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Invariants of Spatial Graphs

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§1. Introduction

The purpose of this paper is to construct invariants of spatial graphs from regular isotopy invariants of non-oriented link diagrams of *knit trace type.* Kauffman's bracket polynomial [4], which is a version of the Jones polynomial, is of knit trace type. The Dubrovnik polynomial [5], which is used in the definition of the Kauffman polynomial, is also of knit trace type [6]. Hence these two invariants are generalized to invariants of spatial graphs by our method. The Yamada polynomial introduced in [10] is the non-trivial simplest one of our invariants. A similar invariants are introduced in [9] for ribbon graphs. They use quasi-triangular Hopf algebras. But we use representations of knit semigroups or braid groups instead of Hopf algebras.

To introduce regular isotopy invariants of link diagrams of *knit trace type*, we need notion of a *Markov knit sequence.* Let $\mathbb{C}$ be the field of complex numbers. Knit semigroups K_n, $(n = 1, 2, \cdots)$ are introduced in [6] defined by the following generators and relations.

$$\begin{aligned} K_n = \langle \tau_1, \cdots, \tau_{n-1},\ \tau_1^{-1}, \cdots, \tau_{n-1}^{-1},\ \varepsilon_1, \cdots, \varepsilon_{n-1} \quad | \\ \tau_i\tau_i^{-1} = \tau_i^{-1}\tau_i = 1, \quad \tau_i\tau_j = \tau_j\tau_i \ (|i-j| \geq 2), \\ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}, \quad \tau_i\varepsilon_j = \varepsilon_j\tau_i \ (|i-j| \geq 2), \\ \varepsilon_i\varepsilon_{i\pm1}\varepsilon_i == \varepsilon_i, \quad \varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i \ (|i-j| \geq 2), \\ \varepsilon_i\tau_{i\pm1} = \varepsilon_i\varepsilon_{i\pm1}\tau_i^{-1}, \quad \varepsilon_i\tau_{i\pm1}^{-1} = \varepsilon_i\varepsilon_{i\pm1}\tau_i, \\ \tau_{i\pm1}\varepsilon_i = \tau_i^{-1}\varepsilon_{i\pm1}\varepsilon_i, \quad \tau_{i\pm1}^{-1}\varepsilon_i = \tau_i\varepsilon_{i\pm1}\varepsilon_i \rangle \end{aligned}$$

The generators of K_n are presented graphically as in Figure 1. In the graphical presentation, the product of two elements of K_n corresponds to the composite of two diagrams as in the case of braid groups. Let

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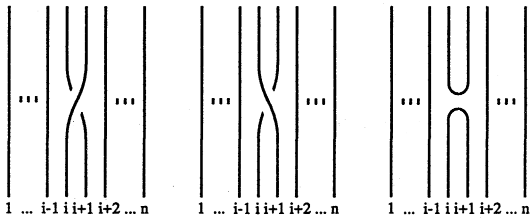

τ_i τ_i^{-1} ε_i

Fig. 1. Generators of K_n.

$\mathbb{C}K_n$ be the semigroup algebra of K_n over $\mathbb{C}$. We regard the braid group B_n as a subsemigroup of K_n generated by $\tau_1, \tau_2, \cdots, \tau_{n-1}$.

Let γ be a non-zero complex number. Knit semigroup algebra with writhe factor γ, denoted by $K_n(\gamma)$, is a quotient algebra of $\mathbb{C}K_n$ defined by the following.

$$K_n(\gamma) = \mathbb{C}K_n/(\tau_i^{\pm 1}\varepsilon_i - \gamma^{\pm 1}\varepsilon_i, \quad \varepsilon_i\tau_i^{\pm 1} - \gamma^{\pm 1}\varepsilon_i \quad (1 \le i \le n-1)).$$

Let A be a semisimple $\mathbb{C}$-algebra. Let $\hat{A}$ be the set of equivalence classes of irreducible representations of A. A $\mathbb{C}$-linear map T from A to $\mathbb{C}$ is called a trace if T is a linear combination of irreducible characters of A, i.e.

$$T(x) = \sum_{\rho \in \hat{A}} a_\rho \operatorname{Trace}(\rho(x)) \quad (a_\rho \in \mathbb{C}) \tag{1.1}$$

The trace T is called *faithful* if all the coefficients a_ρ are not equal to 0. A sequence $A_1, A_2, \cdots, A_n, \cdots$ of semisimple $\mathbb{C}$-algebras are called *a knit type sequence* if they satisfy the following.

(1) There is an algebra epimorphism p_n from $K_n(\gamma)$ to A_n and monomorphism j_n from A_n to A_{n+1} such that $j_n \circ p_n = p_{n+1} \circ i_n$ for $n = 1, 2, \cdots$, where i_n is an inclusion from $K_n(\gamma)$ to $K_{n+1}(\gamma)$ which sends $\tau_i^{\pm 1} \in K_n(\gamma)$ to $\tau_i^{\pm 1} \in K_{n+1}(\gamma)$ and $\varepsilon_i \in K_n(\gamma)$ to $\varepsilon_i \in K_{n+1}(\gamma)$ for $1 \le i \le n-1$.

(2) There are a complex number μ and a faithful trace T_n from A_n to $\mathbb{C}$ which satisfy the following. For any $x \in A_n$, $T_{n+1}(j_n(x)) = \mu T_n(x)$, $T_n(x) = \gamma^{\pm 1} T_{n+1}(j_n(x)\, p_{n+1}(\tau_n^{\pm 1}))$ and $T_n(x) = T_{n+1}(x\, p_{n+1}(\varepsilon_n))$.

For $x \in K_n$, let $\hat{x}$ denote the link diagram obtained from the closure of x (Figure 2). A regular isotopy invariant X of link diagrams is called of *knit trace type* if there is a Markov knit sequence and X is obtained by the traces of it, i.e. $X(\hat{x}) = T_n(p_n(x))$ for $x \in K_n$. Kauffman's bracket polynomial [4] is of knit trace type (see Section 3 of [7]). The Dubrovnik polynomial is also of knit trace type [6].

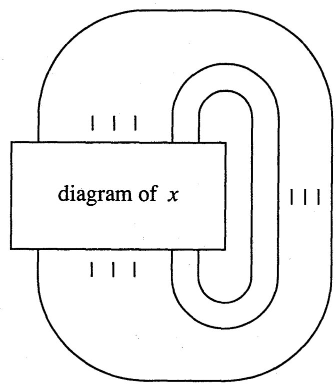

Fig. 2. Closure of $x \in K_n$.

