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On the Deformations of the Geometric Structures on the Seifert 4-Manifolds

Masaaki Ue

We call a closed orientable 4-manifold S *a Seifert* 4-*manifold* if S has a structure of a fibered orbifold $\pi: S \to B$ over some 2-orbifold B with general fiber a 2-torus T^2 where the total space S is a nonsingular manifold. In [10], [11] we discussed the relations between them and certain eight geometries in dimension 4 in the sense of Thurston and also gave their topological classification. Here by a geometric structure of S we mean the structure of the form $\Gamma\backslash X$ diffeomorphic to S where X is a 1-connected complete Riemannian 4-manifold and Γ is a discrete subgroup of the group $\mathrm{Isom}^+ X$ of all orientation-preserving isomorphisms of X acting freely on X. The purpose of this paper is to determine the Teichmüller spaces for their geometric structures in the cases when the base orbifolds are either hyperbolic or euclidean (§1 and §2). Our results are parallel to [5], [6] in which the Teichmüller spaces for the geometric structures on the Seifert 3-manifolds were discussed. But a little more arguments are needed for our cases since we should take account of the nontrivial monodromies. In the meanwhile some of the Seifert 4-manifolds have complex structures compatible to their geometric structures ([12]). In these cases we will also give the relations between the Teichmüller spaces and the deformations of the associated complex structures via the Kodaira Spencer maps. In all cases we treat here these maps are surjective but not injective in general (and hence the Teichmüller spaces are not effectively parametrized as families of complex structures §3). Finally in §4 we also give a remark on the moduli spaces for the geometric structures when the base orbifolds are hyperbolic and show that they are defined as Hausdorff spaces whereas, as is well known, the moduli spaces for the complex structures can not be defined as Hausdorff spaces in general. For simplicity in this paper we only consider the Seifert 4-manifolds over the closed orientable base orbifolds. All the subjects will be considered in the smooth category.

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We will use the same notations for the geometries as in [12], [13], [10] and [11]. The 2-dimensional hyperbolic space will be denoted by H^2 and also by $\mathbf{H}$ (as the complex space). Furthermore $\mathbf{C}^*$ and $\mathbf{R}^+$ will be the set of nonzero complex numbers and the set of positive numbers respectively.

§1. Definitions of Teichmüller spaces and the cases when the bases are hyperbolic

Let S be a closed orientable Seifert 4-manifold over a 2-orbifold B and $\pi: S \to B$ be its fiber map with general fiber T^2. In [10] and [11] we proved that S has a geometric structure if B is either euclidean, spherical or bad and if B is hyperbolic S is geometric if and only if S has a complex structure (and is an elliptic surface). Let $G = \mathrm{Isom}^0 X$ be the identity component of $\mathrm{Isom}^+ X$. For simplicity we only consider the geometric Seifert 4-manifolds over either hyperbolic or euclidean orientable base orbifolds of the form $\Gamma\backslash X$ with $\Gamma \subset G$ and the (G, X) structures on them. We note that any geometric Seifert 4-manifold over the hyperbolic 2-orbifold (we have assumed that the base orbifold is orientable) is of the form $\Gamma\backslash X$ where X is either $H^2 \times E^2$ or $\widetilde{SL}_2 \times E$ and $\Gamma \subset G = \mathrm{Isom}^0 X$ ([11]). Here $\widetilde{SL}_2$ is the universal covering of $SL_2\mathbf{R}$.

Definition 1. Let $\mathcal{R}(\Gamma, G)$ be the set of all faithful discrete co-compact representations from Γ to G with compact open topology.

The group $\mathrm{Inn}\, G$ of the inner automorphisms of G and the group $\mathrm{Aut}\, \Gamma$ of the automorphisms of Γ act on $\mathcal{R}(\Gamma, G)$ by $g \cdot \rho(\gamma) \cdot g^{-1}$ and by $\rho \cdot \phi(\gamma) = \rho(\phi(\gamma))$ respectively where $g \in G$, $\gamma \in \Gamma$, $\phi \in \mathrm{Aut}\, \Gamma$ and $\rho \in \mathcal{R}(\Gamma, G)$. The second action commutes with the first one and induces the action of the group $\mathrm{Out}\, \Gamma$ of the outer automorphisms on the quotient $\mathrm{Inn}\, G\backslash\mathcal{R}(\Gamma, G)$.

Definition 2. We call the quotient space

$$\mathcal{T}(\Gamma, G) = \mathrm{Inn}\, G\backslash\mathcal{R}(\Gamma, G)$$

a Teichmüller space of $S = \Gamma\backslash X$, and the quotient

$$\mathcal{M}(\Gamma, G) = \mathrm{Inn}\, G\backslash\mathcal{R}(\Gamma, G)/\,\mathrm{Out}\,\Gamma$$

a moduli space of $S = \Gamma\backslash X$.

The fundamental group $\pi_1^{\mathrm{orb}} B$ of the base orbifold B has the representation of the form

$$\{\overline{\alpha}_1, \ldots, \overline{\alpha}_{2g}, \overline{q}_1, \ldots, \overline{q}_r \mid$$

$$\overline{q}_1^{m_1} = \cdots = \overline{q}_r^{m_r} = \prod_{i=1}^{g} [\overline{\alpha}_{2i-1}, \overline{\alpha}_{2i}] \prod_{j=1}^{r} \overline{q}_j = 1\}$$

where $\overline{q}_i$ corresponds to a meridian circle around the i-th cone point of cone angle $2\pi/m_i$, and $\overline{\alpha}_1, \ldots, \overline{\alpha}_{2g}$ form a symplectic base of the fundamental group of the underlying space $|B|$ of B of genus g. Here we define $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. Note that $\pi^{\mathrm{orb}} B$ is isomorphic to a discrete subgroup $\overline{\Gamma}$ in Isom $\overline{X}$ and $B = \overline{\Gamma}\backslash\overline{X}$ where $\overline{X} = \mathbf{H}$ if B is hyperbolic and $\overline{X} = \mathbf{E}^2$ if B is euclidean. Then $\pi_1 S$ has the following representation:

$$\begin{aligned}
&\{\alpha_1, \ldots, \alpha_{2g}, q_1, \ldots, q_r, \ell, h \mid \\
&[\ell, h] = 1, [q_j, \ell] = [q_j, h] = 1 \quad \text{for } j = 1, \ldots, r, \\
&(\alpha_i \ell \alpha_i^{-1}, \alpha_i h \alpha_i^{-1}) = (\ell, h) A_i \quad \text{for } i = 1, \ldots, 2g, \\
&q_s^{m_s} \ell^{a_s} h^{b_s} = 1 \quad \text{for } s = 1, \ldots, r, \\
&\prod_{i=1}^{g} [\alpha_{2i-1}, \alpha_{2i}] \prod_{s=1}^{r} q_s = \ell^a h^b\}.
\end{aligned}$$

Here α_i, q_j are the lifts of $\overline{\alpha}_i$ and $\overline{q_j}$ respectively, ℓ and h form a base of the fundamental group $\mathbf{Z}^2$ of the general fiber T^2, $A_i \in SL_2\mathbf{Z}$ is the monodromy matrix corresponding to $\overline{\alpha}_i$ with respect to (ℓ, h), and (m_s, a_s, b_s) is the Seifert invariant of the s-th multiple fiber of multiplicity m_s over the s-th cone point. The Seifert invariants for such S are denoted by

$$\{A_1, \ldots, A_{2g}, (a, b), (m_1, a_1, b_1), \ldots, (m_r, a_r, b_r)\}.$$

First consider $S = \Gamma\backslash X$ when $B = \overline{\Gamma}\backslash\overline{X}$ is hyperbolic. Then by the results in [15] the fibration of S is unique up to fiber-preserving diffeomorphisms and $\pi_1^{\mathrm{orb}} B = \overline{\Gamma}$ is uniquely determined by Γ up to group automorphisms. Moreover all the monodromy matrices are the powers of some common periodic matrix Q ([11], Theorem B). If every monodromy is trivial then the pair $e = (a + \sum a_j/m_j, b + \sum b_j/m_j) \in \mathbf{Q}^2 \bmod GL_2\mathbf{Z}$ is well defined and is called the rational euler class of S. The type of X is $\widetilde{SL_2} \times E$ if every monodromy is trivial and $e \neq (0, 0)$ and $X = H^2 \times E^2$ otherwise. Furthermore we can assume that $a + \sum a_j/m_j = 0$ in the first case by some coordinate change of the fiber ([11]).

Proposition 1. *Let S be a geometric Seifert 4-manifold over a hyperbolic orbifold B. Then for appropriate choices of the lattices of the general fiber and the symplectic basis of the curves on B generating $H_1(|B|,\mathbf{Z})$ we can assume that $A_i = I$ for $i \geq 2$ and A_1 is either* $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \pm I$, *where I is the identity matrix.*

Proof. We can assume that all monodromy matrices A_i of S are powers of a periodic matrix Q. Then by choosing the lattice ℓ, h appropriately we can suppose that Q is either one of the five matrices given above since Q is periodic. In particular all A_i are mutually commutative. If we consider the pullback of S by a self-automorphism ϕ of the base B fixing all cone points and the base point of B then ϕ induces the symplectic isomorphism ϕ_* on $H_1(|B|,\mathbf{Z})$ and the monodromy matrices $A_1,\ldots,A_{2g}$ are transformed to $\phi_*A_1,\ldots,\phi_*A_{2g}$ where if $\phi_*\overline{\alpha}_i = \delta_1\overline{\alpha}_1 + \delta_2\overline{\alpha}_2 + \cdots \delta_{2g}\overline{\alpha}_{2g}$ for the symplectic bases $\overline{\alpha}_1,\ldots,\overline{\alpha}_{2g}$ of $H_1(|B|,\mathbf{Z})$ then $\phi_*A_i = A_1^{\delta_1}A_2^{\delta_2}\cdots A_{2g}^{\delta_{2g}}$. Since all A_i's are powers of Q Euclid algorithm shows that this process simplifies the monodromy matrices (which are still powers of Q after this process) of the Seifert fibration of S induced from the original one by ϕ if ϕ is chosen appropriately. In fact we can see that some automorphism ϕ of B fixing all the cone points induces the isomorphism ϕ_* such that $\phi_*A_i = I$ for $i \geq 2$ as follows. First for any $P_i \in SL_2\mathbf{Z}, (i = 1,\ldots,g)$ there is an automorphism ψ of B such that

$$(\psi_*\overline{\alpha}_{2i-1}, \psi_*\overline{\alpha}_{2i}) = (\overline{\alpha}_{2i-1}, \overline{\alpha}_{2i})P_i$$

since every symplectic isomorphism can be realized by some orinetation preserving self-diffeomorphism of B fixing all the cone points. Using such ψ the monodromy matrices can be transformed so that $A_{2i} = I$ for $i = 1,\ldots,g$. Next consider the symplectic isomorphism ρ satisfying

$$\begin{aligned} \rho(\overline{\alpha}_2) &= \overline{\alpha}_2 - \overline{\alpha}_{2i} \\ \rho(\overline{\alpha}_{2i-1}) &= \overline{\alpha}_1 + \overline{\alpha}_{2i-1} - \overline{\alpha}_{2i} \\ \rho(\overline{\alpha}_j) &= \overline{\alpha}_j \quad \text{otherwise} \end{aligned}$$

which maps A_{2i-1} to $A_{2i-1}A_1$, leaves the other monodromies unchanged and can be realized by the Dehn twists along the curves representing $\overline{\alpha}_1\overline{\alpha}_{2i}{}^{-1}$ and α_1^{-1}. Then applying the isomorphisms of this type or their inverses (or those obtained by exchanging the roles of α_1 and α_{2i-1}) and the isomorphism mapping (α_1, α_2) to $(\alpha_{2i-1}, \alpha_{2i})$ we can see (by Euclid algorithm) that A_{2i-1} $(i \geq 2)$ is reduced to I with all others except for A_1 left unchanged. The desired automorphism ϕ is obtained by the composition of the above automorphisms. Finally consider if necessary the symplectic isomorphism σ satisfying

$$\begin{aligned}\sigma(\overline{\alpha}_1) &= -\overline{\alpha}_1\\ \sigma(\overline{\alpha}_2) &= -\overline{\alpha}_2\\ \sigma(\overline{\alpha}_j) &= \overline{\alpha}_j \quad \text{for } i \geq 3\end{aligned}$$

(which maps A_1 to A_1^{-1}, leaves $A_i = I$ $(i \geq 2)$ unchanged and is also realized by an automorphism of B) and some further change of the lattice of the general fiber we obtain the desired representations of the monodromy matrices since A_1 is still a power of some periodic matrix.

Q.E.D.

In the case of a geometric Seifert 4-manifold $S = \Gamma\backslash X$ over a hyperbolic base orbifold $B = \overline{\Gamma}\backslash\overline{X}$, X is complex analytically equivalent to $\mathbf{H} \times \mathbf{C}$ ([11], [12] and see the proof of Theorem A below) such that the lattice Γ_0 of the general fiber of S acts on the $\mathbf{C}$-factor as translations. Furthermore the lifts of the elements of $\overline{\Gamma}$ to Γ induce the orientation-preserving automorphisms of the $\mathbf{C}$-factor (up to translations) which do not depend on the choices of the lifts and which preserve the lattice in $\mathbf{C}$ defined by Γ_0. Then we have a homomorphism from $\overline{\Gamma}$ to $GL^+(1, \mathbf{C}) = \mathbf{C}^*$ which we call the monodromy representation of $\Gamma\backslash X$ and denote by ϕ. The relation between ϕ and the monodromy matrices of $S = \Gamma\backslash X$ is explained in Theorem A and its proof below. To describe the Teichmüller space of $\Gamma\backslash X$ we introduce the following extra notations.

$\mathcal{T}(\overline{\Gamma}, \mathrm{Isom}^+ \overline{X})$ the Teichmüller space of $B = \overline{\Gamma}\backslash\overline{X}$

$H^1(\overline{\Gamma}, \mathbf{C}^\phi)$ the 1st cohomology group of $\overline{\Gamma}$ with coefficients $\mathbf{C}^\phi$.

Here $\mathbf{C}^\phi$ is $\mathbf{C}$ twisted by ϕ, $\mathcal{T}(\overline{\Gamma}, \mathrm{Isom}^+ \overline{X}) = \mathbf{R}^{2(3g-3+r)} \times \mathbf{Z}_2$ where g is the genus of B and r is the number of the cone points of B. Let $\mathcal{T}_{g,r}$ be the identity component of $\mathcal{T}(\overline{\Gamma}, \mathrm{Isom}^+ \overline{X})$. (It is well known that $\mathcal{T}_{g,r}$ which is also the Teichmüller space of the orbifold B depends only on g and r.)

