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Paths, Maya Diagrams and representations of $\widehat{\mathfrak{sl}}(r, \mathbf{C})$

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Dedicated to Professor Tosihusa Kimura on his 60th birthday

§1. Introduction

Let $\mathfrak{g}$ be the affine Lie algebra $\widehat{\mathfrak{sl}}(r, \mathbf{C})$, let Λ be a dominant integral weight, and let $L(\Lambda)$ be the irreducible $\mathfrak{g}$-module with highest weight Λ. In this article we construct an explicit basis of each weight space $L(\Lambda)_\mu$. As a corollary we prove a new combinatorial formula for the dimensionality of $L(\Lambda)_\mu$, which was conjectured in [1] through the study of corner transfer matrices of solvable lattice models (see Theorem 1.2 below).

The problem of constructing explicit bases goes back to the work of Gelfand and Tsetlin [2] who gave a canonical basis of $L(\Lambda)$ for the classical Lie algebras $\mathfrak{g} = \mathfrak{gl}(r, \mathbf{C}),\ \mathfrak{o}(r, \mathbf{C})$. Analogous results are available in the setting of affine Lie algebras. When Λ is of level 1, $L(\Lambda)$ can be identified with a space of polynomials in infinitely many variables [3,4] or a simple modification thereof [5]. For higher levels, the Z-algebra approach initiated by Lepowsky and Wilson [6] provides a basis in various cases ($\mathfrak{g} = \widehat{\mathfrak{sl}}(2, \mathbf{C})$, arbitrary levels [3],[7], or $\mathfrak{g} = \widehat{\mathfrak{gl}}(r, \mathbf{C}), \widehat{\mathfrak{sp}}(r, \mathbf{C})$, level 2 [8]). Lakshmibai and Seshadri [9] gave a 'standard monomial basis' for $\widehat{\mathfrak{sl}}(2, \mathbf{C})$ using geometric ideas.

A new feature of our approach is the use of an object—*path*, which we now explain. Let $\epsilon_\mu = (0, \cdots, \overset{\mu-\text{th}}{1}, \cdots, 0)\ (0 \le \mu < r)$ denote the standard base vectors of $\mathbf{Z}^r$. We extend the suffixes to $\mathbf{Z}$ by $\epsilon_{\mu+r} = \epsilon_\mu$. Fix a positive integer l.

Definition 1.1. A path is a sequence $\eta = \left(\eta(k)\right)_{k \ge 0}$ consisting of elements $\eta(k) \in \mathbf{Z}^r$ of the form $\epsilon_{\mu_1(k)} + \cdots + \epsilon_{\mu_l(k)}$ $(\mu_1(k), \cdots, \mu_l(k) \in \mathbf{Z})$.

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With a level l dominant integral weight $\Lambda = \Lambda_{\gamma_1} + \cdots + \Lambda_{\gamma_l}$ we associate a path

$$\eta_\Lambda = \left(\eta_\Lambda(k)\right)_{k\geq 0}, \qquad \eta_\Lambda(k) = \epsilon_{\gamma_1+k} + \cdots + \epsilon_{\gamma_l+k}.$$

We call η a Λ-path if $\eta(k) = \eta_\Lambda(k)$ for $k \gg 0$. Let $\mathcal{P}(\Lambda)$ denote the set of Λ-paths. We define the weight λ_η of η by

$$\lambda_\eta = \Lambda - \sum_{k\geq 0} \pi\left(\eta(k) - \eta_\Lambda(k)\right) - \omega(\eta)\delta, \qquad \delta: \text{ the null root,} \tag{1.1}$$

$$\omega(\eta) = \sum_{k\geq 1} k\Big(H\left(\eta(k-1), \eta(k)\right) - H\left(\eta_\Lambda(k-1), \eta_\Lambda(k)\right)\Big). \tag{1.2}$$

Here π is the $\mathbf{Z}$-linear map from $\mathbf{Z}^r$ to the weight lattice of $\widehat{\mathfrak{sl}}(r, \mathbf{C})$ such that $\pi(\epsilon_\mu) = \Lambda_{\mu+1} - \Lambda_\mu$ (we set $\Lambda_r = \Lambda_0$). The H function is given as follows: if $\alpha = \epsilon_{\mu_1} + \cdots + \epsilon_{\mu_l}$ and $\beta = \epsilon_{\nu_1} + \cdots + \epsilon_{\nu_l}$ $(0 \leq \mu_i, \nu_i < r)$, then

$$H(\alpha, \beta) = \min_\sigma \sum_{i=1}^{l} \theta\left(\mu_i - \nu_{\sigma(i)}\right) \tag{1.3}$$

where σ runs over the permutation group on l letters, and

$$\begin{aligned} \theta(\mu) &= 1 \qquad \text{if } \mu \geq 0, \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{1.4}$$

We construct a basis of each weight space $L(\Lambda)_\mu$ indexed by the paths of weight μ:

$$\mathcal{P}(\Lambda)_\mu = \{\eta \in \mathcal{P}(\Lambda) \mid \lambda_\eta = \mu\}.$$

(See Theorem 5.4 for a more precise statement.) In particular we have

Theorem 1.2.

$$\dim L(\Lambda)_\mu = \sharp\mathcal{P}(\Lambda)_\mu. \tag{1.5}$$

This appears as a conjecture in [1].

The paths introduced above arise naturally in the study of solvable lattice models in statistical mechanics, notably the computation of the one point functions. Let us consider a regular square lattice on the plane. To each site i we attach a random variable σ_i that takes its values (called local states) in a set $\mathcal{S}$. With each configuration of four local states a, b, c, d round a face we associate a Boltzmann weight $W(a, b, c, d)$. The probability of the occurrence of a global configuration is then defined to be (up to a normalization factor) the product of $W(a,b,c,d)$ over all faces. The one point function P_a is the probability that a particular site, say site 0, assumes the local state a:

$$P_a = \frac{\sum_{\text{config}} \delta_{\sigma_0\, a} \prod_{\text{face}} W(\sigma_i,\sigma_j,\sigma_k,\sigma_h)}{\sum_{\text{config}} \prod_{\text{face}} W(\sigma_i,\sigma_j,\sigma_k,\sigma_h)}.$$

In [1] we computed P_a in a particular model such that $\mathcal{S}$ is the set of integral weights of $\widehat{\mathfrak{sl}}(r,\mathbf{C})$ of level 1, and a and b can be successive local states only for $b-a=\pi(\epsilon_\mu)$ for some μ. The essential part of the computation was to evaluate a 1 dimensional configuration sum of the form (Λ: level 1)

$$\sum_{\eta\in\mathcal{P}(\Lambda),\lambda_\eta\equiv a \bmod \mathbf{C}\delta} q^{\omega(\eta)}. \tag{1.6}$$

In this paper we treat the general case—Λ has arbitrary level l, as it is formulated in (1.1)$\sim$(1.4). The formula (1.5) states that the sum (1.6) is given by the string function of $L(\Lambda)$

$$\sum_n \dim L(\Lambda)_{\mu-n\delta} q^n.$$

In fact, such a connection between P_a and the Lie algebra character has been encountered repeatedly in our earlier works. When the level l is 1, results of type (1.6) are valid for models based on other types of Lie algebras as well [1]. There are also a family of 'restricted face models' having *dominant* integral weights as local states, where the 1 dimensional configuration sums are equal to the branching coefficients for an affine Lie algebra pair [10]. Previous proofs of these results rely on manipulation of q-series identities, and the connection with characters is understood only at the level of such identities. It is our hope that the treatment in this paper will lead to more structural understanding of this phenomenon.

The text is organized as follows. In Section 2 the basic ideas of the construction is illustrated on the examples $\Lambda=\Lambda_0, 2\Lambda_0$, $\mathfrak{g}=\widehat{\mathfrak{sl}}(2,\mathbf{C})$. This is meant to be an introduction to the more formal construction given in the subsequent Sections. In Section 3 we recall briefly the Fock representation of $\mathfrak{gl}(\infty,\mathbf{C})$ and its subalgebras $\widehat{\mathfrak{gl}}(r,\mathbf{C}), \widehat{\mathfrak{sl}}(r,\mathbf{C})$. Consider the highest weight vector in the tensor product of the Fock spaces $\mathcal{F}[\gamma_1]\otimes\cdots\otimes\mathcal{F}[\gamma_l]$ where $\gamma_1,\cdots,\gamma_l$ denote the charge. We denote its orbit by the action of $\mathfrak{gl}(\infty,\mathbf{C})$ (resp. $\widehat{\mathfrak{gl}}(r,\mathbf{C}), \widehat{\mathfrak{sl}}(r,\mathbf{C})$) by $F(\Lambda)$

(resp. $G(\Lambda), L(\Lambda)$). Our goal is to find a basis of $L(\Lambda)$ (or its dual $L(\Lambda)^*$) in this realization. In Section 4 we give spanning sets of vectors for $F(\Lambda)^*$ and $G(\Lambda)^*$ in terms of the Maya diagrams. They are selected by the Plücker relations. This section has been taken from the lecture notes by Sato [11] and adapted (and slightly generalized) to the present setting. The paths are introduced in Section 5. We show that the spanning vectors for $G(\Lambda)^*$ can be re-labeled by a pair (η, Y) consisting of a path η and a Young diagram Y. In the final Section 6 we prove that among them the vectors ξ_η corresponding to the pairs (η, ϕ), with ϕ being the empty Young diagram, provide a basis of the $\widehat{\mathfrak{sl}}(r, \mathbf{C})$-module $L(\Lambda)^*$. This is done by constructing certain vectors v_η of $L(\Lambda)$ and showing the non-degeneracy of the pairing between $\{\xi_\eta\}$ and $\{v_\eta\}$.

While preparing the manuscript we have received a note from Primc [12] in which he constructs another basis of $L(\Lambda)$ by using the vertex operators. The relation between his construction and the present work is yet to be clarified.

§2. Examples

The aim of the present section is to describe on simple examples the base vectors $\xi_\eta \in L(\Lambda)^*, v_\eta \in L(\Lambda)$ mentioned in Section 1. We shall consider the case $r = 2$, $\mathfrak{g} = \widehat{\mathfrak{sl}}(2, \mathbf{C})$ throughout.

2.1. The case $\Lambda = \Lambda_0$

Let $l = 1, \Lambda = \Lambda_0$.

A Λ_0-path is a binary sequence $\eta = \left(\eta(j)\right)_{j\geq 0}, \eta(j) = 0$ or 1, whose 'tail' is of the form $\cdots, 0, 1, 0, 1, \cdots$ (0 for j even, 1 for j odd). For example,

$$
\begin{aligned}
\eta^{(1)} &= \ 0,\ 1,\ 0,\ 1,\ 0,\ 1, \cdots \equiv \eta_{\Lambda_0}, \\
\eta^{(2)} &= \ 0,\ 0,\ 0,\ 1,\ 0,\ 1, \cdots, \\
\eta^{(3)} &= \ 0,\ 1,\ 1,\ 1,\ 0,\ 1, \cdots, \qquad \textit{etc.}.
\end{aligned}
\tag{2.1}
$$

Let $\mathcal{S}_1 = \{(1-k)\Lambda_0 + k\Lambda_1 \mid k \in \mathbf{Z}\}$ be the set of level 1 integral weights of $\widehat{\mathfrak{sl}}(2, \mathbf{C})$. One can also represent η as a sequence $\mu = \left(\mu(j)\right)_{j\geq 0}$ with $\mu(j) \in \mathcal{S}_1$, such that

$$
\begin{aligned}
\mu(j+1) - \mu(j) &= \Lambda_1 - \Lambda_0 \qquad \text{if} \quad \eta(j) = 0, \\
&= \Lambda_0 - \Lambda_1 \qquad \text{if} \quad \eta(j) = 1,
\end{aligned}
$$

and that $\mu(j) = \Lambda_0$ (j even $\gg 0$) or $= \Lambda_1$ (j odd $\gg 0$). Thus we have (see Fig.2.1)

(2.2)

$$\begin{aligned}
\eta^{(1)} \leftrightarrow \mu^{(1)} &= \Lambda_0,\ \Lambda_1,\ \Lambda_0,\ \Lambda_1,\ \Lambda_0,\ \Lambda_1, \cdots, \\
\eta^{(2)} \leftrightarrow \mu^{(2)} &= 3\Lambda_0 - 2\Lambda_1,\ 2\Lambda_0 - \Lambda_1,\ \Lambda_0,\ \Lambda_1,\ \Lambda_0,\ \Lambda_1, \cdots, \\
\eta^{(3)} \leftrightarrow \mu^{(3)} &= -\Lambda_0 + 2\Lambda_1,\ -2\Lambda_0 + 3\Lambda_1,\ -\Lambda_0 + 2\Lambda_1,\ \Lambda_1,\ \Lambda_0,\ \Lambda_1, \cdots,
\end{aligned}$$

etc..

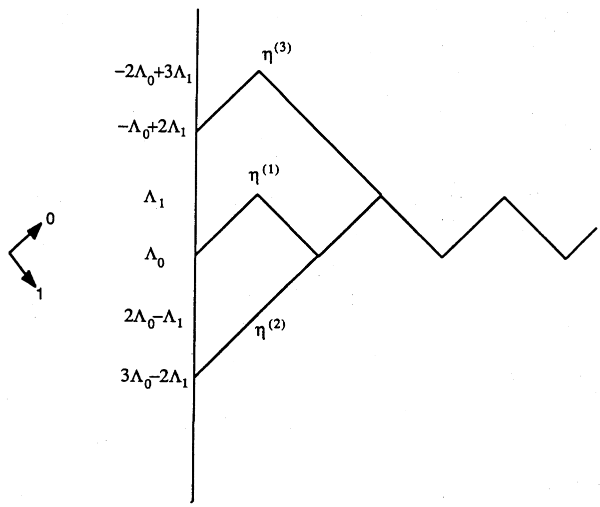


Fig. 2.1 Λ_0-paths as sequences in the weight lattice.

Consider now the vector space $\mathcal{F}[0]$ having all the Young diagrams as base vectors. We equip $\mathcal{F}[0]$ with an inner product (,) with respect to which the Young diagrams are orthonormal. For convenience we color the nodes of each Young diagram Y by white and black alternatingly, the node at the left-top corner being white (Fig.2.2).