Remark. Let X be a regular isotopy invariant of knit trace type with writhe factor γ. For an oriented link diagram x, there are a positive integer n and $y \in K_n$ such that $\hat{y}$ is equal to x without orientation. Let $w(x)$ be the sum of signatures of the crossings of x. Let $X'(x) = \gamma^{w(x)} X(\hat{y})$. Then X' is an invariant of links.

Now we define spatial graphs in S^3. Let $\mathcal{V}$ is a set of 2-disks and $\mathcal{E}$ be a set of edges homeomorphic to $[0,1]$ in S^3. Each edge has an orientation induced by the orientation of $[0,1]$. The terminal points of an edge corresponding to 0 and 1 are called the initial point and the final point of the edge respectively. The pair $\Gamma = (\mathcal{V}, \mathcal{E})$ is called an oriented spatial graph if it satisfies the following. The disks in $\mathcal{V}$ are mutually disjoint and the edges in $\mathcal{E}$ are mutually disjoint. Also assume that the interiors of the disks in $\mathcal{V}$ and edges in $\mathcal{E}$ are mutually disjoint. Terminal points of edges in $\mathcal{E}$ are contained in the boundaries of disks in $\mathcal{V}$. Two spatial graphs Γ and Γ' are called equivalent if there is an isotopy of S^3 which sends Γ to Γ'. A spatial graph Γ is called an embedding of a

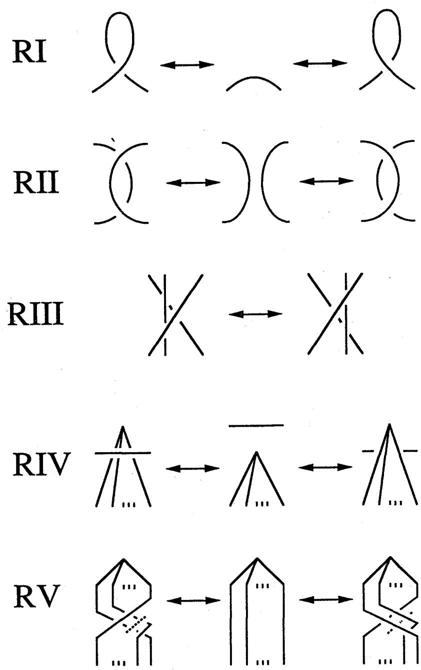

Fig. 3. Reidemeister moves.

tri-valent graph if the degree of all the vertices of Γ are equal to 3. A diagram of a spatial graph is defined as in the case of a link.

Proposition 1. *Two spatial graphs Γ and Γ' are equivalent if and only if there is a sequence of Reidemeister moves of types* (SRI)–(SRV) *sending a diagram of Γ to a diagram of Γ'.*

For a spatial graph Γ, we define a diagram of Γ as in the case of links. Let $A_1, A_2, \cdots$ be a Markov knit sequence. For each edge E of Γ, we associate a non-negative integer $N(E)$, an irreducible representation $R(E) \in \hat{A}_{n(E)}$ and a signature $S(E)$. The triple (N, R, S) is called a coloring of Γ if it satisfies the following. For a vertex v of Γ, let $\mathcal{E}_v$ be a set of edges with terminal point v Then

$$(1.2) \qquad \sum_{E \in \mathcal{E}_v} N(E) = \mathit{even} \text{ and } 2N(E) \leq \sum_{E' \in \mathcal{E}_v} N(E') \text{ for all } E \in \mathcal{E}_v.$$

We construct an invariant of spatial graphs colored as above. First, we generalize link invariants of *braid trace type* to invariants of colored oriented tri-valent graph embeddings in S^3 in §2. And then we generalize invariants of *knit trace type* to invariants of colored spatial graphs in §3. By attaching the same color to all the edges of graphs, we get invariants of spatial graphs. In §4, we give some examples.

§2. Invariants of colored oriented tri-valent graphs

In this section, we generalize link invariants of braid trace type to invariants of embeddings of colored oriented tri-valent graphs in S^3. To introduce link invariants of braid trace type, we need notion of a *Markov braid sequence.*

Definition. A sequence (A_1, T_1), (A_2, T_2), $\cdots$, (A_n, T_n), $\cdots$ of pairs of a semisimple $\mathbb{C}$-algebra and its trace are called a *Markov braid sequence* if they satisfy the following.

(1) There is an algebra homomorphism p_n from $\mathbb{C}B_n$ to A_n and j_n from A_n to A_{n+1} such that $j_n \circ p_n = p_{n+1} \circ i_n$ for $n = 1, 2, \cdots$, where i_n is an inclusion from $\mathbb{C}B_n$ to $\mathbb{C}B_{n+1}$ which sends $\sigma_i \in \mathbb{C}B_n$ to $\sigma_i \in \mathbb{C}B_{n+1}$ for $1 \leq i \leq n-1$.

(2) There is a faithful trace T_n from A_n to $\mathbb{C}$ and $\mu, c \in k \backslash \{0\}$ which satisfy $\mu T_n(x) = T_{n+1}(j_n(x))$, $T_n(x) = c\, T_{n+1}(x\, p_{n+1}(\sigma_n))$ and $T_n(x) = c^{-1}\, T_{n+1}(x\, p_{n+1}({\sigma_n}^{-1}))$ for any $x \in A_n$.

From a Markov braid sequence, we get a $\mathbb{C}$-valued link invariant. For a braid $b = {\sigma_{i(1)}}^{\varepsilon(1)}\, {\sigma_{i(2)}}^{\varepsilon(2)} \cdots {\sigma_{i(r)}}^{\varepsilon(r)} \in B_n$, let $w(b) = \sum_{i=1}^{r} \varepsilon(i)$. Then $w(b)$ is a sum of signatures of all the crossings of b. For a braid b, let $\hat{b}$ denote the link obtained from the closure of b. Let

$$X(\hat{b}) = c^{-w(b)}\, T_n(p_n(b)).$$

Then Alexander's theorem and Markov's theorem ([1], Theorem 2.1 and 2.2) implies that X is an invariant of links. Link invariant obtained from a Markov braid sequence as above is called of *braid trace type.* Jones polynomial, HOMFLY polynomial and Kauffman polynomial are all of braid trace type and the associated braid type sequences are Jones algebras, Iwahori's Hecke algebras and a q-analogue of Brauer's algebras respectively ([2], [3], [6], [8]).

