Onsager's Star-Triangle Equation:
Master Key to Integrability

Helen Au-Yang and Jacques H. H. Perk

Abstract.

After a brief review of the history of the star-triangle equation, we shall illustrate its importance with a few results for the two-dimensional Ising model and its generalization to the chiral Potts model. We shall discuss how the recent solutions in terms of higher-genus Riemann surfaces emerge. We shall finish with some further results for the quantities of interest in these new models. More related work is presented in the talks by Profs. Baxter and McCoy.

§1. Introduction

The present paper, based on two talks at the Taniguchi symposium, is intended to be a review of recent work on higher-genus solutions [1–14] of the star-triangle equations of Onsager [15–22], (alias the Yang-Baxter equations [23–27] as we shall explain below). The discovery of such solutions came about as an unexpected reward for much hard labor. The first result was found in collaboration with B.M. McCoy, S. Tang, and M.L. Yan [1]. The most general result, up to now, has been found in collaboration with R.J. Baxter [5].

Historically, we started out with the belief that there should be integrable models solving the quantum Lax pair equations [28,29] for chiral Potts models, where "chiral" refers to the lifting of a symmetry condition used in [28–33]. This belief was strengthened when we discovered a solution for the selfdual three-state chiral Potts model [1] and when we noted the remarkable properties of certain quantum Potts chains in the works of Howes, Kadanoff, and den Nijs [34] and von Gehlen and Rittenberg [35]. But, when we tried to verify that the transfer matrix of our model was part of a commuting family à la Baxter [26], we noted an inconsistency of treating the boundary of the system in the method of [28,29]. This we could only resolve by showing [4] the equivalence of the

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existence of the quantum version à la [28,29] of the Lax pair [36] and the
existence of a solution of the star-triangle equations à la Onsager [15],
see also [114]. Therefore, we shall restrict ourselves in the following to
these star-triangle equations, as they turn out to be somewhat easier to
use for our models.

Having a selfdual solution, we next pursued the systematic solution
of the three-state nonselfdual case, using brute force algebra, and to
our surprise a plane curve appeared, which was shown to have genus 10
[1] by the mathematicians C.H. Sah and M. Kuga. We continued by
studying four- and five-state selfdual cases making extensive use of the
algebraic manipulation routine SMP. At one point, we (see especially
Tang's thesis [12]) had to factorize complex polynomials that were 80
screens long, in order to solve the five-state case. The results were
leading us to Fermat curves [2–4,12], which we then conjectured to come
out of the general N-state selfdual case [3]. This conjecture was proved
when we, in collaboration with R.J. Baxter, derived a general solution for
the N-state nonselfdual case in terms of the intersection of two Fermat
cylinders [5]. Much of the text shall be devoted to explaining how we
arrived at that solution [5] and how we proved it. We shall say not much
more about the earlier work [1–4], as the lengthy unpublished details of
that work are not needed to derive the most general solution given in
[5].

In spite of the genus not being 0 or 1, progress has been made in
exactly evaluating properties of these models [6–11], also in collaboration
with G. Albertini. The most general results for the three-state case have
been obtained from an exact nonlinear identity [9] for the transfer matrix
that was conjectured on the basis of results of Bazhanov and Reshetikhin
[37] for the restricted SOS model and verified numerically. We expect
that a proof can be given along the lines of [37] and that a generalization
to N states exists. The work of Prof. Baxter [8] is strong support for
this expectation.

In this paper we shall make first some general remarks about the
star-triangle equation in section 2. We shall continue by explaining how
we arrived at the solution of [5] in section 3, postponing the proof to
the appendix. In section 4, we shall make some further remarks on
the thermodynamic quantities obtained so far. For the most recent
developments, we refer to the two talks by Prof. Baxter [10] and Prof.
McCoy [11].

§2. Star-triangle equation

The star-triangle transformation, which is also known as wye-delta

$(Y - \Delta)$, upsilon-delta $(\Upsilon - \Delta)$, or tau-pi $(T - \Pi)$ transformation, originates from the theory of electrical networks. It was apparently first introduced by Kennelly in 1899 [38]. The further development of this theory, including the generalization to "star-mesh transformations", has been reviewed in Starr's textbook [39]. Onsager [15–17] observed in 1944 the importance of star-triangle transformations for the theory of the two-dimensional Ising model. He noted that it relates the Ising model on a honeycomb lattice with one on a triangular lattice and that it is intimately related to the elliptic function parametrization that he used. He also noted [16] that it leads to the commutation of transfer matrices and spin-chain hamiltonians, a fact that he utilized in his derivation of the spontaneous magnetization that he announced in 1948 [16,40], but of which Yang first published a complete derivation in 1952 [41] using a very different method. The star-triangle transformation has since been used for other purposes, see e.g. [18–22].

