (χ-twisted) Betti number, that is the multiplicity of the zero eigenvalue of the restriction of Δ to Ω^q.

Then, in particular, $b_\rho = N(\chi : 0)$. For simplicity, throughout the paper set

$$\Delta^o = \Delta|_{\text{the space of coclosed forms in } \Omega^{2n-1}}.$$

Under these notation, we will define three kinds of infinite determinants which are related to the operator B^o. At least formally, it is quite natural that they are given by the following way :

(The characteristic polynomial of Δ^o)

$$\begin{aligned} \det((B^o)^2 + s^2) &= \prod_{\lambda_j \neq 0} (\lambda_j^2 + s^2)^{N(\chi:\lambda_j)} s^{2b_\rho} \\ &= \prod_{\lambda_j > 0} (\lambda_j^2 + s^2)^{m_j^+} s^{2b_\rho}. \end{aligned} \tag{3-2}$$

(The determinant of the Cayley transform of $\sqrt{\Delta^o}/s$)

$$\begin{aligned} \det\left(\frac{B^o - is}{B^o + is}\right) &= \prod_{\lambda_j \neq 0} \left(\frac{\lambda_j - is}{\lambda_j + is}\right)^{N(\chi:\lambda_j)} (-1)^{b_\rho} \\ &= \prod_{\lambda_j > 0} \left(\frac{\lambda_j - is}{\lambda_j + is}\right)^{m_j^-} (-1)^{b_\rho}, \end{aligned} \tag{3-3}$$

(The characteristic polynomial of $\sqrt{\Delta^o}$)

$$\det(B^o - is) = \prod_{\lambda_j \neq 0} (\lambda_j - is)^{N(\chi:\lambda_j)} (-is)^{b_\rho}. \tag{3-4}$$

§4. Definition of functional determinant

In this section, we devote to regularize various (functional) determinant. First of all, we consider the characteristic polynomial of Δ^o by means of the usual zeta function method (e.g. [DP], [Sar], [V], [D2]). We denote by $\operatorname{Tr} e^{-t\Delta^o}$ the trace of the heat kernel to Δ^o. As is well known from differential equations (see e.g. [G]), the asymptotic law at 0 is given by

$$\operatorname{Tr}' e^{-t\Delta^o} e^{t\lambda_1^2} \sim \sum_{n=0}^{\infty} a_n t^{n-\frac{\ell}{2}} \quad t \downarrow 0. \tag{4-1}$$

where we set $\operatorname{Tr}' e^{-t\Delta^o} = \operatorname{Tr} e^{-t\Delta^o} - b_\rho$. Also we put

$$H(z, w) = \sum_{\lambda_j > 0} \frac{m_j^+}{(\lambda_j^2 + z)^w}.$$

It is well known that the above series converges absolutely and uniformly on compact sets in $\mathrm{Re}\, w > \frac{\ell}{2}$. Therefore in this half plane we see that

$$H(z,w) = \frac{1}{\Gamma(w)} \int_0^\infty \mathrm{Tr}'\, e^{-t\Delta^o} e^{-zt} t^w \frac{dt}{t}. \tag{4-2}$$

Lemma 1. *Suppose that* $\mathrm{Re}\, s^2 > -\lambda_1^2$. *Then* $H(s^2,w)$ *can be meromorphically continued to any half plane* $\mathrm{Re}\, w > -w_0$ $(w_0 \in \mathbb{R})$ *and regular at the integer points which are greater than* $-w_0$. *In particular,* $\frac{\partial H}{\partial w}(s^2,0)$ *is smooth in* s *in this range.*