Theorem A. *$\mathcal{T}(\Gamma, G)$ for a Seifert 4-manifold $S = \Gamma\backslash X$ over a hyperbolic 2-orbifold $B = \overline{\Gamma}\backslash\overline{X}$ has a structure of a trivial fiber bundle of the following form.*

$$\mathcal{F} \to \mathcal{T}(\Gamma, G) \to \mathcal{T}(\overline{\Gamma}, \mathrm{Isom}^+ \overline{X}).$$

Here the fiber $\mathcal{F}$ is isomorphic to $H^1(\overline{\Gamma}, \mathbf{C}^{\phi}) \times \mathbf{T}_1$ where

$$\mathbf{T}_1 = \begin{cases} \mathbf{R}^+ \times \mathbf{H} \times \mathbf{Z}_2, & \text{if } X = H^2 \times E^2 \text{ and } \phi \equiv \pm\mathrm{id} \\ \mathbf{R}^+ \times \mathbf{Z}_2, & \text{if } X = H^2 \times E^2 \text{ and } \phi \not\equiv \pm\mathrm{id} \\ \mathbf{H} \times \mathbf{Z}_2, & \text{if } X = \widetilde{SL}_2 \times E \end{cases}$$

and $\mathbf{T}_1$ corresponds to the deformations of the lattice of the general fiber generated by c and $c\lambda^{-1}$ with $c \in \mathbf{R}^+, \lambda \in \mathbf{C}$ and $\Im\lambda \neq 0$. $\mathbf{T}_1$ has two components according to the sign of $\Im\lambda$. The monodromy representation ϕ satisfies $\phi(\overline{\alpha}_i) = \pm 1$ if the monodromy matrix A_i corresponding to $\overline{\alpha}_i$ is $\pm I$. If $A_i \neq \pm I$ for some i, the lattice Γ_0 in $\mathbf{C}$ of the general fiber is uniquely determined up to scalar multiplication and ϕ is also uniquely determined once the sign of $\Im\lambda$ is fixed. $H^1(\overline{\Gamma}, \mathbf{C}^{\phi})$ satisfies

$$H^1(\overline{\Gamma}, \mathbf{C}^{\phi}) = \begin{cases} \mathbf{C}^{2g} & \text{if } \phi \equiv \mathrm{id} \\ \mathbf{C}^{2g-2} & \text{otherwise.} \end{cases}$$

The identity component $\mathcal{T}_0$ of $\mathcal{T}(\Gamma, G)$ is $\mathcal{T}_{g,r} \times H^1(\overline{\Gamma}, \mathbf{C}^{\phi}) \times \mathbf{T}_0$ where $\mathbf{T}_0$ is the connected component of $\mathbf{T}_1$ and is homeomorphic to a euclidean space.

Proof. We may assume that S satisfies the conditions in Proposition 1. First suppose that the type X of the geometry of $S = \Gamma\backslash X$ is $H^2 \times E^2$ which is identified with $\mathbf{H} \times \mathbf{C}$. In this case $G = \mathrm{Isom}^0 H^2 \times \mathrm{Isom}^0 E^2$. Let $\rho \in \mathcal{R}(\Gamma, G)$ be any element for $S = \Gamma\backslash X$. Then ρ induces the representation $\overline{\rho} \in \mathcal{R}(\overline{\Gamma}, \overline{G})$ with $\overline{G} = \mathrm{Isom}^0 H^2 = PSL_2\mathbf{R}$. $\overline{\rho}$ gives the representation of the base B as the hyperbolic orbifold $B = \overline{\Gamma}\backslash\mathbf{H}$. Then for the coordinates $(z, w) \in \mathbf{H} \times \mathbf{C}$, we have

$$\rho(\ell)(z,w) = (z, w + c) \tag{0}$$

$$\rho(h)(z,w) = (z, w + d) \tag{1}$$

$$\rho(\alpha_i)(z,w) = (\overline{\rho}(\overline{\alpha}_i)z, \lambda_i w + w_i) \quad (i = 1, \ldots, 2g) \tag{2}$$

$$\rho(q_j)(z,w) = (\overline{\rho}(\overline{q}_i)z, w - (a_j c + b_j d)/m_j) \tag{3}$$

where $c, d \in \mathbf{C}$ are linearly independent over $\mathbf{R}$, $\lambda_i \in S^1 \subset \mathbf{C}, w_i \in \mathbf{C}$ which satisfy the following relations. We note that (3) comes from the relations ${q_j}^{m_j} \ell^{a_j} h^{b_j} = 1, [q_j, \ell] = [q_j, h] = 1$. Put $c = u + iv, d = u' + iv'$ and $P = \begin{pmatrix} u & v \\ u' & v' \end{pmatrix} \in GL_2\mathbf{R}$. Then we deduce from $\alpha_i(\ell, h)\alpha_i^{-1} = (\ell, h)A_i$ that

$$PA_iP^{-1} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$$

where $\lambda_i = \exp(\sqrt{-1}\theta_i)$. Thus we have $\lambda_i = 1$ for $i \geq 2$ since $A_i = I$. If $A_1 = \pm I$, then $\lambda_1 = \pm 1$ and there is no further restriction on (c, d). For the remaining cases we have $(c, d) = (c, \lambda^{-1}c)$ with

$$\lambda_1 = \lambda = \exp(\pm 2\pi i/6) \quad \text{if } A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

$$\lambda = \exp(\pm 2\pi i/6), \lambda_1 = \lambda^2 \quad \text{if } A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \tag{5}$$

$$\lambda_1 = \lambda = \pm i \quad \text{if } A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6}$$

The monodromy representation $\phi: \overline{\Gamma} \to \mathbf{C}^*$ is defined by $\phi(\overline{\alpha}_i) = \lambda_i$, $\phi(\overline{q}_i) = 1$. We note that ϕ is uniquely determined if $A_i = \pm I$. If $A_1 \neq \pm I$ we have two choices of λ above according as $\Im\lambda > 0$ or $\Im\lambda < 0$. Hence the parameter space for the lattice of the general fiber (represented by c and λ) has two components and is homeomorphic to $\mathbf{C}^* \times \mathbf{Z}_2$. The monodromy representation ϕ depends only on the choice of λ. If $A_1 = \pm I$, (and if we write $(c, d) = (c, c\lambda^{-1})$) then (c, λ) ranges over $\mathbf{C}^* \times \mathbf{H} \times \mathbf{Z}_2$. In this case we have two components according to the sign of $\Im\lambda$. Finally from the relation $\prod_{j=1}^{g}[\alpha_{2j-1}, \alpha_{2j}] \prod_{j=1}^{r} q_j = \ell^a h^b$ we deduce

$$\begin{aligned} &\sum_{j=1}^{g}(\phi(\overline{\alpha}_{2j-1}) - 1)w_{2j} - \sum_{j=1}^{g}(\phi(\overline{\alpha}_{2j}) - 1)w_{2j-1} \\ &\qquad = (a + \sum_{j=1}^{r} a_j/m_j)c + (b + \sum_{j=1}^{r} b_j/m_j)d. \end{aligned} \tag{7}$$

Clearly we can find w_j satisfying (7) for any $\overline{\rho}$ since the right hand side on (7) is 0 if $\phi \equiv \mathrm{id}$. Hence ρ defined by (0)–(3) satisfying (4)–(7) defines a discrete faithful representation from Γ to G. Conversely every $\overline{\rho} \in \mathcal{R}(\overline{\Gamma}, \overline{G})$ has a lift $\rho \in \mathcal{R}(\Gamma, G)$. Hence the natural projection $p: \mathcal{R}(\Gamma, G) \to \mathcal{R}(\overline{\Gamma}, \overline{G})$ is surjective. Next we describe the fiber $\widetilde{\mathcal{F}}$ of p. Pick up $\overline{\rho}_0 \in \mathcal{R}(\overline{\Gamma}, \overline{G})$ and fix a lift $\rho_0 \in \mathcal{R}(\Gamma, G)$ of $\overline{\rho}$ satisfying (0)–(7) as a base point of the fiber over $\overline{\rho}_0$. Hereafter the parameters in (0)–(7) for ρ_0 are denoted by the same symbols with suffix 0. We can choose w_j^0 in (2) for ρ_0 so that $w_j^0 = 0$ if $\phi \equiv \mathrm{id}$ or $w_j^0 = 0$ for $j \neq 2$ and $w_2^0 = ((a + \sum a_j/m_j)c + (b + \sum b_j/m_j)d)/(\phi(\overline{\alpha_1}) - 1)$ if $\phi \not\equiv \mathrm{id}$ under the assumption in Proposition 1. Take $\rho \in \mathcal{R}(\Gamma, G)$ such that $\overline{\rho} = \overline{\rho}_0, (c, d) = (c_0, d_0)$ and the monodromy representations for ρ and

ρ_0 are the same. Put $m(\overline{\alpha}_j) = w_j - w_j^0$, $m(\overline{q}_j) = 0$. Then $m(\overline{\alpha}_j)$ satisfy

$$(8) \qquad \sum_{j=1}^{g}(\phi(\overline{\alpha}_{2j-1}) - 1)m(\overline{\alpha}_{2j}) - \sum_{j=1}^{g}(\phi(\overline{\alpha}_{2j}) - 1)m(\overline{\alpha}_{2j-1}) = 0$$

and m can be extended to a crossed homomorphism $m: \overline{\Gamma} \to \mathbf{C}$ satisfying $m(\overline{\alpha}\overline{\beta}) = m(\overline{\alpha}) + \phi(\overline{\alpha})m(\overline{\beta})$ for $\overline{\alpha}, \overline{\beta} \in \overline{\Gamma}$. Let $C^i(\overline{\Gamma}, \mathbf{C}^\phi)$ be the i-th cochain of $\overline{\Gamma}$ with coefficients $\mathbf{C}$ twisted by ϕ. Then m is contained in the kernel of the coboundary map $\delta: C^1(\overline{\Gamma}, \mathbf{C}^\phi) \to C^2(\overline{\Gamma}, \mathbf{C}^\phi)$. Conversely if $m \in \ker\delta$ then $m(\overline{q}_j) = 0$ since $\overline{q}_j^{m_j} = 1$ and $\phi(\overline{q}_j) = 1$. Furthermore for any such m, we can define a faithful discrete representation ρ satisfying $\overline{\rho} = \overline{\rho}_0, (c, d) = (c_0, d_0), w_j = w_j^0 + m(\overline{\alpha}_j)$. The choice of m does not depend on the choice of (c_0, d_0) if ϕ is fixed. Thus the fiber $\tilde{\mathcal{F}}$ is homeomorphic to $(\ker\delta) \times \tilde{\mathbf{T}}_1$ where $\tilde{\mathbf{T}}_1 = \mathbf{C}^* \times \mathbf{Z}_2$ if $\phi \not\equiv \pm\,\mathrm{id}$ or $\mathbf{C}^* \times \mathbf{H} \times \mathbf{Z}_2$ if $\phi \equiv \pm\,\mathrm{id}$. We note that the choices of the parameters of $\tilde{\mathbf{T}}_1$ (and the choices of w_j^0 in (2) satisfying (7) for the fixed lift ρ_0 of $\overline{\rho}_0$) do not depend on $\overline{\rho}$. So the projection $p: \mathcal{R}(\Gamma, G) \to \mathcal{R}(\overline{\Gamma}, \overline{G})$ gives a product fibration and $\mathcal{R}(\Gamma, G) = \mathcal{R}(\overline{\Gamma}, \overline{G}) \times \ker\delta \times \tilde{\mathbf{T}}_1$. Next we check the action of Inn G on $\mathcal{R}(\Gamma, G)$. The action of Inn $PSL_2\mathbf{R}$ is nonrivial only on the first factor $\mathcal{R}(\overline{\Gamma}, \overline{G})$ of $\mathcal{R}(\Gamma, G)$ and yields a Teichmüller space $\mathcal{T}(\overline{\Gamma}, \overline{G})$ of the base orbifold B. $\mathcal{T}(\overline{\Gamma}, \overline{G})$ has just two components which correspond to the Teichmüller spaces of the hyperbolic structures on B and B with opposite orientation. Each one is identified with the Teichmüller space $\mathcal{T}_{g,r} = \mathbf{R}^{2(3g-3+r)}$ of r-pointed Riemann surface of genus g. Next we pick up $\mu \in \mathrm{Isom}^+E^2$ defined by $\mu(z, w) = (z, \sigma w + w')$ with $\sigma \in S^1, w' \in \mathbf{C}$ (acting trivially on the first coordinate). If $\rho \in \mathcal{R}(\Gamma, G)$ satisfying (0)–(3), then we have

$$\begin{aligned}
&\mu\rho(\ell)\mu^{-1}(z, w) = (z, w + \sigma c),\\
&\mu\rho(h)\mu^{-1}(z, w) = (w, w + \sigma d),\\
&\mu\rho(\alpha_j)\mu^{-1}(z, w) = (\overline{\rho}(\overline{\alpha}_j)z, \lambda_j w + \sigma w_j + (1 - \lambda_j)w'),\\
&\mu\rho(q_j)\mu^{-1}(z, w) = (\overline{\rho}(\overline{q_j})z, w - (a_i\sigma c + b_i\sigma d)/m_i).
\end{aligned}$$

where $\lambda_j = \phi(\overline{\alpha}_j)$, $d = c\lambda^{-1}$ as in (4)–(6). Then (c, λ) are transformed to $(\sigma c, \lambda)$ and w_j is transformed to $\sigma w_j + (1 - \phi(\overline{\alpha}_j))w'$. Thus if the representative of c in Inn $\mathrm{Isom}^+ E^2\backslash\mathcal{F}$ is fixed so that $c \in \mathbf{R}^+$, then $m(\overline{\alpha}_j)$'s are defined modulo the image of $\delta^0: C^0(\overline{\Gamma}, \mathbf{C}^\phi) \to C^1(\overline{\Gamma}, \mathbf{C}^\phi)$ in Inn $\mathrm{Isom}^+ E^2\backslash\tilde{\mathcal{F}}$. Therefore we obtain

$$\mathcal{T}(\Gamma, G) = \mathcal{T}(\overline{\Gamma}, \overline{G}) \times (\mathbf{T}_1 \times H^1(\overline{\Gamma}, \mathbf{C}^\phi))$$

where $\mathcal{T}(\Gamma, G)$ is parametrized by $\overline{\rho}$ (mod Inn $\overline{G}$) $\in \mathcal{T}(\overline{\Gamma}, \overline{G})$, $(c, \lambda) \in \mathbf{T}_1 = \mathbf{R}^+ \times \mathbf{H} \times \mathbf{Z}_2$ or $\mathbf{R}^+ \times \mathbf{Z}_2$, and $m(\overline{\alpha}_j)$ (mod Imδ^0) $\in H^1(\overline{\Gamma}, \mathbf{C}^\phi)$ as desired. $H^1(\overline{\Gamma}, \mathbf{C}^\phi)$ is a vector space over $\mathbf{C}$ whose dimension is easily computed by (8) under the assumption of Proposition 1 as indicated in Theorem A. We note that $H^1(\overline{\Gamma}, \mathbf{C}^\phi)$ is the same as $H^1(\pi_1|B|, \mathbf{C}^\phi)$ since the coefficient of the cohomology is torsion free and the monodromies along the torsion elements $\overline{q}_j$ in $\overline{\Gamma}$ are trivial.