Onsager's star-triangle equations, defining the above transformation, have since been generalized in a straightforward fashion to many other models, including the chiral Potts model [1–5] to be defined below. We shall see that the Onsager equations are equivalent to a checkerboard generalization of the equations of triangles of McGuire [23] expressing the conditions of factorizability of the S-matrix [23,42], which are also known under the name "Yang-Baxter equations" [24–27]. In fact, Yang observed in 1967 that McGuire's 1964 equations provide the consistency conditions in a nested Bethe-Ansatz approach [24,25,43–45]. Baxter used the same conditions in his famous solution of the eight-vertex model [21,26,27]. A third, "IRF-model" or "interaction-round-a-face" model, version of these equations, which we shall call the equations of hexagons, has been introduced by Baxter in connection with his solution of the hard-hexagon model [21,46]. Earlier work of Baxter, e.g. [47], seems to come close to introducing this version already. The above history is summarized in Fig. 1.
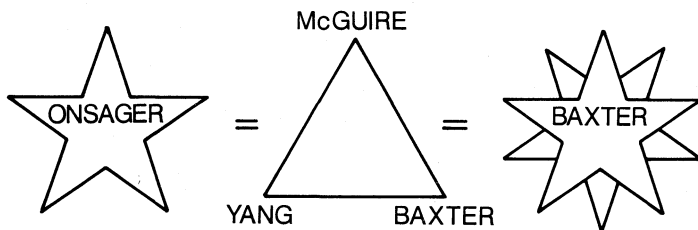

Fig. 1. The inventors of the three versions of the star-triangle relation.

We shall see that the difference between the three points of view is just a matter of language, albeit that the translation between the spin-model, S-matrix or vertex-model, and the IRF-model languages could be rather academic, with one specific choice being preferred depending on the given situation. Still there is one fundamental equation and this star-triangle equation has become a master key to integrability. Many other papers in these proceedings make use of them and the existing literature is enormous. There are works on one-dimensional quantum spin systems [16,21,48–52], on quantum inverse scattering methods (QISM) [53–55], and many general reviews exist, see e.g. [21,53,56–59] and several other works in these proceedings.

At this point, we may also note that in a special limit the braid group relations [60,61] appear. Especially since the work of Jones [62], the star-triangle equation has become of interest in the mathematical theory of knots, for example through the Temperley-Lieb algebra [63]. It is fair to say that Lieb's work [64,65] on the Bethe Ansatz [44,66,67] has played a seminal role. The Bose gas model [64], (that can be shown to be a continuum limit [68] of a spin chain), precedes McGuire's work [23], whereas the ice model work [65], which was guided in turn by work of Yang and Yang [67], precedes Baxter's solution of the eight-vertex model [26].

Let us start by defining a spin model. Such a lattice model is defined on a graph G with vertices v and edges $e \equiv (v_1, v_2)$ being allowed pairs of vertices. A state of the model is given by a function σ_v assigning to each vertex v a variable. In the case that σ_v is ± 1, we have the Ising model. We shall say that we have a true spin model if the interaction energy or Hamiltonian is a sum of edge contributions

$$(2.1) \qquad E = \sum_{(v_1, v_2)} \varepsilon(\sigma_{v_1}, \sigma_{v_2}),$$

where $\varepsilon(\sigma_{v_1}, \sigma_{v_2})$ is the interaction energy of a pair. For the Ising model this reduces to

$$(2.2) \qquad E = - \sum_{e=(v_1, v_2) \in G} J_e \sigma_{v_1} \sigma_{v_2},$$

where J_e is the interaction energy for the edge (or bond) e. The state $\{\sigma_v \mid v \in G\}$ has a statistical weight, called the Boltzmann weight,

$$(2.3) \qquad \rho \equiv e^{-E/k_B T} \Big/ \sum_{\text{all states}} e^{-E/k_B T},$$

where T is the temperature and k_B the Boltzmann constant relating temperature and energy scales. The quantities of interest are the partition function

$$(2.4) \qquad Z = \sum_{\text{all states}} e^{-E/k_B T},$$

the free energy per site

$$(2.5) \qquad f = -\frac{k_B}{|G|} \log Z,$$

$|G|$ being the number of sites or vertices in G, and the n-point correlation function

$$(2.6) \qquad \left\langle \sigma_{v_1} \sigma_{v_2} \cdots \sigma_{v_n} \right\rangle = \sum_{\text{all states}} \rho \, \sigma_{v_1} \sigma_{v_2} \cdots \sigma_{v_n}.$$

We are especially interested in the limit of an infinite system, where $|G| \to \infty$ in some regular fashion. This so-called thermodynamic limit can be shown to exist rigorously in many cases of interest, but that is beyond the scope of this paper. Let us just remark that the boundary grows with a smaller power of the size of the system than does the bulk, leading to the convergence of f. These considerations also hold for the case that

$$(2.7) \qquad \sigma_v = 1, \cdots, N_v,$$

where the number of states at a given site N_v can vary. Here we have only assumed that the number of states for each site v is finite, so that we can enumerate them. Now we can associate Boltzmann factors

$$(2.8) \qquad W(\sigma_v, \sigma_{v'}) = e^{-\epsilon(\sigma_v, \sigma_{v'})/k_B T}$$

to each edge, see Fig. 2, and the partition function Z of (2.4) is a sum over all states of the product of all Boltzmann factors for all edges G.

Having defined spin models, let us continue with vertex models [56]. Now the state variables σ_e are associated with the edges of the graph. The partition function is now the sum over states of the product over vertices of all Boltzmann factors, see also Fig. 2, $\omega(\sigma_e, \sigma_{e'}, \sigma_{e''}, \sigma_{e'''})$, associated with the edges incident at the vertex. Here we have assumed a four-valent graph G for the sake of notation. The number of states,

$$(2.9) \qquad \sigma_e = 1, \cdots, N_e,$$
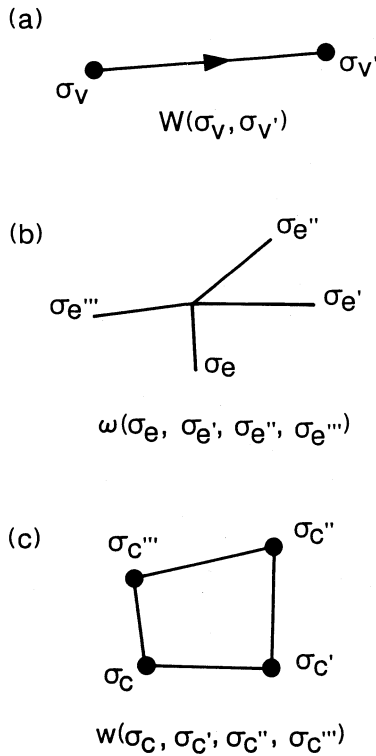
can also vary from edge to edge.