We define the action of $\widehat{\mathfrak{sl}}(2, \mathbf{C})$ on $\mathcal{F}[0]$ as follows. We require that the Young diagrams be weight vectors; if Y has n_0 white nodes and n_1 black nodes, we assign

(2.3) $$\text{the weight of } Y = \Lambda_0 - n_0\alpha_0 - n_1\alpha_1,$$

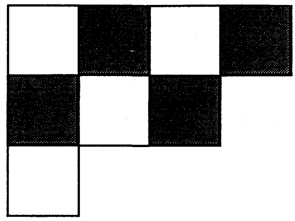
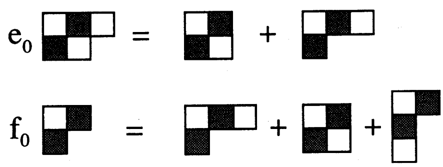


Fig. 2.2 Coloring of nodes.

where α_0, α_1 are the simple roots of $\widehat{\mathfrak{sl}}(2, \mathbf{C})$. Next we define the action of the Chevalley generators e_i, f_i. Put e_0Y (resp. $f_0Y) = \sum_{Y'} Y'$, where Y' runs over the Young diagrams obtained by removing (resp. adjoining) one white node from Y. For instance,



Likewise define e_1, f_1 replacing 'white' by 'black'. We have then

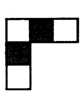
$$(f_iY, Y') = (Y, e_iY'). \tag{2.4}$$

With these definitions the irreducible $\widehat{\mathfrak{sl}}(2, \mathbf{C})$-module $L(\Lambda_0)$ is realized as a subspace of $\mathcal{F}[0]$ spanned by vectors of the form $f_{i_1} \cdots f_{i_k}\phi$, ϕ being the empty Young diagram.

There is a natural map $p_{\Lambda_0} : Y \mapsto \eta$ sending the set of Young diagrams onto that of Λ_0-paths. Let Y be a Young diagram, and let g_j denote the length of its $(j+1)$-th column ($j = 0, 1, \cdots, g_j = 0$ for $j \gg 0$). Then $\eta = p_{\Lambda_0}(Y)$ is defined by

$$\eta(j) \in \{0, 1\}, \quad \eta(j) \equiv j - g_j \bmod 2 \ (j \geq 0).$$

For instance,

$Y =$  gives $\eta = 1,\ 0,\ 1,\ 1,\ 0,\ 1, \cdots.$

Conversely, for each η there exists a unique Young diagram $Y = Y_\eta$

which satisfies the conditions

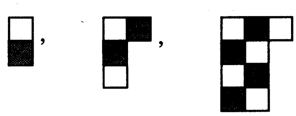
(2.5a) $\quad p_{\Lambda_0}(Y) = \eta,$

(2.5b) $\quad Y$ has the signature $[y_1, y_2, \cdots, y_s]$ with $y_1 > y_2 > \cdots > y_s$.

Thus by (2.5b)

ϕ,  , ... are allowed

but

 , ... are not.

The Young diagram Y_η is called the highest lift of η. It has the property that, for any Y' such that $p_{\Lambda_0}(Y') = \eta$, one has $Y_\eta \subset Y'$.

Our base vectors $\xi_\eta \in L(\Lambda_0)^*$ are defined to be

$$\xi_\eta(v) = (Y_\eta, v), \qquad v \in L(\Lambda_0).$$

Each ξ_η is a weight vector. In the Young diagram picture Y_η, its weight λ_η is simply given by counting the numbers of white and black nodes (2.3). In the path picture η we have

$$\lambda_\eta = \mu(0) - \sum_{k\geq 1} k\Big(H\big(\eta(k-1), \eta(k)\big) - H\big(\eta_\Lambda(k-1), \eta_\Lambda(k)\big)\Big)\delta, \tag{2.6}$$

where $\mu(0)$ is the 'initial point' of the sequence μ (2.2) corresponding to η, $\delta = \alpha_0 + \alpha_1$, and

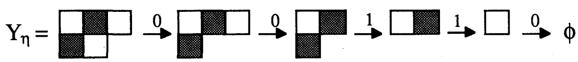
$$\begin{aligned} H(\eta, \eta') &= 0 \qquad \text{if } \eta = 0, \eta' = 1, \\ &= 1 \qquad \text{otherwise.} \end{aligned} \tag{2.7}$$

For example, $\eta = \eta^{(3)}$ in (2.1) has the weight $\lambda_\eta = -\Lambda_0 + 2\Lambda_1 - 3\delta$.

One can also construct a basis $\{v_\eta\}$ of $L(\Lambda_0)$ as follows. Consider the process of removing the nodes from Y_η one by one. At each step we require:

(i) removal of the node produces a Young diagram satisfying (2.5b),
(ii) among the nodes satisfying (i) the rightmost one is removed.

For example:

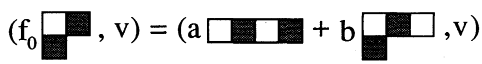


This process gives rise to a sequence $A = (a_0, a_1, \cdots, a_{d-1})$, where $a_i = 0$ or 1 according to whether the removed node at the $(i+1)$-th step is white or black, and $d = \sharp\{$ nodes of $Y\}$. We now define

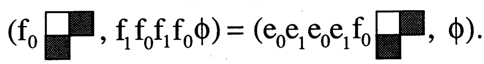
$$v_\eta = f_{a_0} f_{a_1} \cdots f_{a_{d-1}} \phi \quad \in L(\Lambda_0).$$

In the example above, $v_\eta = f_0^2 f_1^2 f_0 \phi$. By construction it is clear that v_η has the same weight λ_η as ξ_η does. In the present case, the v_η coincides with the monomial basis of Lakshmibai-Seshadri [9].

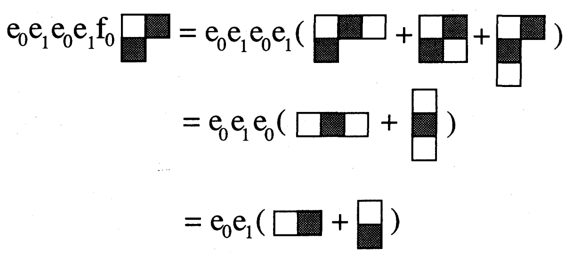
The action of e_i, f_i on ξ_η can in principle be determined from the definition. As an example let us try $f_0 \xi_\eta$ for $\eta = \eta^{(2)}$. From the table below (Fig. 2.5a) one knows that



for $v = f_1 f_0 f_1 f_0 \phi$, $f_0 f_1^2 f_0 \phi$, where $a, b \in \mathbf{C}$ are to be determined. Using (2.4) we have



On the other hand, we have in $\mathcal{F}[0]$



$$= 2\phi.$$

Similar calculations yield the relations

$$2 = a + b, \qquad 6 = 0 \cdot a + 2 \cdot b,$$

giving the result $a = -1, b = 3$.

2.2. The case of $\Lambda = 2\Lambda_0$

Next we consider the case $l = 2, \Lambda = 2\Lambda_0$.

A $2\Lambda_0$-path is a sequence $\eta = \left(\eta(j)\right)_{j\geq 0}$ where $\eta(j)$ takes one of the three possibilities 00, 01 = 10 or 11. We require that for $j \gg 0$ $\eta(j) = 00$ (j even) or $= 11$ (j odd). Thus

$$
\begin{aligned}
\eta^{(1)} &= 00,\ 11,\ 00,\ 11,\ 00,\ \cdots \equiv \eta_{2\Lambda_0},\\
\eta^{(2)} &= 01,\ 01,\ 00,\ 11,\ 00,\ \cdots,\\
\eta^{(3)} &= 11,\ 11,\ 01,\ 11,\ 00,\ \cdots, \qquad \textit{etc..}
\end{aligned} \tag{2.8}
$$

As before we identify η with a sequence of integral weights $\mu = \left(\mu(j)\right)_{j\geq 0}$, where

$$
\begin{aligned}
\mu(j) \ &\in \ \mathcal{S}_2 = \{(2-k)\Lambda_0 + k\Lambda_1 \mid k \in \mathbf{Z}\},\\
\mu(j+1) - \mu(j) &= 2\Lambda_1 - 2\Lambda_0 \qquad \text{if} \quad \eta(j) = 00,\\
&= 0 \qquad\qquad\qquad\ \ \text{if} \quad \eta(j) = 01,\\
&= 2\Lambda_0 - 2\Lambda_1 \qquad \text{if} \quad \eta(j) = 11,\\
\mu(j) = 2\Lambda_0 \ (j \text{ even } &\gg 0), = 2\Lambda_1 \ (j \text{ odd } \gg 0).
\end{aligned}
$$

The μ's for the paths (2.8) are depicted in Fig.2.3.

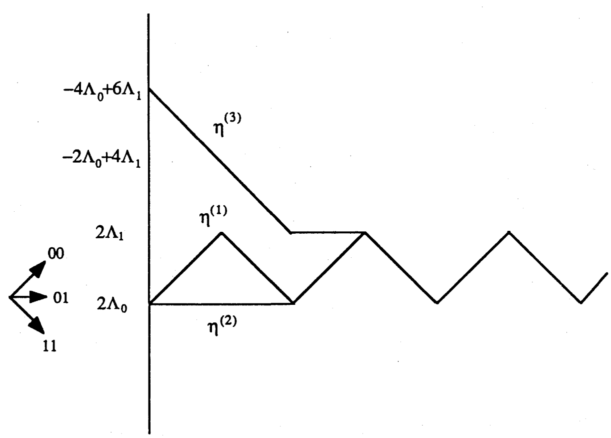


Fig. 2.3 $2\Lambda_0$-paths.

We consider the tensor space $\mathcal{F}[0] \otimes \mathcal{F}[0]$ whose basis is given by an ordered pair $\mathbf{Y} = (Y_1, Y_2)$ of Young diagrams representing $Y_1 \otimes Y_2$. The irreducible module $L(2\Lambda_0)$ is realized as the subspace of $\mathcal{F}[0] \otimes \mathcal{F}[0]$ consisting of elements $f_{i_1} \cdots f_{i_k}(\phi \otimes \phi)$.

A pair $\mathbf{Y} = (Y_1, Y_2)$ gives rise to a $2\Lambda_0$-path $\eta = p_{2\Lambda_0}(\mathbf{Y})$; if $p_{\Lambda_0}(Y_i) = \left(\eta_i(j)\right)_{j\geq 0}$ $(i = 1, 2)$ are the Λ_0-paths corresponding to the components, then we set $\eta = \left(\eta_1(j)\eta_2(j)\right)_{j\geq 0}$. For instance,

$p_{\Lambda_0}($  $) = 0, 0, 0, 1, 0, 1, \ldots,$

$$p_{\Lambda_0}(\square) = 1, 1, 0, 1, 0, 1, \ldots$$

give

$p_{2\Lambda_0}((($  $, \square)) = 01, 01, 00, 11, 00, 11, \ldots .$

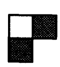
For each $2\Lambda_0$-path η there exists a unique $\mathbf{Y} = \mathbf{Y}_\eta = (Y_1, Y_2)$ (called the highest lift of η) with the following properties:

(i) $p_{2\Lambda_0}(\mathbf{Y}) = \eta$,

(ii) $Y_1 \supset Y_2 \supset Y_1[2]$,

(iii) for any $\mathbf{Y}' = (Y_1', Y_2')$ satisfying (i),(ii) we have $Y_1 \subset Y_1', Y_2 \subset Y_2'$.

Here for a Young diagram $Y = [y_1, \cdots, y_s]$, $Y[2]$ signifies the one obtained by removing the first two rows: $Y[2] = [y_3, y_4, \cdots, y_s]$. As an example let $\eta =$ 01, 01, 00, 11, $\cdots$. Then

$\mathbf{Y} = ($  $, \phi), \qquad \mathbf{Y}' = ($  $, \square)$

are both lifts of η satisfying (i),(ii), and $\mathbf{Y}$ is the highest lift.

We define $\xi_\eta \in L(2\Lambda_0)^*$ by

$$\xi_\eta(v) = (Y_1 \otimes Y_2, v), \qquad v \in L(2\Lambda_0) \subset \mathcal{F}[0] \otimes \mathcal{F}[0]$$

where $\mathbf{Y}_\eta = (Y_1, Y_2)$.

The weight λ_η of ξ_η takes the same form (2.6). The H function is given by

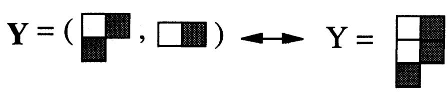
$$H(\eta_1\eta_2, \eta_1'\eta_2') = \min\left(H(\eta_1, \eta_1') + H(\eta_2, \eta_2'), H(\eta_1, \eta_2') + H(\eta_2, \eta_1')\right)$$

where in the right hand side the H signifies the one for Λ_0 (2.7).

$\eta \setminus \eta'$	00	01	11
00	2	1	0
01	2	1	1
11	2	2	2

Table of $H(\eta, \eta')$ for $\Lambda = 2\Lambda_0$.

To construct the base vectors $\{v_\eta\}$ of $L(2\Lambda_0)$, we must define the sequence $A = (a_0, a_1, \cdots)$ from η. Let $\mathbf{Y}_\eta = (Y_1, Y_2)$ be the highest lift. Consider the set of rows of Y_1, Y_2 (taking the coloring into account). Combining and rearranging them in the decreasing order of the length, we get a *single* Young diagram Y. For the highest lifts it can be shown that rows of the same length have the same coloring (Proposition 6.10). Hence Y is uniquely defined. Example:



Now we remove the nodes of Y one by one. The rule is:
(i) removal of a node does not produce rows of same length and different coloring (Fig.2.4),
(ii) among the nodes satisfying (i) the removed node is the rightmost one.

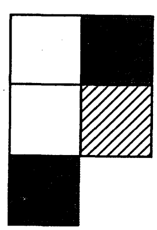
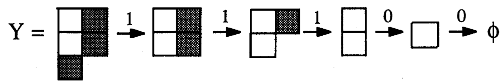


Fig. 2.4 Construction of $(a_0, a_1, \cdots)$ from η. The shaded node cannot be removed because of the condition (i).

For example,



As before we set $v_\eta = f_{a_0} f_{a_1} \cdots (\phi \otimes \phi)$, where $a_i = 0$ or 1 according to whether the removed node is white or black.

Below we give a list of the base vectors $\{\xi_\eta\}, \{v_\eta\}$ for the first few of the weight spaces in the cases $\Lambda = \Lambda_0, 2\Lambda_0$ (Fig. 2.5a,b).

§3. The Fock representation of $\mathfrak{gl}(\infty, \mathbf{C}), \widehat{\mathfrak{gl}}(r, \mathbf{C})$ and $\widehat{\mathfrak{sl}}(r, \mathbf{C})$

Here we recall basic facts about the Fock representation of the Lie algebras $\mathfrak{gl}(\infty, \mathbf{C}), \widehat{\mathfrak{gl}}(r, \mathbf{C})$ and $\widehat{\mathfrak{sl}}(r, \mathbf{C})$. We shall mainly follow the notations of [13].