Next we consider the case when $S = \Gamma\backslash X$ with $X = \widetilde{SL_2} \times E$ and $G = \widetilde{SL_2} \underset{\mathbf{Z}}{\times} \mathbf{R} \times \mathbf{R}$. In this case X is identified with $\mathbf{H} \times \mathbf{C}$ with coordinates $(z, w), z \in \mathbf{H}, w \in \mathbf{C}$ so that w corresponds to $\log dz$ ([13]). Here the imaginary part of $\log dz$ corresponds to the lift of the unit tangent vector at $z \in H^2$ to the fiber of the natural projection $\pi: \widetilde{SL_2} \to H^2$. This projection is defined via the identification of $PSL_2\mathbf{R}$ with the unit tangent bundle T_1H^2 of H^2. The real part of $\log dz$ belongs to the E-factor of X. Then ρ in $\mathcal{R}(\Gamma, G)$ induces the element $\overline{\rho}$ in $\mathcal{R}(\overline{\Gamma}, \overline{G})$ where $\overline{\Gamma} = \pi_1^{\rm orb}B$ and $\overline{G} = PSL_2\mathbf{R}$. Moreover ρ must be of the following form:

$$\begin{aligned}
\rho(\ell)(z,w) &= (z, w+c)\\
\rho(h)(z,w) &= (z, w+d)\\
\rho(\alpha_j)(z,w) &= \tilde{\alpha}_j(z,w) + (0, w_j)\\
\rho(q_j)(z,w) &= \tilde{q}_j(z,w) + (0, y_j)
\end{aligned}$$

where $\tilde{\alpha}_j(z, w)$ is a lift of $\overline{\rho}(\overline{\alpha}_j): z \to (a_j z + b_j)/(c_j z + d_j)$ defined by

$$\tilde{\alpha}_j(z,w) = (\overline{\rho}(\overline{\alpha}_j)z, w - 2\log(c_j z + d_j)).$$

Here the imaginary part of the second factor is chosen so that it is continuous and it coincides with the image of $\Im w$ by the parallel translation from z to $\overline{\rho}(\overline{\alpha_j})z$ along the axis of the hyperbolic element $\overline{\rho}(\overline{\alpha}_j)$ (which is defined as the lift of that on T_1H^2 via the projection $\widetilde{SL_2} \to T_1H^2$) if z lies in this axis. These conditions determine the choice of the branch of log in the image of $\tilde{\alpha}_j$. A lift $\tilde{q}_j$ of $\overline{\rho}(\overline{q}_j)$ is taken so that $\tilde{q}_j^{m_j} = 1$ in G (cf. [11], [13]). Note that $\tilde{q}_j$ is uniquely determined since the $\mathbf{R} \times \mathbf{R}$-factor of G lies in the center of G. Then

$$y_j = -(a_j c + b_j d)/m_j$$

from $q_j^{m_j}\ell^{a_j}h^{b_j} = 1$. We have chosen ℓ, h so that $a + \sum a_j/m_j = 0$, $b + \sum b_j/m_j \neq 0$ and hence we also deduce from $\prod[\alpha_{2j-1}, \alpha_{2j}] \prod q_j =$

$\ell^a h^b$ that

$$d = (2\pi i \chi^{\rm orb} B)/(b + \sum b_j/m_j) \neq 0$$

where $\chi^{\rm orb}$ denotes the orbifold euler characteristic. Therefore the parameters y_j, d are fixed, $c = u + iv$ is an arbitrary complex number with $u \neq 0$ (since c and d must be linearly independent over $\mathbf{R}$), and w_j are arbitrary complex numbers. Then the natural projection $p: \mathcal{R}(\Gamma, G) \to \mathcal{R}(\overline{\Gamma}, \overline{G})$ defined by $p(\rho) = \overline{\rho}$ is surjective and the fiber $\widetilde{\mathcal{F}}$ of p is $C^1(\overline{\Gamma}, \mathbf{C}) \times \mathbf{T}_1$ where $\mathbf{T}_1 = \mathbf{H} \times \mathbf{Z}_2$ which corresponds to c (or equivalently ic). $\mathbf{T}_1$ has two components according to the sign of $\Re c$. Since every translation $(z, w) \to (z, w + s + ti)$ commutes with any element $g \in G$, the action of Inn G on $\mathcal{R}(\Gamma, G)$ yields the following isomorphism;

$$\mathcal{T}(\Gamma, G) = \mathcal{T}(\overline{\Gamma}, \overline{G}) \times H^1(\overline{\Gamma}, \mathbf{C}) \times \mathbf{T}_1$$

which proves Theorem A.

§2. The cases with euclidean base orbifolds

Suppose that S is a Seifert 4-manifold over a closed orientable euclidean 2-orbifold B. In this case S has always a geometric structure of type X where $X = E^4$, $\mathrm{Nil}^3 \times E$, Nil^4 or $\mathrm{Sol}^3 \times E$ ([10]). But in this paper we restrict our attention to the cases when $\pi_1 S = \Gamma$ is a subgroup of $G = \mathrm{Isom}^0 X$. Then by the results in [10] we have only to consider the cases when S is diffeomorphic to one of the followings (note that the fibration of S is not unique when B is not hyperbolic).

(I) $B = T^2$.

(1) $S = T^4$, $X = E^4$;
(2) $S = \{I, I, (a, b)\}$ with $(a, b) \neq (0, 0)$, $X = \mathrm{Nil}^3 \times E$;
(3) $S = \{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, I; (a, b)\}, \lambda \neq 0, b \neq 0$, $X = \mathrm{Nil}^4$;
(4) $S = \{A, I; (a, b)\}$ with $\mathrm{tr}\, A \geq 3$, $X = \mathrm{Sol}^3 \times E$.

(II) B has genus 0.

(1) The rational euler class e of S equals $(0, 0)$, $X = E^4$;
(2) $e \neq (0, 0)$, $X = \mathrm{Nil}^3 \times E$.

Here S is diffeomorphic to a hyperelliptic surface in case (II-1), a primary Kodaira surface in case (I-2), and a secondary Kodaira surface in case (II-2). We note that

$$G = \begin{cases} \mathrm{Isom}^0 E^4 = \mathbf{R}^4 \rtimes SO_4 & \text{if } X = E^4, \\ \mathrm{Nil}^3 \rtimes SO_2 \times \mathbf{R} & \text{if } X = \mathrm{Nil}^3 \times E, \\ \mathrm{Sol}^3 \times \mathbf{R} & \text{if } X = \mathrm{Sol}^3 \times E, \\ \mathrm{Nil}^4 & \text{if } X = \mathrm{Nil}^4. \end{cases}$$

Theorem B. *For the Seifert 4-manifold $S = \Gamma\backslash X$ in the above list we have the list of $\mathcal{T}(\Gamma, G)$ as follows.*

(1) *The cases when $B = T^2$.*

$$\mathcal{T}(\Gamma, G) = \begin{cases} SO_4\backslash GL_4\mathbf{R}, & \textit{in case } \text{(I-1)}, \\ \mathcal{T}_{1,0} \times \mathcal{F}, & \textit{in case } \text{(I-2)}, \\ (\mathbf{R}^*)^2 \times \mathbf{R}^2, & \textit{in case } \text{(I-3)}, \\ (\mathbf{R}^*)^2 \times (\mathbf{Z}_2)^2 \times \mathbf{R}, & \textit{in case } \text{(I-4)}. \end{cases}$$

Here in case (I-2) $\mathcal{T}_{1,0} = SO_2\backslash GL_2\mathbf{R} = \mathbf{R}^+ \times \mathbf{H} \times \mathbf{Z}_2$, $\mathcal{F} = \mathbf{R}^2 \times \mathbf{H} \times \mathbf{Z}_2$.

(2) *The cases with the base orbifolds of genus 0. Let r be the number of the cone points of B ($r = 3$ or 4). Then we have*

$$\mathcal{T}(\Gamma, G) = \mathcal{T}_{0,r} \times \mathbf{T}_1$$

where

$$\mathcal{T}_{0,r} = \begin{cases} \mathbf{R}^{2(r-3)} \times \mathbf{R}^+ \times \mathbf{Z}_2 & \textit{if } X = \mathrm{Nil}^3 \times E \\ \mathbf{R}^{2(r-3)} \times \mathbf{R}^+ & \textit{if } X = E^4 \end{cases}$$

and

$$\mathbf{T}_1 = \begin{cases} \mathbf{H} \times \mathbf{Z}_2 & \textit{if } X = \mathrm{Nil}^3 \times E \\ \mathbf{R}^+ \times \mathbf{H} \times \mathbf{Z}_2 & \textit{if } X = E^4. \end{cases}$$

Proof. In case (I-1) we have $\Gamma = \mathbf{Z}^4$ whose generators α_i are given by translations $\alpha_i x = x + \ell_i$ for $x, \ell_i \in E^4$. Here ℓ_i are mutually linearly independent and hence Γ is parametrized by $GL_4\mathbf{R}$. The action of Inn G is given by $(\ell_1, \dots, \ell_4) = (\sigma\ell_1, \dots, \sigma\ell_4)$ for $\sigma \in SO_4$. Hence $\mathcal{T}(\Gamma, G) = SO_4\backslash GL_4\mathbf{R}$ which has two components and each one is homeomorphic to $\mathbf{R}^{10}$.

Next consider case (I-2). In this case $X = \mathrm{Nil}^3 \times E$. Here we recall the structure of $\mathrm{Nil}^3 \times E$. The point of X is represented by $(w, z) \in \mathbf{C}^2$ such that the action of $G = (\mathrm{Nil}^3 \times \mathbf{R}) \rtimes S^1$ is defined by $(w', z')(w, z) = (w' + w - i\overline{z'}z, z' + z)$ for $(w', z') \in X$, and $t(w, z) = (w, tz)$ for $t \in S^1 \subset \mathbf{C}$. We can assume that $S = \{I, I; (a, 0)\}$ for $a \neq 0$ and $\Gamma = \{\alpha, \beta, \ell, h \mid$

$[\ell, h] = [\alpha, \ell] = [\alpha, h] = [\beta, \ell] = [\beta, h] = 1, [\alpha, \beta] = \ell^a\}$. The subgroup Γ_0 generated by ℓ, h is the center of Γ and the projection $\mathrm{Nil}^3 \times E \to \mathbf{R}^2$ defined by $(w, z) \to z$ induces the structure of a T^2-bundle over T^2 of the form $\Gamma_0 \backslash \mathbf{R} \times E \to \Gamma \backslash \mathrm{Nil}^3 \times E \to \mathbf{Z}^2 \backslash \mathbf{R}^2$ which gives the above fibration ([12]). Thus $\rho \in \mathcal{R}(\Gamma, G)$ must have the following form:

$$\begin{aligned}
\rho(\ell)(w, z) &= (w + \ell_0, z) \\
\rho(h)(w, z) &= (w + h_0, z) \\
\rho(\alpha)(w, z) &= (a_0 + w - i\overline{b_0}z, b_0 + z) \\
\rho(\beta)(w, z) &= (a_1 + w - i\overline{b_1}z, b_1 + z)
\end{aligned}$$

where ℓ_0 and h_0, b_0 and b_1 are linearly independent over $\mathbf{R}$. Since $\rho([\alpha, \beta])(w, z) = (w + i(b_0\overline{b_1} - \overline{b_0}b_1), z)$ must be equal to $\rho(\ell^a)(w, z)$, we have $\ell_0 = i(b_0\overline{b_1} - \overline{b_0}b_1)/a$ and since this is a nonzero real number h_0 is an arbitrary number with $\Im h_0 \neq 0$. Thus we have

$$\mathcal{R}(\Gamma, G) = \widetilde{\mathcal{F}} \times \mathcal{R}(\mathbf{Z}^2, \overline{G})$$

where $\widetilde{\mathcal{F}} = \mathbf{H} \times \mathbf{Z}_2 \times \mathbf{C}^2$ which is represented by h_0, a_0, a_1, $\overline{G} = \mathrm{Isom}^0\, E^2$, and $\mathcal{R}(\mathbf{Z}^2, G) = GL_2\mathbf{R}$ represented by b_0 and b_1. Next taking the conjugation of ρ by $\gamma = (w_0, z_0)$ and $t \in S^1$ we can see that the parameters are transformed as follows:

$$\begin{aligned}
a_0 &\to a_0 + i(\overline{b_0}z_0 - \overline{z_0}b_0) \\
a_1 &\to a_1 + i(\overline{b_1}z_0 - \overline{z_0}b_1) \\
b_0 &\to tb_0 \\
b_1 &\to tb_1.
\end{aligned}$$

Thus to get the representation in $\mathcal{T}(\Gamma, G)$ we can assume that $b_0 \in \mathbf{R}^+$ (choose $t \in S^1$ appropriately). Then $i(\overline{b_0}z_0 - \overline{z_0}b_0) = -2b_0\Im z_0$ and hence choosing z_0 so that $\Im z_0 = \Re a_0/2b_0$ then $a_0 + i(\overline{b_0}z_0 - \overline{z_0}b_0)$ is pure imaginary. On the other hand $i(\overline{b_1}z_0 - \overline{z_0}b_1) = 2((\Im b_1)(\Re z_0) - (\Re b_1)(\Re a_0/2b_0))$. Since $\Im b_1 \neq 0$ by the assumption we can choose z_0 so that both a_0 and a_1 are transformed to pure imaginary numbers. Consequently we have $\mathcal{T}(\Gamma, G) = \mathcal{F} \times \mathcal{T}(\overline{\Gamma}, \overline{G})$ where $\mathcal{F} = \mathbf{R}^2 \times \mathbf{H} \times \mathbf{Z}_2$ represented by ia_0, ia_1, h_0 and $\mathcal{T}(\overline{\Gamma}, \overline{G}) = SO_2 \backslash GL_2\mathbf{R} = \mathbf{R}^+ \times \mathbf{H} \times \mathbf{Z}_2$ represented by b_0 and b_1.