From now on, fix an invariant X of braid trace type and let (A_1, T_1), (A_2, T_2), $\cdots$ be the Markov braid sequence of X. Since A_n is a semisimple algebra, we have

$$A_n = \bigoplus_{\rho\in\hat{A}_n} M_{d(\rho)}(\mathbb{C})$$

where $d(\rho)$ is the degree of ρ. Let q_ρ be an element of A_n such that

$$\nu(q_\rho) = \delta_{\nu\rho}\,\mathrm{id} \in M_{d(\nu)}(\mathbb{C}) \qquad \text{for} \quad \nu \in \hat{A}_n.$$

Let $\tilde{q}_\rho$ be an element of $\mathbb{C}B_n$ such that $p_n(\tilde{q}_\rho) = q_\rho$. Note that $\tilde{q}_\rho$ is not unique. Let $h_n = \sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_1\cdots\sigma_{n-2}\cdots\sigma_1\sigma_2\,\sigma_1$. We call h_n the *half twist* of B_n. Let $f_n = {h_n}^2$ and we call f_n the *full twist* of B_n. It is known that f_n commute with every element of B_n and so $\rho(p_n(f_n))$ is a scalar matrix, i.e. $\rho(p_n(f_n)) = \alpha_\rho$ id.

A formal $\mathbb{C}$-linear combination of link diagrams are called *a virtual link diagram.* We generalize the link invariant X to a function from virtual link diagrams to $\mathbb{C}$ formally as follows. For a virtual link diagram $L = \sum_{i=1}^r a_i\, L_i$ ($a_i \in k$, L_i is a link diagram), let $X(L) = \sum_{i=1}^r a_i\, X(L_i)$.

As in the case of links, we define a diagram of an oriented tri-valent graph embedded in S^3. Let G be an oriented tri-valent graph. We define a *coloring* of G. For each edge E of G, associate a non-negative integer $N(E)$, an irreducible representation $R(E) \in \hat{A}_{n(E)}$ and a signature $S(E) = \pm 1$. The triple (N, R, S) is called a coloring of G if it satisfies the following. For a vertex v of G, let E_v^- be a set of edges with end point v and E_v^+ a set of edges with start point v. Then

$$\sum_{E\in E_v^-} N(E) = \sum_{E\in E_v^+} N(E).$$

Let Γ be a diagram of an embedding of an oriented tri-valent graph G colored by (N, R, S). We identify the edge sets of Γ and G. For an edge E of Γ, let $\beta(E) = \frac{1}{2}\,\tilde{q}_{R(E)}\,(1 + S(E)\,\alpha_{R(E)}^{-1/2}\,h_n) \in \mathbb{C}B_{N(E)}$. Replace every vertices and edges as in Figure 4, we get a virtual link diagram $\Gamma^{(N,R,S)}$. For a edge E of Γ, let $c(E) = S(E)\,\alpha_{R(E)}^{1/2}$.

Theorem 2. *Let Γ and Γ' be equivalent embeddings of an oriented tri-valent graph G colored by (N, R, S). Then, for every edge E of G, there is an integer $d(E)$ such that*

$$X(\Gamma^{(N,R,S)}) = \prod_{E\in\mathcal{E}} c(E)^{d(E)}\, X({\Gamma'}^{(N,R,S)}). \tag{2.1}$$

Proof. We check (2.1) for Reidemeister moves (SRI)–(SRV). Let Γ and Γ' be diagrams of embeddings of G. We identify the sets of edges of Γ and Γ' with that of G.

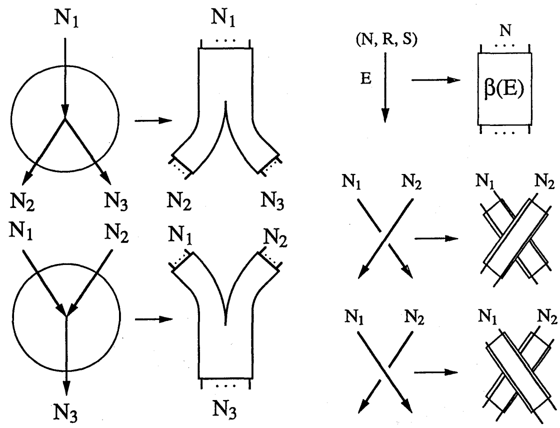

Fig. 4. Replace vertices and edges.

Case 1. Assume that Γ and Γ' are regular isotopic, i. e. there is a sequence of Reidemeister moves of types (SRII), (SRIII), (SRIV) sending Γ to Γ'. Then the associated virtual link diagrams $\Gamma^{(N,R,S)}$ and $\Gamma'^{(N,R,S)}$ are equivalent. Hence we have

$$X(\Gamma^{(N,R,S)}) = X(\Gamma'^{(N,R,S)}). \tag{2.2}$$

Case 2. In this and the next cases, we check (2.1) for (SRI) moves. Assume that Γ and Γ' are identical except within a ball where they are as shown in Figure 5. Let E be the edge of G embedded differently by Γ and Γ'. Let $n = N(E)$, $\rho = R(E)$, $s = S(E)$ and $\beta = \beta(E)$. Then there are positive integer N and a braid $b \in \mathbb{C}B_N$ such that the associated link diagrams $\Gamma^{(N,R,S)}$ and $\Gamma'^{(N,R,S)}$ are equivalent to the closures of $b_1 = b\eta(\beta)$ and $b_2 = b\eta(\beta)\, f_n$ where η is an algebra homomorphism from $\mathbb{C}B_n$ to $\mathbb{C}B_N$ defined by $\eta(\sigma_i) = \sigma_i$ for $1 \le i \le n-1$. Since X is an invariant of trace type, there is an algebra homomorphism J from A_n to A_N such that $p_N \circ \eta = J \circ p_n$. From the definition of trace type invariants, we have

$$X(\hat{b}_2) = T_N(p_N(b_2)) = T_N(p_N(b\,\eta(\beta)\, f_n)).$$

The definitions of q_ρ and β imply that $p_n(\beta\, h_n^{\pm 1}) = (s\,\alpha_\rho^{1/2})^{\pm 1} p_n(\beta)$. Hence we have

$$T_N(p_N(b\,\eta(\beta)\, f_n)) = T_N(p_N(b)\, J(p_n(\beta\, {h_n}^2)))$$

$$=T_N(p_N(b)J(\alpha_\rho p_n(\beta))) = \alpha_\rho T_N(p_N(b)J(p_n(\beta))),$$

and so we get

$$X(\hat{b}_2) = \alpha_\rho X(\hat{b}_1).$$

In other words,

$$X(\Gamma^{(N,R,S)}) = \alpha_\rho X(\Gamma'^{(N,R,S)}). \tag{2.3}$$

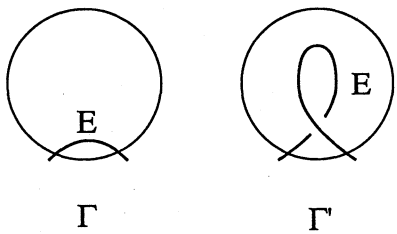

Fig. 5.

