Solvable Lattice Models and Algebras of Face Operators

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§1. Introduction

In this paper, we discuss algebras associated with the solvable lattice models and conformal field theory (CFT). In [7], solutions of Yang-Baxter equations (YBE) associated with the vector representations of simple Lie algebras are introduced. For each solution, we denote by Y_n an algebra of Yang-Baxter operators. On the other hand, solutions of star-triangle relations (or IRF models) associated with the vector representations of classical simple Lie algebras are introduced in [9]. For each solution, an algebra of face operators F_n are introduced in Section 3. Let Y_n and F_n be the above algebras associated with a simple Lie algebra of type A. Then they are both quotients of Iwahori's Hecke algebra $H_n(q)$ of type A. Let $\mathfrak{g}$ be a simple Lie algebra of type B, C or D. For the algebras Y_n and F_n associated with $\mathfrak{g}$, we show in Section 5 that Y_n and F_n are both quotients of a q-analogue of Brauer's centralizer algebra $C_n(a, q)$. The algebras $H_n(q)$ and $C_n(a, q)$ have the following properties.

(1) The algebra of face operators of an IRF model associated with a classical simple Lie algebra is a quotient of $H_n(q)$ or $C_n(a, q)$.

(2) The algebra of Yang-Baxter operators associated with a classical simple Lie algebra is a quotient of $H_n(q)$ or $C_n(a, q)$. ([7], [13])

(3) Let $U_q(\mathfrak{g})$ denote the q-analogue of the universal enveloping algebra of a simple Lie algebra $\mathfrak{g}$. Then the centralizer algebra associated with the vector representation of $U_q(\mathfrak{g})$ is equal to a quotient of $H_n(q)$ if $\mathfrak{g}$ is of type A and a quotient of $C_n(a, q)$ if $\mathfrak{g}$ is of type B or C. The centralizer algebra associated with that of type D contains a quotient of $C_n(a, q)$. (Immediate consequence of (2).)

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(4) The algebra generated by monodromies of the n-point function of CFT associated with a classical simple Lie algebra is a quotient of $H_n(q)$ or $C_n(a, q)$. ([16])

(5) Some knot invariants (the one and two-variable Jones polynomial and the Kauffman polynomial) are linear combinations of irreducible characters of the algebras $H_n(q)$ and $C_n(a, q)$ ([10], [13], [17]).

In Section 2, we reconstruct the IRF models introduced in [9] in terms of the algebra $H_n(q)$ or $C_n(a, q)$. The set of local states is equal to the set of equivalence classes of irreducible representations of the algebra. In Section 3, the algebra F_n of face operators is introduced. In Section 4, we review the solutions of the quantum Yang-Baxter equations associated with the rector representation of Lie algebras of classical types. The algebra Y_n of the Yang-Baxter operators are introduced. In Section 5, we investigate the algebra $C_n(a, q)$. By using the IRF model of type A, we may reconstruct the irreducible representations of Iwahori's Hecke algebras given in [5] and [18]. Similarly, by using the IRF models of types B, C and D, we construct all irreducible representations of the algebra $C_n(a, q)$ (Theorem 5.15). At the last section, we propose some problems.

§2. Solvable lattice models

Solvable lattice models with face interaction are introduced. Each model is a generalization of that in [9] and related to the inductive limit $\varinjlim_n X_n^{(1)}$ of the affine Lie algebra $X_n^{(1)} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$. A sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ is called a partition of a positive integer n if $\lambda_1 \geqq \lambda_2 \geqq \ldots$ and $\sum_i \lambda_i = n$. For a positive integer n, let $\Lambda(n)$ be a set of partitions of n and $\Lambda = \cup_n \Lambda(n)$. For $\lambda \in \Lambda$, let $|\lambda| = \sum_i \lambda_i$. Let $\kappa_i = (0, \ldots, 0, \underset{i}{1}, 0, \ldots)$ and $O = (0, 0, \ldots)$. We list a set $\mathcal{A}$ below. Every element of $\mathcal{A}$ is corresponding to a fundamental weight of the Lie algebra X_n for sufficiently large n.

$$(2.1) \qquad \begin{aligned} \mathcal{A} &= \{\kappa_1, \kappa_2, \ldots\} & \text{for type A,} \\ \mathcal{A} &= \{\pm\kappa_1, \pm\kappa_2, \ldots\} & \text{for types B, C and D.} \end{aligned}$$

Consider now a two dimensional square lattice $\mathcal{L}$. We shall introduce face models on $\mathcal{L}$ that have the following basic features:

(1) The fluctuation variable placed on each lattice site assumes its values in Λ. We call these values local states.