In case (II) we can assume that

$$S = \{(0, 0), (m_1, a_1, b_1), \dots, (m_r, a_r, b_r)\}$$

with $r = 3$ or 4 and

$$\Gamma = \{q_1, \ldots, q_r, \ell, h \mid [\ell, h] = [q_i, \ell] = [q_i, h] = 1,$$

$$q_i^{m_i} \ell^{a_i} h^{b_i} = 1, \prod q_i = 1\}.$$

Here if $r = 3$, then $(m_1, m_2, m_3) = (2,4,4), (2,3,6)$, or $(3,3,3)$ and if $r = 4$ then $(m_1, m_2, m_3, m_4) = (2,2,2,2)$. Furthermore in this case $X = E^4$ or $\mathrm{Nil}^3 \times E$. In either case the subgroup Γ_0 generated by ℓ and h are the center of Γ and the exact sequence $1 \to \Gamma_0 \to \Gamma \to \overline{\Gamma} \to 1$ (where $\overline{\Gamma} = \pi_1^{\mathrm{orb}} B$) yields the original Seifert fibration. First suppose that $X = E^4$, $G = \mathbf{R}^4 \rtimes SO_4$. Pick up $\rho \in \mathcal{R}(\Gamma, G)$. Then the holonomy group of $\rho(\Gamma)$ which is the image $\overline{\rho(\Gamma)}$ of $\rho(\Gamma)$ under the natural map $G \to SO_4$ is cyclic (since S is diffeomorphic to a hyperelliptic surface). Since $\rho(\Gamma_0)$ must be contained in the translation parts of G we can assume that there is a decomposition $\mathbf{C} \times \mathbf{C}$ of E^4 such that $\rho(\Gamma_0)$ acts trivially on the first factor. Since $\rho(\Gamma_0)$ commutes with any element in $\rho(\Gamma)$ we can see that any element of $\overline{\rho(\Gamma)}$ is contained in $SO_2 \times 1 \subset SO_2 \times SO_2 \subset SO_4$. If we take another $\rho' \in \mathcal{R}(\Gamma, G)$, then there exists $\sigma \in SO_4$ such that $\sigma\rho'(\Gamma_0)\sigma^{-1}$ satisfies the above condition for the same decomposition of E^4 and hence the image of $\sigma\rho'(\Gamma)\sigma^{-1}$ under the above map is also contained in the same subgroup $SO_2 \times 1 \subset SO_4$. Therefore it suffices to consider the representation ρ satisfying the above conditions for the fixed decomposition of E^4. Thus ρ projects to $\overline{\rho} \in \mathcal{R}(\overline{\Gamma}, \overline{G})$ where $\overline{G} = \mathrm{Isom}^0 E^2$ and we must have

$$\begin{aligned} \rho(\ell)(z, w) &= (z, w + \ell_0) \\ \rho(h)(z, w) &= (z, w + h_0) \\ \rho(q_i)(z, w) &= (\overline{\rho}(\overline{q_i})z, w + s_i), \quad (1 \le i \le r) \end{aligned}$$

where ℓ_0 and h_0 are linearly independent over $\mathbf{R}$, $\overline{q_i} \in \overline{\Gamma}$ is the image of q_i under the projection $\Gamma \to \overline{\Gamma}$, $s_i \in \mathbf{C}$. Then from the relation $q_i^{m_i} \ell^{a_i} h^{b_i} = 1$ we deduce $s_i = -(a_i \ell_0 + b_i h_0)/m_i$. Next we must see exactly when the two representations ρ and ρ' of the above forms are in the same orbit under the action of $\mathrm{Inn}\, G$. Suppose that there exists $\sigma \in G$ such that $\sigma\rho\sigma^{-1} = \rho'$ with $\sigma\mathbf{x} = \overline{\sigma}\mathbf{x} + s_0$, $\overline{\sigma} \in SO_4$, $s_0 \in \mathbf{R}^4$ and $\mathbf{x} \in \mathbf{R}^4$. Then we can see that $\overline{\sigma}\mathbf{x} = (\sigma_1 z, \sigma_2 w)$ where $\sigma_1, \sigma_2 \in SO_2$ or $\sigma_1, \sigma_2 \in O_2 - SO_2$ and $\mathbf{x} = (z, w)$ with $z, w \in \mathbf{C}$. Suppose that $\sigma(z, w) = (\sigma_1 z + a_0, \sigma_2 w + b_0)$ with $\sigma_1, \sigma_2 \in SO_2$, $a_0, b_0 \in \mathbf{C}$. Then ρ'

satisfies

$$\begin{aligned}
\rho'(\ell)(z,w) &= (z, w+\sigma_2\ell_0)\\
\rho'(h)(z,w) &= (z, w+\sigma_2 h_0)\\
\rho'(q_i)(z,w) &= (\sigma_1\overline{\rho}(\overline{q_i})(\sigma_1^{-1}(z-a_0))+a_0, w+\sigma_2 s_i).
\end{aligned}$$

Next suppose that $\sigma(z,w) = (\sigma_1\overline{z}+a_0, \sigma_2\overline{w}+b_0)$ with $\sigma_1, \sigma_2 \in SO_2$, $a_0, b_0 \in \mathbf{C}$. Then

$$\begin{aligned}
\rho'(\ell)(z,w) &= (z, w+\sigma_2\overline{\ell_0})\\
\rho'(h)(z,w) &= (z, w+\sigma_2\overline{h_0})\\
\rho'(q_i)(z,w) &= (\overline{\rho'(q_i)}(z), w+\sigma_2\overline{s_i})
\end{aligned}$$

where $\overline{\rho'(q_i)}$ is in the component of $\mathcal{R}(\overline{\Gamma},\overline{G})$ different from that containing $\overline{\rho}$. On the other hand the identity component $\mathcal{R}^0(\overline{\Gamma},\overline{G})$ of $\mathcal{R}(\overline{\Gamma},\overline{G})$ is homeomorphic to $\mathbf{R}^{2(r-3)}\times\mathbf{R}^+$ where the first factor coincides with the Teichmüller space for the flat structures of area 1 of the base orbifold B and the second factor corresponds to the area of B. Thus we have $\mathcal{T}(\Gamma,G)=\mathbf{R}^{2(r-3)}\times\mathbf{R}^+\times\mathbf{R}^+\times\mathbf{H}\times\mathbf{Z}_2$ where $\mathbf{R}^+\times\mathbf{H}\times\mathbf{Z}_2 = SO_2\backslash GL_2\mathbf{R}$ corresponds to the deformations of the lattice of the general fiber.

On the other hand in case (II) with $X = \mathrm{Nil}^3\times E$ the natural projection $X\to\mathbf{C}$ represented by $(w,z)\to z$ for the coordinates defined above yields a given fibration for $S = \{(a,b),(m_1,a_1,b_1),\ldots,(m_r,a_r,b_r)\}$. (In fact we can assume that $a=b=0$.) Here we can assume that $a+\sum a_i/m_i = 0$, $b+\sum b_i/m_i \neq 0$. Take an arbitrary representation $\rho\in\mathcal{R}(\Gamma,G)$. Then ρ induces a representation $\overline{\rho}\in\mathcal{R}(\overline{\Gamma},\overline{G})$ where $\overline{\Gamma} = \{\overline{q_1},\ldots,\overline{q_r}\mid \overline{q}_1^{m_1}=\cdots=\overline{q}_r^{m_r}=\overline{q}_1\cdots\overline{q}_r=1\}$ and $\rho(\overline{\Gamma})\subset\overline{G}=\mathrm{Isom}^0 E^2$. Here r and $(m_1,\ldots,m_r)$ satisfy the same conditions as in the case with $X=E^4$. Each $\overline{\rho}(\overline{q_i})$ has the following representation:

$$\overline{\rho}(\overline{q_j})z = \rho_j(z-z_j)+z_j$$

for $\rho_j\in S^1$, $z_j\in\mathbf{C}$ such that the order of ρ_j is m_j and $\prod\rho_j = 1$. Thus ρ must be of the following form:

$$\begin{aligned}
\rho(\ell)(w,z) &= (w+\ell_0, z)\\
\rho(h)(w,z) &= (w+h_0, z)\\
\rho(q_j)(w,z) &= (w+w_j+i\overline{z_j}(z-\rho_j(z-z_j)), \overline{\rho}(\overline{q_j})z)
\end{aligned}$$

where ℓ_0 and h_0 are linearly independent over $\mathbf{R}$, $w_j\in\mathbf{C}$. Then we have

$$\rho(q_j^{m_j})(w,z) = (w+m_j(w_j+i|z_j|^2), z).$$

Thus form $q_j^{m_j}\ell^{a_j}h^{b_j} = 1$ we deduce

$$(1) \qquad w_j = -i|z_j|^2 - a_j\ell_0/m_j - b_jh_0/m_j$$

Suppose that $r = 3$. Then from $\prod \overline{q_j} = 1$ we have

$$(2) \qquad (1-\rho_1)z_1 + (\rho_1 - \rho_1\rho_2)z_2 + (\rho_1\rho_2 - \rho_1\rho_2\rho_3)z_3 = 0.$$

Thus we can see that

$$\rho(q_1q_2q_3)(w,z) = (w + \sum_{j=1}^{3} w_j + iU, z)$$

where

$$U = \sum \rho_j|z_j|^2 + \overline{z_2}z_3(1-\rho_2-\rho_3+\rho_2\rho_3) + \overline{z_1}z_2(1-\rho_1-\rho_2+\rho_1\rho_2) +$$
$$\overline{z_1}z_3(1-\overline{\rho}_1-\overline{\rho}_3+\overline{\rho}_1\overline{\rho}_3).$$

Thus from (1) ρ is well defined if and only if

$$(3) \qquad iV = (a + \sum a_j/m_j)\ell_0 + (b + \sum b_j/m_j)h_0$$

where $V = U - \sum |z_j|^2$.

Claim. *iV is a nonzero real number.*

Easy computation (using (2)) shows that iV is invariant under translations along the real line $(z_1, z_2, z_3) \to (z_1+\lambda, z_2+\lambda, z_3+\lambda)$ for $\lambda \in \mathbf{R}$ and the rotations in the origin. Therefore to prove Claim we can assume that $z_1 = 0$ and z_2 is a nonzero real number r. Then again by easy computation we can see that $V = r^2(\rho_1 - 1)(\rho_2 - 1)/(\rho_1\rho_2 - 1)$ and V is a nonzero pure imaginary number. Thus we deduce that $h_0 = iV/(b + \sum b_j/m_j) \neq 0$ from the normalization $a + \sum a_j/m_j = 0$. Then ℓ_0 is an arbitrary number with $\Im \ell_0 \neq 0$. The case with $r = 4$ ($m_i = 2, \rho_j = -1$ for all j) can be treated by a similar computation and we can see that h_0 is some fixed real number and ℓ_0 is also an arbitrary number with $\Im \ell_0 \neq 0$. In any case $\overline{\rho} \in \mathcal{R}(\overline{\Gamma}, \overline{G})$ can be lifted to some $\rho \in \mathcal{R}(\Gamma, G)$ and we have $\mathcal{R}(\Gamma, G) = \mathcal{R}(\overline{\Gamma}, \overline{G}) \times \mathbf{H} \times \mathbf{Z}_2$ where the factor $\mathbf{H} \times \mathbf{Z}_2$ corresponds to ℓ_0. The action of Inn G on $\mathcal{R}(\Gamma, G)$ can be determined as in case (I-2). For any element $\gamma \in G$ the conjugation by γ acts on the factor $\overline{\rho}(\overline{q_i})$ as the inner automorphism of $\overline{G}$ and acts trivially on $\rho(\ell), \rho(h)$. Since these parameters determine the remaining ones uniquely and the natural map Inn $G \to$ Inn $\overline{G}$ is surjective we have

$$\mathcal{T}(\Gamma, G) = \mathcal{T}(\overline{\Gamma}, \overline{G}) \times \mathbf{H} \times \mathbf{Z}_2.$$

The proofs for the other cases are done by similar methods and hence are omitted.

§3. The Teichmüller spaces and the complex structures

In this section we consider the Seifert 4-manifolds over closed orientable hyperbolic or euclidean 2-orbifolds which admit complex structures. The arguments in [12, §7] show that any elliptic surface S with $c_2 = 0$ and with $\kappa = 0$ or 1 is biholomorphic to $\Gamma\backslash X$ where a geometry X has a complex structure such that any element of $G = \mathrm{Isom}^0 X$ acts on X as a biholomorphism and $\Gamma \subset G$. (We need to restrict G to $U(2)$ when $X = E^4$.) Here we start with such a Seifert 4-manifold $S = \Gamma\backslash X$ with a given compatible complex structure. Let g be the genus of B and r be the number of the cone points (with the prescribed cone angles) of $B = \overline{\Gamma}\backslash\overline{X}$. In this section let $\mathcal{R} = \mathcal{R}^0(\Gamma, G)$ and $\mathcal{T} = \mathcal{T}^0(\Gamma, G)$ be the connected components of $\mathcal{R}(\Gamma, G)$ and of $\mathcal{T}(\Gamma, G)$ containing $S = \Gamma\backslash X$ with given geometric structure $\rho_0 \in \mathcal{R}$ and its equivalence class $[\rho_0] \in \mathcal{T}$ respectively.

First suppose that B is hyperbolic. In this case $X = H^2 \times E^2$ or $\widetilde{SL_2} \times E$ each of which is identified with $\mathbf{H} \times \mathbf{C}$ as in §1. Hereafter we adopt the same notations for Γ, $\overline{\Gamma}$ and S as in §1 and we assume that the monodromies of S satisfy the conditions in Proposition 1. Now we will describe $\mathcal{T}^0(\Gamma, G)$ as a differentiable family of the complex structures on S. Let S_ρ be the Seifert 4-manifold S with the geometric structure corresponding to $\rho \in \mathcal{R}$ and $[\rho]$ be its equivalence class in $\mathcal{T}$ respectively. For simplicity put $S_0 = S_{\rho_0}$ and let Θ_0 be the sheaf of germs of holomorphic tangent vector fields on S_0. First recall that $\mathcal{T} = \overline{\mathcal{T}} \times \mathbf{T}_0 \times H^1(\overline{\Gamma}, \mathbf{C}^\phi)$. Here $\overline{\mathcal{T}}$ is (the connected component of) the Teichmüller space of $B = \overline{\Gamma}\backslash\mathbf{H}$ or equivalently of the Fuchsian group $\overline{\Gamma}$, ϕ is the monodromy representation for S, and $\mathbf{T}_0$ is the identity component of $\mathbf{T}_1$ in §1. Put

$$\widetilde{\mathbf{T}}_0 = \begin{cases} \mathbf{H} & \text{if } \phi \equiv \pm\,\mathrm{id} \\ 1 & \text{otherwise} \end{cases}$$

and let $\widetilde{\mathcal{T}} = \overline{\mathcal{T}} \times \widetilde{\mathbf{T}}_0 \times H^1(\overline{\Gamma}, \mathbf{C}^\phi)$. Then $\widetilde{\mathcal{T}}$ denotes the Teichmüller space of the geometric structures which fix the area of the general fiber (if $X = H^2 \times E^2$). On the other hand the Teichmüller space $\overline{\mathcal{T}}$ for the Fuchsian group $\overline{\Gamma}$ is realized as a bounded domain in $\mathbf{C}^{3g-3+r}$. Moreover there is a fiber space $\overline{\mathcal{C}} = \{(z, \tau) \in \mathbf{C} \times \overline{\mathcal{T}} \mid z \in D(\tau)\}$ over $\overline{\mathcal{T}}$ where $D(\tau)$ is a domain in $\mathbf{C}$ on which a quasi-Fuchsian group $\overline{\Gamma}^\tau$ corresponding to τ acts

([1]). $\overline{\mathcal{C}}$ has a complex structure such that the map $(z, \tau) \to (\tau(\alpha)z, \tau)$ for any $\alpha \in \overline{\Gamma}$ is holomorphic. Here $\tau(\alpha)$ acts on $z \in D(\tau)$ via $\overline{\Gamma}^{\tau}$. It follows that for any $\tau \in \overline{\mathcal{T}}$ we have a biholomorphism $h_\tau : \mathbf{H} \to D(\tau)$ and a representation $\overline{\rho} \in \overline{\mathcal{R}}$ with $[\overline{\rho}] = \tau$ such that $\tau(\alpha) \cdot h_\tau(z) = h_\tau(\overline{\rho}(\alpha)z)$ for any $z \in \mathbf{H}, \alpha \in \overline{\Gamma}$.