Proof. We may assume that $w_0 \in \mathbb{Z}$ and $w_0 \geq -\frac{\ell+1}{2}$. The asymptotic expansion enables us to write

$$H(z,w) = H_1(z,w) + H_2(z,w) + H_3(z,w),$$

where we put

$$\begin{aligned}
H_1(z,w) &= \frac{1}{\Gamma(w)} \int_0^1 [\mathrm{Tr}'\, e^{-t\Delta^o} e^{t\lambda_1^2} - \theta_{w_0}(t)] e^{-t(z+\lambda_1^2)} t^w \frac{dt}{t} \\
H_2(z,w) &= \frac{1}{\Gamma(w)} \int_0^1 \theta_{w_0}(t) e^{-t(z+\lambda_1^2)} t^w \frac{dt}{t} \\
H_3(z,w) &= \frac{1}{\Gamma(w)} \int_1^\infty \mathrm{Tr}'\, e^{-t\Delta^o} e^{-zt} t^w \frac{dt}{t},
\end{aligned}$$

Here we put

$$\theta_{w_0}(t) = \sum_{n=0}^{\frac{\ell+1}{2}+w_0} \alpha_n t^{n-\frac{\ell}{2}}.$$

For $t \geq 1$, the elementary estimate shows that

$$(0 <)\, \mathrm{Tr}'\, e^{-t\Delta^o} \leq c e^{-t\lambda_1^2}$$

for some constant c. Therefore, since $\mathrm{Re}\, s^2 > 0$, it is obvious to see that $H_3(s^2,w)$ is analytic in w. It is also clear that the same assertion holds for $\frac{\partial H_1}{\partial w}(s^2,w)$. As for the term $H_2(s^2,w)$, a little manipulation shows that

$$\begin{aligned}
H_1(s^2,w) = \sum_{n=0}^{\frac{\ell+1}{2}+w_0} & \frac{\alpha_n}{\Gamma(w)(s^2+\lambda_1^2)^{(2w-\ell+2n)/2}} [\Gamma(w - \frac{\ell}{2} + n) \\
& - \int_{s^2+\lambda_1^2}^\infty e^{-y} y^{(2w-\ell+2n)/2} \frac{dy}{y}].
\end{aligned}$$

Since ℓ is odd, this completes the proof of the lemma.

Formally,

$$\frac{\partial H}{\partial w}(s^2,0) = -\sum_{\lambda_j>0} m_j^+ \log(\lambda_j^2+s^2),$$

so, by means of Lemma 1, we define

$$\det(\Delta^o+s^2) = \exp[-\frac{\partial H}{\partial w}(s^2,0)]s^{2b_\rho}. \tag{4-3}$$

Next, we shall define a functional determinant of the Cayley transform of $\sqrt{\Delta^o}/s$. At the first place we have the following.

Lemma 2. *Put*

$$I(s) = \int_0^\infty e^{-ts^2}\,\mathrm{Tr}(B^o e^{-t\Delta^o})dt.$$

Then the function $I(s)$ is analytic in the half plane $\mathrm{Re}\, s^2 > -\lambda_1^2$ and can be meromorphically continued to the whole s plane. The poles of $I(s)$ occur at the points $\pm i\lambda_j\ (j=1,2,\cdots)$. These poles are all simple and the residues at $\pm i\lambda_j$ are $\pm\frac{m_j^-}{2i}$, respectively.

Proof. It is known [BF] that

$$\mathrm{Tr}(B^o e^{-t\Delta^o}) = O(t^{1/2}) \quad t\downarrow 0.$$

On the other hand, for fixed $t_0>0$, it is easy to see that there is a constant c such that the following estimate holds:

$$|\,\mathrm{Tr}(B^o e^{-t\Delta^o})| \le ce^{-t\lambda_1^2} \quad (t\ge t_0).$$

Hence the function $I(s)$ is analytic for $\mathrm{Re}\, s^2 > -\lambda_1^2$. Using again both estimates and Fubini's theorem we see that

$$I(s) = \lim_{\varepsilon\downarrow 0}\mathrm{Tr}\int_\varepsilon^\infty B^o e^{-t(\Delta^o+s^2)}dt = \lim_{\varepsilon\downarrow 0}\mathrm{Tr}(\frac{B^o}{\Delta^o+s^2}e^{-\varepsilon(\Delta^o+s^2)}). \tag{4-4}$$

It should be noted that, for each $\varepsilon>0$, $\mathrm{Tr}(\frac{B^o}{\Delta^o+s^2}e^{-\varepsilon(\Delta^o+s^2)})$ is a meromorphic function with simple poles at $\pm i\lambda_j\ (j=1,2,\cdots)$ and residue of it at the point $\pm i\lambda_j$ are $\pm\frac{m_j^-}{2i}$, respectively. Hence (4-4) implies the desired results.