3.1. The Fock space

Let $\mathcal{W}$ be a complex vector space with a distinguished basis indexed by integers $\{\psi_i, \psi_i^*\}_{i \in \mathbf{Z}}$. Let $\mathcal{A} = T(\mathcal{W})/\mathcal{J}$ be the Clifford algebra over $\mathcal{W}$, where $T(\mathcal{W})$ signifies the free associative algebra over $\mathcal{W}$ and $\mathcal{J}$ is the two-sided ideal generated by

$$[\psi_i, \psi_j]_+, \qquad [\psi_i, \psi_j^*]_+ - \delta_{ij}, \qquad [\psi_i^*, \psi_j^*]_+ \qquad (i, j \in \mathbf{Z}).$$

Here $[X, Y]_+$ signifies the anti-commutator $XY + YX$. Let $\mathcal{W} = \mathcal{W}_{cre} \oplus \mathcal{W}_{ann}$ be a splitting into two subspaces given by $\mathcal{W}_{cre} = (\oplus_{i \geq 0} \mathbf{C}\psi_i) \oplus (\oplus_{i<0} \mathbf{C}\psi_i^*)$ and $\mathcal{W}_{ann} = (\oplus_{i<0} \mathbf{C}\psi_i) \oplus (\oplus_{i \geq 0} \mathbf{C}\psi_i^*)$. To this decomposition we associate the right and the left $\mathcal{A}$-modules

$$\mathcal{F}^* = \mathcal{W}_{cre}\mathcal{A} \backslash \mathcal{A} = \langle 0|\mathcal{A}, \qquad \mathcal{F} = \mathcal{A}/\mathcal{A}\mathcal{W}_{ann} = \mathcal{A}|0\rangle.$$

Here the 'vacuum vectors' $\langle 0| \underset{\text{def}}{=} 1 \bmod \mathcal{W}_{cre}\mathcal{A}$, $|0\rangle \underset{\text{def}}{=} 1 \bmod \mathcal{A}\mathcal{W}_{ann}$ enjoy the properties

$$\begin{aligned} &\langle 0|\psi_i = 0 \quad (i \geq 0), \qquad \langle 0|\psi_i^* = 0 \quad (i < 0), \\ &\psi_i|0\rangle = 0 \quad (i < 0), \qquad \psi_i^*|0\rangle = 0 \quad (i \geq 0). \end{aligned} \tag{3.1}$$

We call $\mathcal{F}$ the Fock space and $\mathcal{F}^*$ the dual Fock space. Denote by τ the involutive anti-automorphism of $\mathcal{A}$ such that $\tau(\psi_i) = \psi_i^*$. Then $\tau(\mathcal{W}_{cre}) = \mathcal{W}_{ann}$, and we have an isomorphism of vector spaces $\mathcal{F}^* \xrightarrow{\sim} \mathcal{F}$ given by $\langle 0|a \mapsto \tau(a)|0\rangle$.

There exists on $\mathcal{A}$ a unique linear form $\langle\ \rangle : \mathcal{A} \longrightarrow \mathbf{C}$ such that

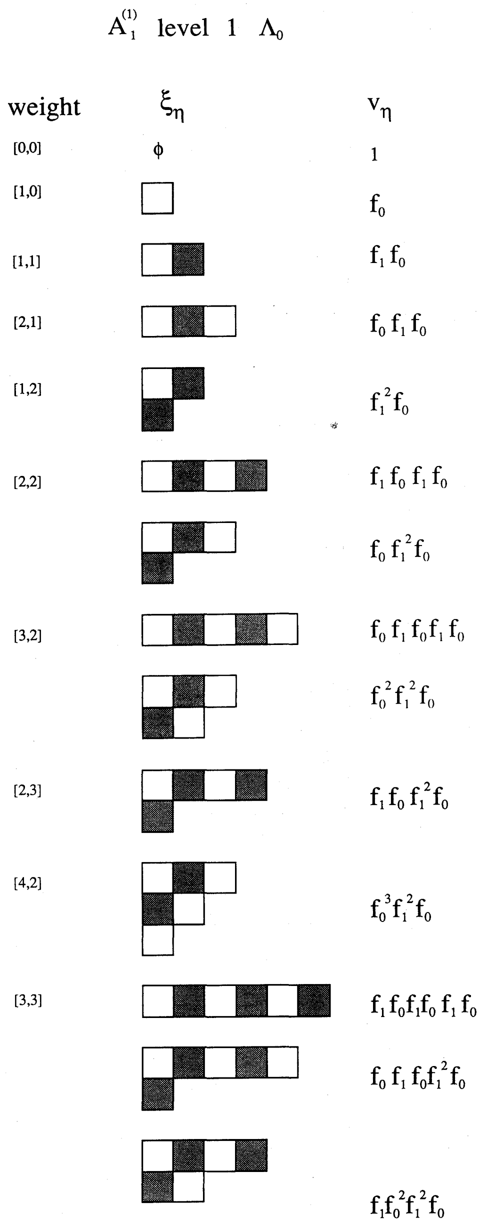


Fig. 2.5a The base vectors of level 1.

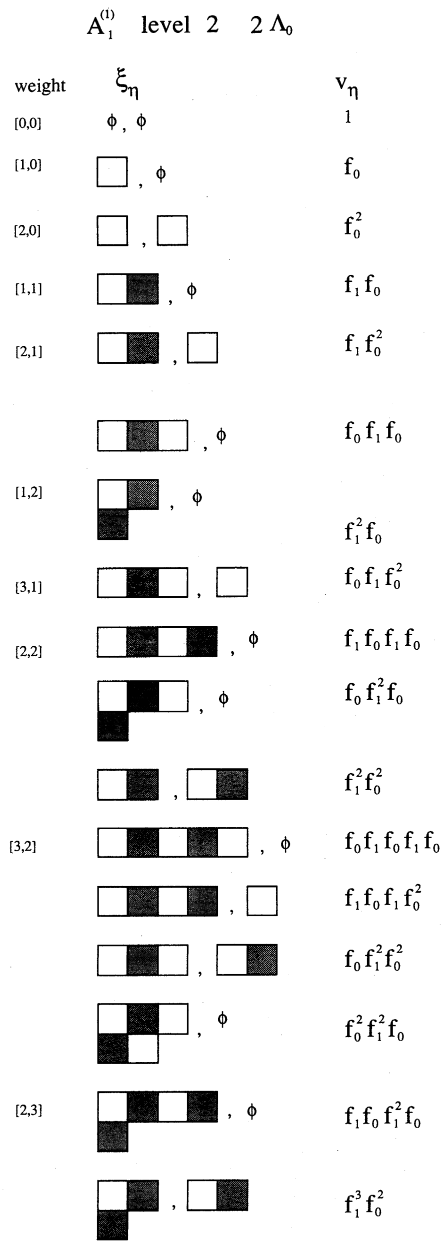


Fig. 2.5b The base vectors of level 2.

(i) $\langle 1 \rangle = 1$,

(ii) $\langle \psi_i \psi_j \rangle = 0, \quad \langle \psi_i^* \psi_j^* \rangle = 0, \quad \langle \psi_i \psi_j^* \rangle = \theta(-i-1)\delta_{ij}$, where θ is defined in (1.4),

(iii) for any $w_1, \cdots, w_s \in \mathcal{W}$

$$\begin{aligned} \langle w_1 \cdots w_s \rangle &= 0 && \text{if } s \text{ is odd,} \\ &= \textstyle\sum_\sigma \operatorname{sgn} \sigma \, \langle w_{\sigma(1)} w_{\sigma(2)} \rangle \cdots \langle w_{\sigma(s-1)} w_{\sigma(s)} \rangle && \text{if } s \text{ is even.} \end{aligned}$$

In (iii) the sum is over permutations σ satisfying $\sigma(1) < \sigma(2), \cdots, \sigma(s-1) < \sigma(s)$ and $\sigma(1) < \sigma(3) < \cdots < \sigma(s-1)$. The form $\langle \ \rangle$ gives rise to a non-degenerate $\mathcal{A}$-invariant bilinear pairing betweeen $\mathcal{F}^*$ and $\mathcal{F}$

$$\mathcal{F}^* \otimes_{\mathcal{A}} \mathcal{F} \longrightarrow \mathbf{C}, \qquad \langle 0 | a \otimes b | 0 \rangle \mapsto \langle ab \rangle.$$

By identifying $\mathcal{F}^*$ with $\mathcal{F}$ this pairing translates to a bilinear form $(\ , \ )$ on $\mathcal{F}$ satisfying

$$(av, w) = (v, \tau(a)w) \qquad \text{for } v, w \in \mathcal{F}, \ a \in \mathcal{A}. \tag{3.2}$$

The algebra $\mathcal{A}$ carries a gradation by integers $\mathcal{A} = \oplus_{\gamma \in \mathbf{Z}} \mathcal{A}[\gamma]$, $\mathcal{A}[\gamma] = \{a \in \mathcal{A} \mid \deg a = \gamma\}$, through the assignment

$$\deg \psi_i = 1, \qquad \deg \psi_i^* = -1.$$

Setting $\deg(\langle 0 | a) = -\deg a$ and $\deg(a | 0 \rangle) = \deg a$ for $a \in \mathcal{A}$, one has the induced grading $\mathcal{F}^* = \oplus_{\gamma \in \mathbf{Z}} \mathcal{F}^*[\gamma]$, $\mathcal{F} = \oplus_{\gamma \in \mathbf{Z}} \mathcal{F}[\gamma]$, where $\mathcal{F}^*[\gamma] = \{v^* \in \mathcal{F}^* \mid \deg v^* = \gamma\}, \mathcal{F}[\gamma] = \{v \in \mathcal{F} \mid \deg v = \gamma\}$. We shall refer to the degree as *charge*. Each *charge* γ *sector* $\mathcal{F}^*[\gamma]$ or $\mathcal{F}[\gamma]$ has a canonical vector $\langle \gamma |$ or $| \gamma \rangle$ such that

$$\begin{aligned} \langle \gamma | &= \langle \gamma' | \psi_{\gamma'}^* \psi_{\gamma'+1}^* \cdots \psi_{\gamma-1}^*, \\ | \gamma \rangle &= \psi_{\gamma-1} \cdots \psi_{\gamma'+1} \psi_{\gamma'} | \gamma' \rangle \end{aligned}$$

for all $\gamma' < \gamma$. We have $\langle \gamma | \gamma' \rangle = \delta_{\gamma\gamma'}$, $\mathcal{F}^*[\gamma] = \langle \gamma | \mathcal{A}[0]$ and $\mathcal{F}[\gamma] = \mathcal{A}[0] | \gamma \rangle$. The annihilation condition (3.1) generalizes to

$$\begin{aligned} &\langle \gamma | \psi_i = 0 \quad (i \geq \gamma), \qquad \langle \gamma | \psi_i^* = 0 \quad (i < \gamma), \\ &\psi_i | \gamma \rangle = 0 \quad (i < \gamma), \qquad \psi_i^* | \gamma \rangle = 0 \quad (i \geq \gamma). \end{aligned} \tag{3.3}$$

3.2. $\mathfrak{gl}(\infty, \mathbf{C}), \widehat{\mathfrak{gl}}(r, \mathbf{C})$ and $\widehat{\mathfrak{sl}}(r, \mathbf{C})$

Let $A = (a_{ij})_{i\,j \in \mathbf{Z}}$ be an infinite matrix with $a_{ij} \in \mathbf{C}$, satisfying the condition

(3.4) there exists an integer $N > 0$ such that $a_{ij} = 0$ for $|i - j| > N$.

Put

$$X_A = \sum_{i,j \in \mathbf{Z}} a_{ij} : \psi_i \psi_j^* : .$$

Here the symbol : : signifies the normal ordering defined by

$$: ww' := ww' - \langle ww' \rangle \qquad \text{for } w, w' \in \mathcal{W}.$$

By definition, $\mathfrak{gl}(\infty, \mathbf{C})$ is the following Lie algebra:

$$\mathfrak{gl}(\infty, \mathbf{C}) = \{X_A \mid A \text{ satisfies (3.4)}\} \oplus \mathbf{C}c,$$

where c belongs to the center and the bracket is defined to be

$$[X_A, X_{A'}] = X_{[A,A']} + c(A, A')c,$$
$$c(A, A') = \sum_{j \geq 0 > i} \left(a_{ij} a'_{ji} - a'_{ij} a_{ji} \right).$$

The Lie subalgebra of $\mathcal{A}[0]$ consisting of quadratic elements

$$\mathfrak{gl}_{\text{fin}}(\infty, \mathbf{C}) = \{X_A \mid a_{ij} = 0 \text{ for all but a finite number of } i, j\} \oplus \mathbf{C}1$$

can be regarded as a subalgebra of $\mathfrak{gl}(\infty, \mathbf{C})$, where we identify $1 \in \mathfrak{gl}_{\text{fin}}(\infty, \mathbf{C})$ with $c \in \mathfrak{gl}(\infty, \mathbf{C})$. There is a well-defined action of $\mathfrak{gl}(\infty, \mathbf{C})$ on $\mathcal{F}^*$ or $\mathcal{F}$ extending that of $\mathfrak{gl}_{\text{fin}}(\infty, \mathbf{C})$. Set

$$H_j = \sum_{i \in \mathbf{Z}} : \psi_i \psi_{i+j}^* :, \qquad j \in \mathbf{Z}.$$

Then one has $[H_j, H_k] = j\delta_{j+k,0}c$, so that the subalgebra $\mathcal{H} = (\oplus_{j \neq 0} \mathbf{C}H_j) \oplus \mathbf{C}c$ of $\mathfrak{gl}(\infty, \mathbf{C})$ becomes a Heisenberg subalgebra. The element H_0 is central in $\mathfrak{gl}(\infty, \mathbf{C})$, and acts as $\gamma \cdot \text{id}$ on $\mathcal{F}^*[\gamma], \mathcal{F}[\gamma]$. It is known that $\mathcal{H}$ acts irreducibly on each $\mathcal{F}^*[\gamma], \mathcal{F}[\gamma]$, and hence so does $\mathfrak{gl}(\infty, \mathbf{C})$.

Let $\mathcal{A}_N$ be the finite dimensional subalgebra of $\mathcal{A}$ generated by ψ_i, ψ_i^* ($|i| \leq N$), and let $\overline{\mathbf{G}_N}(\subset \mathcal{A}_N)$ be the closure (in the linear topology) of the elements of the form ae^X where $a \in \mathbf{C} \setminus \{0\}$ and $X = \sum_{|i|,|j| \leq N} a_{ij} \psi_i \psi_j^*$. We set

$$\overline{\mathbf{G}} = \varinjlim_N \overline{\mathbf{G}_N}.$$

The following is a version of Wick's theorem [14].