If $X = H^2 \times E^2$ then h_τ is lifted to $h_\tau \times \mathrm{id}: \mathbf{H} \times \mathbf{C} \to D(\tau) \times \mathbf{C}$ such that it commutes with the translations in the $\mathbf{C}$-factor. On the other hand we can choose the elements $m_1, \ldots, m_d$ of $C^1(\overline{\Gamma}, \mathbf{C}^\phi)$ which maps to the basis of $H^1(\overline{\Gamma}, \mathbf{C}^\phi)$ where $d = \dim_{\mathbf{C}} H^1(\overline{\Gamma}, \mathbf{C}^\phi)$. Thus we have the family of representations of Γ on $\overline{\mathcal{C}} \times \mathbf{C} \times \mathbf{C}^d \times \mathbf{T}_0$ as follows. Let $\tau_0 \in \overline{\mathcal{T}}$ be the equivalence class of the element $\overline{\rho_0} \in \overline{\mathcal{R}}$ determined by ρ_0. Then S_0 is biholomorphic to $\overline{\Gamma}^{\tau_0} \backslash (D(\tau_0) \times \mathbf{C})$ where ρ_0 is represented (via $\overline{\Gamma}^{\tau_0}$) as follows;

$$
\begin{aligned}
\rho_0(\ell)(z, w) &= (z, w + r_0), \\
\rho_0(h)(z, w) &= (z, w + r_0 h_0), \\
\rho_0(\alpha_i)(z, w) &= (\tau_0(\overline{\alpha_i})z, \phi(\overline{\alpha_i})w + w_j^0), \\
\rho_0(q_i)(z, w) &= (\tau_0(\overline{q_i})z, w - a_i r_0/m_i - b_i r_0 h_0/m_i)
\end{aligned}
$$

where $(z, w) \in D(\tau_0) \times \mathbf{C}$, $r_0 \in \mathbf{R}^+$, $h_0 \in \mathbf{H}$ and $w_j^0 \in \mathbf{C}$ is defined as in the proof of Theorem A. Then we have the following representations $\rho = \rho(\tau, r, h, s)$ from $\overline{\Gamma}$ to the group of the biholomorphisms of $D(\tau) \times \mathbf{C}$ where $s = (s_1, \ldots, s_d) \in \mathbf{C}^d$, $\tau \in \overline{\mathcal{T}}$, $(r, h) \in \mathbf{T}_0$ ($h = h_0$ if $\phi \not\equiv \pm\,\mathrm{id}$).

$$
\begin{aligned}
\rho(\ell)(z, w) &= (z, w + r), \\
\rho(h)(z, w) &= (z, w + rh), \\
\rho(\alpha_i)(z, w) &= (\tau(\overline{\alpha_i})z, \phi(\overline{\alpha_i})w + w_j^0 + \sum_{j=1}^{d} s_j m_j(\overline{\alpha_i})), \\
\rho(q_i)(z, w) &= (\tau(\overline{q_i})z, w - a_i r/m_i - b_i rh/m_i).
\end{aligned}
$$

Here $H^1(\overline{\Gamma}, \mathbf{C}^\phi)$ is identified with $\mathbf{C}^d$. Thus we get the fiber space $\mathcal{C}$ over $\mathcal{T}$ obtained from $\overline{\mathcal{C}} \times \mathbf{C} \times \mathbf{T}_0 \times H^1(\overline{\Gamma}, \mathbf{C}^\phi)$ by the actions of ρ defined above such that the fiber of $\mathcal{C}$ over $\tilde{\tau} \in \mathcal{T}$ is an elliptic surface corresponding to $\tilde{\tau}$. Also we have the fiber space $\widetilde{\mathcal{C}}$ over $\widetilde{\mathcal{T}}$ by restricting the above representations to the cases with $r = r_0$ (the constant). The above representation depends holomorphically on all the parameters except for $r \in \mathbf{R}^+$ (if $\mathbf{T}_0$ has the $\mathbf{R}^+$-factor). Therefore we have the differentiable family $\mathcal{C} \to \mathcal{T}$ and the complex analytic family $\widetilde{\mathcal{C}} \to \widetilde{\mathcal{T}}$ of the complex structures on S respectively.

If $X = \widetilde{SL_2} \times E$ then X is identified with $\mathbf{H} \times \mathbf{C}$ with coordinates (z, w) so that w corresponds to $\log dz$. Hence in this case $h_\tau: \mathbf{H} \to D(\tau)$ is lifted to the biholomorphism $\widetilde{h_\tau}: \mathbf{H} \times \mathbf{C} \to D(\tau) \times \mathbf{C}$ defined by $\widetilde{h_\tau}(z, w) = (h_\tau(z), w + \log(\partial h_\tau/\partial z)(z))$ where the branch of the log is chosen so that $\widetilde{h_\tau}$ depends holomorphically on $\tau \in \overline{\mathcal{T}}$. Again $\widetilde{h_\tau}$ commutes with the translation $(z, w) \to (z, w + c)$ with $c =$ a constant. On the other hand we can define $\tilde{\rho}(\alpha): \mathbf{H} \times \mathbf{C} \to \mathbf{H} \times \mathbf{C}$ for $\alpha \in \Gamma$ such that $\widetilde{\alpha}_i = \tilde{\rho}(\alpha_i)$ (where $\overline{\alpha}_i \in \overline{\Gamma}$ is hyperbolic), $\widetilde{q}_i = \tilde{\rho}(q_i)$ (where $\overline{q}_i \in \overline{\Gamma}$ is elliptic) satisfy the same conditions as $\widetilde{\alpha}_i$, $\widetilde{q}_i$ stated in the proof of Theorem A. Thus for $\overline{\rho} \in \overline{\mathcal{R}}$ with $[\overline{\rho}] = \tau \in \overline{\mathcal{T}}$, we can define $\tilde{\tau}(\alpha): D(\tau) \times \mathbf{C} \to D(\tau) \times \mathbf{C}$ for $\alpha = \alpha_i$ or q_i which covers $\tau(\overline{\alpha})$ such that $\widetilde{h_\tau}\tilde{\tau}(\alpha) = \tilde{\rho}(\alpha)\widetilde{h_\tau}$ where $\overline{\alpha}$ is the image of α in $\overline{\Gamma}$. Using this lift we can define the family of representations parametrized by $\mathcal{T}$ (in this case $\widetilde{\mathcal{T}} = \mathcal{T}$ since there is no $\mathbf{R}$-factor in $\mathcal{T}$) as in the case with $X = H^2 \times E^2$. Thus we also get the analogous family of complex structures on S.

Next we will describe the Kodaira-Spencer's infinitesimal deformation map

$$\Phi: \mathbf{T}_0\mathcal{T} \to H^1(S_0, \Theta_0)$$

where $\mathbf{T}_0\mathcal{T}$ is the tangent space of $\mathcal{T}$ at S_0 (or equivalently at $[\rho_0]$). Since the base orbifold B of S is hyperbolic in our case here $\mathcal{T}$ is homeomorphic to a euclidean space and is homeomorphic to $\mathbf{T}_0\mathcal{T}$. To describe $H^1(S_0, \Theta_0)$ we recall some results in [8]. Let $\mathbf{T}^1$ be the complex torus of dimension 1. For a holomorphic Seifert fibering $S = \Gamma_\rho \backslash X$ with $\Gamma_\rho = \rho(\Gamma) \subset G$, the base orbifold B is naturally a nonsingular curve B_ρ of the form $\overline{\Gamma}_\rho \backslash \overline{X}$ where $\overline{X} = \mathbf{H}$ or $\mathbf{C}$, $\overline{\Gamma}_\rho = \pi_1^{\mathrm{orb}} B$. (In [8], $S, \overline{X}, B, \overline{\Gamma}$ are denoted by M, W, V, N respectively.) Then $\widetilde{S} \to \overline{X}$ induced by the covering projection $\overline{X} \to B$ is a principal $\mathbf{T}^1$-bundle and $\widetilde{S} = \overline{X} \times \mathbf{T}^1$ ([8, §1,§7] in which $\widetilde{S}$ is denoted by B). Let $\mathcal{Z}^2$, $\mathcal{O}$, $\mathcal{T}^1$ be the sheaves over $\overline{X}$ of germs of local holomorphic maps from $\overline{X}$ into $\mathbf{Z}^2$, $\mathbf{C}$, $\mathbf{T}^1$ respectively. Then the action of $\overline{\Gamma}$ on $\widetilde{S}$ is defined by the element $m \in H^1(\overline{\Gamma}, \mathcal{T}^1)$. Here $H^1(\overline{\Gamma}, \mathcal{T}^1) = H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{T}^1)^\phi)$ where the coefficients $H^0(\overline{X}, \mathcal{T}^1)$ on the right hand side is the space of global holomorphic maps on $\overline{X}$ and is twisted by the monodromy representation $\phi: \overline{\Gamma} \to GL_1\mathbf{C}$. The element m is represented by a crossed homomorphism $m: \overline{\Gamma} \to H^0(\overline{X}, \mathcal{T}^1)$ such that $m(x, \alpha)$ for a fixed $\alpha \in \overline{\Gamma}$ is a holomorphic map from $\overline{X}$ to $\mathbf{T}^1$ $(x \in \overline{X})$ satisfying

$$m(x, \alpha\beta) = \phi(\alpha)m(\alpha^{-1}x, \beta) + m(x, \alpha) \quad \text{for } \alpha, \beta \in \overline{\Gamma}$$

where $\overline{\Gamma}$ acts on $\overline{X}$ via $\overline{\rho}$ and $\phi(\alpha)$ gives the automorphism of $\mathbf{T}^1$ since it preserves the lattice of the fiber. The action of $\overline{\Gamma}$ on $\widetilde{S} = \overline{X} \times \mathbf{T}^1$ is given by

$$\alpha(x,t) = (\alpha x, \phi(\alpha)t + m(\alpha x, \alpha)) \quad \text{for } \alpha \in \overline{\Gamma}, x \in \overline{X}, t \in \mathbf{T}^1$$

(see [8, 7.2]). From the exact sequence

$$0 \to \mathcal{Z}^2 \to \mathcal{O} \to \mathcal{T}^1 \to 0$$

we have the exact sequence

$$H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{O})^\phi) \xrightarrow{\eta} H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{T}^1)^\phi) \xrightarrow{c} H^2(\overline{\Gamma}, \mathcal{Z}^{2\phi})$$

where $c(m)$ represents the extension $1 \to \mathbf{Z}^2 \to \Gamma \to \overline{\Gamma} \to 1$. On the other hand $H^1(S_0, \Theta_0)$ is described by the following exact sequences ([8, §2]).

$$(1) \qquad 0 \to D \to H^1(S_0, \Theta_0) \to G \to 0$$

$$(2) \qquad 0 \to F \to G \to H \to 0$$

$$(3) \qquad 0 \to C \to D \to E \to 0$$

where the exact sequence (3) splits ([8, §4]). Here

$$(4) \qquad E = \{a \in \mathbf{C}; a\overline{\phi(\alpha)} = \phi(\alpha)a \quad \text{for all } \alpha \in \overline{\Gamma}\}$$

corresponds to fiber deformations ([8, Theorem 7.10, §4]). Note that in our cases $\phi(\alpha)$ is a root of unity for any α when the base is hyperbolic (§1) and it suffices to consider the cases with trivial monodromies when the base is not hyperbolic (in the case of euclidean base orbifolds, we have only to consider the cases (I-1), (I-2), (II-1) and (II-2) in §2). Thus the kernel of ϕ has finite index in $\overline{\Gamma}$ and then the assumption in [8, §7] is automatically satisfied. Hence we have

$$E = \begin{cases} \mathbf{C}, & \text{if } \phi(\alpha) = \pm 1 \text{ for any } \alpha \in \overline{\Gamma} \\ 0, & \text{otherwise.} \end{cases}$$

The subspace C in (3) corresponds to the twist deformations coming from the the complex analytic family of the form $m + \eta(s\ell) \in H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{T}^1)^\phi)$ for $\ell \in H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{O})^\phi), s \in \mathbf{C}$ ([8, §3]). In our case by [8, §7] we have

$$(5) \qquad C = H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{O})^\phi)$$

unless $g = 1, r = 0$. The subspace F in (2) corresponds to the base deformations and by [8, §7], we have

$$F = H^1(\overline{\Gamma}, \Theta_{\overline{X}}) = H^1(B, \Theta_{B|d}) \tag{6}$$

where $B = B_{\rho_0}$ is considered as a nonsingular curve, $d = \sum_{i=1}^{r} p_i$ is the divisor corresponding to the cone point p_i and $\Theta_{B|d}$ is the sheaf of germs of holomorphic tangent vector fields on B which vanish on d. Finally the element of the space H represents (if it is not obstructed) a deformation of S which destroys the fiber structure. By [8, Theorem 7.13] we have

$$H = 0 \quad \text{unless } g = 1, r = 0 \text{ or } g = 0, r < 3. \tag{7}$$

Now we consider the Kodaira Spencer map $\Phi\colon \mathbf{T}_0\mathcal{T} \to H^1(S_0, \Theta_0)$ for the case with $X = H^2 \times E^2$. The tangent space $\mathbf{T}_0\mathcal{T}$ is homeomorphic to $\mathbf{T}_0\overline{\mathcal{T}} \times \mathcal{F}_0$ where $\mathcal{F}_0 = \mathbf{T}_0 \times H^1(\overline{\Gamma}, \mathbf{C}^{\phi})$ and $\mathbf{T}_0 = \mathbf{R}^+ \times \mathbf{H}$ if $\phi \equiv \pm\,\mathrm{id}$, $\mathbf{T}_0 = \mathbf{R}^+$ if $\phi \not\equiv \pm\,\mathrm{id}$. The derivatives of the family of representations defined above span $\mathbf{T}_0\mathcal{T}$ and the discussions in §3–§5 in [8] show that the Kodaira Spencer map preserves the fiber structure as is indicated by the following commutative diagram.

$$\begin{array}{ccc} \mathbf{T}_0\mathcal{T} & \xrightarrow{\Phi} & H^1(S_0, \Theta_0) \\ \downarrow & & \downarrow \\ \mathbf{T}_0\overline{\mathcal{T}} & \xrightarrow{\overline{\Phi}} & F \end{array} \tag{8}$$

Here the vertical maps are the projections of the fibrations and $\overline{\Phi}$ gives the infinitesimal deformation map for the Teichmüller space of the r-pointed Riemann surface of genus g at $B = B_{\rho_0}$. We have $\mathbf{T}_0\overline{\mathcal{T}} = \mathbf{R}^{2(3g-3+r)}$ ([6]), $\dim_{\mathbf{C}} F = 3g - 3 + r$ ([8, Lemma 7.3]) and $\overline{\Phi}$ is a homeomorphism. Here we note that if two geometric Seifert 4-manifolds $S = \Gamma\backslash X, S' = \Gamma'\backslash X$ (with $X = \mathbf{H} \times \mathbf{C}$) over the hyperbolic 2-orbifolds B, B' are biholomorphic then B and B' are isometric. For, any biholomorphism $\varphi\colon S \to S'$ is lifted to a biholomorphism $\tilde{\varphi}$ from $\mathbf{H} \times \mathbf{C}$ to itself such that there is an automorphism $\psi\colon \Gamma \to \Gamma'$ satisfying $\tilde{\varphi}(\gamma(z, w)) = \psi(\gamma)(\tilde{\varphi}(z, w))$ for $(z, w) \in \mathbf{H} \times \mathbf{C}$, $\gamma \in \Gamma$. Here Γ and Γ' have the exact sequences $1 \to \Gamma_0 \to \Gamma \to \overline{\Gamma} \to 1$ and $1 \to \Gamma'_0 \to \Gamma' \to \overline{\Gamma}' \to 1$ such that $\overline{\Gamma} = \pi_1^{\mathrm{orb}} B$, $\overline{\Gamma}' = \pi_1^{\mathrm{orb}} B'$ and Γ_0, Γ'_0 correspond to the fundamental groups of the general fibers respectively. Moreover ψ induces the isomorphism between Γ_0 and Γ'_0 and also induces the isomorphism $\overline{\psi}\colon \overline{\Gamma} \to \overline{\Gamma}'$. On the other hand Γ_0 and Γ'_0 act on $\mathbf{H} \times \mathbf{C}$ by the translations in the $\mathbf{C}$-factor since S and S' are geometric (of type $H^2 \times E^2$ or $\widetilde{SL}_2 \times E$).