Case 3. Let Γ and Γ' be diagrams of colored tri-valent graphs identical except within a ball where they are as shown in Figure 6. Then, as in Case 2, we have

$$X(\Gamma^{(N,R,S)}) = \alpha_{R(E)}^{-1} X(\Gamma'^{(N,R,S)}). \tag{2.4}$$

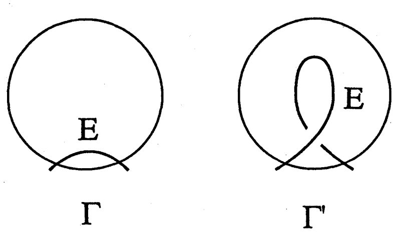

Fig. 6.

Case 4. To check (SRV), it is suffice to verify the theorem for moves illustrated in Figures 7–10. Assume that Γ and Γ' are identical except within a ball where they are as shown in Figure 7. Let $n(i) = N(E_i)$, $\rho(i) = R(E_i)$, $s(i) = S(E_i)$, $\tilde{q}_i = \tilde{q}_{\rho(i)}$, $q_i = q_{\rho(i)}$, $p_i = p_{n(i)}$, $h_i = h_{n(i)}$ and $\beta_i = \beta(E_i)$ for $i = 1, 2, 3$. Then there are positive integer N and $b \in \mathbb{C}B_N$ such that the associated link diagrams $\Gamma^{(N,R,S)}$ and $\Gamma'^{(N,R,S)}$ are equivalent to the closures of

$$\begin{aligned} b_1 =& b\eta_1(\beta_1)\,\eta_2(\beta_2)\,\eta_3(\beta_3), \\ b_2 =& b\eta_1(\beta_1)\,\eta_2(f_{n(2)}\,\beta_2)\,\sigma_{n(1),n(2)}\,\eta_3(\beta_3), \end{aligned}$$

where $\sigma_{n(1),n(2)} = \sigma_{n(1)}\sigma_{n(1)+1}\cdots\sigma_{n(1)+n(2)-1}\ \sigma_{n(1)-1}\cdots\sigma_{n(1)+n(2)-2}$ $\cdots\sigma_1\sigma_2\cdots\sigma_{n(2)}$ and η_1, η_2 η_3 are algebra homomorphisms from $\mathbb{C}B_{n(1)}$, $\mathbb{C}B_{n(2)}$ $\mathbb{C}B_{n(3)}$ to $\mathbb{C}B_N$ defined by the following. $\eta_1(\sigma_i) = \sigma_i$ for $1 \leq i \leq n(1) - 1$, $\eta_2(\sigma_i) = \sigma_{n(1)+i}$ for $1 \leq i \leq n(2) - 1$ and $\eta_3(\sigma_i) = \sigma_i$ for $1 \leq i \leq n(3) - 1$. We know that $\eta_1(h_{n(1)})\eta_2(h_{n(2)}\sigma_{n(1),n(2)}) = \eta_3(h_{n(3)})$. Hence we have

$$b_2 = b\,\eta_1(\beta_1\,h_1^{-1})\eta_2(\beta_2\,h_2)\eta_3(h_3\beta_3).$$

Since X is an invariant of trace type, there are algebra homomorphisms J_1, J_2 and J_3 from $A_{n(1)}$, $A_{n(2)}$ and $A_{n(3)}$ to A_N such that $p_N \circ \eta_s = J_s \circ p_{n(s)}$ for $s = 1, 2, 3$. From the definition of the trace type, we have

$$\begin{aligned} X(\hat{b}_2) =& T_N(p_N(b_2)) \\ =& T_N(p_N(b\eta_1(\beta_1\,h_1^{-1})\,\eta_2(\beta_2\,h_2)\,\eta_3(h_3\,\beta_3)) \\ =& T_N(p_N(b)\,J_1(p_1(\beta_1\,h_1^{-1}))\,J_2(p_2(\beta_2\,h_2))\,J_3(p_3(h_3\,\beta_3))). \end{aligned}$$

The definition of q_R and $\beta(E)$ implies that

$$p_t(\beta(t)\,h_t^{\pm 1}) = S(t)\,\alpha_{\rho(t)}^{\pm 1/2}\,p_t(\beta_t) \qquad (t = 1, 2, 3).$$

Hence we have

$$\begin{aligned} & T_N(p_N(b)\,J_1(p_1(\beta_1\,h_1^{-1}))\,J_2(p_2(\beta_2\,h_2))\,J_3(p_3(h_3\,\beta_3))\,) \\ =& (\prod_{t=1}^{3} s(t))\,\alpha_{\rho(1)}^{-1/2}\,\alpha_{\rho(2)}^{1/2}\,\alpha_{\rho(3)}^{1/2}\,T_N(p_N(b)\,J_1(p_1(\beta_1))\,J_2(p_2(\beta_2))\,J_3(p_3(\beta_3))), \end{aligned}$$

and so we get

$$X(\hat{b}_2) = s(1)\,\alpha_{\rho(1)}^{-1/2}\,s(2)\,\alpha_{\rho(2)}^{1/2}\,s(3)\,\alpha_{\rho(3)}^{1/2}\,X(\hat{b}_1).$$

In other words,

$$X(\Gamma^{(N,R,S)}) = s(1)\,\alpha_{\rho(1)}^{-1/2}\,s(2)\,\alpha_{\rho(2)}^{1/2}\,s(3)\,\alpha_{\rho(3)}^{1/2} X(\Gamma'^{(N,R,S)}). \tag{2.5}$$

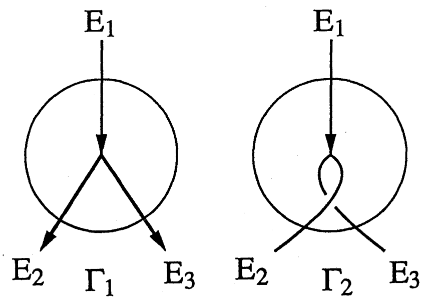

Fig. 7.