Lemma 3.1. *Let $g \in \overline{\mathbf{G}}$, and let γ be a negative integer such that $\langle\gamma|g|\gamma\rangle \neq 0$. Then we have*

$$\frac{\langle\gamma|\psi^*_{i_1}\cdots\psi^*_{i_m}g\psi_{j_m}\cdots\psi_{j_1}|\gamma\rangle}{\langle\gamma|g|\gamma\rangle} = \det\left(\frac{\langle\gamma|\psi^*_{i_\mu}g\psi_{j_\nu}|\gamma\rangle}{\langle\gamma|g|\gamma\rangle}\right)_{1\le\mu,\nu\le m}$$

Now let ι denote the automorphism of $\mathcal{A}$ (resp. $\mathcal{F}^*, \mathcal{F}$) given by

$$\begin{aligned} &\iota(\psi_i) = \psi_{i-1}, && \iota(\psi^*_i) = \psi^*_{i-1}, \\ &\iota(\langle\gamma|) = \langle\gamma-1|, && \iota(|\gamma\rangle) = |\gamma-1\rangle. \end{aligned} \tag{3.5}$$

Fix a positive integer r. Denoting by the same letter ι the induced automorphism of $\mathfrak{gl}(\infty,\mathbf{C})$ we define a Lie subalgebra of $\mathfrak{gl}(\infty,\mathbf{C})$

$$\begin{aligned} \widetilde{\mathfrak{gl}}(r,\mathbf{C}) &= \{X \in \mathfrak{gl}(\infty,\mathbf{C}) \mid \iota^r(X) = X\}, \\ &= \{X_A \in \mathfrak{gl}(\infty,\mathbf{C}) \mid a_{i+r\,j+r} = a_{ij} \text{ for all } i,j\} \oplus \mathbf{C}c. \end{aligned}$$

It can be split into the sum of two commuting subalgebras

$$\widetilde{\mathfrak{gl}}(r,\mathbf{C}) = \widetilde{\mathfrak{sl}}(r,\mathbf{C}) + \mathcal{H}_r, \qquad \widetilde{\mathfrak{sl}}(r,\mathbf{C}) \cap \mathcal{H}_r = \mathbf{C}c \tag{3.6a}$$

where

$$\begin{aligned} \widetilde{\mathfrak{sl}}(r,\mathbf{C}) &= \{X_A \in \widetilde{\mathfrak{gl}}(r,\mathbf{C}) \mid \sum_{i=0}^{r-1} a_{i\,i+kr} = 0 \text{ for all } k \in \mathbf{Z}\} \oplus \mathbf{C}c, \\ \mathcal{H}_r &= (\oplus_{j\equiv 0 \bmod r} H_j) \oplus \mathbf{C}c. \end{aligned} \tag{3.6b}$$

Let further $d \in \mathfrak{gl}(\infty,\mathbf{C})$ be defined by

$$d = -\sum_{i\in\mathbf{Z}}\left[\frac{i}{r}\right] : \psi_i\psi^*_i :$$

with $[x]$ denoting the largest integer not exceeding x. Set

$$\widehat{\mathfrak{gl}}(r,\mathbf{C}) = \widetilde{\mathfrak{gl}}(r,\mathbf{C}) \oplus \mathbf{C}d, \qquad \widehat{\mathfrak{sl}}(r,\mathbf{C}) = \widetilde{\mathfrak{sl}}(r,\mathbf{C}) \oplus \mathbf{C}d.$$

The subalgebra $\widehat{\mathfrak{sl}}(r,\mathbf{C})$ is isomorphic to the affine Lie algebra $A^{(1)}_{r-1}$. Along with d its Chevalley generators are given by

$$\begin{aligned} e_i &= \sum_{j\equiv i \bmod r} e^\infty_j, \qquad f_i = \sum_{j\equiv i \bmod r} f^\infty_j, \\ h_i &= \sum_{j\equiv i \bmod r} h^\infty_j \qquad (0 \le i < r) \end{aligned} \tag{3.7a}$$

where

(3.7b) $e_i^\infty = \psi_{i-1}\psi_i^*, \qquad f_i^\infty = \psi_i\psi_{i-1}^*, \qquad h_i^\infty = \psi_{i-1}\psi_{i-1}^* - \psi_i\psi_i^*.$

We have

$$[d, e_i] = \delta_{i0}e_i, \qquad [d, f_i] = -\delta_{i0}f_i, \qquad [d, h_i] = 0 \qquad (0 \le i < r)$$

and

$$[d, H_j] = \frac{j}{r}H_j \qquad \text{for} \quad j \equiv 0 \bmod r.$$

3.3. Fundamental weights and highest weight modules

Let $\mathfrak{h}^\infty \subset \mathfrak{gl}(\infty, \mathbf{C})$ be the subspace spanned by the elements $\sum_{i\in\mathbf{Z}} b_i h_i^\infty = \sum_{i\in\mathbf{Z}} a_i \ : \psi_i\psi_i^* : \ + a\,c$, where $a_i = b_{i+1} - b_i, a = b_0$. Define the linear form $\Lambda_i : \mathfrak{h}^\infty \to \mathbf{C}$ by $\Lambda_i(h_j^\infty) = \delta_{ij}$. Let $\mathcal{U}(\mathfrak{gl}(\infty, \mathbf{C}))$ be the universal enveloping algebra of $\mathfrak{gl}(\infty, \mathbf{C})$. The $\mathfrak{gl}(\infty, \mathbf{C})$-module $\mathcal{F}[\gamma] = \mathcal{U}(\mathfrak{gl}(\infty, \mathbf{C}))|\gamma\rangle$ is a highest weight module with highest weight Λ_γ. We have

$$e_i^\infty|\gamma\rangle = 0, \qquad h_i^\infty|\gamma\rangle = \Lambda_\gamma(h_i^\infty)|\gamma\rangle \qquad \text{for all } i \in \mathbf{Z}.$$

Let $\mathfrak{h} = (\oplus_{0\le i<r}\mathbf{C}h_i) \oplus \mathbf{C}c \oplus \mathbf{C}d$ be the Cartan subalgebra of $\widehat{\mathfrak{sl}}(r, \mathbf{C})$. Restricting Λ_i to $\mathfrak{h}$ we get the fundamental weights of $\widehat{\mathfrak{sl}}(r, \mathbf{C})$

$$\Lambda_i(h_j) = \delta_{ij}, \qquad \Lambda_i(c) = 1, \qquad \Lambda_i(d) = 0 \qquad (0 \le i, j < r).$$

More generally $\Lambda_{kr+i}\big|_{\mathfrak{h}} = \Lambda_i\big|_{\mathfrak{h}} - \big(k(k-1)r/2 + ki\big)\delta$ for $k \in \mathbf{Z}, 0 \le i < r$, where δ is the null root (in particular $\Lambda_r\big|_{\mathfrak{h}} = \Lambda_0\big|_{\mathfrak{h}}$). We shall often use the same letter Λ_i to mean $\Lambda_i\big|_{\mathfrak{h}}$ for $0 \le i \le r$. Henceforth we set

$$\Lambda = \Lambda_{\gamma_1} + \cdots + \Lambda_{\gamma_l} \qquad (0 \le \gamma_1 \le \cdots \le \gamma_l < r).$$

Since the bilinear form (3.2) is non-degenerate, the tensor module $\mathcal{F}[\gamma_1]\otimes\cdots\otimes\mathcal{F}[\gamma_l]$ is completely reducible. The $\mathfrak{gl}(\infty, \mathbf{C})$- (resp. $\widehat{\mathfrak{gl}}(r, \mathbf{C})$-, $\widehat{\mathfrak{sl}}(r, \mathbf{C})$-) submodule generated by the vector $v_\Lambda = |\gamma_1\rangle \otimes \cdots \otimes |\gamma_l\rangle$ is necessarily irreducible; we denote it by $F(\Lambda)$ (resp. $G(\Lambda), L(\Lambda)$).

(3.8) $$\mathcal{F}[\gamma_1] \otimes \cdots \otimes \mathcal{F}[\gamma_l] \supset F(\Lambda) \supset G(\Lambda) \supset L(\Lambda).$$

The last one is the irreducible highest weight $\widehat{\mathfrak{sl}}(r, \mathbf{C})$-module with highest weight Λ. As we have noted before, when $l = 1$, $\mathcal{F}[\gamma] = F(\Lambda_\gamma)$ is irreducible under the action of the Heisenberg subalgebra $\mathcal{H} \subset \widehat{\mathfrak{gl}}(r, \mathbf{C})$. This implies that one has the explicit realizations

$$\begin{aligned} F(\Lambda_\gamma) &= G(\Lambda_\gamma) = \mathbf{C}[x_1, x_2, \cdots], \\ L(\Lambda_\gamma) &= \mathbf{C}[\, x_j \mid j \in \mathbf{N}, j \not\equiv 0 \bmod r\,], \end{aligned}$$

which implies in particular

$$G(\Lambda_\gamma) \cong L(\Lambda_\gamma) \otimes \mathbf{C}[x_r, x_{2r}, \cdots].$$

In general, we have

Proposition 3.2.

$$G(\Lambda) \cong L(\Lambda) \otimes \mathbf{C}[x_r, x_{2r}, \cdots]. \tag{3.9}$$

Proof. This is a consequence of the decomposition (3.6). □

§4. Maya diagrams and Plücker relations

In this section we give a spanning set of $G(\Lambda)^*$, the dual of the irreducible highest weight module $G(\Lambda)$ of $\widehat{\mathfrak{gl}}(r, \mathbf{C})$. We shall follow Sato [11], in which the Plücker relations are extensively studied in the language of Maya diagrams.

4.1. Maya diagrams

A *Maya diagram* is a sequence consisting of white or black squares, labeled by integers and arranged on a horizontal line, such that to the far left (resp. right) the squares are all black (resp. white) (Fig.4.1).

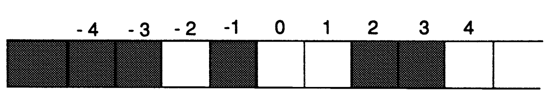


Fig. 4.1 A Maya diagram (charge=1).

Alternatively, a Maya diagram can be represented by a bijection $m : \mathbf{Z} \longrightarrow \mathbf{Z}$ such that $\big(m(j)\big)_{j<0}$ and $\big(m(j)\big)_{j\geq 0}$ are both increasing. Here $\big(m(j)\big)_{j<0}$ (resp. $\big(m(j)\big)_{j\geq 0}$) correspond to the positions of the black (resp. white) squares. For each Maya diagram there exists a unique

$\gamma \in \mathbf{Z}$ such that $m(j) - j = \gamma$ for $|j| \gg 0$. The integer γ is called the *charge* of m. Let $\mathcal{M}[\gamma]$ denote the set of Maya diagrams of charge γ. For $m \in \mathcal{M}[\gamma]$ we put

$$m[r] = (m(j) + r)_{j \in \mathbf{Z}} \in \mathcal{M}[\gamma + r].$$

A Maya diagram can be visualized also by a Young diagram as follows. Consider a lattice on the right half plane with sites $\{(i,j) \in \mathbf{Z}^2 \mid i \geq 0\}$. We consider edges on the lattice as oriented, starting from (i,j) and ending at $(i+1,j)$ or $(i,j+1)$, and as numbered by the integer $i+j$. Given a Maya diagram, draw a path on the lattice. We here mean by a path a map e from $\mathbf{Z}$ to the set of edges on the lattice, $j \mapsto e(j)$, such that $e(j)$ has number j and the ending site of $e(j)$ coincides with the starting site of $e(j+1)$. (This has nothing to do with the definition of paths given in Definition 1.1.) The condition that fixes the path is as follows:

(i) For $j \ll 0$, $e(j)$ is the edge joining $(0,j)$ and $(0,j+1)$.
(ii) The edge $e(m(j))$ is vertical (resp. horizontal) if and only if $j < 0$ (resp. $j \geq 0$).

Note that if $j \gg 0$ the edge $e(j)$ is from $(j-\gamma,\gamma)$ to $(j-\gamma+1,\gamma)$.

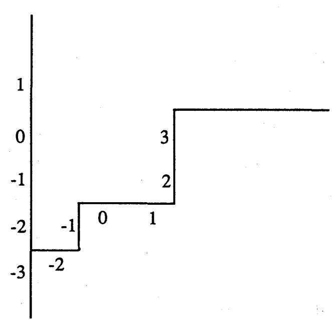


Fig. 4.2 The Young diagram corresponding to Fig. 4.1.

The resulting path divides the right half plane into two components. The upper half is an *infinite Young diagram* $\mathcal{Y}$, namely a diagram consisting of a quadrant and a (finite) Young diagram Y attached together along a horizontal line determined by the charge γ (Fig.4.2). In this way m is in one to one correspondence with the pair (Y,γ).

Lemma 4.1. *Let $m \in \mathcal{M}[\gamma], m' \in \mathcal{M}[\gamma']$, and let $\mathcal{Y}, \mathcal{Y}'$ be the corresponding infinite Young diagrams. Then the following are equivalent.*

(i) $m(j) \le m'(j)$ *for* $j \ge 0$,

(ii) $\gamma \le \gamma'$ *and* $m(j-\gamma) \ge m'(j-\gamma')$ *for* $j < \gamma$,

(iii) $\mathcal{Y} \supset \mathcal{Y}'$.

Proof. Let e be the path corresponding to a Maya diagram m. The horizontal edge $e(m(j))(j \ge 0)$ is from $(j, m(j))$ to $(j+1, m(j))$ and the vertical edge $e(m(j))(j < 0)$ is from $(\gamma + j, m(j) - j - \gamma)$ to $(\gamma + j + 1, m(j) - j - \gamma)$. The equivalence follows from this. □

Definition 4.2. We write $m \le m'$ if the conditions in Lemma 4.1 hold.

Let m be a Maya diagram. We put for $j \ll 0$

$$\langle m| = \langle m(j)|\psi^*_{m(j)}\psi^*_{m(j+1)}\cdots\psi^*_{m(-1)},$$
$$|m\rangle = \psi_{m(-1)}\cdots\psi_{m(j+1)}\psi_{m(j)}|m(j)\rangle.$$

These vectors do not depend on the choice of j if it is sufficiently negative, and one sees immediately:

Proposition 4.3. *We have* $\langle m|m'\rangle = \delta_{mm'}$.

We denote by $\mathcal{F}^*_{\mathbf{Z}}[\gamma]$ (resp. $\mathcal{F}_{\mathbf{Z}}[\gamma]$) the free $\mathbf{Z}$-module generated by the vectors $\{\langle m|\}_{m\in\mathcal{M}[\gamma]}$ (resp. $\{|m\rangle\}_{m\in\mathcal{M}[\gamma]}$). Then we have $\mathcal{F}^*[\gamma] = \mathcal{F}^*_{\mathbf{Z}}[\gamma] \otimes_{\mathbf{Z}} \mathbf{C}, \mathcal{F}[\gamma] = \mathcal{F}_{\mathbf{Z}}[\gamma] \otimes_{\mathbf{Z}} \mathbf{C}$.