Hence if we write $\tilde{\varphi}(z,w) = (h(z,w), k(z,w)) \in \mathbf{H} \times \mathbf{C}$, then $h(z,w)$ is invariant under the action of Γ_0 in the w-coordinate. Since Γ_0 has rank 2, $h(z,w)$ depends only on z, i.e., $h(z,w) = h(z)$ which gives a biholomorphism from $\mathbf{H}$ to itself and $\tilde{\varphi}$ descends to an isometry $h: \mathbf{H} \to \mathbf{H}$. Since we have $h(\overline{\gamma}(z)) = \overline{\psi}(\overline{\gamma})h(z)$ for $z \in \mathbf{H}$, $\overline{\gamma} \in \overline{\Gamma}$ we can see that B and B' are isometric. Thus by the fact that the action of Aut $\overline{\Gamma}$ on the Teichmüller space of B is properly discontinuous we can see directly that the kernel of $\overline{\Phi}$ induced from Φ above is zero. Thus $\overline{\Phi}$ is an isomorphism (compare the dimensions of the spaces in (8)). Moreover Φ induces the map between the fibers of the projections in (8) of the form

$$H^1(\overline{\Gamma}, \mathbf{C}^{\phi}) \times \mathbf{T}_0 \xrightarrow{\Phi_1 \times \Phi_2} C \times E. \tag{9}$$

If $\phi \equiv \pm\,\mathrm{id}$ then $E = \mathbf{C}$ such that small $s \in \mathbf{C}$ determines the complex structure of the general fiber whose period matrix is given by $\Omega(s) = (1+s, h_0 + s\overline{h_0})$ where $\Omega(0)$ corresponds to that for the original S_0 ([8, Lemma 4.5]). We have the representations ρ parametrized by s in the above family whose $\mathbf{T}_0$-component $h(s)$ satisfies $h(s) = (h_0 + s\overline{h_0})/(1+s)$ and $\Phi(\partial/\partial s)$ corresponds to 1 ([8, §4]). Thus Φ_2 maps $\mathbf{T}_0$ onto E whose kernel is the $\mathbf{R}^+$-component of $\mathbf{T}_0$ represented by the parameter detecting the deformation of the area of the general fiber. On the other hand $C = H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{O})^{\phi})$ and Φ_1 is the map $H^1(\overline{\Gamma}, \mathbf{C}^{\phi}) \to H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{O})^{\phi})$ induced by the natural inclusion $\mathbf{C} \subset H^0(\overline{X}, \mathcal{O})$. Moreover by the naturality of the spectral sequences (used in [8]) we have the following commutative diagram.

$$\begin{array}{ccc} H^1(\overline{\Gamma}, \mathbf{C}^{\phi}) & \xrightarrow{\Phi_1} & H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{O})^{\phi}) \\ \uparrow{\scriptstyle \varphi_1} & & \uparrow{\scriptstyle \varphi_2} \\ H^1(B, \mathbf{C}(Q)) & \xrightarrow{\Phi'} & H^1(B, \mathcal{O}(Q)) \end{array}$$

Here Q is the flat $\mathbf{C}$-bundle over $B = B_{\rho_0}$ determined by the monodromy representation ϕ which can be considered as the representation of $\pi_1(B_{\rho_0})$ (see §1). The coefficient $\mathbf{C}(Q)$ (resp. $\mathcal{O}(Q)$) is the sheaf of the germs of locally constant (resp. holomorphic) sections of Q and φ_1 and φ_2 are the isomorphisms ([8, §7]). The map Φ' is the part of the following exact sequence (in which the base B is omitted)

$$\begin{aligned} 0 &\to H^0(\mathbf{C}(Q)) \to H^0(\mathcal{O}(Q)) \to H^0(\Omega^1(Q)) \\ &\to H^1(\mathbf{C}(Q)) \xrightarrow{\Phi'} H^1(\mathcal{O}(Q)) \to H^1(\Omega^1(Q)) \end{aligned}$$

which comes from the exact sequence

$$0 \to \mathbf{C}(Q) \to \mathcal{O}(Q) \xrightarrow{\partial} \Omega^1(Q) \to 0.$$

Here Ω^1 is the sheaf of the germs of holomorphic 1 forms on the non-singular curve B. By the Riemann-Roch theorem (and since $c_1(Q) = 0$) $\dim_{\mathbf{C}} H^0(\Omega^1(Q)) = \dim_{\mathbf{C}} H^1(\mathcal{O}(Q))$ equals g if $\phi = \mathrm{id}$ and equals $g-1$ otherwise (cf. [8, §7]). Comparing this with $\dim_{\mathbf{C}} H^1(\mathbf{C}(Q)) = \dim_{\mathbf{C}} H^1(\overline{\Gamma}, \mathbf{C}^{\phi})$ which equals $2g$ if $\phi \equiv \mathrm{id}$ and $2g-2$ otherwise we deduce the exact sequence

$$0 \to H^0(\Omega^1(Q)) \to H^1(\mathbf{C}(Q)) \xrightarrow{\Phi'} H^1(\mathcal{O}(Q)) \to 0.$$

Hence the kernel of Φ_1 is isomorphic to $H^0(B, \Omega^1(Q))$. The same argument holds for the case with $X = \widetilde{SL_2} \times E$ except for the fact that the kernel of Φ_2 is trivial since there is no $\mathbf{R}^+$-component in $\mathbf{T}_0$. Thus we have

Theorem C-1. *The Kodaira Spencer map Φ for the Teichmüller space $\mathcal{T}$ for the Seifert 4-manifold S over the closed orientable hyperbolic 2-orbifold B with any given representation $\rho \in \mathcal{T}$ is surjective and the kernel of $\Phi: \mathbf{T}_0\mathcal{T} \to H^1(S, \Theta)$ at $S = S_\rho$ is homeomorphic to $H^0(B, \Omega^1(Q)) \times \mathbf{R}^+$ (if $X = H^2 \times E^2$) or $H^0(B, \Omega^1(Q))$ (if $X = \widetilde{SL_2} \times E$) for the base curve B determined by ρ. The subspace $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ defined above gives a locally complete complex analytic family of the complex structures on S.*

The last statement comes from [4]. We can see directly that any deformation in the subspace $H^0(B, \Omega^1(Q))$ of $\mathbf{T}_0\mathcal{T}$ (which depends on the choice of $\rho \in \mathcal{T}$) does not change the complex structure as follows. Take any $w \in H^0(B, \Omega^1(Q))$. Lift w to the 1-form on $\mathbf{H}$ which is represented as $d\psi$ for some holomorphic function ψ on $\mathbf{H}$ satisfying $d\psi(\alpha z) = \phi(\alpha) d\psi(z)$ for any $\alpha \in \overline{\Gamma}, z \in \mathbf{H}$. Taking the integral we deduce that $b(\alpha) = \psi(\alpha z) - \phi(\alpha)\psi(z)$ is a constant. Furthermore we have $b(\alpha\beta) = b(\alpha) + \phi(\alpha) b(\beta)$. If α is a torsion then we can choose the fixed point of α as z and hence $b(\alpha) = 0$. The image of $b(\alpha)$ in $H^1(\overline{\Gamma}, \mathbf{C}^{\phi})$ maps to 0 in $H^1(\overline{\Gamma}, H^0(\mathbf{H}, \mathcal{O})^{\phi})$ since $b(\alpha) = \psi(z) - \phi(\alpha)\psi(\alpha^{-1}z)$ and conversely any element in the kernel of Φ_1 can be represented by the above way. Then the biholomorphic automorphism of $\mathbf{H} \times \mathbf{C}$ defined by $(z, w) \to (z, w + s\psi(z))$ for $s \in \mathbf{C}$ descends to the biholomorphism between S_ρ and $S_{\rho'}$ such that the difference $m_{\rho'} - m_\rho$ of the parameters in $H^1(\overline{\Gamma}, \mathbf{C}^{\phi})$ is sb and all the other parameters are the same.

Next consider the case when B is euclidean and S has a complex structure. If $g = 1$, $r = 0$ (B is a torus) then we can assume that S is either T^4 (a complex torus) or a primary Kodaira surface. In the first case $X = E^4$, $G = \mathrm{Isom}^0 E^4$. However since the complex structure of X is not preserved by G but is preserved by $G' = E^4 \rtimes U(2)$ we consider $\mathcal{T}'$ = the identity component of $\mathcal{T}(\Gamma, G')$ in this case. Then $\mathcal{T}' = U(2)\backslash GL_4^+\mathbf{R}$ and this is realized by the family of translations ρ defined by

$$\rho(\alpha_i)(z, w) = (z + w_{i1}, w + w_{i2})$$

for the generators α_i, $(i = 1, \ldots, 4)$ in $\Gamma = \mathbf{Z}^4$ such that

$$w_{12} = 0, w_{11}, w_{22} \in \mathbf{R}^+, \det(\Re w_{ij}, \Im w_{ij}) > 0$$

where $(\Re w_{ij}, \Im w_{ij})$ is the matrix of rank 4 defined by $\Re w_{ij}$, $\Im w_{ij}$ for $i, j = 1, \ldots, 4$. Thus we have a differentiable family $\mathcal{C}$ over $\mathcal{T}'$ of the complex structures on S. $\mathcal{T}'$ contains the subfamily (which is complex analytic) consisting of the representations with

$$w_{11} = w_{22} = 1, w_{12} = w_{21} = 0, \det(\Im(w_{ij})_{i,j=3,4}) > 0$$

which is complete and effectively parametrized ([3]). It follows that the Kodaira Spencer map for $\mathcal{T}'$ at any point is surjective. In the second case $X = \mathrm{Nil}^3 \times E$ and $\mathcal{T}$ is homeomorphic to $\mathbf{R}^+ \times \mathbf{H} \times \mathbf{R}^2 \times \mathbf{H}$. Here the $\mathbf{R}^+$-factor, the first $\mathbf{H}$-factor, the last $\mathbf{H}$-factor and the $\mathbf{R}^2$-factor correspond to the area of the base, the period of the base, the period of the fiber (the image of one of the lattices of the fiber is uniquely determined and not deformed) and the twisting parameters for the fibrations respectively (see §2). Hence we have a differentiable family $\mathcal{C} \to \mathcal{T}$ of the complex structures. On the other hand in the decomposition of $H^1(S, \Theta)$ we have $E = F = \mathbf{C}$ ([8]). Since the canonical divisor K of S is trivial there is an isomorphism $\Theta \cong \Omega^1$ and hence $\dim_{\mathbf{C}} H^1(S, \Theta) = h^{1,1} = 2$ since the Hodge numbers satisfy $h^{0,2} = h^{2,0} = 1$ and $b_2 = h^{2,0} + h^{0,2} + h^{1,1} = 4$. It follows that $C = H = 0$ and as in the cases when B is hyperbolic the Kodaira Spencer map is surjective with kernel $= \mathbf{R}^+ \times \mathbf{R}^2$.

Finally consider the case when B is euclidean of genus 0. In this case $\mathcal{T} = \mathcal{T}_{0,r} \times \mathbf{H} \times \mathbf{R}^+$ if $X = E^4$ and $\mathcal{T} = \mathcal{T}_{0,r} \times \mathbf{H}$ if $X = \mathrm{Nil}^3 \times E$. Here $\mathcal{T}_{0,r} = \mathbf{R}^{2(r-3)} \times \mathbf{R}^+$ (with $r = 3, 4$) denotes the Teichmüller space of the base orbifold B where the first factor corresponds to the Teichmüller space of an r-pointed Riemann surface of genus 0 (with fixed area) and the $\mathbf{R}^+$-factor corresponds to the area of the base B. The other factor in $\mathcal{T}$ corresponds to the deformations of the lattices of the fiber (in the case with $X = \mathrm{Nil}^3 \times E$ one of the lattices of the fiber has the fixed image and hence there is no $\mathbf{R}^+$-factor). If $r = 3$ then the base B is parametrized by the area only and if $r = 4$ then the $\mathbf{R}^{2(r-3)}$-factor is identified with the Teichmüller space of the double covering torus of B which is isomorphic to $\mathbf{H}$. In either case we have the differentiable family $\mathcal{C} \to \mathcal{T}$ of the complex structures of S as in the arguments in §2. (In the case with $X = E^4, g = 0$ we can choose the representatives ρ for $\mathcal{T}$ such that the image of ρ lies in $E^4 \rtimes U(1)$ and hence we do not need to restrict G to the subgroup $E^4 \rtimes U(2)$.) On the other hand in the decomposition of $H^1(S_0, \Theta_0)$ we have $C = H = 0, E = \mathbf{C}, F = \mathbf{C}^{r-3}$ and we can argue as in the case when B is hyperbolic. Thus we obtain

Theorem C-2. *Let S be a Seifert 4-manifold over some orientable hyperbolic or euclidean 2-orbifold B which admits a complex structure. Then S has a geometric structure of type (X, G) with $X = H^2 \times E^2$, $\widetilde{SL_2} \times E$, E^4 or $\mathrm{Nil}^3 \times E$ and $G = \mathrm{Isom}^0 X$. Let $\mathcal{T}$ be the identity component of the Teichmüller space $\mathcal{T}(\Gamma, G)$ where $\Gamma = \pi_1 X$. (In the case when $S = T^4$ restrict G to $E^4 \rtimes U(2)$.) Then $\mathcal{T}$ gives a differentiable family of complex structures on S such that the infinitesimal deformation map at any point in $\mathcal{T}$ is surjective.*

Remark. The statements in Theorem C-2 do not hold in general for a Seifert 4-manifold S over a closed orientable spherical or bad 2-orbifold B. In this case S is either a ruled surface of genus 1 (with $X = S^2 \times E^2$) or a Hopf surface (with $X = S^3 \times E$). In either case not every complex structure on S comes from the geometric one nor every differentiable family of the complex structures containing the geometric one comes from the Teichmüller space of the geometric structures. In general the dimension of $H^1(S, \Theta)$ (which is not constant) can be greater than that of the Teichmüller space (cf. [9], [14], [2], [12]).