(a)

$$W(\sigma_v, \sigma_{v'})$$

(b)

$$\omega(\sigma_e, \sigma_{e'}, \sigma_{e''}, \sigma_{e'''})$$

(c)

$$w(\sigma_c, \sigma_{c'}, \sigma_{c''}, \sigma_{c'''})$$

Fig. 2.   Boltzmann weights: (a) Spin language, (b) Vertex
language, (c) IRF language.

Finally, we can also assign our state variables σ_c to the corners of the faces of a graph, taking on values

(2.10) $\sigma_c = 1, \cdots, N_c.$

The IRF model is defined by a total Boltzmann weight that is the product of the Boltzmann factors for each face, see Fig. 2, $w(\sigma_c, \sigma_{c'}, \sigma_{c''}, \sigma_{c'''})$. This type of model emerges quite naturally if one thinks of finite range interactions. If one groups the state variables together in "block-spin" variables such that state variables that share interactions appear either in the same block-spin or in block-spin variables belonging to a common face, one arrives at an IRF model. One may naively think that the IRF language is the more general one. But this is not so.

In fact, with an IRF model on a bipartite graph we can associate an equivalent vertex model on an even-valenced graph. (The condition of bipartite is not serious, as we can insert trivial dummy spins $\sigma_e \equiv 1$.) A general mapping was given in [69], see Fig. 3, where for a pair of adjacent corner states a, b of the IRF model we take an edge state $\{a, b\}$ on the dual graph. Not all vertex model configurations are globally consistent. We get a one-to-one map between IRF states and allowed vertex model states, so the equivalence requires

(2.11) ω(allowed vertex state) $= w$(IRF state),

ω(not allowed) $= 0$.

This mapping goes, if N_c is fixed, from an N_c-state model to an $N_c{}^2$-model. In certain cases there are more economic equivalences [70,71], e.g. relating the eight-vertex model with an Ising model with four-spin interaction, but these maps are not always one-to-one. The above map has the feature that it transforms the equation of hexagons in the equation of triangles. But there are also other maps [72].
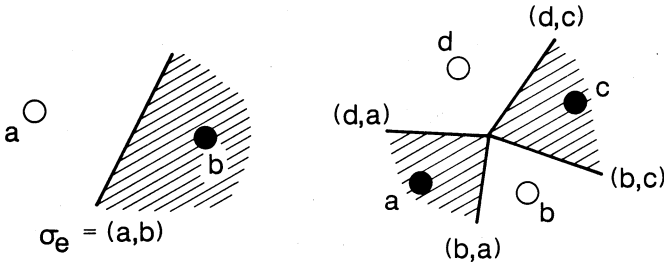

Fig. 3.   Mapping from IRF model to vertex model.

Secondly, we can also map a vertex model on an even-valenced graph to a spin model. The faces of the vertex model are two-colorable in black and white, see also [47,73]. With the black faces we shall associate a spin variable that is just the set of all surrounding edge states, starting to count counterclockwise, say, at 12 o'clock, see Fig. 4. Again, this is not a very elegant map, but it transforms the equation of triangles in to an Onsager star-triangle equation. The map is one-to-one between allowed spin model states and vertex model states, that is

(2.12) W(allowed spin state) $= \omega$(vertex state),
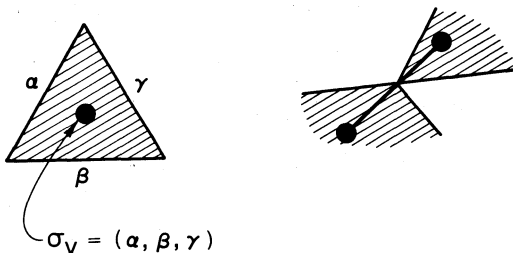
W(not allowed) $= 0$.

Fig. 4.   Mapping from vertex model to spin model.

Finally, the mapping from spin model to IRF model can be con-
structed by putting trivial spins $\sigma_c \equiv 1$ on the dual lattice and identi-
fying $w = W$, see Fig. 5. Another way is to put two spin models, one
on the lattice and one on the dual lattice, on top of each other and tak-
ing product weights. What all these mappings show, is that the space
of exactly solvable models is quite complicated and that there is some
self-similar structure for the "space" of infinite integrable systems.
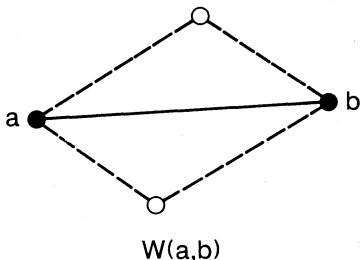

W(a,b)

Fig. 5.   Mapping from spin model to IRF model, adding
         trivial spins on dual lattice.