(2)   Adjacent local states differ by an element in $\mathcal{A}$. More precisely, this means that the Boltzmann weights $W\begin{pmatrix} a & b \\ d & c \end{pmatrix}$ describing the interaction of four fluctuation variables round a face [1] satisfy the condition

(2.2) $W\begin{pmatrix} a & b \\ d & c \end{pmatrix} = 0$ unless $b - a,\ c - b,\ d - a,\ c - d \in \mathcal{A}.$

Under the setting above we have found a system of Boltzmann weights $W\begin{pmatrix} a & b \\ d & c \end{pmatrix} u$ that depend on the spectral parameter $u \in \mathbb{C}$ and solve the star-triangle relation (STR)

(2.3) $$\sum_g W\begin{pmatrix} b & g \\ c & d \end{pmatrix} u\bigg) W\begin{pmatrix} a & f \\ b & g \end{pmatrix} u+v\bigg) W\begin{pmatrix} f & e \\ g & d \end{pmatrix} v\bigg)$$

$$= \sum_g W\begin{pmatrix} a & g \\ b & c \end{pmatrix} v\bigg) W\begin{pmatrix} g & e \\ c & d \end{pmatrix} u+v\bigg) W\begin{pmatrix} a & f \\ g & e \end{pmatrix} u\bigg).$$
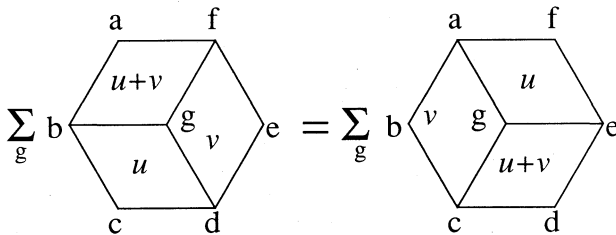

Fig. 1.   Star-triangle relation (STR).

The solutions are parametrized in terms of the elliptic theta function

$$[u] = \theta_1\left(\frac{\pi u}{L}, p\right),$$

(2.4)

$$\theta_1(u, p) = 2p^{1/8} \sin u \prod_{k=1}^{\infty} (1 - 2p^k \cos 2u + p^{2k})(1 - p^k),$$

where $L \neq 0$ is an arbitrary complex parameter. We need the following notations. Let α be an arbitrary complex parameter. For $\lambda \in \Lambda$ and

$\mu \in \mathcal{A}$, $\lambda(\mu)$ denotes the following:

(2.5)
$$\lambda(\mu) = \lambda_i + \alpha_1 - i + 1 \quad \text{for} \quad \mu = \kappa_i,$$
$$\lambda(\mu) = -\lambda_i - \alpha_1 + i - 1 \quad \text{for} \quad \mu = -\kappa_i,$$

where

(2.6)
$$\alpha_1 = \begin{cases} \alpha & \text{for types A and C,} \\ \alpha - 1/2 & \text{for type B,} \\ \alpha - 1 & \text{for type D.} \end{cases}$$

We also use the following:

(2.7)
$$\alpha_2 = \begin{cases} \alpha & \text{for types A and C,} \\ \alpha - 1 & \text{for type B,} \\ \alpha - 2 & \text{for type D.} \end{cases}$$

The parameter ω is fixed as follows:

(2.8)
$$\omega = \begin{cases} -(\alpha+1)/2 & \text{for type A,} \\ -(2\alpha-1)/2 & \text{for type B,} \\ -(\alpha+1) & \text{for type C,} \\ -(\alpha-1) & \text{for type D.} \end{cases}$$

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell, 0, 0, \ldots) \in \Lambda$, let

(2.9) $h_\lambda(i,j) = \lambda_i - i - j + \max\{k | \lambda_k \geq j\} \qquad (1 \leq i \leq \ell,\ 1 \leq j \leq \lambda_i),$

and

(2.10)
$$g_\lambda(i) = \begin{cases} \displaystyle\prod_{j=i+1}^{\lambda_i+i} [\lambda(\kappa_i) + \lambda(\kappa_j)] & (\lambda_i + i \geq \ell), \\[2ex] \dfrac{\prod_{j=i+1}^{\ell} [\lambda(\kappa_i) + \lambda(\kappa_j)]}{\prod_{j=1}^{\ell-\lambda_i-i} [2\alpha_1 + 2 - 2i - j]} & (\lambda_i + i < \ell). \end{cases}$$

The factor $G_{\lambda\mu}$ is given by the following:

(2.11)
$$G_{\lambda\mu} = G_{\lambda+\mu}/G_\lambda, \qquad G_{\lambda O} = 1,$$

$$(2.12) \quad G_\lambda = \begin{cases} \displaystyle\prod_{i=1}^{\ell}\prod_{j=1}^{\lambda_i} \frac{[j-i+\alpha]}{[h_\lambda(i,j)+1]} & \text{for type A,} \\[2.5em] \displaystyle\prod_{i=1}^{\ell} \frac{g_\lambda(i)\,[\lambda(\kappa_i)]}{[\alpha_1+1-i]\prod_{j=1}^{\lambda_i}[h_\lambda(i,j)+1]} & \text{for type B,} \\[2.5em] \displaystyle\prod_{i=1}^{\ell} \frac{g_\lambda(i)\,[2\lambda(\kappa_i)]\,[\alpha-i+1]}{[2\alpha_1+2-2i]\,[\lambda_i-i+\alpha+1]\prod_{j=1}^{\lambda_i}[h_\lambda(i,j)+1]} & \\[0.5em] & \text{for type C,} \\[1.5em] \displaystyle\prod_{i=1}^{\ell} \frac{g_\lambda(i)\,[\lambda_i-i+\alpha]}{[\alpha-i]\prod_{j=1}^{\lambda_i}[h_\lambda(i,j)+1]} & \text{for type D.} \end{cases}$$