Formally we have

$$I(s)=\sum_{\lambda_j>0}\frac{\lambda_j m_j^-}{\lambda_j^2+s^2}=\frac{1}{2i}\sum_{\lambda_j>0}m_j^-\left(\frac{i}{\lambda_j+is}+\frac{i}{\lambda_j-is}\right).$$

Therefore we may define $\det(\frac{B^o-is}{B^o+is})$ as the meromorphic function which is uniquely determined by the following way:

(1) $\frac{d}{ds}\log\det(\frac{B^o-is}{B^o+is})=\frac{2}{i}I(s)$,
(2) $\det(\frac{B^o-is}{B^o+is})|_{s=0}=(-1)^{b_\rho}$.

Here log stands for the principal branch of the logarithmic function. Since $I(s)$ is even, by the definition it is clear that

$$\det\left(\frac{B^o-is}{B^o+is}\right)\det\left(\frac{B^o+is}{B^o-is}\right)=1. \tag{4-5}$$

Lastly we want to define $\det(B^o-is)$. Let $d(s)$ be the unique function defined for $\operatorname{Re}s^2>-\lambda_1^2$ such that

(1) $d(s)^2=\det((B^o)^2+s^2)s^{-2b_\rho}\det(\frac{B^o-is}{B^o+is})(-1)^{b_\rho}$,
(2) $d(0)=\exp[-\frac{1}{2}\frac{\partial H}{\partial w}(0,0)]$.

Now we put

$$\det(B^o-is)=d(s)(is)^{b_\rho}. \tag{4-6}$$

Eventually, since $(\mathrm{Spec}B^o)^2=\mathrm{Spec}\,\Delta^o$, under these normalization, it is obvious to see that

$$\det((B^o)^2+s^2)\det\left(\frac{B^o-is}{B^o+is}\right)^{\pm1}=(\det(B^o\mp is))^2. \tag{4-7}$$

§5. Formulas for the determinants

The closed geodesics γ are in a one to one correspondence to the nontrivial conjugacy classes in Γ. So, we can write $\mathrm{Tr}\,\chi(\gamma)$ for a geodesic γ. Further we write m_γ for the holonomy map of the parallel displacement along γ. This m_γ also may be considered as an element of M. We denote by $\ell(\gamma)$ the length of γ and γ_p the primitive closed geodesic underlying γ. More precisely, since all elements $\gamma\in\Gamma$ are semisimple and Γ has no elements of finite order, it follows that every element $\gamma\in\Gamma$

is conjugate in $G = SO_0(4n-1,1)$ to an element of the Cartan subgroup $A = A_{\mathfrak{k}}A_{\mathfrak{p}}$ $(A_{\mathfrak{k}} \subset M = SO(4n-2))$. Choose an element $h(\gamma)$ of A to which γ is conjugate. Then we can write $h(\gamma) = m_\gamma a(\gamma)$ $(a(\gamma) \in A_{\mathfrak{p}})$. We further demand that $h(\gamma)$ be chosen so that $a(\gamma)$ lies in $A_{\mathfrak{p}}^+ = \exp \mathfrak{a}_{\mathfrak{p}}^+$, the positive Weyl chamber in $A_{\mathfrak{p}}$. Of course, we see that $\ell(\gamma) = \alpha(\log a(\gamma))$, where α stands for the unique positive restricted root with respect to $\mathfrak{a}_{\mathfrak{p}}$. For any linear form λ on $\mathfrak{a}_{\mathfrak{p},\mathbb{C}}$, let ξ_λ denote the character of the Cartan subgroup A defined by $\xi_\lambda(h) = \exp \lambda(\log h)$ $(h \in A)$. Let P_+ be the subset of Δ^+, the set of positive roots relative to the pair $(G,\, A)$, so that the restriction to $\mathfrak{a}_{\mathfrak{p}}$ are all α. We now enumerate the roots in P_+ as $\alpha_1, \dots, \alpha_{2n-1}$.