The last transformation does also map the star-triangle equation
[15] nicely. But now we are led to checkerboard generalizations of the
McGuire-Yang-Baxter equation. This is illustrated in Fig. 6. In the
first line of the figure the external sites have a given configuration, but
the left-hand side has a summation over the configuration of the internal
spin. The equation, to be written out algebraically in the next section,
expresses the equivalence of the partition functions for a star and for a
triangle. In principle, a normalization factor R may be needed which
should be independent of the three external spins.

The second line of the figure expresses the equivalence of two triple
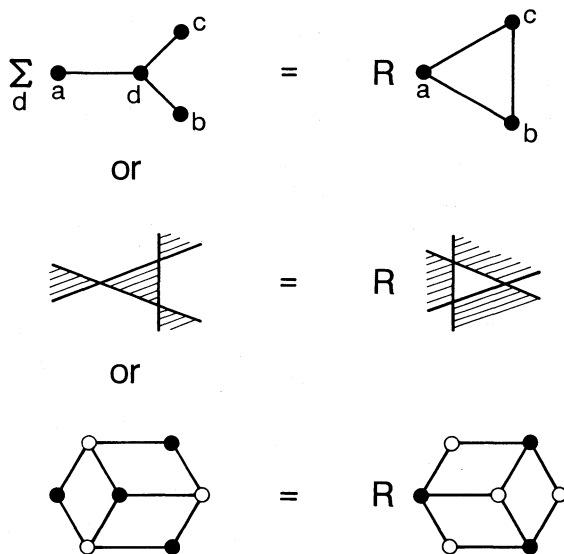sums, as there are three internal edges. Now the Boltzmann weight fac-

Fig. 6. Star-triangle equation (checkerboard Yang-Baxter equation) in spin, vertex, and IRF languages.

tors of the vertices may depend on how the faces are shaded. In the S-matrix picture of McGuire [23], the lines correspond to world lines of particles. The equation expresses the factorization of the three-body S-matrix in two-body contributions and that the order of the collisions does not affect the final outcome. McGuire realized [23] that this condition is all you need for the consistency of factoring the n-body S-matrix. Yang [24,25] needed precisely this in his nested Bethe Ansatz calculation. Baxter [26,27] also discovered this principle, which he has called Z-invariance as it expresses an invariance of the partition function Z [47]. In Fig. 6, we have an additional staggering, which is not part of the usual McGuire-Yang-Baxter equation. In the S-matrix picture one requires (Galilean or) Lorentz invariance. So with the particles are associated (momenta or) rapidities u_i, where i labels the line. These are exchanged in the collisions and in the field-theory case one imposes that the weights depend on the differences of the rapidities, corresponding to the conservation of energy-momentum in the collisions.

One can arrange it in such a way that the second line of Fig. 6 becomes an operator equation acting on a tensor product space, $V_1 \otimes V_2 \otimes \cdots \otimes V_N$. The vertex weights then become operators acting on

two consecutive factors in the tensor product, say the operators $R_{i,i+1}$ acting on $V_i \otimes V_{i+1}$. In this notation, for the difference variable case, the star-triangle equation becomes

$$(2.13) \quad R_{i,i+1}(u_1 - u_2)R_{i+1,i+2}(u_1 - u_3)R_{i,i+1}(u_2 - u_3)$$
$$= R_{i+1,i+2}(u_2 - u_3)R_{i,i+1}(u_1 - u_3)R_{i+1,i+2}(u_1 - u_2).$$

If the limit $u_i - u_j \to \infty$ makes sense, so that the "spectral parameters" u_i drop out of the problem, then these become the braid relations

$$(2.14) \qquad\qquad R_{i,i+1}R_{i+1,i+2}R_{i,i+1}$$
$$= R_{i+1,i+2}R_{i,i+1}R_{i+1,i+2},$$

$$(2.15) \qquad\qquad [R_{i,i+1}, R_{j,j+1}] = 0, \quad \text{if} \quad |i - j| \geq 2.$$

Finally, the last line of Fig. 6 expresses the equality of two single sums, as the central spin state is summed over. The six external spin states are arbitrary but fixed; they serve as boundary conditions for the partition functions given by the pictures. This equation is a checkerboard variation of Baxter's condition [21,46].

§3. Chiral Potts model

Before giving definitions it may be good to explain roughly why we became interested in the chiral Potts model. The Potts model [21,74] is a generalization of the Ising model and all kinds of interesting results exist for it, but not yet for its correlation functions. For the Ising model, on the contrary, many results for the correlations exist. One particularly intriguing class of Ising models is the Z-invariant inhomogeneous class of models [47,75]. As an example, (see [76], also for further references), we derived that the two-point correlation function at the critical temperature is given as the ratio of two determinants

$$(3.1) \qquad\qquad \langle \sigma\sigma' \rangle = \frac{\det_{1 \leq j,k \leq l} f(u_{l+j} - u_k)}{\det_{1 \leq j,k \leq l} g(u_{l+j} - u_k)}.$$

where the u's are rapidity-type variables. This expression becomes more complicated in the uniform case! In more general Z-invariant models we may not expect such a nice form as (3.1), although we may want to anticipate infinite determinants like (3.1) for particular generalizations of the Ising model. However, Z-invariance should teach us something more about correlations in "exactly solvable models". We also expect

that Z-invariance ties in with the lattice generalizations of conformal invariance and deformation theory, which are also mutually related [77], at least for the Ising case.