4.2. Plücker relations

There exist natural surjective maps

$$(\mathcal{F}[\gamma_1] \otimes \cdots \otimes \mathcal{F}[\gamma_\ell])^* \longrightarrow F(\Lambda)^* \longrightarrow G(\Lambda)^*$$

dual to (3.8). Here and in what follows, for a Lie algebra module V we denote by V^* its restricted dual, i.e., the direct sum of the dual spaces of the (finite dimensional) weight spaces. The images of the base vectors of $(\mathcal{F}[\gamma_1] \otimes \cdots \otimes \mathcal{F}[\gamma_\ell])^*$ under these surjections obey quadratic relations in $F(\Lambda)^*$ or $G(\Lambda)^*$, known as the *Plücker* or *r-reduced Plücker relations.*

To state these relations we prepare several notations. Let μ be an integer. A *μ-index* is a sequence of integers $I = (i_k)_{k<\mu}$ such that $i_k = k$ for $k \ll 0$. Let $J = (j_1, \cdots, j_\nu)$ be a finite sequence of integers. We set $|J| = \nu$. By $I' = IJ$ we denote a $(\mu + \nu)$-index $(i'_k)_{k<\mu+\nu}$ given by

$i'_k = i_k$ for $k < \mu$ and $i'_k = j_{k-\mu+1}$ for $\mu \le k < \mu + \nu$. For a Maya diagram m of charge γ, we denote by $I(m)$ the γ-index

$$I(m) = \big(m(j-\gamma)\big)_{j<\gamma}.$$

Let I be a μ-index. We define a vector ξ_I in $\mathcal{F}^*[\mu]$ by

$$\xi_I = \langle i_k | \psi^*_{i_k} \cdots \psi^*_{i_{\mu-2}} \psi^*_{i_{\mu-1}}, \qquad k \ll 0.$$

The right hand side does not depend on the choice of k if it is sufficiently negative, and is skew symmetric with respect to the permutations of elements of I. When $I = I(m)$, $m \in \mathcal{M}[\gamma]$, the vector $\xi_{I(m)}$ coincides with $\langle m|$.

Proposition 4.4 (Plücker relations). *Let γ_1, γ_2 be two integers such that $\gamma_1 \le \gamma_2$. Let J be a finite sequence and let I (resp. K) be a (γ_2+1)- (resp. $(\gamma_1 - |J| - 1)$-) index. We have then*

$$\sum_{\substack{I = I' \cup I'' \\ |I''| = |J|+1+\gamma_2-\gamma_1}} \operatorname{sgn}(I'I'') \xi_{I'J} \otimes \xi_{KI''} \big|_{F(\Lambda_{\gamma_1}+\Lambda_{\gamma_2})} = 0. \tag{4.1}$$

Here the sum is over all the partitions (I', I'') of I such that $|I''| = |J| + 1 + \gamma_2 - \gamma_1$ and $\operatorname{sgn}(I'I'')$ *is the signature of the permutation which sends I to $I'I''$.*

Remark. In (4.1) $I'J$ is a γ_1-index and KI'' is a γ_2-index. Note also that given I, K there are only a finite number of I'' for which $\xi_{KI''} \ne 0$ holds.

Example. Let $\gamma_1 = \gamma_2 = 0$, and take $I = (\cdots -5\ -4\ -3\ -2\ -1\ 0)$, $J = (1)$ and $K = (\cdots -5\ -4\ -3)$. The possible choices of I'' for which $\xi_{KI''} \ne 0$ are $I'' = (-1\ 0), (-2\ 0), (-2\ -1)$. We have thus on $F(2\Lambda_0)$

$$\begin{aligned} 0 = {} & \xi_{\cdots -3\ -2\ 1} \otimes \xi_{\cdots -3\ -1\ 0} - \xi_{\cdots -3\ -1\ 1} \otimes \xi_{\cdots -3\ -2\ 0} \\ & + \xi_{\cdots -3\ 0\ 1} \otimes \xi_{\cdots -3\ -2\ -1}. \end{aligned}$$

Proof of Proposition 4.4. Because of (3.3), any element of $F(\Lambda_{\gamma_1} + \Lambda_{\gamma_2})$ belongs to the linear hull of $\overline{\mathbf{G}_N}(|\gamma_1\rangle \otimes |\gamma_2\rangle)$ for some N. Hence it suffices to show

$$0 = \sum_{\substack{I = I' \cup I'' \\ |I''| = |J|+1+\gamma_2-\gamma_1}} \operatorname{sgn}(I'I'') \big(\xi_{I'J} \otimes \xi_{KI''}\big) \big(g|\gamma_1\rangle \otimes g|\gamma_2\rangle\big) \qquad \text{for } g \in \overline{\mathbf{G}}. \tag{4.2}$$

Without loss of generality we may assume $\langle\gamma|g|\gamma\rangle \neq 0$ for some $\gamma \ll 0$. Set $\mu = |J|+1+\gamma_2-\gamma_1$ and $\nu = |J|$. Suppose that $I' = (i_\alpha)_{\alpha<\gamma_1-\nu}, I'' = (l_1,\cdots,l_\mu), J = (j_1,\cdots,j_\nu)$ and $K = (k_\alpha)_{\alpha<\gamma_1-\nu-1}$. For an index, say K, let $\overline{K}$ signify the finite part $(k_\gamma,\cdots,k_{\gamma_1-\nu-2})$ obtained by dropping k_α with $\alpha < \gamma$. The right hand side of (4.2) can be written as

$$\begin{aligned}(4.3)\quad \sum_{\substack{\overline{I}=\overline{I}'\cup I''\\|I''|=\mu}} &\operatorname{sgn}(\overline{I}'I'')\langle\gamma|\psi^*_\gamma\cdots\psi^*_{i_{\gamma_1-\nu-1}}\psi^*_{j_1}\cdots\psi^*_{j_\nu}g\psi_{\gamma_1-1}\cdots\psi_\gamma|\gamma\rangle\\ &\times\langle\gamma|\psi^*_\gamma\cdots\psi^*_{k_{\gamma_1-\nu-2}}\psi^*_{l_1}\cdots\psi^*_{l_\mu}g\psi_{\gamma_2-1}\cdots\psi_\gamma|\gamma\rangle.\end{aligned}$$

Now set $P = (\gamma,\cdots,\gamma_1-1), Q = (\gamma,\cdots,\gamma_2-1)$ and consider the following $(\gamma_1+\gamma_2-2\gamma)\times(\gamma_1+\gamma_2-2\gamma)$ matrix A:

$$A = \begin{pmatrix} A_{JP} & 0 \\ A_{\overline{I}P} & A_{\overline{I}Q} \\ 0 & A_{\overline{K}Q}\end{pmatrix},$$

where

$$A_{XY} = \left(\langle\gamma|\psi^*_i g\psi_j|\gamma\rangle\right)_{i\in X, j\in Y}.$$

Thanks to Lemma 3.1, the Laplace expansion of the determinant of A gives (4.3) up to a trivial factor. On the other hand, writing $Q = P \sqcup R$, we have

$$\begin{aligned}\det A &= \begin{vmatrix} A_{JP} & 0 & 0 \\ A_{\overline{I}P} & A_{\overline{I}P} & A_{\overline{I}R} \\ 0 & A_{\overline{K}P} & A_{\overline{K}R}\end{vmatrix} = \begin{vmatrix} A_{JP} & 0 & 0 \\ 0 & A_{\overline{I}P} & A_{\overline{I}R} \\ -A_{\overline{K}P} & A_{\overline{K}P} & A_{\overline{K}R}\end{vmatrix}\\ &= \pm\begin{vmatrix} A_{JP} & 0 & 0 \\ -A_{\overline{K}P} & A_{\overline{K}P} & A_{\overline{K}R} \\ 0 & A_{\overline{I}P} & A_{\overline{I}R}\end{vmatrix}.\end{aligned}$$

Noting that $|J\cup\overline{K}| = \gamma_1-\gamma-1$ and $|P| = \gamma_1-\gamma$, we have $\det A = 0$. $\square$

Next we consider the case of $G(\Lambda)^*$. For a μ-index $I = (i_k)_{k<\mu}$ (resp. a finite sequence $J = (j_1,\cdots,j_\nu)$) we define its s-shift by $I[s] = (i_{k-s}+s)_{k<\mu+s}$ (resp. $J[s] = (j_1+s,\cdots,j_\nu+s)$).

Proposition 4.5 (r-reduced Plücker relations). *Let γ_1, γ_2 be two integers such that $\gamma_1 \leq \gamma_2 + r$. Let J be a finite sequence and I (resp.*

K) a (γ_2+r+1)- (resp. $(\gamma_1-r-|J|-1)$-) index. We have then

$$\sum_{\substack{I=I'\cup I''\\ |I''|=|J|+1+\gamma_2+r-\gamma_1}} \operatorname{sgn}(I'I'')\xi_{I'J}\otimes\xi_{K\,I''[-r]}\big|_{G(\Lambda_{\gamma_1}+\Lambda_{\gamma_2})}=0. \tag{4.4}$$

Example. Let $r=2, \gamma_1=\gamma_2=0$, and take $I=(\cdots-2\ -1\ 0\ 2\ 3)$, $J=\phi, K=(\cdots-5\ -4)$. We have

$$\begin{aligned}0=&\xi_{\cdots-4\ -3\ -2\ -1}\otimes\xi_{\cdots-4\ -2\ 0\ 1}-\xi_{\cdots-4\ -3\ -2\ 0}\otimes\xi_{\cdots-4\ -3\ 0\ 1}\\ &+\xi_{\cdots-4\ -3\ -2\ 2}\otimes\xi_{\cdots-4\ -3\ -2\ 1}-\xi_{\cdots-4\ -3\ -2\ 3}\otimes\xi_{\cdots-4\ -3\ -2\ 0}.\end{aligned}$$

Proof of Proposition 4.5. Let $a\in\mathcal{U}(\widehat{\mathfrak{gl}}(r,\mathbf{C}))$. We are to prove

$$\sum_{\substack{I=I'\cup I''\\ |I''|=|J|+1+\gamma_2+r-\gamma_1}} \operatorname{sgn}(I'I'')\big(\xi_{I'J}\otimes\xi_{K\,I''[-r]}\big)\big(a(|\gamma_1\rangle\otimes|\gamma_2\rangle)\big)=0. \tag{4.5}$$

Note that a commutes with $1\otimes\iota^r$ where ι denotes the automorphism (3.5). Using the property

$$v^*(v)=\iota(v^*)\big(\iota(v)\big)\qquad (v^*\in\mathcal{F}^*, v\in\mathcal{F}),$$

we have

$$\begin{aligned}&\big(\xi_{I'J}\otimes\xi_{KI''[-r]}\big)\big(a(|\gamma_1\rangle\otimes|\gamma_2\rangle)\big)\\ &=\big(\xi_{I'J}\otimes\iota^{-r}(\xi_{KI''[-r]})\big)\big(a(|\gamma_1\rangle\otimes\iota^{-r}(|\gamma_2\rangle))\big)\\ &=\big(\xi_{I'J}\otimes\xi_{K[r]I''}\big)\big(a(|\gamma_1\rangle\otimes|\gamma_2+r\rangle)\big).\end{aligned}$$

Therefore (4.5) is reduced to the ordinary Plücker relation (4.1) with K,γ_2 replaced by $K[r],\gamma_2+r$. □

4.3. Spanning sets of $F(\Lambda)^*$ and $G(\Lambda)^*$

Let $I=(i_j)$ and $I'=(i'_j)$ be increasing μ-index and μ'-index, respectively. We denote $I<I'$ if $\mu\le\mu'$ and there exists $j_0<\mu$ such that $i_j=i'_j$ for $j<j_0$ and $i_{j_0}>i'_{j_0}$. By virtue of Lemma 4.1, for Maya diagrams m,m' we have $m<m'$ if and only if $I(m)<I(m')$.

For a Maya diagram m, we define its type $t(m)=\big(t(m)_i\big)_{i\in\mathbf{Z}}$ by

$$\begin{aligned}t(m)_i=&1\quad\text{if } i\in I(m),\\ =&0\quad\text{otherwise.}\end{aligned}$$

Extend this definition to an s-tuple of Maya diagrams $M=(m_1,\cdots,m_s)$ by setting $t(M)=t(m_1)+\cdots+t(m_s)$. We call this the total type of

M. Observe that the terms entering the Plücker relation (4.1) have the same total type. As for the μ-indices, we introduce a linear order in M, namely we set $t(M) < t(M')$ if there exists a j_0 such that $t(M)_i = t(M')_i$ for $i < j_0$ and $t(M)_{j_0} > t(M')_{j_0}$.

Proposition 4.6. *The set*

$$(4.6) \qquad \{\, \xi_{I(m_1)} \otimes \cdots \otimes \xi_{I(m_l)}\big|_{F(\Lambda)} \quad | \, m_j \in \mathcal{M}[\gamma_j], \; m_1 \le \cdots \le m_l \,\}$$

gives a spanning set of $F(\Lambda)^*$.

Proof. First let us consider the case $l = 2$.

Let m and m' be two Maya diagrams of charges γ and γ', respectively. Assume that $\gamma \le \gamma'$ and $m \not\le m'$. By Lemma 4.1 there exists a $j < \gamma$ such that $m(j-\gamma) < m'(j-\gamma')$. Let j_0 be the smallest among such, and define a $(\gamma'+1)$- (resp. j_0-) index $I = (i_j)$ (resp. $K = (k_j)$) by

$$\begin{aligned} i_j &= m(j-\gamma) && \text{for} \quad j \le j_0, \\ &= m'(j-\gamma'-1) && \text{for} \quad j_0 < j < \gamma'+1, \\ k_j &= m'(j-\gamma') && \text{for} \quad j < j_0. \end{aligned}$$

Further put $J = \big(m(j_0+1-\gamma), \cdots, m(-1)\big)$. With these choices of I, J, K the Plücker relation on $F(\Lambda_\gamma + \Lambda_{\gamma'})$ has the following structure (Fig.4.3):

$$0 = \xi_{I(m)} \otimes \xi_{I(m')} + \sum_{L,L'} \pm \xi_L \otimes \xi_{L'}.$$

Here the index L (resp. L') is an increasing γ- (resp. γ'-) index such that $I(m) > L$ (resp. $I(m') < L'$).

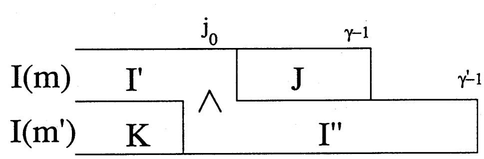


Fig. 4.3 Reduction by the Plücker relation.

Therefore, assuming $m \not\le m'$ one can reduce $\xi_{I(m)} \otimes \xi_{I(m')}$ into a linear combination of $\xi_L \otimes \xi_{L'}$'s with $I(m) > L, I(m') < L'$. Moreover, since the total type is invariant, L' cannot exceed an upper bound determined from $I(m), I(m')$. The above procedure strictly increases the order of the second index, hence it terminates after a finitely many steps. This implies that we can express $\xi_{I(m)} \otimes \xi_{I(m')}$ as a linear combination of elements in (4.7).