Case 5. Assume that Γ and Γ′ are identical except within a ball where they are as shown in Figure 8. Then, as in Case 4, we have

$$X(\Gamma^{(N,R,S)}) = s(1)\,\alpha_{\rho(1)}^{1/2}\,s(2)\,\alpha_{\rho(2)}^{-1/2}\,s(3)\,\alpha_{\rho(3)}^{-1/2}\,X(\Gamma'^{(N,R,S)}). \tag{2.6}$$

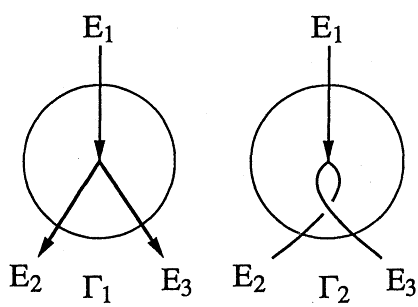

Fig. 8.

Case 6. Assume that Γ and Γ′ are identical except within a ball where they are as shown in Figure 9. Then, as in Case 4, we have

$$X(\Gamma^{(N,R,S)}) = s(1)\,\alpha_{\rho(1)}^{1/2}\,s(2)\,\alpha_{\rho(2)}^{1/2}\,s(3)\,\alpha_{\rho(3)}^{-1/2}\,X(\Gamma'^{(N,R,S)}). \tag{2.7}$$

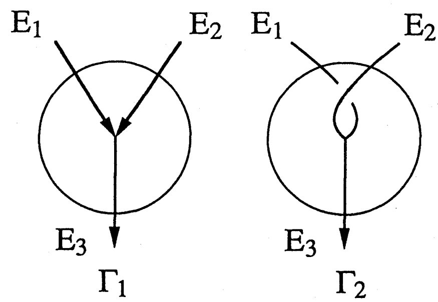

Fig. 9.

Case 7. Let Γ and Γ' be diagrams of colored tri-valent graphs identical except within a ball where they are as shown in Figure 10. Then, as in Case 4, we have

$$X(\Gamma^{(N,R,S)}) = s(1)\,\alpha_{\rho(1)}^{-1/2}\,s(2)\,\alpha_{\rho(2)}^{-1/2}\,s(3)\,\alpha_{\rho(3)}^{1/2}\,X(\Gamma'^{(N,R,S)}). \tag{2.8}$$

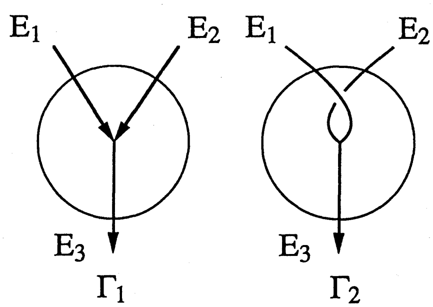

Fig. 10.

The above formulas (2.2)–(2.8) implies Theorem 2. Q.E.D.

§3. Invariants of non-oriented spatial graphs

Let X be a regular isotopy invariant of link diagrams of knit trace type with writhe factor γ. Let G be an abstract graph. For each edge E of G, we attach a non-negative integer $N(E)$, an irreducible representation $R(E) \in \check{A}_{N(E)}$ and a signature $S(E) = \pm 1$. If these data satisfy

(1.2) in §1, they are called a *coloring* of G and denoted by (N, R, S). Let $\mathcal{E}_v$ be the subset of edges of G with a terminal point v.

From now on, fix an invariant X of knit trace type and let (A_1, T_1), (A_2, T_2), $\cdots$ be the Markov knit sequence of X. Since A_n is a semisimple algebra, we have

$$A_n = \bigoplus_{\rho \in \check{A}_n} M_{d(\rho)}(\mathbb{C})$$

where $d(\rho)$ is the degree of ρ. Let q_ρ be an element of A_n such that

$$\nu(q_\rho) = \delta_{\nu\rho} \text{ id} \in M_{d(\nu)}(\mathbb{C}) \qquad \text{for} \quad \nu \in \check{A}_n.$$

Let $\tilde{q}_\rho$ be an element of $\mathbb{C}K_n$ such that $p_n(\tilde{q}_\rho) = q_\rho$. Note that $\tilde{q}_\rho$ is not unique. Let $h_n = \tau_1\tau_2 \cdots \tau_{n-1}\,\tau_1 \cdots \tau_{n-2} \cdots \tau_1\tau_2\,\tau_1$. We call h_n the *half twist* of K_n. Let $f_n = {h_n}^2$ and we call f_n the *full twist* of K_n. It is known that f_n commute with every element of K_n and so $\rho(p_n(f_n))$ is a scalar matrix, i.e. $\rho(p_n(f_n)) = \alpha_\rho$ id.

Let G be an abstract graph colored by (N, R, S). Let Γ be a colored non-oriented spatial graph equal to G as an abstract graph. We identify the sets of edges of Γ and G. Let v be a vertex of Γ. Let E_1, E_2, $\cdots$, E_r be the edges with a terminal point v. Let ξ_1, ξ_2, $\cdots$, ξ_r be the terminal points of E_1, E_2, $\cdots$, E_r on the boundary of v and $N(i) = N(E_i)$ for $i = 1, 2, \cdots, r$. Replace these points by $\zeta_1^{(1)}$, $\zeta_1^{(2)}$, $\cdots$, $\zeta_1^{(N(1))}$, $\zeta_2^{(1)}$, $\cdots$, $\zeta_2^{(N(2))}$, $\cdots$, $\zeta_r^{(1)}$, $\cdots$, $\zeta_r^{(N(r))}$ as in Figure 11. Let $n_v = (\sum_{i=1}^r N(i))/2$. A diagram D on v is a set of mutually disjoint n_v curves connecting $\gamma_{i(1)}^{j(1)}$ to $\gamma_{i(2)}^{j(2)}$. Two diagrams D and D' on v are called equivalent if there is an isotopy of v sending D to D' which fixes the boundary of v. A diagram D on v is called *essential* if D satisfies the following.