Looking for other models we did not want to make too many assumptions at the beginning. We did want to give up parity invariance, as studies on Potts-like models assuming it, see e.g. [28–30,78], do not seem to find enough solvable cases. Next, we did not want to a priori assume that the solutions of the star-triangle equations, that we were after, would always be expressible in terms of functions of differences of line-variables (or rapidities). In fact, this is also not the case for Shastry's model [79]. Therefore, we decided to study the chiral Potts (or chiral clock) model for which there is much interest in connection with commensurate-incommensurate phase transitions [34,35,80–107] and which is also introduced as a possible explanation [108] of the "ripple phase" in lipid-bilayer biological membranes. There are many approximate calculations using perturbation theory, low-temperature expansions, mean-field methods, random-walk methods, renormalization-group techniques, Bethe-lattice calculations, exact diagonalization of finite-size transfer matrices, studies of related quantum chains, and Monte Carlo computer runs. But it is clear that exact results would resolve some of the controversies, as the models are sufficiently complex to have features that escape even refined studies. However, we were particularly struck by the results in [34,35], that show features very similar to Ising model features even though the model is definitely more complicated. Therefore, we were encouraged to find further exact solutions.
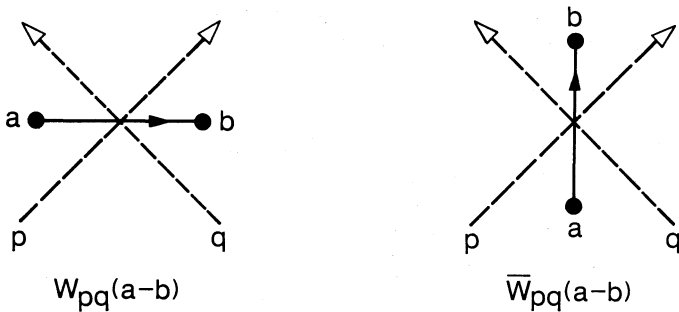

Fig. 7. Boltzmann weights for the chiral Potts model.

The chiral Potts model is a spin model in the sense of the previous

section. We have state variables or "spins" at each site (or vertex) of the graph, that take on values $a, b = 1, 2, \cdots, N$, (mod N). In order to fully specify the model, it is sufficient to give the Boltzmann weights associated with the pair interactions, see Fig. 7. We assume that there are two types of such weights W and $\overline{W}$, that on a square lattice would correspond to horizontal and vertical interactions. We assume also that the weights only depend on the difference modulo N of the two spin states a and b. But the chiral character is connected with the breakdown of parity, $W(a - b) \not\equiv W(b - a)$, which can only occur if $N \neq 2$. At this point we shall assume more structure, that we did not assume originally [1–4], but that we arrived at on the basis of much work [5]. We shall assume that there are oriented straight lines, (which we shall somewhat improperly call "rapidity lines"), on the medial graph that is obtained by connecting the middles of all pairs of edges that are incident to a single vertex and share a common face. No more than two rapidity lines meet at any given point. These lines carry variables $p, q, \cdots$, chosen from some manifold, and arrows specifying their orientations. The choice of W or $\overline{W}$ depends on the relative positioning of the edge with respect to the orientations of the two rapidity lines, see Fig. 7. Here the identification is invariant under rotation of the figure.
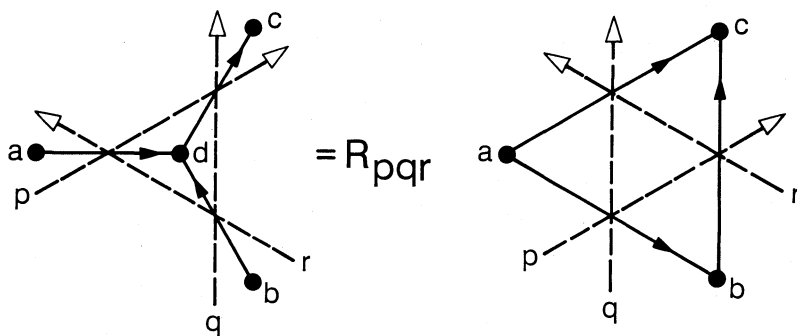

Fig. 8.   Star-triangle relations for the chiral Potts model. The three "rapidity lines" are also indicated using dashed lines.

Onsager's star-triangle equation is, in this notation, graphically ex-

pressed by Fig. 8 and algebraically by

$$(3.2) \qquad \sum_{d=1}^{N} \overline{W}_{qr}(b-d) W_{pr}(a-d) \overline{W}_{pq}(d-c)$$

$$= R_{pqr} W_{pq}(a-b) \overline{W}_{pr}(b-c) W_{qr}(a-c).$$

Here we would have had to find six independent sets of weights if we would not have made the assumption on the line-variable dependence. Now we are looking for two functions depending on two line variables and one state variable $a - b$.

At this point we are going to assume a product form for the weights as was found first by Fateev and Zamolodchikov [30], inspired by the "fishnet diagram" paper [73]. At the workshop it was explained by Zamolodchikov that the result of [30] arose after making a q-basic generalization of a result in [73]. We arrived, after much labor [1–3], at a more general product form for the selfdual model, where $W(n)$ and $\overline{W}(n)$ are each other's discrete Fourier transform. In [5] we found a general solution also for the nonselfdual case, on the basis of comparing the product form and the result for the Ising case [21,75].