Now we define two Selberg zeta functions which are exclusively attached to only one spin representation $\sigma^\pm$ of M, respectively, by the following Euler product:

$$z(\chi : \sigma^\pm : s) = \prod_{\gamma_p} \prod_{\lambda \in \Lambda} \det(\mathrm{I} - \chi(\gamma_p)\,\mathrm{Tr}(\sigma^\pm(m_{\gamma_p})^{-1})\xi_\lambda(h(\gamma_p))^{-1} e^{-s\ell(\gamma_p)})$$

where Λ denotes the semi-lattice defined by

$$\Lambda = \{\sum_{i=1}^{2n-1} m_i\alpha_i\,;\, m_i \geq 0,\, m_i \in \mathbb{Z}\}.$$

It should be noted that any $\lambda \in \Lambda$ is written uniquely as a non-negative linear combination of α_i's.

Further on, we put

$$Z_\chi^\pm(s) = z(\chi : \sigma^+ : s) z(\chi : \sigma^- : s)^{\pm 1}. \tag{5-1}$$

Z_χ^+ is an ordinary type and Z_χ^- is Millson's type Selberg zeta function.

Now we describe the the Selberg trace formula (STF). Put $J^\varepsilon = \{z \in \mathbb{C}\,;\, |\operatorname{Im} z| < \rho + \varepsilon\}$, where $\rho = \frac{\ell-1}{2}$. For any $\varepsilon > 0$, by $\mathcal{A}^\varepsilon$ we denote the set of all functions h which are holomorphic in the strip J^ε and satisfy the growth condition $h(z) = O(|z|^{-\ell-\varepsilon})$ on this strip. For such functions we put $\hat{h}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(s)e^{isu}ds$. In this situation, using the results in [F1] concerning the trace of the heat kernel we have the following two types of Selberg trace formulas. The first one is due to Deitmar [D1].

(STF for Even Type) Let $h \in \mathcal{A}^\varepsilon$ be even. Then we have with absolute convergence of both sides of sums

$$
\begin{aligned}
&B_\rho h(0) + \sum_{\lambda_j>0} m_j^+ h(\lambda_j) \\
=&2 \operatorname{Tr}\chi(1) \operatorname{vol}(X) \int_{-\infty}^{\infty} h(x)\mu(x)dx + \sum_{\gamma} \varepsilon^+_{\gamma,\chi} e^{-\rho\ell(\gamma)} \hat{h}(\ell(\gamma)).
\end{aligned}
$$

where the summation is taken over all closed geodesics in X and B_ρ stands for the alternating sum of Betti numbers, namely

$$B_q = \sum_{j=0}^{q} (-1)^{j+q} b_j.$$

Also, under the suitable normalization of the Haar measure on $H = SO_0(4n-1,1)/SO(4n-1)$ (e.g. [Wak1]), the density of the Plancherel measures associated to the spin representations $\sigma^\pm$ (both of them coincide with each other) are given by

$$\mu(x) = \frac{\pi}{2^{8n-6}\Gamma(2n-\frac{1}{2})^2} \binom{4n-2}{2n-1} \prod_{j=1}^{2n-1} [x^2+j^2]. \tag{5-2}$$

(STF for Odd Type) Suppose that $h \in \mathcal{A}^\varepsilon$ is odd. Then we have with absolute convergence of both sides of sums

$$\sum_{\lambda_j>0} m_j^- h(\lambda_j) = \sum_{\gamma} \varepsilon^-_{\gamma,\chi} e^{-\rho\ell(\gamma)} \hat{h}(\ell(\gamma)).$$

Here we put

$$\varepsilon^\pm_{\gamma,\chi} = \operatorname{Tr}\chi(\gamma)[\operatorname{Tr}\sigma^+(m_\gamma) \pm \operatorname{Tr}\sigma^-(m_\gamma)]\ell(\gamma_p)\det(\mathrm{I} - m_\gamma e^{-\ell(\gamma)})^{-1}.$$

Proposition 1.

$$\det((B^o)^2+s^2) = s^{2b_\rho}\exp[P_\chi(s)] Z^+_\chi(s+\rho),$$

where $P_\chi(s)$ is an odd polynomial of degree ℓ given by

$$P_\chi(s) = 4\pi \operatorname{Tr}\chi(1)\operatorname{vol}(X)\int_0^s \mu(it)dt.$$

Remark. The above result is derived also in [D2] except for the deciding "integration constants".