The general case can be shown by applying this procedure to the adjacent pairs $\xi_{I(m_i)} \otimes \xi_{I(m_{i+1})}$ repeatedly. □

Proposition 4.7. *The set*

(4.7)
$$\{\, \xi_{I(m_1)} \otimes \cdots \otimes \xi_{I(m_l)}\big|_{G(\Lambda)} \quad | \, m_j \in \mathcal{M}[\gamma_j],\ m_1 \le \cdots \le m_l \le m_1[r] \,\}$$

is a spanning set of $G(\Lambda)^*$

Proof. The proof of Proposition 4.6 shows that the linear hull of vectors $\xi_{I(m_1)} \otimes \cdots \otimes \xi_{I(m_l)}$ in $F(\Lambda)^*$ with fixed total type $t(m_1, \cdots, m_l)$ is spanned by those satisfying $m_1 \le \cdots \le m_l$. Let us show that if $m_l \not\le m_1[r]$, then on $G(\Lambda)$ such a vector can be written as a linear combination of those with higher total types. Let j_0 be the smallest of $j < \gamma_l$ such that $m_l(j - \gamma_l) < m_1[r](j - \gamma_1 - r)$. We apply the r-reduced Plücker relation by taking $I = (i_j)_{j<\gamma_1+r+1}, K = (k_j)_{j<j_0}$ where (Fig.4.4)

$$\begin{aligned} i_j &= m_l(j - \gamma_l) && \text{for} \quad j \le j_0, \\ &= m_1[r](j - \gamma_1 - r - 1) && \text{for} \quad j_0 < j < \gamma_1 + r + 1, \\ k_j &= m_1(j - \gamma_1 - r) && \text{for} \quad j < j_0, \end{aligned}$$

and $J = \big(m_l(j_0 + 1 - \gamma_l), \cdots, m_l(-1)\big)$.

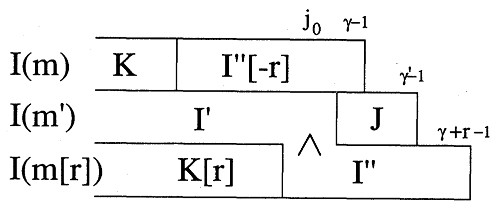


Fig. 4.4 Reduction by the r-reduced Plücker relation.

Arguing as in the proof of Proposition 4.6, we can write $\xi_{I(m_1)} \otimes \cdots \otimes \xi_{I(m_l)}$ as a linear combination of terms $\xi_{I(n_1)} \otimes \xi_{I(m_2)} \otimes \cdots \otimes \xi_{I(m_{l-1})} \otimes$

$\xi_{I(n_l)}$ such that $t(m_1, \cdots, m_l) < t(n_1, m_2, \cdots, m_{l-1}, n_l)$. The total type $t(m_1, \cdots, m_l)$ is bounded from above if the charges $\gamma_1, \cdots, \gamma_l$ are fixed. Hence after a finitely many steps the resulting terms eventually satisfy both $m_1 \le \cdots \le m_l$ and $m_l \le m_1[r]$. $\square$

§5. Paths and lifts

In this section we establish the relation between paths and Maya diagrams. Using this relation we then define a set of vectors $\{\xi_\eta\}$ of $L(\Lambda)^*$ labeled by the Λ-paths η.

5.1. Lifts of a path

As before, we fix $l \ge 1$ and $\Lambda = \Lambda_{\gamma_1} + \cdots + \Lambda_{\gamma_l} (0 \le \gamma_1 \le \cdots \le \gamma_l < r)$. The set of Λ-paths $\mathcal{P}(\Lambda)$, the weight λ_η of a path η and the set $\mathcal{P}(\Lambda)_\mu$ of Λ-paths of weight μ have been introduced in Section 1.

Definition 5.1. Let η be a Λ-path. An element $M = (m_1, \ldots, m_l) \in \mathcal{M}[\gamma_1] \times \cdots \times \mathcal{M}[\gamma_l]$ is called a lift of η if and only if it satisfies the condition

$$m_1 \le \cdots \le m_l \le m_1[r]$$

and

$$\eta(j) = \epsilon_{m_1(j)} + \cdots + \epsilon_{m_l(j)}.$$

Let $M = (m_1, \ldots, m_l), M' = (m'_1, \ldots, m'_l)$ be lifts of a Λ-path η. We denote $M \ge M'$ if and only if $m_j \ge m'_j$ for $1 \le j \le l$.

Proposition 5.2. *For each Λ-path η there exists a unique highest lift M of η such that $M \ge M'$ for any lift M' of η.*

Proof. We define a set of integers $t_{jk} (j, k \in \mathbf{Z}, k \ge 0)$ as follows. For each k we require that $t_{jk} \le t_{j+1k}, t_{j+lk} = t_{jk} + r$ and $\eta(k) = \sum_{j \bmod l} \epsilon_{t_{jk}+k}$. These conditions determine $t_{jk} (j \in \mathbf{Z})$ up to a shift of the first index, $t_{jk} \to t_{j-s(k)k}$. For $k \gg 0$ we fix t_{jk} by the condition $t_{jk} = \gamma_j (1 \le j \le l)$. Suppose that $t_{jk+1} (j \in \mathbf{Z})$ is already given. We then fix $t_{jk} (j \in \mathbf{Z})$ by the condition that $t_{j-sk} \le t_{jk+1}$ is valid for arbitrary j if and only if $s \ge 0$. Set $m_j(k) = t_{jk} + k$ for $1 \le j \le l$ and $k \ge 0$. An element $M = (m_1, \ldots, m_l) \in \mathcal{M}[\gamma_1] \times \cdots \times \mathcal{M}[\gamma_l]$ is determined by this assignment. The above conditions on t_{jk} imply that M is the highest lift of η. $\square$

Definition 5.3. Let $(m_1, \ldots, m_l)$ be the highest lift of a Λ-path η. We denote by ξ_η the image of $\langle m_1| \otimes \cdots \otimes \langle m_l|$ by the projection $\mathcal{F}^*[\gamma_1] \otimes \cdots \otimes \mathcal{F}^*[\gamma_l] \longrightarrow L(\Lambda)^*$.

Our goal is to prove

Theorem 5.4. *The set* $\{\xi_\eta \mid \eta \in \mathcal{P}(\Lambda)_\mu\}$ *is a basis of* $L(\Lambda)^*_\mu$.

The proof follows from Theorem 5.7, Proposition 5.11 and Theorem 6.14 below.

We start with the computation of the $\widehat{\mathfrak{sl}}(r, \mathbf{C})$-weight of the vector ξ_η. Consider the case $l = 1$. Suppose that m is the highest lift of a Λ_γ-path η. Then we have

$$m(k) \geq m(k+1) - r \qquad \text{for all} \quad k \geq 0. \tag{5.1}$$

Conversely, if a Maya diagram m satisfies (5.1), then m is the highest lift of a path.

Proposition 5.5. *Let γ be an integer such that $0 \leq \gamma < r$, and let m be a Maya diagram of charge γ such that m is the highest lift of the Λ_γ-path $\eta = \left(\epsilon_{m(k)}\right)_{k \geq 0}$. Then the $\widehat{\mathfrak{sl}}(r, \mathbf{C})$-weight of $\langle m|$ is equal to λ_η.*

Proof. Set

$$\begin{aligned} \delta^{(r)}_{ab} &= 1 \qquad \text{if } a \equiv b \bmod r, \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{5.2}$$

The weight of $\langle m|$ is given by

$$\Lambda_\gamma - \sum_{k=0}^{\infty} \sum_{m(k) < \mu \leq \gamma + k} \alpha_\mu = \Lambda_\gamma - \sum_{k=0}^{\infty} \sum_{m(k) < \mu \leq \gamma + k} \left(\pi\left(\epsilon_{\mu-1} - \epsilon_\mu\right) + \delta^{(r)}_{\mu 0} \delta\right),$$

where we set $\alpha_{\mu+r} = \alpha_\mu$. Since

$$\sum_{m(k) < \mu \leq \gamma + k} \pi\left(\epsilon_{\mu-1} - \epsilon_\mu\right) = \pi\left(\eta(k) - \eta_{\Lambda_\gamma}(k)\right),$$

it is sufficient to show that

$$\sum_{k=0}^{\infty} \sum_{m(k) < \mu \leq \gamma + k} \delta^{(r)}_{\mu 0} = \sum_{k=1}^{\infty} k\Big(H\left(\eta(k-1), \eta(k)\right) - H\left(\eta_{\Lambda_\gamma}(k-1), \eta_{\Lambda_\gamma}(k)\right)\Big). \tag{5.3}$$

Set

$$X = \{(k, \mu) | k, \mu \in \mathbf{Z}, k \geq 0, m(k) < \mu \text{ and } \mu \equiv 0 \bmod r\}.$$

This set decomposes into a disjoint union of $X_i (i \in \mathbf{Z})$ where $X_i = X \bigcap \{(k, ir) | k \in \mathbf{Z}, k \geq 0\}$ (Fig.5.1).

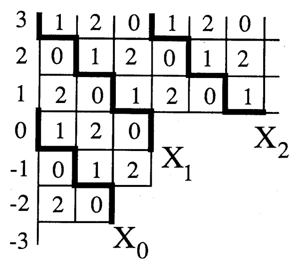


Fig. 5.1 Counting the number of 0-boxes.

Take a positive integer k and assume that $m(k-1) < ir \leq m(k)$. Then we have $\sharp(X_i) = k$ and $H(\eta(k-1), \eta(k)) = 1$. Conversely, if $H(\eta(k-1), \eta(k)) = 1$ there exists a unique i such that $m(k-1) < ir \leq m(k)$. Argue similarly with $m(k)$ and η replaced by $\gamma + k$ and η_Λ, respectively, and consider the difference. Then the equality (5.3) follows immediately. □

Now we consider the general case.

Proposition 5.6. *Let $(m_1, \ldots, m_l)$ be the highest lift of a Λ-path η. Then the Maya diagram m_j is the highest lift of the Λ_{γ_j}-path $\eta_j = (\epsilon_{m_j(k)})_{k \geq 0}$. The weight of η satisfies the additivity*

$$\lambda_\eta = \sum_{j=1}^{l} \lambda_{\eta_j}.$$

Proof. For the first half, it is sufficient to show that $m_j(k) \geq m_j(k+1) - r$, or equivalently that $t_{jk} > t_{jk+1} - r$, for $1 \leq j \leq l, k \geq 0$. Assume that $t_{jk} \leq t_{jk+1} - r$ for some j, k. For $1 \leq i < j$ we have $t_{i+1k} \leq t_{jk} \leq t_{jk+1} - r = t_{j-lk+1} \leq t_{ik+1}$, and for $j \leq i \leq l$ we have $t_{i+1k} \leq t_{j+lk} = t_{jk} + r \leq t_{jk+1} \leq t_{ik+1}$. This contradicts the choice of t_{jk}.

For $t \in \mathbf{Z}$ denote by $\bar{t}$ the integer satisfying $0 \le \bar{t} < r$ and $\bar{t} \equiv t$ mod r. For the second half, it is sufficient to show that

$$H\big(\eta(k), \eta(k+1)\big) = \sum_{j=1}^{l} H\big(\eta_j(k), \eta_j(k+1)\big),$$

or equivalently that

$$\min_{\sigma} \sum_{j=1}^{l} \theta\big(\overline{m_j(k)} - \overline{m_{\sigma(j)}(k+1)}\big)$$

is attained by $\sigma = \mathrm{id}$. Note that $t_j = m_j(k)(1 \le j \le l)$ attains the maximum of $\sum_{j=1}^{l} t_j$ under the condition that $m_j(k+1) - r \le t_j < m_j(k+1)$ $(1 \le j \le l)$ and $\overline{t_j} = \overline{m_{\sigma(j)}(k)}$ for some σ. Since

$$m_j(k+1) - t_j = \overline{m_j(k+1)} - \overline{t_j} + r\theta\big(\overline{t_j} - \overline{m_j(k+1)}\big),$$

$t_j = m_j(k)$ $(1 \le j \le l)$ attains the minimum of $\sum_{j=1}^{l} \theta\big(\overline{t_j} - \overline{m_j(k+1)}\big)$. □

From Propositions 5.5 and 5.6 we have

Theorem 5.7. *Let η be a Λ-path and let $(m_1, \ldots, m_l)$ be its highest lift. The weight of $\langle m_1| \otimes \cdots \otimes \langle m_l|$ is equal to λ_η.*

5.2. Lifts and Young diagrams

Let us determine the totality of lifts of a given Λ-path η. From the proof of Proposition 5.2 it follows immediately that

Proposition 5.8. *An element $(n_1, \ldots, n_l) \in \mathcal{M}[\gamma_1] \times \cdots \times \mathcal{M}[\gamma_l]$ is a lift of η if and only if there exists a unique sequence $\big(s(k)\big)_{k \ge 0}$ such that $s(k) \ge s(k+1)$ for any k, $s(k) = 0$ for $k \gg 0$ and*

$$n_j(k) = t_{j-s(k)k} + k \quad \textit{for} \quad k \ge 0. \tag{5.4}$$

Let $N = (n_1, \ldots, n_l)$ be a lift of η satisfying (5.4). We denote by $Y(N)$ the Young diagram of signature $[y_1, \ldots, y_s]$ where

$$y_j = \sharp\{k | s(k) \ge j\}.$$

The following is also immediate.

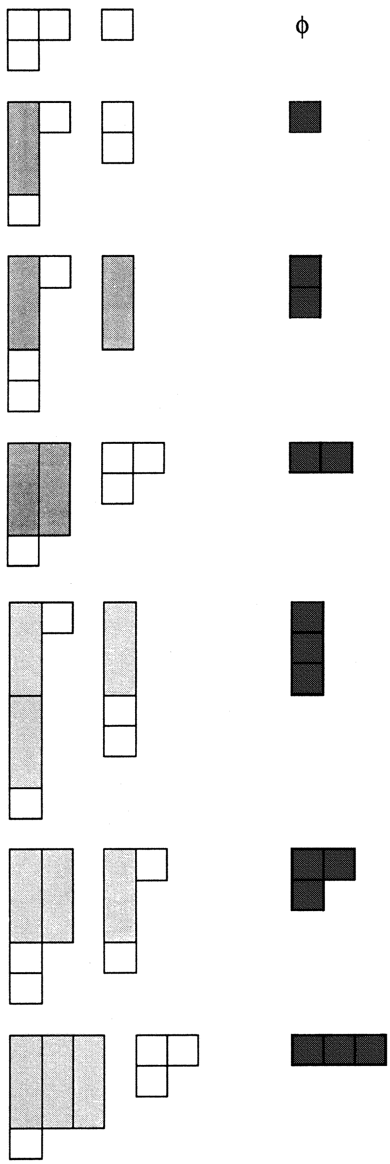


Fig. 5.2 Lifts and Young diagrams.

Proposition 5.9. *Fix a Λ-path η. The correspondence $N \mapsto Y(N)$ from the set of lifts of η to the set of Young diagrams is bijective.*

Let Y be a Young diagram with n nodes. We define the degree of Y by $d(Y) = n$. If $Y = Y(N)$ then $d(Y) = \sum_{j=1}^{s} y_j = \sum_{k=0}^{\infty} s(k)$.