Hence, we are first going to analyze the products

$$(3.3) \qquad \boxed{f(n) = f(0) \prod_{j=1}^{n} \left(\frac{\omega x_1 - x_2 \omega^j}{x_4 - x_3 \omega^j} \right),}$$

where

$$(3.4) \qquad \omega \equiv e^{2\pi i/N}, \quad i \equiv \sqrt{-1}.$$

For consistency we must demand that

$$(3.5) \qquad f(n+N) = f(n),$$

leading to

$$\frac{x_1^N - x_2^N}{x_4^N - x_3^N} = 1,$$

or

$$(3.6) \qquad \boxed{x_1^N + x_3^N = x_2^N + x_4^N.}$$

We see that the product form (3.3) is closely linked with the Fermat curves of [2,3]. The Fourier transform of (3.3) is defined as

$$(3.7) \qquad \widehat{f}(m) \equiv \sum_{n=0}^{N-1} \omega^{mn} f(n).$$

This can be calculated most easily by rewriting (3.3) as a linear recurrence relation

$$(x_4 - x_3\omega^n)f(n) = (\omega x_1 - x_2\omega^n)f(n-1),$$

leading via

(3.8) $\qquad x_4\widehat{f}(m) - x_3\widehat{f}(m+1) = \omega^{m+1}x_1\widehat{f}(m) - \omega^{m+1}x_2\widehat{f}(m+1)$

to

(3.9) $\qquad \boxed{\dfrac{\widehat{f}(n)}{\widehat{f}(0)} = \prod_{j=1}^{n}\left(\dfrac{x_4 - x_1\omega^j}{x_3 - x_2\omega^j}\right),}$

This is of the same form as (3.3), replacing

(3.10) $\qquad x_p \to x_{p-1}, \quad (x_0 \equiv \omega^{-1}x_4).$

The second ingredient is the two-dimensional Ising model, for which $\omega = -1$ and the spin variable a is now replaced by a spin $\sigma = \omega^a = \pm 1$. For the Boltzmann weight factors of horizontal and vertical edges we now have

(3.11) $\qquad W(\sigma, \sigma') = e^{J\sigma\sigma'/k_BT}, \quad \overline{W}(\sigma, \sigma') = e^{\overline{J}\sigma\sigma'/k_BT},$

or

(3.12) $\qquad \dfrac{W(1)}{W(0)} = e^{-2J/k_BT}, \quad \dfrac{\overline{W}(1)}{\overline{W}(0)} = e^{-2\overline{J}/k_BT}.$

If we utilize Onsager's elliptic function parametrization [15], (see in particular also p.18, table I, of [75]), we have the elliptic moduli

(3.13) $\qquad k = \sqrt{1 - k'^2}, \quad \frac{1}{k'} = \sinh\frac{2J}{k_BT}\sinh\frac{2\overline{J}}{k_BT}.$

For the weights we then can read off

(3.14) $\qquad \dfrac{W(1)}{W(0)} = \mathrm{nc}(p-q) - \mathrm{sc}(p-q) = \dfrac{\mathrm{cn}(q) - \mathrm{sn}(p)\mathrm{dn}(q)}{\mathrm{cn}(p) - \mathrm{dn}(p)\mathrm{sn}(q)},$

(3.15) $\qquad \dfrac{\overline{W}(1)}{\overline{W}(0)} = \dfrac{\mathrm{ds}(p-q) - \mathrm{cs}(p-q)}{k'} = \dfrac{k'\left(\mathrm{sn}(p) - \mathrm{sn}(q)\right)}{\mathrm{dn}(p)\mathrm{cn}(q) + \mathrm{cn}(p)\mathrm{dn}(q)},$

where we are using Glaisher's notation, (ds≡dn/sn, etc.). The last steps in these identities can be easily verified evaluating

$$\left(\frac{\partial}{\partial p} + \frac{\partial}{\partial q}\right)\text{RHS} = 0,$$

for the right-hand sides of both (3.14) and (3.15), which shows that they are both functions of $p - q$. Therefore, we can rewrite (3.14), (3.15) as

(3.16)
$$\frac{W_{pq}(1)}{W_{pq}(0)} = \frac{d_p b_q + a_p c_q}{b_p d_q + c_p a_q},$$

(3.17)
$$\frac{\overline{W}_{pq}(1)}{\overline{W}_{pq}(0)} = \frac{-a_p d_q + d_p a_q}{c_p b_q + b_p c_q},$$

where

(3.18a)
$$(a_p, b_p, c_p, d_p) = \left(-\,\mathrm{H}(p), \mathrm{H}_1(p), \Theta_1(p), \Theta(p)\right),$$

(3.18a)
$$(a_q, b_q, c_q, d_q) = \left(-\,\mathrm{H}(q), \mathrm{H}_1(q), \Theta_1(q), \Theta(q)\right).$$

In (3.18) we just have the Jacobi theta functions. Note that in (3.16) and (3.17) the quantities in the right-hand side do only depend on either p or q. We have separated the p and q dependences.