In order to prove the proposition, we need some elementary lemmas.

Lemma 3. *Suppose that* $\mathrm{Re}\, s > 0$. *Then for* $(m,w) \in \mathbb{C}^2$ *satisfying* $\mathrm{Re}\, m > -\frac{1}{2}$, $\mathrm{Re}(w-m-\frac{1}{2}) > 0$, *the following formula holds:*

$$\int_0^\infty e^{-s^2 t} t^w \{\int_{-\infty}^\infty e^{-r^2 t} r^{2m} dr\} \frac{dt}{t} = s^{2(m-w)+1} \Gamma(w-m-\frac{1}{2}) \Gamma(m+\frac{1}{2}).$$

We omit the proof.

Lemma 4. *Put*

$$\det{}'(\Delta^o + s^2) = \exp[-\frac{\partial H}{\partial w}(s^2, 0)].$$

Then

$$\frac{1}{2s}\frac{\partial}{\partial s} \log \det{}'(\Delta^o + s^2) = H(s^2, 1).$$

Proof. It is easy to see that Lemma 1 implies the regularity of $H(z,w)$ as well as the smoothness (for s, $\mathrm{Re}\, s^2 > -\lambda_1^2$) of $\frac{\partial}{\partial w} H(s^2, w)$ at $w = 0, 1$, so by the definition , we have

$$\frac{\partial}{\partial z} H(z,w) = -wH(z, w+1),$$

hence also

$$\frac{1}{2s}\frac{\partial}{\partial s}\frac{\partial}{\partial w} H(s^2, w) = -H(s^2, w+1) - w\frac{\partial}{\partial w} H(s^2, w+1).$$

Therefore letting $w \longrightarrow 0$ we get the desired formula.

Now applied to the function $h(\lambda) = e^{-t\lambda^2}$ $(t > 0)$, the STF of even type gives

$$B_\rho + \mathrm{Tr}'(e^{-t\Delta^o}) = 2\,\mathrm{Tr}\,\chi(1)\,\mathrm{vol}(X) \int_{-\infty}^\infty e^{-tr^2} \mu(r) dr + L(t) \tag{5-3}$$

where we put

$$L(t) = \sum_\gamma \varepsilon_{\gamma,\chi}^+ e^{-\rho\ell(\gamma)} e^{-\ell(\gamma)^2/4t}. \tag{5-4}$$

Using Lemma 3 and the definition of the function $H(s^2, w)$, if $\operatorname{Re} w \gg 0$ then (4-3) implies that

$$\frac{1}{\Gamma(w)} \int_0^\infty L(t) e^{-s^2 t} t^w \frac{dt}{t} = H(s^2, w) + \frac{B_\rho}{s^2}$$

(5-5)

$$- \frac{2 \operatorname{Tr} \chi(1) \operatorname{vol}(X)}{\Gamma(w)} \sum_{k=0}^{2n-1} c_k s^{2(k-w)+1} \Gamma(w - k + \frac{1}{2}) \Gamma(k + \frac{1}{2}).$$

Here the constants c_k is defined by the formula

$$\mu(r) = \sum_{k=0}^{2n-1} c_k r^{2k}.$$

Since the right hand side of (5-5) is meromorphic, hence defines a meromorphic continuation of the left hand side. Hence using Lemma 4 we see that

$$\lim_{w \downarrow 1} \frac{1}{\Gamma(w)} \int_0^\infty L(t) e^{-s^2 t} t^w \frac{dt}{t}$$