Proposition 5.10. *Let $N = (n_1, \ldots, n_l)$ be a lift of a Λ-path η. Then the weight of $\langle n_1| \otimes \cdots \otimes \langle n_l|$ is equal to $\lambda_\eta - d\big(Y(N)\big)\delta$.*

Proof. Let $(m_1, \ldots, m_l)$ be the highest lift of η. The difference of the weights of $\langle n_1| \otimes \cdots \otimes \langle n_l|$ and $\langle m_1| \otimes \cdots \otimes \langle m_l|$ is given by

$$-\sum_{j=1}^{l}\sum_{k=0}^{\infty}\sum_{n_j(k)<\mu\le m_j(k)} \alpha_\mu. \tag{5.5}$$

The interval $n_j(k) < \mu \le m_j(k)$ is equal to $t_{j-s(k)k} + k < \mu \le t_{jk} + k$. Suppose that integers p_1, p_2, q_1, q_2 satisfy $\min(p_1, p_2) \ge \max(q_1, q_2)$. Then the following is obvious.

$$\Big(\sum_{q_1<\mu\le p_1} + \sum_{q_2<\mu\le p_2}\Big)\alpha_\mu = \Big(\sum_{q_1<\mu\le p_2} + \sum_{q_2<\mu\le p_1}\Big)\alpha_\mu. \tag{5.6}$$

Note that $t_{l-s\,k} = t_{1-(s+1)\,k} + r$. Therefore, by a repeated use of (5.6), we see that the increment of $s(k)$ by 1 induces the change of (1.1) by $-\delta$. This implies that the weight (5.5) is equal to $-\sum_{k=0}^{\infty} s(k)\delta = -d(Y)\delta$. □

For $\mu \in \mathfrak{h}^*$ we set

$$G(\Lambda)^*_\mu = \big\{\langle v| \in G(\Lambda)^* \mid \langle v|h = \langle v|\mu(h) \text{ for } h \in \mathfrak{h}\big\},$$
$$L(\Lambda)^*_\mu = \big\{\langle v| \in L(\Lambda)^* \mid \langle v|h = \langle v|\mu(h) \text{ for } h \in \mathfrak{h}\big\}.$$

For an integer t we denote by $\mathbf{C}[x_r, x_{2r}, \cdots]_t$ the subspace of $\mathbf{C}[x_r, x_{2r}, \cdots]$ consisting of the polynomials of degree t in the sense that the degree of x_{jr} is j. Its dimension is given by $\dim \mathbf{C}[x_r, x_{2r}, \cdots]_t = p(t)$ where $p(t)$ is the number of partitions of t.

Proposition 5.11. *Let μ be a $\widehat{\mathfrak{sl}}(r, \mathbf{C})$-weight. Then we have*

$$\sum_t p(t) \dim L(\Lambda)_{\mu+t\delta} \le \sum_t p(t)\ \sharp\mathcal{P}(\Lambda)_{\mu+t\delta}.$$

Proof. From Proposition 3.2 the weight space $G(\Lambda)_\mu$ decomposes into

$$G(\Lambda)_\mu = \sum_t L(\Lambda)_{\mu+t\delta} \otimes \mathbf{C}[x_r, x_{2r}, \cdots]_t.$$

From Theorem 5.7 and Proposition 5.10 it follows that

$$\dim G(\Lambda)^*_\mu \leq \sharp\{N \mid N \text{ is a lift of a } \Lambda\text{-path } \eta \\ \text{such that } \lambda_\eta - d(Y(N))\delta = \mu\}.$$

Now the assertion follows from Proposition 5.9. □

§6. Paths and divisors

We now carry out the final step in the proof of Theorem 5.4 to show that the vectors $\{\xi_\eta \mid \eta \in \mathcal{P}(\Lambda)\}$ are linearly independent and span $L(\Lambda)^*$.

6.1. Reduction of divisors

Consider a free abelian group $\mathcal{D}$ in multiplicative form with generators $(k,a)(k \geq 1, a \in \mathbf{Z}_r)$. An element of $\mathcal{D}$ is written as

$$D = \prod_{k \geq 1, a \in \mathbf{Z}_r} (k,a)^{D_{k,a}},$$

and is called a *divisor.* We set $(0,a) = 1$, the unit element in $\mathcal{D}$. A divisor D is called positive if $D_{k,a} \geq 0$ for all k, a. We denote by $\mathcal{D}_+$ the set of positive divisors and by $\mathbf{Z}[\mathcal{D}_+]$ the $\mathbf{Z}$-free module generated by the positive divisors. For $D \in \mathcal{D}_+$ we use the following notations:

$$\sharp(D) = \sum_{k,a} D_{k,a}, \qquad n(D) = \max\{\, k \mid D_{k,a} > 0 \text{ for some } a \,\},$$

$$L(D,k) = \{\, a \in \mathbf{Z}_r \mid D_{k,a} = 0 \,\},$$
$$N(D,k) = \{\, a \in \mathbf{Z}_r \mid D_{k,a} > 0, D_{k,a-1} = 0\}.$$

We set formally $L(D,0) = \mathbf{Z}_r$. Let us denote by $\mathfrak{b}(D)$ the Young diagram of signature $[n^{d_n} \cdots 2^{d_2} 1^{d_1}]$ where $n = n(D)$ and $d_j = \sum_a D_{j,a}$. We also set

$$\mathfrak{b}_i(D) = \mathfrak{b}\Big(\prod_{k \geq i, a \in \mathbf{Z}_r} (k-i+1,a)^{D_{k,a}} \Big).$$

We write $\mathfrak{b}(D) < \mathfrak{b}(D')$ if and only if there exists an integer i such that $\mathfrak{b}_i(D) \subsetneq \mathfrak{b}_i(D')$ in the usual sense of inclusion.

$$\begin{array}{|c|c|c|}
\hline
0 & 1 & 2 \\
\hline
0 & 1 & 2 \\
\hline
2 \\
\cline{1-1}
1 \\
\cline{1-1}
\end{array}$$

Fig. 6.1 The divisor $(3,2)^2(1,2)(1,1)$.

Definition 6.1. A divisor D is called *reduced* if and only if it is positive and $L(D,k) \neq \phi$ for any k.

Proposition 6.2. *Let D be a reduced divisor, and let $n = n(D)$. Then we have the following alternative:*

Case 1. *There exists $a \in \mathbf{Z}_r$ such that*

$$a \in N(D,n), \qquad L(D,n-1) \neq \{\, a-1 \,\}. \tag{6.1}$$

Case 2. *There exists $a \in \mathbf{Z}_r$ and an integer k such that*

$$\begin{aligned}
1 \leq k &\leq n-1, \\
N(D,n) &= \{\, a \,\}, \\
L(D,j) &= \{\, a-1 \,\} \qquad \textit{for} \quad k \leq j \leq n-1, \\
L(D,k-1) &\neq \{\, a-1 \,\}.
\end{aligned}$$

Proof. If it is not Case 1, then there exists $a \in \mathbf{Z}_r$ such that

$$N(D,n) = \{\, a \,\}, \qquad L(D,n-1) = \{\, a-1 \,\}.$$

Take the smallest integer k such that $L(D,j) = \{\, a-1 \,\}$ for $k \leq j \leq n-1$. Then Case 2 is valid. □

Let us follow the notation of Proposition 6.2. Define $k = n$ in Case 1. If we set

$$D' = D \cdot (k-1, a-1)/(k,a),$$

then D' is reduced. We write $D \xrightarrow{ka} D'$ to indicate this relation, and call it a reduction from D to D' occurring at (k,a). In Case 1 there may be more than one choice of a. In Case 2 the choice of k, a is unique. Note that $\sharp(D') = \sharp(D) - 1$. Therefore, by repeating the above procedure we get a sequence

$$D = D^{(0)} \xrightarrow{k_0 a_0} D^{(1)} \xrightarrow{k_1 a_1} \cdots \xrightarrow{k_{d-1} a_{d-1}} D^{(d)} = 0 \tag{6.2}$$

where $d = \sharp(D)$. We set

$$A = (a_0, a_1, \cdots, a_{d-1}) \in (\mathbf{Z}_r)^d$$

and write $D \longrightarrow A$ if A is obtained from D in this way.

Let us examine the sequence (6.2) more closely.

Lemma 6.3. *Suppose that $D \xrightarrow{ka} D'$. Then we have $D_{n,a} > 0$.*

Proof. This is immediate from Lemma 6.4 below. □

Lemma 6.4. *Suppose that $D \xrightarrow{ka} D' \xrightarrow{k'a'} D''$ and $k < n(D)$. Then $k' = k$ or $k+1$ and $a' = a$.*

Proof. The reduction $D \xrightarrow{ka} D'$ is Case 2. If $D_{ka} > 1$ then Case 2 occurs for D' with the same k and a. Therefore $k' = k$ and $a' = a$. If $D_{ka} = 1$ and $k \leq n-2$, then Case 2 occurs for D' with $k' = k+1$ and $a' = a$. If $D_{ka} = 1$ and $k = n-1$, then (6.1) is valid for D' with a being the unique choice. Therefore $k' = k+1\,(= n)$ and $a' = a$. □

Proposition 6.5. *Consider the sequence (6.2). It contains a divisor $D^{(t)}$ such that*

$$D^{(t)}_{j,a} = D_{j+1,a+1} \qquad \text{for} \quad j \geq k_0, \forall a. \tag{6.3}$$

We choose t to be the smallest integer with this property. Then we have

$$\begin{aligned} & a_{i'} \neq a_i - 1 \qquad \text{if} \quad 0 \leq i < i' \leq t-1, \\ & k_t \leq k_0. \end{aligned} \tag{6.4}$$

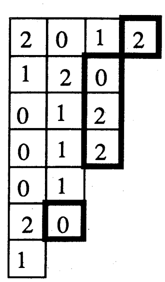


Fig. 6.2 Reduction from D to $D^{(t)}$.

Proof. We set $n = n(D), a = a_0$ and $k = k_0$ for short.

Suppose that $k = n$. Then we choose t to be the smallest integer such that $b_n(D^{(t)}) = 0$. The conditions (6.3) and (6.4) are obvious. Suppose that $0 \leq i < i' \leq t-1$. Since $a_i \in N(D^{(i)}, n)$, we have $D^{(i)}_{n,a_i-1} = 0$. The order $i < i'$ implies $D^{(i)}_{n,a_i-1} \geq D^{(i')}_{n,a_i-1} = 0$. From Lemma 6.3 we have $D^{(i')}_{n,a_{i'}} > 0$. Therefore $a_{i'}$ is not equal to $a_i - 1$.

Next consider the case $k < n$. Then we have

$$L(D,n) = \{\, a-1, a-2, \cdots, a-s \,\} \quad (1 \leq s \leq r-1). \tag{6.5}$$

Now let us start again from scratch assuming (6.2) and (6.5). Let i_0 be the smallest integer such that $L(D,j) = \{a-1\}$ for $i_0 \leq j \leq n-1$. If there is no such j, we set $i_0 = n$. (In fact i_0 is equal to k.) Let t_1 be the largest integer such that $a_j = a$ for $0 \leq j \leq t_1 - 1$, and set $D_1 = D^{(t_1)}$. From Lemma 6.3 it is easy to see that

$$D_1 = D \prod_{j=k}^{n} \left((j-1, a-1)/(j,a)\right)^{D_{j,a}}.$$

We refer to the subsequence of (6.2) from D to D_1 as an a-cycle. We consider two different cases. First, suppose that $s \neq r-1$. Then $n(D_1)$ is equal to n and the following are valid.

$$\begin{aligned}
L(D_1, n) &= \{\, a, a-1, \cdots, a-s \,\}, \\
N(D_1, n) &= \{\, a+1 \,\}, \\
L(D_1, j) &= \{\, a \,\} \qquad \text{for} \quad k \leq j \leq n-1.
\end{aligned}$$

If we denote by i_1 the smallest integer such that $L(D_1, j) = \{\, a \,\}$ for $i_1 \leq j \leq n-1$, then $i_1 \leq i_0 = k$ and Case 2 is valid for D_1. Thus, we return to a similar situation with D, i_0, a replaced by $D_1, i_1, a+1$. Therefore there exists an $(a+1)$-cycle starting from D_1 and ending at, say D_2. By repeating this process we reach the second case; we now suppose that $s = r-1$ in (6.5). In this case $n(D_1) = n-1$, and

$$L(D_1, j) = \{\, a \,\} \qquad i_0 \leq j \leq n-1.$$

Therefore, there exists an $(a+1)$-cycle starting from D_1 such that the first reduction occurs at $(i_1, a+1)$ with $i_1 \leq i_0$. In conclusion, a succession of cycles, the a-cycle from D to D_1, the $(a+1)$-cycle from D_1 to D_2, etc., continues to the $(a-2)$-cycle from D_{r-2} to D_{r-1}. Note also that $i_0 \geq i_1 \geq \cdots \geq i_{r-2}$. The choice $D^{(t)} = D_{r-1}$ meets the requirement of Proposition 6.5. $\square$

From the above construction it is easy to see the following.

Lemma 6.6. *Suppose that for $b \in \mathbf{Z}_r$ there exists an integer i satisfying*

$$0 \leq i \leq t-1, \quad a_i = b, \quad k_i = k_0 - 1.$$

Then we have

$$D^{(t)}_{k_0-1,b-1} = D_{k_0,b}.$$

6.2. Reconstruction of divisors

So far we have considered the reduction process $D \longrightarrow A$. Conversely, given a sequence $A = (a_0, a_1, \cdots, a_{d-1}) \in (\mathbf{Z}_r)^d$, we construct a positive divisor D' as follows. Consider a subsequence $I = \big(I(0), \cdots, I(p-1)\big)$ of $(0, 1, \cdots, d-1)$ such that $a_{I(j+1)} = a_{I(j)} - 1$ for $0 \leq j \leq p-2$. We refer to such I as a p-string. Given a decomposition of $(0, 1, \cdots, d-1)$ into strings

$$(0, 1, \cdots, d-1) = \coprod_{\text{disjoint}} I_\lambda, \tag{6.6}$$

a positive divisor is correspondingly defined by

$$D' = \prod_\lambda (l_\lambda, b_\lambda),$$

where l_λ is the length of I_λ and $b_\lambda = a_{I_\lambda(0)}$. We write $A \Longrightarrow D'$ indicating this construction of D' from A. It is obvious that $D \longrightarrow A$ implies $A \Longrightarrow D$. The aim of this paragraph is to show that D can be reconstructed from A as the 'maximal' divisor (see Proposition 6.8 for the precise meaning).