Comparing the Ising case with the selfdual case, having the crazy idea that the product can be guessed on the basis of the case with at most one factor, we arrived at

(3.19)
$$\boxed{\frac{W_{pq}(n)}{W_{pq}(0)} = \prod_{j=1}^{n}\left(\frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j}\right),}$$

(3.20)
$$\boxed{\frac{\overline{W}_{pq}(n)}{\overline{W}_{pq}(0)} = \prod_{j=1}^{n}\left(\frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j}\right),}$$

where

(3.4)
$$\omega \equiv e^{2\pi\sqrt{-1}/N}.$$

Here, the parameters

(3.21)
$$\mathbf{x}_p \equiv (a_p, b_p, c_p, d_p), \quad \mathbf{x}_q \equiv (a_q, b_q, c_q, d_q),$$

are restricted by the consistency conditions (3.5), which gives the periodicity requirements

(3.22) $$W_{pq}(N+n) = W_{pq}(n),$$

(3.23) $$\overline{W}_{pq}(N+n) = \overline{W}_{pq}(n),$$

giving

(3.24) $$\frac{a_p^N \pm b_p^N}{c_p^N \pm d_p^N} = \lambda_\pm = \text{ independent of } p, q, r, \cdots.$$

Fixing, for all p's, the relative normalization of the a_p, b_p with respect to c_p, d_p by $\lambda_+\lambda_- = -1$, we get

(3.25) $$\boxed{a_p^N + k'b_p^N = kd_p^N,}$$

(3.26) $$\boxed{k'a_p^N + b_p^N = kc_p^N,}$$

(3.27) $$\boxed{ka_p^N + k'c_p^N = d_p^N,}$$

(3.28) $$\boxed{kb_p^N + k'd_p^N = c_p^N,}$$

where

(3.29) $$k^2 + k'^2 = 1.$$

(In the notation of [1], we must identify $k = \lambda'$, $k' = \lambda$.) Any two of the equations (3.25)–(3.28) determine the other two. The equations describe a complex curve. This is the intersection of two "Fermat surfaces", generalizing the elliptic curve for the Ising model ($N = 2$). The genus of this curve is

(3.30) $$g = N^2(N-2) + 1,$$

as was found by C.H. Sah using the Riemann-Hurwitz formula and in [7] by counting the abelian integrals of the first kind. For $N = 2$ we have $g = 1$, as we just said. But, for $N = 3$ we already have $g = 10$ [1], for $N = 4$, $g = 33$, etc. The selfdual limit is given by

(3.31) $$k' = 1, \quad c_p = d_p = 1, \quad a_p^N + b_p^N = I.$$

Here the choice $c_p = d_p = 1$ can be made without loss of generality once the choice $c_p = d_p$ is made. This is the Fermat curve for $I \neq 0$, which has genus $g = (N-1)(N-2)/2$. The genus zero case of $I = 0$ is the case of Fateev and Zamolodchikov [30].

Our proof that (3.19), (3.20) indeed solves the star-triangle equation (3.2) is given in the appendix. What remains to be determined is the factor R_{pqr} in (3.2). We were again guided by the Ising case. Using the abbreviations $pr \to 1$, $pq \to 2$, $qr \to 3$,

$$(3.32) \qquad K_n \equiv \frac{J_n}{k_B T}, \quad \overline{K}_n \equiv \frac{\overline{J}_n}{k_B T}, \quad \text{for } n = 1, 2, 3,$$

we arrive, for example from Baxter's textbook [21], at

$$(3.33) \qquad R^2 = \frac{f_2 f_3}{f_1},$$

where

$$
\begin{aligned}
(3.34) \qquad f_n &\equiv 2 \sinh 2\overline{K}_n \\
&= 2 \cosh \overline{K}_n \cdot 2 \sinh \overline{K}_n \\
&= \frac{\left(e^{\overline{K}_n} + e^{-\overline{K}_n} \right) \left(e^{\overline{K}_n} - e^{-\overline{K}_n} \right)}{e^{K_n} \qquad e^{-K_n}} \\
&= \frac{\widehat{\overline{W}}_n(0) \widehat{\overline{W}}_n(1)}{W_n(0) W_n(1)}
\end{aligned}
$$

where apparently trivial denominators are written down so that the formula also applies to other normalizations.

In order to generalize this we need to make a couple of replacements,

$$2 \to N, \quad \prod_{m=0}^{1} \to \prod_{m=0}^{N-1}.$$

The conjectured generalization of (3.33), (3.34) is

$$(3.35) \qquad \boxed{R_{pqr} = \frac{f_{pq} f_{qr}}{f_{pr}},}$$

$$(3.36) \qquad \boxed{f_{pq} = \left[\prod_{m=1}^{N} \widehat{\overline{W}}_{pq}(m) \bigg/ \prod_{m=1}^{N} W_{pq}(m) \right]^{1/N},}$$

with

$$(3.37) \qquad \widehat{\overline{W}}_{pq}(m) \equiv \sum_{k=1}^{N} \omega^{mk} \overline{W}_{pq}(k).$$

This result (3.36) exhibits the full Z_N symmetry; that means that under $m \to m + \text{const} \pmod{N}$ the answer remains the same. Next, besides the Ising case $N = 2$, also the selfdual case works out with $R = \sqrt{N}$ for $\widehat{\overline{W}}(m) = \sqrt{N} W(m)$. Finally, using FORTRAN about 50 cases chosen at random worked out to the numerical accuracy of the computer. The conjecture has been proved, noting that (3.2) is a product identity for diagonal and cyclic matrices and taking the determinant [115].

     There is a domain, where all the Boltzmann weights of the solution (3.19), (3.20) are real and positive. There are three, equivalent conditions (modulo some powers of ω):

$$(3.38) \qquad a_p^* c_p = \omega^{\frac{1}{2}} b_p^* d_p,$$

$$(3.39) \qquad |a_p| = |d_p|,$$

$$(3.40) \qquad |b_p| = |c_p|.$$

For real positive weights we can make the choice

$$(3.41) \qquad \frac{b_p}{c_p} = \omega^{\frac{1}{2}} e^{i\theta_p}, \quad 0 < \theta_p < \theta_q < \theta_r < \frac{\pi}{N},$$

for $0 < k, k' < 1$. At this point, it may be good to note that in general the reality of the Boltzmann weights does not coincide with the hermiticity of the associated quantum chain, just like what happened with the original observation of McCoy and Wu [48,57] for the relation of the six-vertex model in electric fields and the Heisenberg-Ising quantum spin chain. In that case the thermodynamic limit of the spin chain with Dzyaloshinsky-Moriya interactions is trivially equivalent to one without [109], whereas the six-vertex model in fields is very different from the one without [56].