The solutions are given precisely by the following formulas. For $\kappa,\mu,\nu,\sigma \in \mathcal{A}$ with $\mu+\nu=\kappa+\sigma$, we write

$$(2.13) \qquad \mu\,{\overset{\kappa}{\underset{\nu}{\square}}}\,\sigma = W\!\left(\begin{matrix} \lambda & \lambda+\kappa \\ \lambda+\mu & \lambda+\mu+\nu \end{matrix} \,\bigg|\, u\right).$$

$\varinjlim_n A_n^{(1)}$:

$$(2.14a) \qquad \mu\,{\overset{\mu}{\underset{\mu}{\square}}}\,\mu = \frac{[1+u]}{[1]},$$

$$(2.14b) \qquad \mu\,{\overset{\mu}{\underset{\nu}{\square}}}\,\nu = \frac{[\lambda(\mu)-\lambda(\nu)-u]}{[\lambda(\mu)-\lambda(\nu)]} \qquad (\mu\neq\nu),$$

$(2.14c)$

$$\mu\,{\overset{\nu}{\underset{\nu}{\square}}}\,\mu = \frac{[1+u]}{[1]}\left(\frac{[\lambda(\mu)-\lambda(\nu)+1]\,[\lambda(\mu)-\lambda(\nu)-1]}{[\lambda(\mu)-\lambda(\nu)]^2}\right)^{1/2} \qquad (\mu\neq\nu).$$

$\varinjlim_n B_n^{(1)},\ \varinjlim_n C_n^{(1)},\ \varinjlim_n D_n^{(1)}$:

$$(2.14d) \qquad \mu\,{\overset{\mu}{\underset{\mu}{\square}}}\,\mu = \frac{[\omega-u]\,[1+u]}{[\omega]\,[1]},$$

$$(2.14e) \qquad \mu\,{\overset{\mu}{\underset{\nu}{\square}}}\,\nu = \frac{[\omega-u]\,[\lambda(\mu)-\lambda(\nu)-u]}{[\omega]\,[\lambda(\mu)-\lambda(\nu)]} \qquad (\mu\neq\pm\nu),$$

$$(2.14\mathrm{f}) \quad \mu \underset{\nu}{\overset{\nu}{\square}} \mu = \frac{[\omega - u][u]}{[\omega][1]} \left(\frac{[\lambda(\mu) - \lambda(\nu) + 1][\lambda(\mu) - \lambda(\nu) - 1]}{\cdot \quad [\lambda(\mu) - \lambda(\nu)]^2} \right)^{1/2}$$

$$(\mu \neq \pm\nu),$$

(2.14g)

$$\mu \underset{-\mu}{\overset{\nu}{\square}} -\nu = \sigma \frac{[u][\lambda(\mu) + \lambda(\nu) + 1 + \omega - u]}{[\omega][\lambda(\mu) + \lambda(\nu) + 1]} (G_{\lambda\mu} G_{\lambda\nu})^{1/2} \qquad (\mu \neq \nu),$$

(2.14h)

$$\mu \underset{-\mu}{\overset{\mu}{\square}} -\mu = \frac{[\omega - u][2\lambda(\mu) + 1 - u]}{[\omega][2\lambda(\mu) + 1]} + \sigma \frac{[u][2\lambda(\mu) + 1 + \omega - u]}{[\omega][2\lambda(\mu) + 1]} G_{\lambda\mu},$$

where $\sigma = 1$ for types B and D, and $\sigma = -1$ for type C.

Theorem 2.15. *The Boltzmann weight given above satisfies the STR.*

Proof. Let r be a positive integer. By specializing the parameter α to r in the above models, we get Boltzmann weights of the models in [9]. A local state $(\lambda_1, \ldots, \lambda_r, 0, \ldots) \in \Lambda$ of our model corresponds to the following local state of [9].

$$(\lambda_1 + r, \ldots, \lambda_i + r - i + 1, \ldots, \lambda_r + 1) \qquad \text{for types A and C,}$$
$$(2.16) \quad (\lambda_1 + r - 1/2, \ldots, \lambda_i + r - i + 1/2, \ldots, \lambda_r + 1/2) \quad \text{for type B,}$$
$$(\lambda_1 + r - 1, \ldots, \lambda_i + r - i, \ldots, \lambda_r) \qquad \text{for type D.}$$

We assume that the parameter L is generic. For $a, b, c, d, e, f \in \Lambda$, let

$$(2.17) \quad S(a, b, c, d, e, f; u, v; \alpha) =$$

$$\sum_g W\left(\begin{matrix} b & g \\ c & d \end{matrix} \middle| u \right) W\left(\begin{matrix} a & f \\ b & g \end{matrix} \middle| u + v \right) W\left(\begin{matrix} f & e \\ g & d \end{matrix} \middle| v \right)$$

$$- \sum_g W\left(\begin{matrix} a & g \\ b & c \end{matrix} \middle| v \right) W\left(\begin{matrix} g & e \\ c & d \end{matrix} \middle| u + v \right) W\left(\begin{matrix} a & f \\ g & e \end{matrix} \middle| u \right).$$

For W's of types C and D, from the proof of (2.2) in [9] we have $S(a, b, c, d, e, f; u, v; r) = 0$ for a positive integer r such that $a_r = b_r = c_r = d_r = e_r = f_r = 0$. There are infinitely many integers which satisfy the above condition. Hence we have $S(a, b, c, d, e, f; u, v; \alpha) = 0$ for arbitrary α since the parameter L is generic. Hence W's given in (2.14) satisfy STR. From the proof of (2.2) and Figure 2 in [9], Boltzmann weight W of type B also satisfies STR.