§4. A remark on the moduli spaces

In this section we give a remark on the moduli space $\mathcal{M}(\Gamma, G)$ for a geometric Seifert 4-manifold $S = \Gamma \backslash X$ over a closed orientable hyperbolic base orbifold $B = \overline{\Gamma} \backslash \overline{X}$ with $\Gamma \subset G = \mathrm{Isom}^0 X$. We adopt the representations of Γ given in §1 and also assume that the monodromy matrices $A_1, \ldots, A_{2g}$ satisfy the conditions in Proposition 1. In this case the fibration of S is unique and then every element φ of Aut Γ induces the automorphism $\overline{\varphi}$ of $\overline{\Gamma}$ and also induces the automorphism of $\mathbf{Z}^2$ generated by ℓ, h. Put $\pi_* = \prod[\alpha_{2j-1}, \alpha_{2j}] \prod q_j$.

Proposition 2. *Any element $\varphi \in$ Aut Γ must be of the following* *form.*

$$\varphi(\alpha_i) = \tilde{\varphi}(\alpha_i)\ell^{s_i}h^{t_i}$$
$$\varphi(q_j) = \tilde{\varphi}(q_j)\ell^{u_j}h^{v_j}$$
$$(\varphi(\ell), \varphi(h)) = (\ell, h)P.$$

Here $P \in GL_2\mathbf{Z}$ and $\tilde{\varphi}(\alpha_i), \tilde{\varphi}(q_j)$ are the words of $\alpha_1, \dots, \alpha_{2g}, q_1, \dots, q_r$ satisfying

$$\tilde{\varphi}(q_j) = \mu_j q^{\sigma}_{\nu(j)} \mu_j^{-1}$$
$$\tilde{\varphi}(\pi_*) = \mu \pi_*^{\sigma} \mu^{-1}$$

where $\sigma = \pm 1$, μ, μ_j are some words of $\alpha_1, \dots, \alpha_{2g}, q_1, \dots, q_r$ and $\nu: (1, \dots, r) \to (\nu(1), \dots, \nu(r))$ is a permutation. We have further conditions on the above parameters and the words as follows. Let ϵ_i, η_j, η be the exponent sums of α_1 in $\tilde{\varphi}(\alpha_i)$, μ_j, μ respectively.

(0) $m_{\nu(i)} = m_i$

(1) $P^{-1} A_1^{\epsilon_i} P = A_i$

(2) $\begin{pmatrix} u_j \\ v_j \end{pmatrix} = \sigma A^{\eta_j} \begin{pmatrix} a_{\nu(j)}/m_{\nu(j)} \\ b_{\nu(j)}/m_{\nu(j)} \end{pmatrix} - P \begin{pmatrix} a_j/m_j \\ b_j/m_j \end{pmatrix}$

(3) $\sigma A_1^{\eta} \begin{pmatrix} a \\ b \end{pmatrix} + \sigma \sum A_1^{\eta_j} \begin{pmatrix} a_{\nu(j)}/m_{\nu(j)} \\ b_{\nu(j)}/m_{\nu(j)} \end{pmatrix} + (A_1 - I) \begin{pmatrix} s_2 \\ t_2 \end{pmatrix} = P \begin{pmatrix} a + \sum a_i/m_i \\ b + \sum b_i/m_i \end{pmatrix}$

Sketch of Proof. The proof is similar to that of [7, §5, Lemma 4, Theorem 5] . (1) and (2) are derived from the relations $\alpha_i(\ell, h)\alpha_i^{-1} = (\ell, h)A_i$, $q_j^{m_j} \ell^{a_j} h^{b_j} = 1$. (3) comes from (1), (2) and the remaining relation $\prod[\alpha_{2j-1}, \alpha_{2j}] \prod q_j = \ell^a h^b$.

The map $\varphi \to \overline{\varphi}$ induces the homomorphism q: Aut $\Gamma \to$ Aut $\overline{\Gamma}$ which descends to a homomorphism $\overline{q}$: Out $\Gamma \to$ Out $\overline{\Gamma}$. Let Aut $(\overline{\Gamma}, q)$ and Out $(\overline{\Gamma}, \overline{q})$ be the images of q and $\overline{q}$ respectively. Also put $K = q^{-1}(\text{Inn}\,\overline{\Gamma})$. Then since q maps Inn Γ onto Inn $\overline{\Gamma}$ the natural projection π: Aut $\Gamma \to$ Out Γ maps K onto $\overline{K} = \ker \overline{q}$ and we have the following commutative diagram with exact rows and columns.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}\,\Gamma & \longrightarrow & K & \xrightarrow{\pi} & \overline{K} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}\,\Gamma & \longrightarrow & \mathrm{Aut}\,\Gamma & \xrightarrow{\pi} & \mathrm{Out}\,\Gamma & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow \overline{q} & & \\
 & & & & \mathrm{Out}\,(\overline{\Gamma}, \overline{q}) & = & \mathrm{Out}\,(\overline{\Gamma}, \overline{q}) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1. & &
\end{array}
$$

It is easy to see that the action of Aut Γ (resp. Out Γ) preserves the product fibration $\widetilde{\mathcal{F}} \to \mathcal{R} \to \overline{\mathcal{R}}$ for $\mathcal{R} = \mathcal{R}(\Gamma, G)$, $\overline{\mathcal{R}} = \overline{\mathcal{R}}(\overline{\Gamma}, \overline{G})$ (resp. $\mathcal{F} \to \mathcal{T} \to \overline{\mathcal{T}}$ for $\mathcal{T} = \mathcal{T}(\Gamma, G)$, $\overline{\mathcal{T}} = \overline{\mathcal{T}}(\overline{\Gamma}, \overline{G})$) (given in §1) and induces the natural action of Aut $(\overline{\Gamma}, q)$ (resp. Out $(\overline{\Gamma}, q)$) on $\overline{\mathcal{R}}$ (resp. $\overline{\mathcal{T}}$).

Now we will check the action of $\overline{K}$ on $\mathcal{F}$ (note that $\overline{K}$ acts trivially on $\overline{\mathcal{T}}$). First suppose that $X = H^2 \times E^2$. Define $\rho_0 = \rho_0(\ell_0, \lambda) \in \mathcal{R}$ by

$$
\begin{aligned}
&\rho_0(\ell)(z,w) = (z, w + \ell_0)\\
&\rho_0(h)(z,w) = (z, w + \ell_0\lambda^{-1})\\
&\rho_0(\alpha_1)(z,w) = (\overline{\rho}_0(\overline{\alpha}_1)z, \phi(\overline{\alpha}_1)w + w_1^0)\\
&\rho_0(\alpha_i)(z,w) = (\overline{\rho}_0(\overline{\alpha}_i)z, w + w_i^0) \quad (i \geq 2)\\
&\rho_0(q_j)(z,w) = (\overline{\rho}_0(\overline{q}_j)z, w - (a_j\ell_0 + b_i\ell_0\lambda^{-1})/m_j)
\end{aligned}
$$

where

$$\ell_0 \in \mathbf{R}^+, \Im\lambda \neq 0,$$

$$w_j^0 = 0 \text{ for any } j \quad \text{if } A_1 = I$$

and if $A_1 \neq I$

$$
w_j^0 = \begin{cases} \ell_0((a + \sum a_i/m_i) + (b + \sum b_i/m_i)\lambda^{-1})/(\phi(\overline{\alpha}_1) - 1) & \text{if } j = 2 \\ 0 & \text{if } j \neq 2. \end{cases}
$$

(We can choose such w_j^0. See the proof of Theorem A in §1.) Furthermore by the assumption in Proposition 1 we have

$$\lambda = \phi(\overline{\alpha}_1) = \exp(\pm 2\pi i/6) \quad \text{if } A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\lambda = \phi(\overline{\alpha}_1) = \exp(\pm 2\pi i/4) \quad \text{if } A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\lambda = \exp(\pm 2\pi i/6), \phi(\overline{\alpha}_1) = \lambda^2 \quad \text{if } A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

There is no further restriction on λ if $A_1 = \pm I$. Then the image $[\rho_0] \in \mathcal{T}$ of ρ_0 belongs to the fiber $(\cong \mathcal{F} = \mathbf{T}_1 \times H^1(\overline{\Gamma}, \mathbf{C}^\phi))$ of $\mathcal{T}$ over the image $[\overline{\rho}_0] \in \overline{\mathcal{T}}$ of $\overline{\rho}_0$. Its $\mathbf{T}_1$-coordinates are detected by (ℓ_0, λ) and it corresponds to 0 in the $H^1(\overline{\Gamma}, \mathbf{C}^\phi)$-component. Now we take the subfamily $\mathcal{R}_0$ of $\mathcal{R}$ with a fixed image $\overline{\rho}_0$ in $\overline{\mathcal{R}}$ whose elements $\rho = \rho(\ell_0, \lambda, m)$ are defined by

$$\begin{aligned}
&\rho(\ell)(z, w) = (z, w + \ell_0) \\
&\rho(h)(z, w) = (z, w + \ell_0\lambda^{-1}) \\
&\rho(\alpha_1)(z, w) = (\overline{\rho}_0(\overline{\alpha}_1)z, \phi(\overline{\alpha}_1)w + w_1) \\
&\rho(\alpha_i)(z, w) = (\overline{\rho}_0(\overline{\alpha}_i)z, w + w_i) \quad (i \geq 2) \\
&\rho(q_j)(z, w) = (\overline{\rho}_0(\overline{q}_j)z, w - (a_j\ell_0 + b_j\ell_0\lambda^{-1})/m_j).
\end{aligned}$$

Here (ℓ_0, λ) satisfies the same conditions as before and

$$w_i = w_i^0 + m(\overline{\alpha}_i)$$

where m is a crossed homomorphism from $\overline{\Gamma}$ to $\mathbf{C}^\phi$ (with $m(\overline{q}_j) = 0$) satisfying

$$m(\overline{\alpha}_1) = m(\overline{\alpha}_2) = 0 \quad \text{if } A_1 \neq I.$$

(There are no restrictions on $m(\overline{\alpha}_j)$ if $A_1 = I$.) Hence we have

$$\begin{aligned}
&w_1 = 0, w_2 = w_2^0, w_j = m(\overline{\alpha}_j) \ (j \geq 3) \quad \text{if } \phi \not\equiv \text{id} \\
&w_j = m(\overline{\alpha}_j) \text{ for any } j \quad \text{if } \phi \equiv \text{id}.
\end{aligned}$$

We note that if $A_1 \neq I$ then $w_2 = w_2^0$ is fixed once (ℓ_0, λ) is fixed by the relations (7), (8) in the proof of Theorem A (§1) and any crossed homomorphism $n\colon \overline{\Gamma} \to \mathbf{C}^\phi$ with $n(\overline{\alpha}_j) = 0$ for $j \geq 2$ is contained in the image of $\delta\colon C^0(\overline{\Gamma}, \mathbf{C}^\phi) \to C^1(\overline{\Gamma}, \mathbf{C}^\phi)$. Therefore the m's satisfying the above conditions descend isomorphically onto $H^1(\overline{\Gamma}, \mathbf{C}^\phi)$. Taking these facts into account we can see that the family $\mathcal{R}_0$ generated by $\rho(\ell_0, \lambda, m)$ whose parameters satisfy the above conditions is the subfamily of the fiber over $\overline{\rho}_0 \in \mathcal{R}$ whose $\widetilde{\mathcal{F}}$-components give the representatives of $\mathcal{F}$. To check the action of $\overline{K}$ on $\overline{\mathcal{F}}$, it suffices to find the element $\mu \in \mathrm{Inn}\, G$ for given $\rho \in \mathcal{R}_0$, $\varphi \in K$ (μ may depend on ρ and φ) such that $\mu \cdot \rho \cdot \varphi \in \mathcal{R}_0$ and examine the action of $\mu \cdot \rho \cdot \varphi$ (which is independent of $\overline{\rho}_0$ in the w-coordinate). Since the action of K commutes with that of $\mathrm{Inn}\, G$, it suffices to consider the element $\varphi \in K$ of the following form.

$$\begin{aligned} \varphi(\alpha_i) &= \alpha_i \ell^{s_i} h^{t_i} \\ \varphi(q_j) &= q_j \ell^{u_j} h^{v_j} \\ (\varphi(\ell), \varphi(h)) &= (\ell, h)P \end{aligned} \tag{4}$$

where $s_i, t_i, u_i, v_i \in \mathbf{Z}, P \in GL_2\mathbf{Z}$ satisfy

$$PA_1P^{-1} = A_1 \tag{5}$$

$$\begin{pmatrix} u_j \\ v_j \end{pmatrix} = (I - P)\begin{pmatrix} a_j/m_j \\ b_j/m_j \end{pmatrix} \tag{6}$$

$$(A_1 - I)\begin{pmatrix} s_2 \\ t_2 \end{pmatrix} = (P - I)\begin{pmatrix} a + \sum a_i/m_i \\ b + \sum b_i/m_i \end{pmatrix}. \tag{7}$$