(*) Let $\gamma_{i(1)}^{j(1)}$ and $\gamma_{i(2)}^{j(2)}$ be distinct boundary points of a curve of D. Then $i(1) \neq i(2)$.

We denote by $\mathcal{D}_v$ the set of equivalence classes of essential diagrams on v. If the valency of v is equal to 3, then $\mathcal{D}_v$ has only one element. If the valency of v is equal to 4 and $N(E_i) = 2$ for $i = 1, \cdots, 4$, then $\mathcal{D}_v$ consists of 3 elements as in Figure 12.

Let $\beta(E) = \frac{1}{2}\,\tilde{q}_{R(E)}\,(1 + S(E)\,\alpha_{R(E)}^{-1/2}\,h_n) \in \mathbb{C}B_{N(E)}$. Let $\Gamma^{(N,R,S)}$ be the virtual link diagram obtained by replacing each vertex v by a sum of the all elements of $\mathcal{D}_v$ and each edge E by $\beta(E)$ as in the case of embeddings of oriented tri-valent graphs. For a edge E of Γ, let $c(E) = S(E)\,\alpha_{R(E)}^{1/2}$.

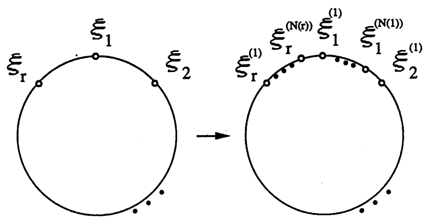

Fig. 11. Replace $\xi_1, \cdots, \xi_r$ by $\zeta_1^{(1)}, \cdots, \zeta_1^{(N(1))}, \zeta_2^{(1)}, \cdots, \zeta_2^{(N(2))}, \cdots, \zeta_r^{(1)}, \cdots, \zeta_r^{(N(r))}$.

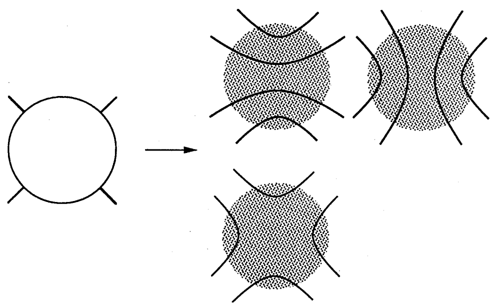

Fig. 12. Elements of $\mathcal{D}_v$.

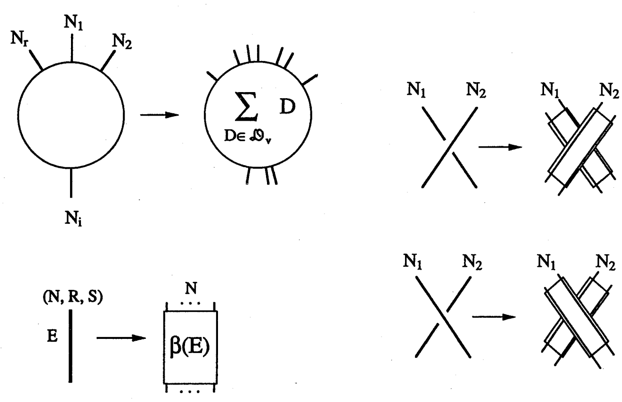

Fig. 13. Replace edges and vertices.

Theorem 3. *Let Γ and Γ' be colored spatial graphs isomorphic to* *a graph G colored by (N, R, S) as abstruct graphs. Identify the sets of edges of Γ and Γ' with that of G. If Γ and Γ' are equivalent as spatial graphs, then there are integers d and $d(E)$ for every edge E of G such that*

$$X(\Gamma^{(N,R,S)}) = \gamma^d \prod_{E\in\mathcal{E}} c(E)^{d(E)} X(\Gamma'^{(N,R,S)}). \tag{3.1}$$

Proof. We check (3.1) for Reidemeister moves (SRI)–(SRV). Let Γ and Γ' be diagrams of colored spatial graphs isomorphic to G. We identify the sets of edges of Γ and Γ' with that of G.

Case 1. Assume that Γ and Γ' are regular isotopic, i. e. there is a sequence of reidemeister moves of types (SRII), (SRIII), (SRIV) sending Γ to Γ'. Then the associated virtual link diagrams $\Gamma^{(N,R,S)}$ and $\Gamma'^{(N,R,S)}$ are equivalent and we have

$$X(\Gamma^{(N,R,S)}) = X(\Gamma'^{(N,R,S)}). \tag{3.2}$$

Case 2. In this and the next cases, we check (2.1) for (SRI) moves. Assume that Γ and Γ' are identical except within a ball where they are as shown in Figure 5. Let $n = N(E)$, $\rho = R(E)$, $s = S(E)$ and $\beta = \beta(E)$. Then there are positive integer N and $b \in \mathbb{C}K_N$ such that the associated link diagrams $\Gamma^{(N,R,S)}$ and $\Gamma'^{(N,R,S)}$ are equivalent to the closures of $b_1 = b\,\eta(\beta)$ and $b_2 = b\,\eta(\beta)\,h_n^2$ where η is an algebra homomorphism from $\mathbb{C}K_n$ to $\mathbb{C}K_N$ defined by $\eta(\sigma_i) = \sigma_i$ for $1 \leq i \leq n-1$. Since X is a regular isotopy invariant of knit trace type, there is an algebra homomorphism J from A_n to A_N such that $p_N \circ \eta = J \circ p_n$. From the definition of trace type invariants, we have

$$X(\hat{b}_2) = T_N(p_N(b_2)) = T_N(p_N(b\,\eta(\beta)\,{h_n}^2)).$$

The definition of β implies that

$$p_n(\beta\, h_n^{\pm 1}) = s\,\alpha_\rho^{\pm 1/2} p_n(\beta).$$

Hence we have

$$\begin{aligned} T_N(p_N(b\,\eta(\beta)\,{h_n}^2)) =& T_N(p_N(b)\,J(p_n(\beta\,{h_n}^2))) \\ =& T_N(p_N(b) J(\alpha_\rho\, p_n(\beta))) \\ =& \alpha_\rho\, T_N(p_N(b) J(p_n(\beta))), \end{aligned}$$

and so we get

$$X(\hat{b}_2) = \alpha_\rho\, X(\hat{b}_1).$$

In other words,

$$X(\Gamma^{(N,R,S)}) = \alpha_\rho X(\Gamma'^{(N,R,S)}).$$

Case 3. Assume that Γ and Γ' are identical except within a ball where they are as shown in Figure 6. Then, as in Case 2, we have

$$X(\Gamma^{(N,R,S)}) = \alpha_\rho^{-1} X(\Gamma'^{(N,R,S)}). \tag{3.3}$$

Case 4. Assume that Γ and Γ' are identical except within a ball where they are as shown in Figure 14. Let E_1, E_2, $\cdots$, E_r be edges around the vertex v. Let $n(i) = N(E_i)$ for $i = 1, 2, \cdots, r$ and $n = \sum_{i=1}^r n(i)$. Let $\varepsilon_{1,n} = \varepsilon_1\varepsilon_3\cdots\varepsilon_{2n-1} \in K_n$.