(5-6)

$$= \frac{1}{2s} \frac{\partial}{\partial s} \log \det{}'(\Delta^o + s^2) + \frac{B_\rho}{s^2} - \frac{2\pi \operatorname{Tr} \chi(1) \operatorname{vol}(X)}{s} \mu(is).$$

On the other hand, as in [Wak1], by means of the dominated convergence theorem we obtain

$$\lim_{w \downarrow 1} \frac{1}{\Gamma(w)} \int_0^\infty L(t) e^{-s^2 t} t^w \frac{dt}{t} = \int_0^\infty L(t) e^{-s^2 t} \frac{dt}{t}$$

(5-7)

$$= \frac{1}{2s} \frac{\partial}{\partial s} \log Z_\chi^+(s + \rho).$$

Here we used the well known integral formula:

$$\int_0^\infty \frac{1}{\sqrt{4\pi t}} \exp[-(x^2 + y^2/4t)] dt = \frac{1}{2x} e^{-xy}.$$

Hence if we put

$$R(s) = \log \det{}'(\Delta^o + s^2) - P_\chi(s) + 2B_\rho \log s - \log Z_\chi^+(s + \rho)$$

then

$$\frac{d}{ds} R(s) = 0$$

as a meromorphic function. Therefore $R(s)$ is equal to a constant C. We now claim that

$$B_\rho = C = 0.$$

As in (4-1) we can write

$$\mathrm{Tr}'\, e^{-t(\Delta^o)} \sim \sum_{n=0}^{\infty} \beta_n t^{n-\frac{\ell}{2}} \quad t \downarrow 0.$$

If we split the integral (4-2) as we did in the proof of Lemma 1, since ℓ is odd we get

$$\frac{\partial H}{\partial w}(s^2, 0) = -\sum_{n=0}^{\frac{\ell-1}{2}} \beta_n \Gamma(n - \frac{\ell}{2}) s^{\ell-2n} + o(1) \quad \text{as} \quad s \to \infty.$$

On the other hand, since $\log Z_\chi^+(s;\rho) \to 0$ as $s \to \infty$ we see that

$$\lim_{s\to\infty} (2B_\rho \log s - C) = 0.$$

This completes the proof of the proposition.

We turn our attention to the determinant of the Cayley transform. Namely, we have also a following result which is considered as an odd version of the preceding proposition.

Proposition 2.

$$\det(\frac{B^o - is}{B^o + is}) = e^{\pi i(b_\rho - \eta(\chi))} Z_\chi^-(s+\rho).$$

If we put the function $h(\lambda) = \lambda e^{-\lambda^2 t}$ $(t > 0)$ into the STF of odd type, then

$$\mathrm{Tr}(B^o e^{-t\Delta^o}) = \frac{i}{2} \sum_{\gamma} \varepsilon^-_{\gamma,\chi} e^{-\rho\ell(\gamma)} e^{-\ell(\gamma)^2/4t} \frac{\ell(\gamma)}{\sqrt{4\pi t}t}.$$

Using the similar argument as we did in (5-7), we obtain

Lemma 5 ([Mil]). *For* $\mathrm{Re}\, s^2 > -\lambda_1^2$ *we have*

$$I(s) = \int_0^\infty e^{-ts^2}\, \mathrm{Tr}(B^0 e^{-t\Delta^o}) dt = \frac{i}{2}\frac{d}{ds} \log Z_\chi^-(s+\rho).$$

This lemma implies that there exist a constant K such that

$$\det(\frac{B^o - is}{B^o + is}) = KZ_\chi^-(s+\rho).$$

By the way, by (2-4) we get

$$\begin{aligned}\eta(\chi : s) &= \sum_{\lambda_j > 0} m_j^- \frac{\lambda_j}{|\lambda_j|^{s+1}} \\ &= \frac{i}{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})} \int_0^\infty x^{-s} \frac{\partial}{\partial x} \log Z_\chi^-(x+\rho)dx\end{aligned}$$

for $\mathrm{Re}\, s^2 > -\lambda_1^2$, $\mathrm{Re}\, s < 1$. But since $\lim_{x\to+\infty} Z_\chi^-(x) = 1$, $\Gamma(1/2) = \sqrt{\pi}$, we see that ([Mil])

$$\eta(\chi) = \eta(\chi : 0) = \frac{1}{\pi i} \log Z_\chi^-(\rho). \tag{4-8}$$

Hence, by the definition of the normalization of $\det(\frac{B^o - is}{B^o + is})$ at 0 we obtain

$$K = (-1)^{b_\rho} e^{-i\pi\eta(\chi)}.$$

The proposition now follows.