The Boltzmann weights enjoy the following properties.

Initial condition

$$(2.18) \qquad W\!\left(\begin{array}{cc} a & b \\ d & c \end{array}\middle| \, 0 \right) = \delta_{bd}.$$

Reflection symmetry

$$(2.19) \qquad W\!\left(\begin{array}{cc} a & b \\ d & c \end{array}\middle| \, u \right) = W\!\left(\begin{array}{cc} a & d \\ b & c \end{array}\middle| \, u \right).$$

Rotational symmetry (valid except for A_r)

$$(2.20) \qquad W\!\left(\begin{array}{cc} a & b \\ d & c \end{array}\middle| \, u \right) = \left(\frac{G_b G_d}{G_a G_c}\right)^{1/2} W\!\left(\begin{array}{cc} d & a \\ c & b \end{array}\middle| \, \omega - u \right).$$

Inversion relations

$$(2.21) \qquad \begin{aligned} \sum_g W\!\left(\begin{array}{cc} a & g \\ d & c \end{array}\middle| \, u \right) W\!\left(\begin{array}{cc} a & b \\ g & c \end{array}\middle| \, -u \right) &= \delta_{bd}\varrho_1(u), \\[2mm] \sum_g \overline{W}\!\left(\begin{array}{cc} a & b \\ d & g \end{array}\middle| \, \omega - u \right) \overline{W}\!\left(\begin{array}{cc} c & d \\ b & g \end{array}\middle| \, \omega + u \right) &= \delta_{ac}\varrho_2(u). \end{aligned}$$

Here we have set

$$\overline{W}\!\left(\begin{array}{cc} a & b \\ d & c \end{array}\middle| \, u \right) = \left(\frac{G_a G_c}{G_b G_d}\right)^{1/2} W\!\left(\begin{array}{cc} d & a \\ c & b \end{array}\middle| \, \omega - u \right),$$

$$\varrho_1(u) = \frac{[1+u]\,[1-u]}{[1]^2}, \quad \varrho_2(u) = \frac{[\omega + u]\,[\omega - u]}{[1]^2} \quad \text{for } A_n,$$

$$\varrho_1(u) = \varrho_2(u) = \frac{[\omega + u]\,[\omega - u]\,[1+u]\,[1-u]}{[\omega]^2\,[1]^2} \quad \text{for } B_r, C_r, D_r.$$

To prove the above relations, we can apply the argument used in the proof of STR. Corresponding relations are given in [9].

§3. An algebra of face operators

In this section, we define an algebra of face operators of the models in the last section. We denote by $V^{\otimes n}$ the n-fold tensor product of a vector space V. Let $V = \varinjlim_n \mathbb{C}\,[\cup_{i=1}^n \Lambda(i)]$ and v_λ the base of V corresponding

to $\lambda \in \Lambda$. A face operator $T_i(u)$ $(i = 1, 2, \ldots, n-1, u \in \mathbb{C})$ is an element of $\mathrm{End}(V^{\otimes(n+1)})$ defined by the following:

$$(3.1) \quad T_i(u)(v_{\lambda_0} \otimes \cdots \otimes v_{\lambda_{i-1}} \otimes v_{\lambda_i} \otimes v_{\lambda_{i+1}} \otimes \cdots \otimes v_{\lambda_n})$$

$$= \sum_{\lambda' \in \Lambda} W\begin{pmatrix} \lambda_{i-1} & \lambda_i \\ \lambda' & \lambda_{i+1} \end{pmatrix} u \right) (v_{\lambda_0} \otimes \cdots \otimes v_{\lambda_{i-1}} \otimes v_{\lambda'} \otimes v_{\lambda_{i+1}} \otimes \cdots \otimes v_{\lambda_n}).$$

In the rest of this paper, we restrict our models to their trigonometric limit. In this case, $[u]$ stands for $2\sin(\pi u/L)$. Let $x = \exp(\pi i u/L)$. Then the trigonometric limits of the Boltzmann weights of our models are Laurent polynomials on x, and so $T_i(u)$'s are Laurent polynomials on x.

Let T_i $(i = 1, \ldots, n-1)$ be the coefficient of the highest degree of $T_i(u)$ with respect to x.

Proposition 3.2. *The operator $T_i(u)$ is expressed by T_i for $i = 1, \ldots, n-1$.*

Proof. For type A,

$$(3.3a) \qquad\qquad T_i(u) = xT_i - x^{-1}[1]^{-2}T_i^{-1}.$$

For types B, C and D,

$$(3.3b) \quad T_i(u) = x^2 T_i - T_i + 1 - [\omega]^{-2}[1]^{-2}T_i^{-1} + x^{-2}[\omega]^{-2}[1]^{-2}T_i^{-1}.$$

For any case, the algebra F_n is generated by $T_1, \ldots, T_{n-1}$.