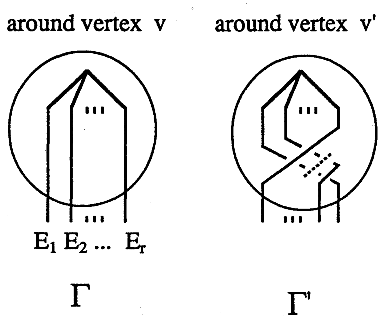

Fig. 14.

Let h_v, e_v and e'_v be the element of K_n corresponding to the diagram in Figure 15. Let $\eta_{i,j,k}$ $(i, j > 0,\ k \geq 0,\ i + k \leq j)$ be a semigroup homomorphism from K_i to K_j which sends $\tau_i^{\pm 1}$, $\varepsilon_i \in K_i$ to $\tau_{i+k}^{\pm 1}$, $\varepsilon_{i+k} \in K_j$ and $\phi_{i,j} = \eta_{n(i),j,n(1)+n(2)+\cdots+n(i-1)}$. Note that

$$h_n e_v = \gamma^n e_v,$$

$$h_n e_v = e'_v\, \phi_{1,n}(h_{n(1)})\, \phi_{2,n}(h_{n(2)}) \cdots \phi_{r,n}(h_{n(r)}),$$

and so we have

$$e'_v = \gamma^n\, e_v\, \phi_{1,n}(h_{n(1)}^{-1})\, \phi_{2,n}(h_{n(2)}^{-1}) \cdots \phi_{r,n}(h_{n(r)}^{-1}). \tag{3.4}$$

Let $\rho(i) = R(E_i)$, $s(i) = S(E_i)$ and $\beta(i) = \beta(E_i)$ for $i = 1, 2, \cdots, r$. Then there are an integer N and an element $b \in \mathbb{C}K_n$ such that the

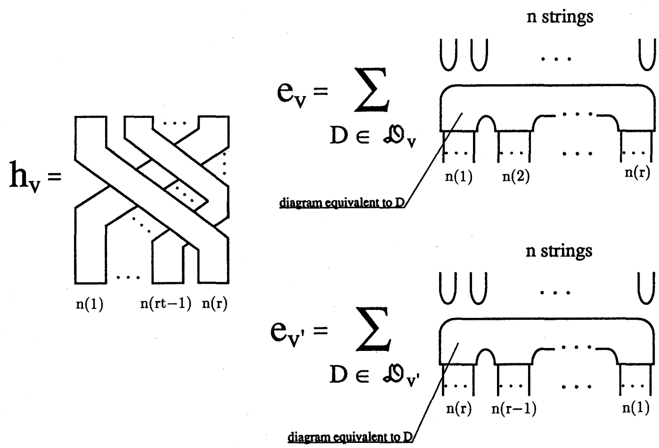

Fig. 15. Diagrams of h_v, e_v and e'_v.

associated link diagrams $\Gamma^{(N,R,S)}$ and $\Gamma'^{(N,R,S)}$ are equivalent to the closures of

$$\begin{aligned} b_1 =& b\,\eta_{n,N,0}(e_v)\,\phi_{1,N}(\beta(1)\tilde{q}_{\rho(1)})\,\phi_{2,N}(\beta(2)\tilde{q}_{\rho(2)})\,\cdots\,\phi_{r,N}(\beta(r)\tilde{q}_{\rho(r)}). \\ b_2 =& b\,\eta_{n,N,0}(e'_v h_v)\,\phi_{1,N}(\beta(1)\tilde{q}_{\rho(1)})\,\phi_{2,N}(\beta(2)\tilde{q}_{\rho(2)})\,\cdots\,\phi_{r,N}(\beta(r)\tilde{q}_{\rho(r)}). \end{aligned}$$

From (3.4), we have

$$\text{(3.5)}\quad b_2 = \gamma^n\, b\,\eta_{n,N,0}(e_v)\,\phi_{1,N}(h_{n(1)}^{-1}\beta(1)\tilde{q}_{\rho(1)})\,\cdots\,\phi_{r,N}(h_{n(r)}^{-1}\beta(r)\tilde{q}_{\rho(r)}).$$

Recall that the definition of $q_{R(E)}$ and $\beta(E)$ implies that

$$q_{\rho(t)}p_{n(t)}(\beta(t)\,{h_{n(t)}}^{\pm 1}) = s(t)\,\alpha_{\rho(t)}^{\pm 1/2}\,q_{\rho(t)}\,p_{n(t)}(\beta(t))$$

for $t = 1, 2, \cdots, r$. Hence formula (3.5) implies

$$X(\hat{b}_2) = \prod_{i=1}^{r} S(t)\,\alpha_{\rho(t)}^{-1/2}\,X(\hat{b}_1), \tag{3.6}$$

because X is of knit trace type.

Case 5. Assume that Γ and Γ' are identical except within a ball where they are as shown in Figure 16. Then, as in Case 4, we have

$$X(\hat{b}_2) = \prod_{i=1}^{r} s(t)\,\alpha_{\rho(t)}^{1/2}\,X(\hat{b}_1). \tag{3.7}$$

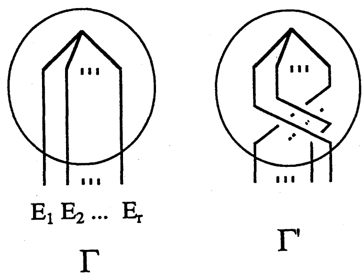

Fig. 16.

Formulas (3.2), (3.3), (3.6), (3.7) show Theorem 3. Q.E.D.