Combining Propositions 1, 2 with the proof of Proposition 1 we can easily show that

$$\log \frac{d(\pm s)}{z(\chi : \sigma^\pm : s+\rho)} = \frac{1}{2}((P_\chi(s) \mp i\pi\eta(\chi)).$$

Hence we obtain

Theorem.

$$\det(B^o \mp is) = (\pm is)^{b_\rho} \exp \frac{1}{2}(P_\chi(s) \mp i\pi\eta(\chi))z(\chi : \sigma^\pm : s+\rho).$$

We use the notations $B^o(\chi)$, $b_\rho(\chi)$ instead of B^o, b_ρ to express the dependence on χ precisely. Then we have the functional equation.

Corollary 1.

$$z(\chi : \sigma^+ : -s+\rho) = \exp(P_\chi(s) + i\pi\eta(\chi))z(\chi : \sigma^- : s+\rho).$$

Proof. Since the operator $B^o(\chi)$ is formally self adjoint, it is clear that $b_{\overline{\rho}}(\chi) = b_\rho(\chi)$, $\eta(\overline{\chi}) = \eta(\chi)$. Also, note the fact that the odd polynomial $P_\chi(s)$ has real coefficients and satisfies the relations $P_\chi(s) = P_{\overline{\chi}}(s)$. On the other hand, by the Euler product expression of $z(\chi : \sigma^\pm : s+\rho)$ we have

$$\overline{z(\chi : \sigma^+ : \overline{s}+\rho)} = z(\overline{\chi} : \sigma^- : s+\rho).$$

The result now follows immediately from those facts.

According to [APS3], we define the reduced η-invariant $\widetilde{\eta}(\chi)$ by the formula

$$\widetilde{\eta}(\chi) = \eta(\chi) - \chi(1)\eta(I)$$

where I denotes the trivial representation of Γ. Then, since $P_\chi(s) = \chi(1)P_I(s)$ we obtain

Corollary 2.

$$e^{\frac{i\pi}{2}\widetilde{\eta}(\chi)}\frac{\det(B^o(\chi)-is)}{\det(B^o(I)-is)^{\chi(1)}} = (is)^{b_\rho(\chi)-\chi(1)b_\rho(I)}\frac{z(\chi : \sigma^+ : s+\rho)}{z(I : \sigma^+ : s+\rho)^{\chi(1)}}.$$

The author closes this paper with some comment on the study of the determinant of the Cayley transform of the square root of Laplacian. Until Professor Stanton pointed out to the author during the Conference on Zeta Functions in Geometry held at Tokyo Institute of Technology, he did not know their study in [MS1], for his lack of care. Thus, unfortunately, there was no reference on their work in the first draft and his talk at this conference. More precisely, in spite of some minor differences between the two methods, Lemma 2 and accordingly, Proposition 2 of this paper is already proved in more general situation . In fact, our operator B^o is the most important one but the special type of the Dirac operator investigated in their paper [MS1]. The author would like to thank Professor Stanton for his magnanimity and kind advice.

References

[APS1] M.F. Atiyah, V.K. Patodi and I.M. Singer, Spectral asymmetry and Riemannian geometry I, Proc. Camb. Phil. Soc., **77** (1975), 43–69.

[APS2] ———, Spectral asymmetry and Riemannian geometry II, Proc. Camb. Phil. Soc., **78** (1975), 405–432.

[APS3] ———, Spectral asymmetry and Riemannian geometry III, Proc. Camb. Phil. Soc., **79** (1975), 71-99.