Definition 3.4. Let F_n be a subalgebra generated by the operators $T_1, T_2, \ldots, T_{n-1}$. We call F_n *the algebra of face operators.*

For type A, it is known that the algebra of face operators F_n is isomorphic to the Iwahori's Hecke algebra of type A_{n-1}. For types B, C and D, the algebra F_n is isomorphic to a q-analogue of Brauer's centralizer algebra (see Theorem 5.9). We give finite dimensional F_n-submodules of $V^{\otimes(n+1)}$. Fix an element λ in Λ. Let V_λ be a subspace of $V^{\otimes(n+1)}$ spanned by a part of basis of $V^{\otimes(n+1)}$:

$$(3.5) \qquad\qquad V_\lambda = \sum_{\substack{\lambda_0, \ldots, \lambda_n \in \Lambda, \\ \lambda_{i+1}-\lambda_i \in \mathcal{A}, \\ \lambda_0=(0,0,\ldots), \\ \lambda_n=\lambda}} \mathbb{C}\, v_{\lambda_0} \otimes v_{\lambda_1} \cdots \otimes v_{\lambda_n}.$$

The following proposition is a direct consequence of the definition of the face operators.

Proposition 3.6. *The subspace V_λ is invariant under the action of the algebra F_n of face operators.*

§4. An algebra of Yang-Baxter operators

Let V be a finite dimensional vector space over $\mathbb{C}$ and n be a positive integer. For $R \in \mathrm{End}(V \otimes V)$, R_i $(i = 1, 2, \ldots, n-1)$ denote the element in $\mathrm{End}(V^{\otimes n})$

$$(4.1) \qquad R_i = \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \underset{i,i+1}{\underbrace{R}} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id},$$

which acts by R on the i and $i+1$ components and acts by identity on the other components. Quantum Yang-Baxter equation (YBE) is the following equation for a one-parameter family $\check{R}(x) \in \mathrm{End}(V \otimes V)$;

$$(4.2) \qquad \check{R}_1(x)\check{R}_2(x+y)\check{R}_1(y) = \check{R}_2(y)\check{R}_1(x+y)\check{R}_2(x).$$

Let $\mathfrak{g}$ be a Lie algebra of type A_r, B_r, C_r or D_r and $\mathcal{U}_q(\mathfrak{g})$ the q-analogue of the universal enveloping algebra of $\mathfrak{g}$. Let V be the vector representation of $\mathcal{U}_q(\mathfrak{g})$ and Δ be the comultiplication of $\mathcal{U}_q(\mathfrak{g})$. In [7], solutions $\check{R}(x)$ of YBE with $\left[\check{R}(x), \Delta(g)\right] = 0$ for $g \in \mathcal{U}_q(\mathfrak{g})$ are constructed explicitly. In the following, we rewrite these solutions. By convention the indices α, β run over 1, 2, ..., N, where $N = \dim V$. $N = r+1$, $2r+1$, $2n$, $2n$ for types A_r, B_r, C_r, D_r. We put $\alpha' = N+1-\alpha$. Let $E_{\alpha\beta}$ be the matrix $(\delta_{i\alpha}\delta_{j\beta})$. Let further $\varepsilon_\alpha = 1$ $(1 \leq \alpha \leq r)$, $= -1$ $(r+1 \leq \alpha \leq 2r)$ for type C_r and $\varepsilon_\alpha = 1$ in the remaining cases. Let k be a non-zero complex parameter associated with the quantization $(k = e^{-2\hbar})$,

$$(4.3) \quad \xi = k^{2r-1} \text{ for type } B_r, \ k^{2r+2} \text{ for type } C_r, \ k^{2r-2} \text{ for type } D_r,$$

$$(4.4\text{a}) \qquad \bar{\alpha} = \begin{cases} \alpha - 1/2 & (1 \leq \alpha \leq r) \\ \alpha + 1/2 & (r+1 \leq \alpha \leq 2r) \end{cases}$$

for type C_r, and

$$(4.4\text{b}) \qquad \bar{\alpha} = \begin{cases} \alpha + 1/2 & \left(1 \leq \alpha < \frac{N+1}{2}\right) \\ \alpha & \left(\alpha = \frac{N+1}{2}\right) \\ \alpha - 1/2 & \left(\frac{N+1}{2} < \alpha \leq N\right) \end{cases}$$

for types B_r and D_r. Under these notations the solutions are given as follows: Type A_r:

(4.5a)
$$\check{R}(x) = (x - k^2) \sum E_{\alpha\alpha} \otimes E_{\alpha\alpha} + k(x - 1) \sum_{\alpha \neq \beta} E_{\beta\alpha} \otimes E_{\alpha\beta}$$

$$- (k^2 - 1) \left(\sum_{\alpha < \beta} + x \sum_{\alpha > \beta} \right) E_{\beta\beta} \otimes E_{\alpha\alpha}.$$

Types B_r, C_r and D_r:

(4.5b)
$$\check{R}(x) = (x - k^2)(x - \xi) \sum_{\alpha \neq \alpha'} E_{\alpha\alpha} \otimes E_{\alpha\alpha} + k(x - 1)(x - \xi) \sum_{\alpha \neq \beta, \beta'} E_{\beta\alpha} \otimes E_{\alpha\beta}$$

$$- (k^2 - 1)(x - \xi) \left(\sum_{\alpha < \beta, \alpha \neq \beta'} + x \sum_{\alpha > \beta, \alpha \neq \beta'} \right) E_{\beta\beta} \otimes E_{\alpha\alpha}$$

$$+ \sum a_{\alpha\beta}(x) E_{\alpha'\beta} \otimes E_{\alpha\beta'},$$

where

(4.6)
$$a_{\alpha\beta}(x) = \begin{cases} (k^2 x - \xi)(x - 1) & (\alpha = \beta, \alpha \neq \alpha') \\ k(x - \xi)(x - 1) + (\xi - 1)(k^2 - 1)x & (\alpha = \beta, \alpha = \alpha') \\ (k^2 - 1)(\varepsilon_\alpha \varepsilon_\beta \xi k^{\bar{\alpha} - \bar{\beta}}(x - 1) - \delta_{\alpha\beta'}(x - \xi)) & (\alpha < \beta) \\ (k^2 - 1)x(\varepsilon_\alpha \varepsilon_\beta \xi k^{\bar{\alpha} - \bar{\beta}}(x - 1) - \delta_{\alpha\beta'}(x - \xi)) & (\alpha > \beta). \end{cases}$$

Let $\check{R}$ denote the highest degree of $\check{R}(x)$ with respect to x.

Proposition 4.7. *The operator* $\check{R}_i(u)$ *is expressed by* $\check{R}_i$ *for* $i = 1,$ *...,* $n - 1$.

Proof. For type A,

(4.8a)
$$\check{R}_i(x) = x\check{R}_i - k^2 \check{R}_i^{-1}.$$

For types B, C and D,

(4.8b) $\quad \check{R}_i(x) = x^2 \check{R}_i - x(\check{R}_i - (1 - k^2)(1 - \xi) + k^2 \xi \check{R}_i^{-1}) + k^2 \xi \check{R}_i^{-1}.$

For any case, the algebra Y_n is generated by $\check{R}_1, \ldots, \check{R}_{n-1}$.

Definition 4.9. Let Y_n be a subalgebra of $\text{End}(V^{\otimes n})$ generated by the operators $\check{R}_1, \check{R}_2, \ldots, \check{R}_{n-1}$. We call Y_n *the algebra of Yang-Baxter operators*.

For type A_r, it is known that the algebra of Yang-Baxter operators Y_n is isomorphic to a quotient of the Iwahori's Hecke algebra of type A_{n-1}. For types B, C and D, the algebra Y_n is a quotient of a q-analogue of Brauer's centralizer algebra (see Theorem 5.20). Details are given in the next section. This fact follows from [17].

§5. Brauer's centralizer algebra and its q-analogue

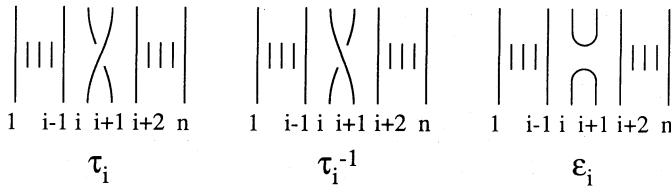At first we define a knit semigroup. A knit semigroup is a generalization of a braid group.

Fig. 2. Generators of a knit semigroup.

Definition 5.1. Let n be a positive integer. The semigroup K_n defined by the following generators and relations are called *a knit semigroup*.

(5.2)
$$K_n = \langle \tau_1, \ldots, \tau_{n-1}, \tau_1^{-1}, \ldots, \tau_{n-1}^{-1}, \varepsilon_1, \ldots, \varepsilon_{n-1} \mid$$
$$\tau_i \tau_i^{-1} = \tau_i^{-1} \tau_i = 1, \quad \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \quad \varepsilon_i \varepsilon_{i\pm 1} \varepsilon_i = \varepsilon_i,$$
$$\tau_i \tau_j = \tau_j \tau_i, \quad \tau_i \varepsilon_j = \varepsilon_j \tau_i, \quad \varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i \quad (|i - j| \geqq 2),$$
$$\varepsilon_i \varepsilon_{i+1} \tau_i^{\pm 1} = \varepsilon_i \tau_{i+1}^{\mp 1}, \quad \varepsilon_i \varepsilon_{i-1} \tau_i^{\pm 1} = \varepsilon_i \tau_{i-1}^{\mp 1},$$
$$\tau_i^{\pm 1} \varepsilon_{i+1} \varepsilon_i = \tau_{i+1}^{\mp 1} \varepsilon_i, \quad \tau_i^{\pm 1} \varepsilon_{i-1} \varepsilon_i = \tau_{i-1}^{\mp 1} \varepsilon_i \rangle.$$

The generators τ_i, τ_i^{-1} and ε_i are presented graphically as in Figure 2. Some relations of K_n are graphically presented in Figure 3.
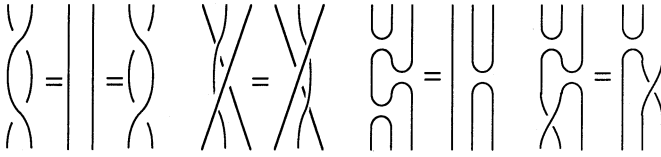
Fig. 3.   Relations of the knit semigroup.