Lemma 6.7. *Assume that $D \longrightarrow A \Longrightarrow D'$. We retain the notation in the proof of Proposition* 6.5. *Then we have the following alternative:*

Case 1. $\quad D_{j,a} = D'_{j,a} \quad$ *for* $\; j \geq k_0, \; \forall a.$

Case 2. *There exists an integer s $(k_0 \leq s \leq n(D))$ such that*

$$D_{j,a} = D'_{j,a} \quad \textit{for} \;\; j \geq s+1, \; \forall a,$$
$$D_{s,a} \geq D'_{s,a} \quad \textit{for} \;\; \forall a,$$
$$b_s(D) \neq b_s(D').$$

Proof. We prove by induction on $n(D)$. If $n(D) = 1$, it is clear that Case 1 holds. Suppose $n(D) > 1$ and set $\overline{D} = D^{(t)}$. Then we have $n(\overline{D}) = n(D) - 1$. Set $\overline{A} = (a_t, a_{t+1}, \cdots, a_{d-1})$. Then we have $\overline{D} \longrightarrow \overline{A}$. We now construct a reduced divisor $\overline{D}'$ satisfying $\overline{A} \Longrightarrow \overline{D}'$ by contracting $A \Longrightarrow D'$. Consider the decomposition (6.6). From (6.4) it follows that each $j (0 \le j \le t-1)$ must be the top of one of the strings, say I_{λ_j}, i.e., $j = I_{\lambda_j}(0)$. Set

$$\overline{D}' = D' \cdot \prod_{\lambda=\lambda_0,\cdots,\lambda_{t-1}} (l_\lambda - 1, b_\lambda - 1)/(l_\lambda, b_\lambda).$$

Then we have $\overline{A} \Longrightarrow \overline{D}'$. Now we are to show that either Case 1 or Case 2 holds. By the induction hypothesis one of the following is valid.

Case 1′. $\overline{D}_{j,a} = \overline{D}'_{j,a}$ for $j \ge i_1$, $\forall a$.

Case 2′. There exists an integer $\overline{s}$ $(i_1 \le \overline{s} \le n(D) - 1)$ such that

$$\overline{D}_{j,a} = \overline{D}'_{j,a} \quad \text{for} \quad j \ge \overline{s} + 1, \ \forall a,$$
$$\overline{D}_{\overline{s},a} \ge \overline{D}'_{\overline{s},a} \quad \text{for} \quad \forall a,$$
$$b_{\overline{s}}(\overline{D}) \ne b_{\overline{s}}(\overline{D}').$$

Note that $i_1 \le i_0 = k_0$. From the construction of $\overline{D}$ we have

$$D_{j,a} = \overline{D}_{j-1,a-1} \quad \text{for} \quad j \ge k_0 + 1. \tag{6.7}$$

Suppose that Case 2′ occurs for $\overline{s} \ge k_0$. We refer to this case as Case 2′A. Let s_0 be the smallest integer such that for any $j \ge s_0 + 1$ and $a \in \mathbf{Z}_r$

$$\sharp\{\, i \mid 0 \le i \le t-1, (k_{\lambda_i}, b_{\lambda_i}) = (j, a) \,\} = D'_{j,a}.$$

Then using (6.7) we can deduce that Case 2 occurs for $s = \max(\overline{s}+1, s_0)$. Suppose that it is not Case 2′A. Then we have $\overline{D}_{j,a} = \overline{D}'_{j,a}$ for $j \ge k_0, \forall a$ and

$$\overline{D}_{k_0-1,a} \ge \overline{D}'_{k_0-1,a} \quad \text{for} \quad \forall a. \tag{6.8}$$

If $s_0 \ge k_0 + 1$, then Case 2 occurs for $s = s_0$. If $s_0 \le k_0$, then we have $D_{j,a} = D'_{j,a}$ for $j \ge k_0 + 1, \forall a$. Using (6.8) and Lemma 6.6 we can deduce that $D_{k_0,a} \ge D'_{k_0,a}$ for $\forall a$. Thus Case 1 or Case 2 with $s = k_0$ occurs. □

Proposition 6.8. *Assume that $D \longrightarrow A \Longrightarrow D'$. Then we have $b(D) > b(D')$ or $D = D'$.*

Proof. We prove by induction on $\sharp(D)$. Suppose that $b(D) \leq b(D')$. Then Case 1 of Lemma 6.7 must hold. We are to show that $D = D'$.

Let t_1 be as in the proof of Proposition 6.5. First we show the following.

(i) Let I_λ be one of the strings of (6.6) satisfying $0 \leq I_\lambda(0) \leq t_1 - 1$. We refer to such I_λ as a string of the first class. Then $l_\lambda \geq k_0$.

Assume that $l_\lambda < k_0$. Define $A^{(1)}$ and D'' by

$$A^{(1)} = (a_1, \cdots, a_{d-1}),$$
$$D'' = D' \cdot (l_\lambda - 1, a_0 - 1)/(l_\lambda, a_0).$$

Then we have $D^{(1)} \longrightarrow A^{(1)} \Longrightarrow D''$. On the other hand, we have

$$D^{(1)} = D \cdot (k_0 - 1, a_0 - 1)/(k_0, a_0). \tag{6.9}$$

By the induction hypothesis $b(D^{(1)}) \geq b(D'')$. From this follows that $b(D) > b(D')$, since $l_\lambda < k_0$. This is a contradiction.

Next we prove

(ii) There exists a k_0-string of the first class.

Let n_j be the number of j-strings of the first class. For $j \geq k_0$ we have $n_j \leq D'_{j,a_0} = D_{j,a_0}$. On the other hand from (i) we have $\sum_{j \geq k_0} n_j = t_1 = \sum_{j \geq k_0} D_{j,a_0}$. Therefore we get $n_{k_0} = D_{k_0,a_0} > 0$.

Set

$$D''' = D' \cdot (k_0 - 1, a_0 - 1)/(k_0, a_0).$$

From (ii) we have $D^{(1)} \longrightarrow A^{(1)} \Longrightarrow D'''$. The assertion follows from (6.9) by induction. $\square$

6.3. Paths and divisors

Let $\mathcal{F}^*_{\mathbf{Z}}[\gamma]$ denote the $\mathbf{Z}$-span of the base vectors $\{\langle m|\}_{m \in \mathcal{M}[\gamma]}$. We define a $\mathbf{Z}$-linear map $\Delta : \mathcal{F}^*_{\mathbf{Z}}[\Lambda] = \mathcal{F}^*_{\mathbf{Z}}[\gamma_1] \otimes_{\mathbf{Z}} \cdots \otimes_{\mathbf{Z}} \mathcal{F}^*_{\mathbf{Z}}[\gamma_l] \longrightarrow \mathbf{Z}[\mathcal{D}_+]$ by

$$\Delta\big(\langle m_1| \otimes \cdots \otimes \langle m_l|\big) = \prod_{j<0, i=1,\ldots,l} \big(m_i(j) - j - \gamma_i, m_i(j) \bmod r\big).$$

Lemma 6.9. *Suppose that $\Delta\big(\langle m_1| \otimes \cdots \otimes \langle m_l|\big) = \prod (k,a)^{D_{k,a}}$. Then we have*

$$D_{k,a} = \sum_{i=1}^{l} \sum_{m_i(k-1) < b < m_i(k)} \delta^{(r)}_{ab}, \tag{6.10}$$

where $\delta_{ab}^{(r)}$ *is defined in* (5.2).

Proof. If $j < 0$ and $m_i(j) < m_i(0)$, then $m_i(j) - j - \gamma_i = 0$. If $k \geq 1, j < 0$ and $m_i(k-1) < m_i(j) < m_i(k)$, then we can show inductively that $m_i(j) - j - \gamma_i = k$. Therefore we obtain (6.10). □

Proposition 6.10. *The divisor* $\Delta(\xi_\eta)$ *is reduced.*

Proof. Recall the construction of the highest lift. From the proof of Proposition 5.2 we have

$$\begin{array}{ccc} m_1(k-1) & < & m_1(k) \\ \wedge\mid & & \wedge\mid \\ \vdots & & \vdots \\ \wedge\mid & & \wedge\mid \\ m_l(k-1) & < & m_l(k) \\ \wedge\mid & & \wedge\mid \\ m_1(k-1)+r & < & m_1(k)+r. \end{array}$$

Furthermore for each k there exists an integer $s = s(k)$ $(2 \leq s \leq l+1)$ such that $m_s(k-1) \geq m_{s-1}(k)$. (We set $m_{l+1} = m_1[r]$.) Therefore there exists $\alpha \in \mathbf{Z}$ such that $m_{s-1}(k) \leq \alpha \leq m_s(k-1)$. For this α we have $\Delta(\xi_\eta)_{k,\alpha \bmod r} = 0$. □

Let v_Λ be the highest weight vector in $L(\Lambda)$. We choose a reduction $\Delta(\xi_\eta) \longrightarrow A = (a_0, a_1, \cdots, a_{d-1}) \in (\mathbf{Z}_r)^d$, and set

$$v_\eta = f_{a_0} \cdots f_{a_{d-1}} v_\Lambda.$$

Notice that for a given path η the choice of A and hence of v_η is not unique in general; we choose one and fix it for all. In Section 5 we constructed the vector ξ_η in $L(\Lambda)^*_\mu$ for a Λ-path η of weight μ. Here we show that the vectors $\{\, \xi_\eta \mid \eta \in \mathcal{P}(\Lambda)_\mu \,\}$ are independent and span the vector space $L(\Lambda)^*_\mu$. Because of Proposition 5.11 it suffices to show that the pairing between $\{\, \xi_\eta \mid \eta \in \mathcal{P}(\Lambda)_\mu \,\}$ and $\{\, v_\eta \mid \eta \in \mathcal{P}(\Lambda)_\mu \,\}$ is non-degenerate. We shall see that the matrix $\left(\xi_\eta(v_{\eta'})\right)_{\lambda_\eta = \mu}$ is triangular with positive diagonal entries (Theorem 6.14 below).

An element of $\mathcal{F}^*_{\mathbf{Z}}[\Lambda]$ is called a monomial if it is of the form $\langle m_1| \otimes \cdots \otimes \langle m_l|$. An element $M \in \mathcal{F}^*_{\mathbf{Z}}[\Lambda]$ is called positive if it is a sum of monomials. For $M, N \in \mathcal{F}^*_{\mathbf{Z}}[\Lambda]$ we write $M \geq N$ if and only if $M - N$ is positive.

An element $X \in \mathbf{Z}[\mathcal{D}_+]$ is called positive if it is a sum of positive divisors. For $X, Y \in \mathbf{Z}[\mathcal{D}_+]$ we write $X \geq Y$ if and only if $X - Y$ is positive.

For $a \in \mathbf{Z}_r$ we define a $\mathbf{Z}$-linear derivation $F_a : \mathbf{Z}[\mathcal{D}_+] \longrightarrow \mathbf{Z}[\mathcal{D}_+]$ by

$$F_a\big((k,b)\big) = \delta_{ab}(k-1,a-1) \qquad \text{for} \quad k \geq 1.$$

Lemma 6.11. *With the above notations we have*

$$\Delta\big((\langle m_1| \otimes \cdots \otimes \langle m_l|)f_a\big) \leq F_a\Big(\Delta\big(\langle m_1| \otimes \cdots \otimes \langle m_l|\big)\Big). \tag{6.11}$$

Proof. Let m be a Maya diagram. Then we have

$$\langle m|f_a = \sum_{m'} \langle m'| \tag{6.12}$$

where the sum is over m' such that there exist $j' \geq 0$ and $j < 0$ satisfying $m(j') + 1 = m(j) \equiv a \bmod r$ and

$$\begin{aligned} m'(k) &= m(k) \qquad && \text{if } k \neq j, j', \\ &= m(j) && \text{if } k = j', \\ &= m(j') && \text{if } k = j. \end{aligned} \tag{6.13}$$

Therefore we have

(6.14)

$$\begin{aligned} &\Delta\big((\langle m_1| \otimes \cdots \otimes \langle m_l|)f_a\big)/\Delta\big(\langle m_1| \otimes \cdots \otimes \langle m_l|\big) \\ &= \sum_{j<0,k=1,\ldots,l} \delta^{(r)}_{m_k(j),a}\big(1 - \delta_{m_k(j),m_k(j-1)+1}\big)\frac{\big(m_k(j) - j - \gamma_k - 1, a-1\big)}{\big(m_k(j) - j - \gamma_k, a\big)}. \end{aligned}$$

On the other hand

$$\begin{aligned} F_a\Big(\Delta\big(\langle m_1| \otimes \cdots \otimes \langle m_l|\big)\Big)&/\Delta\big(\langle m_1| \otimes \cdots \otimes \langle m_l|\big) \\ &= \sum_{j<0,k=1,\ldots,l} \delta^{(r)}_{m_k(j),a}\frac{\big(m_k(j) - j - \gamma_k - 1, a-1\big)}{\big(m_k(j) - j - \gamma_k, a\big)}. \end{aligned} \tag{6.15}$$

Comparing (6.14) and (6.15) we obtain (6.11). □

Lemma 6.12. *Let* $A = (a_0, a_1, \cdots, a_{d-1}) \in (\mathbf{Z}_r)^d$ *and let* $D \in \mathcal{D}_+$ *be a positive divisor such that* $\natural(D) = d$. *Then*

$$F_{a_{d-1}} \cdots F_{a_0}(D) \neq 0$$

if and only if $A \Longrightarrow D$.

Proof. This is immediate from the definitions. □

From Lemma 6.11 and Lemma 6.12 follows

Corollary 6.13. *Let η and η' be Λ-paths. Suppose that $v_\eta = f_{a_0} \cdots f_{a_{d-1}} v_\Lambda$. Then $\xi_{\eta'}(v_\eta) = 0$ unless $A = (a_0, a_1, \cdots, a_{d-1}) \Longrightarrow \Delta(\xi_{\eta'})$.*

Finally we have

Theorem 6.14. *For all Λ-path η we have $\xi_\eta(v_\eta) > 0$. If η' is a Λ-path such that $b(\Delta(\xi_{\eta'})) \leq b(\Delta(\xi_\eta))$ and $\eta' \neq \eta$, then we have $\xi_\eta(v_{\eta'}) = 0$.*

Proof. The latter half is obvious from Proposition 6.8 and Corollary 6.13. The first half follows from the following lemma. □

Lemma 6.15. *Suppose that $D = \Delta(\langle m_1| \otimes \cdots \otimes \langle m_l|)$ is reduced. If $D \xrightarrow{ka} D'$, then there exists $\langle m_1'| \otimes \cdots \otimes \langle m_l'| \in \mathcal{F}_{\mathbf{Z}}^*[\Lambda]$ satisfying*

$$\Delta(\langle m_1'| \otimes \cdots \otimes \langle m_l'|) = D' \tag{6.16a}$$

$$\langle m_1'| \otimes \cdots \otimes \langle m_l'| \leq (\langle m_1| \otimes \cdots \otimes \langle m_l|) f_a. \tag{6.16b}$$

Proof. Note that $a \in N(D, k)$. Therefore, from (6.12) we have $i (1 \leq i \leq l)$ and $j (< 0)$ such that

$$m_i(j) - j - \gamma_i = k,$$
$$m_i(j) \equiv a \bmod r,$$
$$m_i(j-1) \neq m_i(j) - 1.$$

Suppose that $m_i(j') = m_i(j) - 1$. Then we have $j' \geq 0$. Define m_i' by (6.13) and set $m_k' = m_k$ for $k \neq i$. Then (6.16) is valid. □

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