[AS] M.F. Atiyah and I.M. Singer, The index of elliptic operators III, Ann. of Math., **87** (1968), 546–604.

[BF] J.M. Bismut and D.S. Freed, The analysis of elliptic families, II, Commun. Math. Phys., **107** (1986), 103–163.

[BW] A. Borel and N. Wallach, "Continuous cohomology, discrete subgroups, and representations of reductive groups", Ann. of Math. Study 94, Princeton Univ. Press, New Jersey, 1980.

[D1] A. Deitmar, The Selberg trace formula and the Ruelle Zeta function for compact hyperbolics, Abh. Math. Sem. Univ. Hamburg, **59** (1989), 101–106.

[D2] ———, The Selberg zeta function is a characteristic polynomial, pre print.

[DP] E. D'Hoker and D.H. Phong, On determinants of Laplacians on Riemann surfaces, Commun. Math. Phys., **104** (1986), 537–545.

[Do] H. Donnelly, On the analytic torsion and eta invariant for negatively curved manifolds, Amer. J. Math., **101** (1979), 1365–1379.

[F1] D. Fried, Analytic torsion and closed geodesics on hyperbolic manifolds, Invent. math., **84** (1986), 523–540.

[F2] ———, The zeta functions of Ruelle and Selberg I, Ann. scient. Éc. Norm. Sup. 4^e série, t., **19** (1986), 491–517.

[G] P. Gilkey, "The index theorem and the heat equation", Publish or Perish, Math. Lecture Series 4, New York, 1974.

[H] D. Hejhal, "The Selberg trace formula for $PSL(2, \mathbb{R})$, I", Lecture Notes in Math 548, Springer-Verlag, Berlin, 1976.

[MM] Y. Matsushima and S. Murakami, On vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds, Ann. of Math., **78** (1963), 417–449.

[Mia] R.J. Miatello, The Minakshisundaram-Pleijel coefficients for the vector valued heat kernel on compact locally symmetric spaces of negative curvature, Trans. Amer. Math. Soc., **260** (1980), 1–33.

[Mil] J.J. Millson, Closed geodesics and the η-invariant, Ann. of Math., **108** (1978), 1–39.

[MS1] H. Moscovici and R.J.Stanton, Eta invariants of Dirac operators on locally symmetric manifolds, Invent. math., **95** (1989), 629–666.

[MS2] ———, R-torsion and zeta functions for locally symmetric manifolds, preprint.

[PS] R. Phillips and P. Sarnak, Geodesics in homology classes, Duke Math. J., **55** (1987), 287–297.

[Q] D. Quillen, Determinant of Cauchy-Riemann operators on Riemann surfaces, Funct. Anal. Appl., **19** (1985), 31–34.

[RS] D.B. Ray and I.M. Singer, Analytic torsion for complex manifolds, Ann. of Math., **98** (1973), 154–177.

[Sar] P. Sarnak, Determinants of Laplacians, Commun. Math. Phys., **110** (1987), 113–120.

[Se] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc., **20** (1956), 47–87.

[Sin] I.M. Singer, Eigenvalues of the Laplacian and Invariants of Manifolds, in "Proc. Int. Cong. Math. Vancouver", 1974, pp. 187–200.

[V] A. Voros, Spectral functions, special functions and the Selberg zeta function, Commun. Math. Phys., **110** (1987), 439–465.

[Wak1] M. Wakayama, Zeta functions of Selberg's type associated with homogeneous vector bundles, Hiroshima Math. J., **15** (1985), 235–295.

[Wak2] ———, A formula for the logarithmic derivative of Selberg's zeta function, J. Math. Soc. Japan, **41** (1989), 463–471.

[Wak3] ———, A note on the Selberg zeta function for compact quotients of hyperbolic spaces, to appear in Hiroshima Math. J., **21** (1991).

[Wal] N. Wallach, On the Selberg trace formula in the case of compact quotient, Bull. Amer. Math. Soc., **82** (1976), 171–195.

Department of Mathematics
Tottori University
Tottori 680, Japan