The knit semigroup K_n contains the braid group B_n on n-strings. B_n is the subsemigroup of K_n generated by τ_i, τ_i^{-1} $(i = 1, 2, \ldots, n-1)$. We denote by $\mathbb{C}K_n$ the semigroup algebra of K_n. We call $\mathbb{C}K_n$ a *knit algebra*. Let a be a non-zero complex number. Let

$$(5.3) \qquad K_n(a) = K_n/<\tau_i\varepsilon_i = \varepsilon_i\tau_i = a\varepsilon_i \ (i = 1, \ldots, n-1)>.$$

We call $K_n(a)$ *a knit algebra* with twist factor a.

Let β be an arbitrary complex parameter and $D_n(\beta)$ the algebra defined by the following:

$$(5.4) \qquad D_n(\beta) = K_n(1)/<\tau_i^2 = 1, \ \varepsilon_i^2 = \beta \quad (i = 1, \ldots, n-1)>.$$

$D_n(\beta)$ is introduced in [3] and is called *Brauer's centralizer algebra*. Centralizer algebras associated with the vector representations of Lie algebras of types B_r and C_r are quotients of $D_n(\beta)$ for suitable β.

Let a and q be arbitrary complex parameters and $C_n(a, q)$ the algebra defined by the following:

(5.5)
$$C_n(a, q) = K_n(a)/<\tau_i - \tau_i^{-1} - (q - q^{-1})(1 - \varepsilon_i) \quad (i = 1, \ldots, n-1)>.$$

We call $C_n(a, q)$ *a q-analogue of Brauer's centralizer algebra*, since it is a one-parameter deformation of the algebra $D_n(\beta)$. We have

$$(5.6) \qquad \lim_{q \to 1} C_n(q^\alpha, q) = D(1 - \alpha), \quad \lim_{q \to 1} C_n(-q^\alpha, q) = D(1 + \alpha),$$

since the relations of (5.3) and (5.5) imply that

$$(5.7) \qquad \varepsilon_i^2 = \left(1 - \frac{a - a^{-1}}{q - q^{-1}}\right)\varepsilon_i \quad (i = 1, \ldots, n-1).$$

By putting $\varepsilon = 0$ in $C_n(a, q)$, we obtain Iwahori's Hecke algebra

$H_n(q)$ of type A, which is defined by the following:

(5.8)
$$H_n(q) = <\tau_1, \ldots, \tau_{n-1} \mid$$
$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \quad \tau_i \tau_j = \tau_j \tau_i \; (|i - j| \geqq 2)$$
$$\tau_i^2 - (q - q^{-1})\tau_i - 1 = 0 \; (1 \leqq i \leqq n - 1) > .$$

The algebra $H_n(q)$ is a one-parameter deformation of the group ring of the symmetric group $\mathfrak{S}_n$ of degree n. In other words, $H_n(q)$ is a q-analogue of the centralizer algebra associated with the vector representation of the Lie algebra of type A_r.

Theorem 5.9. *For any algebra F_n of face operators of types B, C and D, there is a pair (a, q) such that F_n is isomorphic to the algebra $C_n(a, q)$.*

Proof. An isomorphism $\varphi : F_n \to C_n(a, q)$ is given by the following:

(5.10)
$$\varphi(T_i) = \frac{-q^{-\omega}}{(q^\omega - q^{-\omega})(q - q^{-1})} \tau_i \quad (i = 1, \ldots, n - 1),$$

where ω is given by (2.8). The relation between parameters (α, L) and (a, q) is given by

(5.11)
$$q = \exp(2\pi L), \qquad a = \begin{cases} q^{-2\alpha} & \text{for type B,} \\ -q^{-2\alpha-1} & \text{for type C,} \\ q^{-2\alpha+1} & \text{for type D.} \end{cases}$$

For the type C, the well-definedness of the map *phi* is given in [14]. Proofs for the other types are similar.

Remark 5.12. The classical limit of the algebra F_n are given as follows:

(5.13)
$$\lim_{q \to 1} F_n = \begin{cases} D_n(2\alpha + 1) & \text{for type B,} \\ D_n(-2\alpha) & \text{for type C,} \\ D_n(2\alpha) & \text{for type D.} \end{cases}$$

Fix a positive integer n. We construct all the irreducible representations of $C_n(a, q)$ for generic parameters a and q. We may consider the

F_n-modules $V^{\otimes n}$ and V_λ ($\lambda \in \Lambda$) introduced in Section 2 as $C_n(a, q)$-modules. Let

$$(5.14) \qquad \Lambda_n = \Lambda(n) \cup \Lambda(n-2) \cup \ldots \cup \begin{cases} \Lambda(0) & \text{if } n \text{ is even,} \\ \Lambda(1) & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 5.15. *For any of solvable models of types B, C and D given in Section 1 with generic parameters α and L, we have the following.*

   (i) *For $\lambda \in \Lambda_n$, V_λ is a simple F_n-module.*

   (ii) *For two distinct elements λ, $\lambda' \in \Lambda_n$, F_n-modules V_λ and $V_{\lambda'}$ are not equivalent.*

   (iii) *For a finite dimensional simple F_n-module M, there is $\lambda \in \Lambda_n$ such that M is equivalent to V_λ.*