Let N be a positive even number. Let R be an irreducible representation of the algebra A_N associated with the link invariant X. Let S be 1 or -1. For a spatial graph Γ, let (N', R', S') be the coloring of Γ defined by $N'(E) = E$, $R'(E) = R$ and $S'(E) = S$ for every edge E of Γ. Let $X^{(N,R,S)}(\Gamma) = X(\Gamma^{(N',R',S')})$. Then $X^{(N,R,S)}$ is a regular isotopy invariant of diagrams of spatial graphs.

Corollary 4. *Let Γ and Γ' be diagrams of the same spatial graph G. Then, there are integers d and d' such that*

$$X^{(N,R,S)}(\Gamma) = \gamma^d\,{\alpha_R}^{d'}\,X^{(N,R,S)}(\Gamma').$$

The proof is similar to that of Theorem 2.

§4. Examples

KauffmanUs bracket polynomial $\langle\,.\,\rangle$ is a regular isotopy invariant of knit trace type and the Jones polynomial is obtained from $\langle\,.\,\rangle$ as in Remark in §1. To fix the notation, we give the definition of the bracket polynomial $\langle\,.\,\rangle$ [4]. Let $A \in \mathbb{C}\setminus\{0\}$ which is not equal to any roots of unity. The bracket polynomial with parameter A is a regular isotopy invariant of non-oriented link diagrams defined by the following relations.

$$\langle L_O\rangle = 1,$$
$$\langle L_x\rangle = A\langle L_{||}\rangle + A^{-1}\langle L_\infty\rangle,$$

where L_O is a trivial knot and L_x, $L_{||}$, L_∞ are link diagrams identical except within a ball where they are as shown in Figure 17.

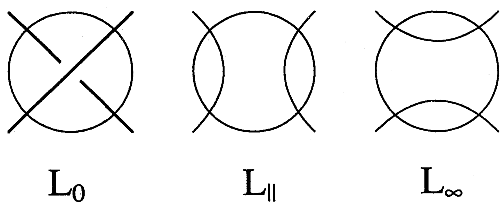

Fig. 17. Diagrams of L_x, $L_{||}$, L_∞.

Let A be a non-zero complex number which is not equal to any roots of unity. Let $J_n(A)$ be the Jones algebra defined over $\mathbb{C}$ by the following.

$$J_n(A) = \langle e_1, e_2, \cdots, e_{n-1} \mid e_i\, e_{i\pm1}\, e_i = e_i,\ e_i\, e_j = e_j\, e_i\ (|i-j| \geq 2),$$
$$e_i^2 = -(A^2 + A^{-2})\, e_i\rangle.$$

The Markov knit sequence of KauffmanUs bracket polynomial $\langle\,.\,\rangle$ is $J_1(A)$, $J_2(A)$, $\cdots$. The algebra homomorphism p_n from $\mathbb{C}K_n$ to $J_n(A)$ is defined by $p_n(\varepsilon_i) = e_i$, $p_n(\tau_i) = A + A^{-1}\, e_i$ and $p_n(\tau_i^{-1}) = A^{-1} + A\, e_i$. Let ρ_n be the linear representation of $J_n(A)$ sending e_1, e_2, $\cdots$, e_{n-1} to 0. Since $\rho_n(p_n(\tau_i)) = A$, we have

$$\rho_n(h_n) = A^{n(n-1)/2}. \tag{4.1}$$

Let $\alpha_n = A^{n(n-1)}$ and $\sqrt{\alpha_n} = A^{n(n-1)/2}$. The Yamada polynomial in [10] is coming from $\langle\,.\,\rangle$ as in Corollary 4 with $N = 2$, $R = \rho_2$ and $S = 1$.

Let Γ_1 and Γ_2 be two diagrams of spatial graphs as in Figure 18.

The diagrams Γ_1 and Γ_2 are colored as in the figure. Let C_1, C_2 denote the above coloring for Γ_1 and Γ_2 respectively. Since $p_2(1 + (A^2 +$

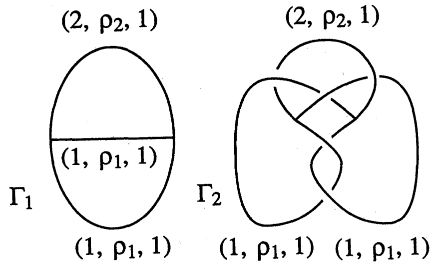

Fig. 18. Diagrams of spatial graphs Γ_1 and Γ_2.

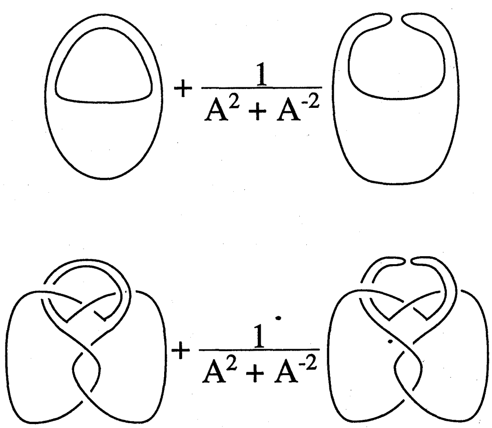

Fig. 19. Virtual link diagrams $\Gamma_1^{C_1}$ and $\Gamma_2^{C_2}$.

$A^{-2})^{-1}\varepsilon_1) = q_{R_2}$, the virtual diagrams $\Gamma_1^{C_1}$ and $\Gamma_2^{C_2}$ associated to the colorings are given in Figure 19.

Hence we have

$$\langle\Gamma_1\rangle^{C_1} = -\frac{A^8 + A^4 + 1}{A^2(A^4+1)}$$

and

$$\langle\Gamma_2\rangle^{C_2} = -\frac{-A^{32} + A^{28} + A^{20} + A^8 + 1}{A^{13}(A^4+1)}.$$

By (4.1) and Theorem 3, we know that Γ_1 and Γ_2 are not equivalent as spatial graphs.

To investigate the invariants associated with the Jones polynomial more closely, Section 4 of [7] may be helpful.

The HOMFLY polynomial P is an oriented link invariant of trace type. Hence we get invariants of colored oriented tri-valent graph embeddings from the HOMFLY polynomial.

The Kauffman polynomial F is an oriented link invariant obtained from the Dubrovnik polynomial [5], which is a regular isotopy invariant of unoriented link diagrams. It is shown in [2], [7], [8] that the Dubrovnik polynomial is of knit trace type. Hence we get invariants of spatial graphs from the Dubrovnik polynomial. To investigate properties of these invariants, Section 5 of [7] may be helpful.

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