For such $\varphi \in K$ and $\rho \in \mathcal{R}_0$ denote the first and the second factors of $\rho \cdot \varphi(\alpha)(z, w)$ by $\rho \cdot \varphi(\alpha)(z, w)_i$ $(i = 1, 2)$ respectively for $\alpha \in \Gamma$. Then we have

$$\rho \cdot \varphi(\alpha)(z, w)_1 = \begin{cases} \overline{\rho}_0(\overline{\alpha})z & \text{if } \alpha = \alpha_i \text{ or } q_j \\ z & \text{if } \alpha = \ell \text{ or } h \end{cases} \tag{8}$$

and

$$\begin{aligned} \rho \cdot \varphi(\alpha_1)(z, w)_2 &= \phi(\overline{\alpha}_1)(w + \ell_0(s_1 + t_1\lambda^{-1})) + w_1 \\ \rho \cdot \varphi(\alpha_i)(z, w)_2 &= w + w_i + \ell_0(s_i + t_i\lambda^{-1}) \ (i \geq 2) \\ \rho \cdot \varphi(q_j)(z, w)_2 &= w - \ell_0(a_j + b_j\lambda^{-1})/m_j + \ell_0(u_j + v_j\lambda^{-1}) \\ \rho \cdot \varphi(\ell)(z, w)_2 &= w + \ell_0' \\ \rho \cdot \varphi(h)(z, w)_2 &= w + h_0' \end{aligned} \tag{9}$$

where $(\ell_0', h_0') = (\ell_0, \ell_0\lambda^{-1})P$. Let K_1 be the subgroup of K consisting of the elements satisfying (4)–(7) with $P = I$ and let $\overline{K}_1$ be its image in $\overline{K}$. For any $\varphi \in K_1$ we deduce $u_j = v_j = 0$ and if $A_1 \neq I$, $s_2 = t_2 = 0$. Also let K_0 be the subgroup of K_1 generated by the elements of the above forms with $s_j = t_j = 0$ for $j \geq 2$ and $\overline{K}_0$ be its image in $\overline{K}$. Note that $\overline{K}_1 \cong K_1/(K_1 \cap \mathrm{Inn}\,\Gamma)$, $\overline{K}_0 \cong K_0/(K_0 \cap \mathrm{Inn}\,\Gamma)$ and $K_1 \cap \mathrm{Inn}\,\Gamma = K_0 \cap \mathrm{Inn}\,\Gamma = \mathrm{Inn}\,\mathbf{Z}^2$ where $\mathbf{Z}^2$ is the subgroup of Γ generated by ℓ and h. (This comes from the fact that $\overline{\Gamma}$ is centerless.) The element $\varphi \in K_0$ comes from $\mathrm{Inn}\,\mathbf{Z}^2$ if $\begin{pmatrix} s_1 \\ t_1 \end{pmatrix} = (A_1^{-1} - I)\begin{pmatrix} s \\ t \end{pmatrix}$ and hence

$$\overline{K}_0 \text{ is finite if } A_1 \neq I \text{ and moreover } \overline{K}_0 = 1 \text{ if } A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Given $\rho \in \mathcal{R}_0$ and $\varphi \in K_1$ we take an inner automorphism μ by the element $(z, w) \to (z, w + \phi(\overline{\alpha}_1)(s_1\ell_0 + t_1\ell_0\lambda^{-1})/(\phi(\overline{\alpha}_1) - 1))$ if $A_1 \neq I$. Then we can see by the conditions on w_i above for (8), (9) that the correspondence $\rho \to \rho \cdot \varphi$ (if $A_1 = I$) or $\rho \to \mu \cdot \rho \cdot \varphi$ (if $A_1 \neq I$) gives a map from $\mathcal{R}_0$ to itself such that the parameters are changed as follows.

$$(\ell_0, \lambda) \to (\ell_0, \lambda)$$
$$m(\overline{\alpha_i}) \to m(\overline{\alpha_i}) + s_i\ell_0 + t_i\ell_0\lambda^{-1}$$

for $i \geq 1$ if $A_1 = I$ and for $i \geq 3$ if $A_1 \neq I$. (We can see from (7) that the w_2-parameter of $\mu \cdot \rho \cdot \varphi$ is the same as that for ρ if $A_1 \neq I$.) Since s_i, t_i ($i \geq 1$ if $A_1 = I$ and $i \geq 3$ if $A_1 \neq I$) are arbitrary integers this gives the action of $\overline{K}_1$ on $\mathcal{F}$. Hence

$$\mathcal{F}/\overline{K}_1 \cong \mathbf{T}_1 \times H^1(\overline{\Gamma}, \mathbf{C}^{\phi})/H^1(\overline{\Gamma}, \mathbf{Z}^{2^{\phi}})$$

with

$$H^1(\overline{\Gamma}, \mathbf{C}^{\phi})/H^1(\overline{\Gamma}, \mathbf{Z}^{2^{\phi}}) \cong \begin{cases} (\mathbf{T}^1)^{2g} & \text{if } A_1 = I \\ (\mathbf{T}^1)^{2g-2} & \text{if } A_1 \neq I \end{cases}$$

where $\mathbf{T}^1 \cong \mathbf{C}/\mathbf{Z}^2$ is the complex torus of dimension 1 whose lattice is generated by ℓ_0 and $\ell_0\lambda^{-1}$. Here we note that if $A_1 \neq I$ then $\overline{K}_0$ is the subgroup of $\overline{K}_1$ which acts trivially on $\mathcal{F}$. Hence $\overline{K}_1$ (or $\overline{K}_1/\overline{K}_0$ if $A_1 \neq I$) acts effectively and properly discontinuously on $\mathcal{F}$. Next consider the action of $\varphi \in K$ not decending to $\overline{K}_1$.

Case (1). $A_1 = \pm I$. Put $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL_2\mathbf{Z}$ for $\varphi \in K$ defined in (4). Here u_j, v_j, s_2, t_2 for φ must be defined as elements in $\mathbf{Z}$ according to (6) and (7) (if $A_1 = I$ then (7) becomes obvious since the right hand side of (7) is 0 in case $X = H^2 \times E^2$). Then considering (8), (9) for $\rho \in \mathcal{R}_0$ and φ we can take an inner automorphism μ of the form $(z, w) \to (z, \sigma w + c)$ for some $c \in \mathbf{C}$ with $\sigma = |p + r\lambda^{-1}|/(p + r\lambda^{-1})$ such that the the correspondence $\rho \to \mu \cdot \rho \cdot \varphi$ gives a map from $\mathcal{R}_0$ to itself of the form

$$\begin{aligned} &\lambda \to (p\lambda + r)/(q\lambda + s) \\ &\ell_0 \to |p + r\lambda^{-1}|\ell_0 \\ &m(\overline{\alpha}_i) \to \sigma(m(\overline{\alpha}_i) + s_i\ell_0 + t_i\ell_0\lambda^{-1}) \end{aligned}$$

for $i \geq 1$ if $A_1 = I$ and for $i \geq 3$ if $A_1 = -I$.

Case (2). $A_1 \neq \pm I$. In this case we deduce from the conditions on λ above that

$$(\ell_0, \ell_0\lambda^{-1})A_1' = (\lambda\ell_0, \ell_0) \quad \text{for } A_1' = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

if $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ and

$$(\ell_0, \ell_0\lambda^{-1})A_1 = (\lambda\ell_0, \ell_0)$$

otherwise. Furthermore for the presentation of φ in (4) we must have $P = A_1'^k$ (in case $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$) or $P = A_1^k$ (otherwise) for some $k \in \mathbf{Z}$ by the condition (5) and

$$u_j, v_j, s_2, t_2 \in \mathbf{Z}$$

where these numbers are defined by (6) and (7) for the above P (we have assumed that $P \neq I$ since $\varphi \notin K_1$.) Then in the presentation (9) for $\rho \in \mathcal{R}_0$, $\varphi \in K$ we have

$$(\ell', h') = (\ell_0, \ell_0\lambda^{-1})P = (\lambda^k\ell_0, \lambda^{k-1}\ell_0)$$

for $P = A_1^k$ or $P = A_1'^k$ as above. Hence taking an inner automorphism μ by the element $(z, w) \to (z, \lambda^{-k}w + c)$ where

$$c = \lambda^{-k}\phi(\overline{\alpha}_1)(s_1\ell_0 + t_1\ell_0\lambda^{-1})/(\phi(\overline{\alpha}_1) - 1)$$

we can see that the correspondence $\rho \to \mu \cdot \rho \cdot \varphi$ gives a map from $\mathcal{R}_0$ to itself of the form

$$\begin{aligned} &(\ell_0, \lambda) \to (\ell_0, \lambda) \\ &m(\overline{\alpha}_i) \to \lambda^{-k}(m(\overline{\alpha}_i) + s_i\ell_0 + t_i\ell_0\lambda^{-1}) \quad (i \geq 3). \end{aligned}$$

In the cases when $A_1 = \pm I$ the above correspondence shows that the action of $\overline{K}$ on $\mathcal{F}$ preserves the product fibration of the form $H^1(\overline{\Gamma}, \mathbf{C}^{\phi}) \times \mathbf{R}^+ \to \mathcal{F} \to \mathbf{H} \times \mathbf{Z}_2$ (where $\mathbf{T}_1 = \mathbf{R}^+ \times \mathbf{H} \times \mathbf{Z}_2$) which induces the properly discontinuous action on $\mathbf{H} \times \mathbf{Z}_2$ (which is identified with $\{\lambda \in \mathbf{C} \mid \Im\lambda \neq 0\}$) of the form $\lambda \to (p\lambda + r)/(q\lambda + s)$ for some matrix $\begin{pmatrix} p & r \\ q & s \end{pmatrix} \in GL_2\mathbf{Z}$. In the cases when $A_1 \neq \pm I$ we have at most finite number of possible choices for P in the presentation of φ above since A_1 (or A_1') is periodic. Hence by the above correspondences and the action of $\overline{K}_1$ (and taking the fact that $\overline{K}_0$ is finite if $A_1 \neq I$ into account) we can easily see that $\overline{K}$ (or $\overline{K}/\overline{K}_0$ if $\phi \not\equiv \mathrm{id}$) acts properly discontinuously on $\mathcal{F}$ in either case. Finally consider the action of Out Γ on $\mathcal{T}$. The group Out Γ acts on $\mathcal{T}$ so that it preserves the product fibration $\mathcal{F} \to \mathcal{T} \to \overline{\mathcal{T}}$ and induces the action of Out $(\overline{\Gamma}, \overline{q})$ on $\overline{\mathcal{T}}$ which is properly discontinuous since the action of Out $(\overline{\Gamma})$ on $\overline{\mathcal{T}}$ has, as is well known, the same property. Since $\overline{K}$ (which is the subgoup of Out Γ which induces the identity on $\overline{\mathcal{T}}$) acts properly discontinuously on the fiber of $\mathcal{T}$ as above we can see that Out Γ acts also properly discontinuously on $\mathcal{T}$. The cases with $X = \widetilde{SL}_2 \times E$ (in this case $A_1 = I$) can be treated similarly and we omit the details. Thus we have

Proposition 3. *Let $S = \Gamma\backslash X$ be a geometric Seifert 4-manifold over a closed orientable hyperbolic orbifold B with $\Gamma \subset G = \mathrm{Isom}^0 X$. Then* Out Γ (*or* Out $\Gamma/\overline{K}_0$ *in case the monodromy representation ϕ of S is not trivial where $\overline{K}_0$ is a finite subgroup of $\overline{K}$ defined above*) *acts on $\mathcal{T}(\Gamma, G)$ properly discontinuously and the muduli space $\mathcal{M}(\Gamma, G)$ is Hausdorff.*

On the other hand if $\overline{K}_1 \neq \overline{K}$ then we must have $\varphi \in K$ of the form (4) satisfying (5)–(7) with $P \neq I$. In particular s_2, t_2, u_j, v_j defined by (5) and (6) for some appropriate $P \neq I$ satisfying (4) must be integers. From these conditions we can deduce some extra conditions on the Seifert invariants of S and derive the following proposition. Here we omit the details of the computations.

Proposition 4. *Let $S = \Gamma\backslash X$ be a geometric Seifert 4-manifold as in Proposition 3. Then if the monodromies of S satisfy the conditions in Proposition 1 and if the Seifert invariants of S do not satisfy the conditions below then $\overline{K} = \overline{K}_1$ and $\mathcal{M}(\Gamma, G)$ is a Seifert fibration over $\overline{\mathcal{T}}(\overline{\Gamma}, \overline{G})/$* Out $(\overline{\Gamma}, \overline{q})$ (*which is defined above*) *with general fiber* $\mathbf{T}_1 \times H^1(\overline{\Gamma}, \mathbf{C}^{\phi})/H^1(\overline{\Gamma}, \mathbf{Z}^{2\phi})$.

(I) $A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$.

(1) *There are no multiple fibers*;
(2) $(m_i, a_i, b_i) = (3, \epsilon_i, \epsilon_i)$, $\epsilon_i = \pm 1$ *for any* i;
(3) $m_i = 2$ *for any* i.

(II) $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.

(1) *There are no multiple fibers and* $a \equiv b \mod 3$;
(2) $(m_i, a_i, b_i) = (3, \epsilon_i, \epsilon_i)$ *with* $\epsilon_i = \pm 1$ *for any* i *and* $\sum \epsilon_i \equiv 0 \mod 3$;
(3) $m_i = 2$ *for any* i *and* $\sum a_i - 2a \equiv \sum b_i - 2b \mod 3$.

(III) $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(1) *There are no multiple fibers*;
(2) $m_i = 2$ *for every* i *and* $\sum a_i \equiv \sum b_i \mod 2$.

(IV) $A_1 = \pm I$.

References

[1] L. Bers, Fiber spaces over Teichmüller spaces, Acta Math., **130** (1973), 89–126.

[2] K. Dabrowsky, Moduli spaces for Hopf surfaces, Math. Ann., **259** (1982), 201–226.

[3] K. Kodaira, D. C. Spencer, On deformations of complex analytic structures, I, II, Ann. Math., **67** (1958), 328–466.

[4] ———, A theorem of completeness for complex analytic fiber spaces, Acta. Math., **100** (1958), 281–294.

[5] R. Kulkarni, K. B. Lee, F. Raymond, Deformation spaces for Seifert manifolds, in "Geometry and Topology, Lect. Notes in Math. vol. 1167, Springer", 1985, pp. 180–216.

[6] K. Ohshika, Teichmüller spaces of Seifert fibered manifolds with infinite π_1, Topology and its appl., **27** (1987), 75–93.

[7] P. Orlik, "Seifert Manifolds", Springer, Berlin–Heidelberg–New York, 1972.

[8] T. Suwa, Deformations of Holomorphic Seifert Fiber Spaces, Invent. math., **51** (1979), 77–102.

[9] ———, On ruled surfaces of genus 1, J. Math. Soc. Japan, **21** (1969), 291–311.

[10] M. Ue, Geometric 4-manifolds in the sense of Thurston and Seifert 4-manifold I, J. Math. Soc. Japan, **42** (1990), 511–540.

[11] ———, Geometric 4-manifolds in the sense of Thurston and Seifert 4-manifolds II, J. Math. Soc. Japan, **43** (1991), 149–183.

[12] C. T. C. Wall, Geometric structures on compact complex surfaces, Topology, **25** (1986), 119–153.

[13] ———, Geometries and geometric structures in real dimension 4 and complex dimension 2, in "Geometry and Topology. Lect. Notes in Math., 1167, Springer", 1985, pp. 268–292.

[14] J. Wehler, Versal deformations for Hopf surfaces, J. Reine Angew. Math., **328** (1982), 22–32.

[15] H. Zieschang, On toric fiberings over surfaces, Math. Notes, **5** (1967), 341–345.

Institute of mathematics
Yoshida College
Kyoto University
Kyoto 606
Japan