Proof. The idea of proof is given in [14]. This theorem can be proved similarly as that for Theorem 2.3 in [18]. The proofs of the parts (i) and (ii) are done by induction on n. The essential facts used in the proofs are the followings. The first fact is that the Boltzmann weight $W\begin{pmatrix} a & b \\ d & c \end{pmatrix} u$ is not identically equal to 0 if $a - b, b - c, a - d, d - c \in \mathcal{A}$. We identify the algebra F_{n-1} with the subalgebra of F_n generated by $T_1, T_2, \ldots, T_{n-2}$. Then the second fact is the following:

Lemma 5.16. *The multiplicity of any simple F_{n-1}-module in V_λ is at most 1.*

Proof. From the definition of the F_n-module V_λ ($\lambda \in \Lambda$), we have

$$(5.17) \qquad\qquad V_\lambda = \bigoplus_{\substack{\mu \in \Lambda, \\ \lambda - \mu \in \mathcal{A}}} V_\mu$$

as an F_{n-1}-module. Hence the multiplicity of any simple F_{n-1}-module in V_λ is at most 1 because of the induction hypothesis.

By using the above two facts, we can apply the argument of the proof of Theorem 2.3 in [18] for the algebra $C_n(a, q)$.

It remains to show the part (iii). By Theorem 3.7 of [2], we know that the irreducible representations of $C_n(a, q)$ are parametrized by Λ_n. Hence, by using the parts (i) and (ii), we get the part (iii).

Now we get a method to construct all the irreducible representations of $C_n(a, q)$ for generic parameters. The following gives such method for Brauer's centralizer algebra $D_n(\beta)$.

Corollary 5.18. *For $\lambda \in \Lambda_n$ and the F_n-module V_λ, define t_i, $e_i \in \mathrm{End}(V_\lambda)$ for $i = 1, \ldots, n - 1$ by*

$$(5.19) \qquad t_i = \lim_{q \to 1} T_i, \ e_i = \lim_{q \to 1} (q^\omega - q^{-\omega})(q^\omega T_i - q^{-\omega} T_{i-1}) + 1.$$

If the parameter α is generic, there is a representation of Brauer's centralizer algebra D_n to $\mathrm{End}(V_\lambda)$ sending τ_i, ε_i to t_i, e_i respectively.

Proof. Check finiteness of all the matrix elements of $\lim_{q \to 1} T_i$ with respect to the basis $\{v_{\lambda_0} \otimes v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n}\}$ of V_λ.

Theorem 5.20. *For the algebra Y_n of Yang-Baxter operators of types A_r introduced in Section 4, there is q such that Y_n is a quotient of Iwahori's Hecke algebra $H_n(q)$. For any algebra Y_n of types B_r, C_r and D_r, there is a pair (a, q) such that Y_n is a quotient of the algebra $C_n(a, q)$.*

Proof. For the case of type A, it is shown in [6]. A homomorphism $\varphi : H_n(k^{-1}) \to Y_n$ is given by the following:

$$(5.21) \qquad\qquad \varphi(\tau_i) = k^{-1} \check{R}_i \quad (i = 1, \ldots, n - 1).$$

A homomorphism $\varphi : C_n(a, q) \to Y_n$ is given by the following:

$$(5.22) \qquad\qquad \varphi(\tau_i) = k^{-1} \check{R}_i \quad (i = 1, \ldots, n - 1),$$

where ω is given by (2.8). The relation between parameters (a, q) and (r, k) is given by

$$(5.23) \qquad q = k^{-1}, \qquad a = \begin{cases} q^{-2r} & \text{for type B,} \\ -q^{-2r-1} & \text{for type C,} \\ q^{-2r+1} & \text{for type D.} \end{cases}$$

To show the well-definedness of the map φ, we need the following two facts. Theorem 4.3.4 of [17] shows the relation between the Yang-Baxter operator $\check{R}_i$ and the Kauffman polynomial, which is an invariant of knots and links. Articles [2] and [13] give the relation between the Kauffman polynomial and the algebra $C_n(a, q)$. By combining the above results, we get the well-definedness of the map φ. We omit the detail.

§6. Problems

In this section, we propose three problems.

The first problem is to give the correspondence among the algebras of face operators, those of Yang-Baxter operators and those of monodromies of CFT associated with any representation of a Lie algebra. Such models are constructed for all the irreducible representations of the Lie algebras of type A. IRF-models are given in [8] and vertex models are given in [4]. The algebras of Yang-Baxter operators of these models are discussed in [15]. CFT of this case is studied in [16].

The second problem is to describe the generic properties of the algebras obtained as above. For example, they are finite dimensional quotients of the group ring of the braid group B_n or the semigroup ring of the knit semigroup K_n. In [12], some properties of algebras of monodromies of CFT are given. At first construct an algebra with those properties. Then it might be possible to construct models from the algebra. In this manner, we might get models not associated with the representations of Lie algebras.

The last question is about the irreducible representations of the algebra $C_n(a, q)$. The elements of representation matrices of our construction are not Laurent polynomials in a and q. But it seems to exist a construction of each representation of $C_n(a, q)$ whose matrix elements are Laurent polynomials. For Iwahori's Hecke algebra, such construction is known. For example, see [11].

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