     We conclude this section by noting that the Riemann surface of "rapidities" given by (3.25)–(3.28) has a large number of automorphisms. To be more precise, there are $4N^3$ automorphisms, generated by:

$$(3.42) \quad \text{R} : \mathbf{x}_p = (a, b, c, d) \quad \longmapsto \quad \text{R}\mathbf{x}_p = (b, \omega a, d, c),$$

$$(3.43) \quad \text{S} : \mathbf{x}_p = (a, b, c, d) \quad \longmapsto \quad \text{S}\mathbf{x}_p = (\omega^{-\frac{1}{2}} c, d, a, \omega^{-\frac{1}{2}} b),$$

$$(3.44) \quad \text{T} : \mathbf{x}_p = (a, b, c, d) \quad \longmapsto \quad \text{T}\mathbf{x}_p = (\omega a, b, \omega c, d),$$

$$(3.45) \quad \text{U} : \mathbf{x}_p = (a, b, c, d) \quad \longmapsto \quad \text{U}\mathbf{x}_p = (\omega a, b, c, d),$$

The $4N^2$ automorphisms generated by R, S, and T preserve reality of the Boltzmann weights. They play a role, for example, in connection with twisted boundary conditions. Consider the case that a square lattice of size $L \times L$ is wrapped around a torus. There are N^2 such twisted boundary conditions, as one may pick up an arbitrary power of ω for each of the two directions that one can go around the torus once. One way to generate such phase factors is to apply a T or a T', where

$$(3.46) \qquad \mathrm{T}' : \mathbf{x}_p = (a, b, c, d) \quad \longmapsto \quad \mathrm{T}'\mathbf{x}_p = (\omega a, b, c, \omega d),$$

to just one of the rapidity lines.

To conclude this section, we would like to state that it is our belief that the above exactly solvable model describes the wetting line for the 3-state chiral clock model. There has been much controversy in the literature about this and the nature of the phase diagram [106,107]. So a complete analysis of our model should resolve some of these issues as it gives exact solutions for a submanifold of the phase diagram.
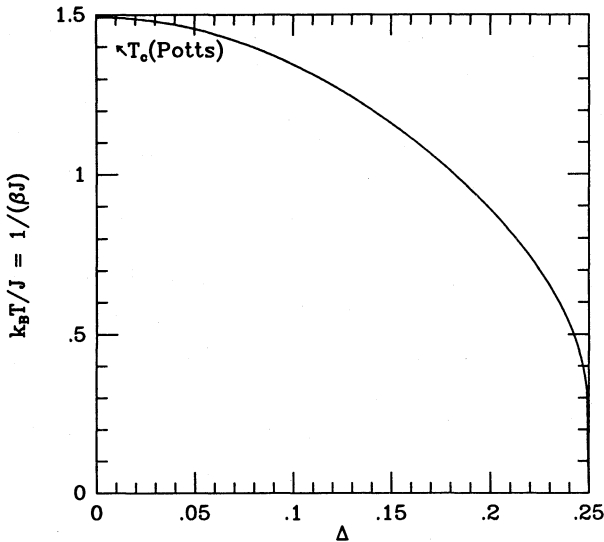

Fig. 9. The exact solution manifold for the 3-state chiral clock model with $K = \overline{K}$ and $\overline{\Delta} = 0$.

To be more specific, we have analyzed the three-state chiral clock

model, which is described by

$$(3.47) \qquad E/k_B T = -K \sum_{i,j} \cos \frac{2\pi}{3}(n_{i,j} - n_{i,j+1} + \Delta)$$

$$- \overline{K} \sum_{i,j} \cos \frac{2\pi}{3}(n_{i,j} - n_{i+i,j} + \overline{\Delta})$$

with $n_{i,j} = 0, 1, 2$. Within the four-dimensional parameter space given by $(K, \overline{K}, \Delta, \overline{\Delta})$, we find a three-dimensional surface on which the model is integrable. In other words, if $K, \overline{K}, \Delta, \overline{\Delta}$ satisfy

$$(3.48) \qquad \qquad \Omega \overline{\Omega} = 1,$$

where

$$(3.49) \qquad \Omega \equiv \frac{2(\sinh \delta_0 + \sinh \delta_1 + \sinh \delta_2)}{e^{-\delta_0} + e^{-\delta_1} + e^{-\delta_2} - 3},$$

$$\delta_n \equiv 3K \cos \frac{2\pi}{3}(n + \Delta), \quad (n = 0, 1, 2),$$

(and two similar equations with bars), then the star-triangle equations are satisfied. For the case considered by Huse and Ostlund [80,81], we have $\overline{\Delta} = 0$, so we find

$$(3.50) \qquad \qquad \Omega = \overline{\Omega}^{-1} = \frac{2 + e^{-\frac{3}{2}\overline{K}}}{2 \sinh(\frac{3}{2}\overline{K})}.$$

For $T \to 0$, we have $\overline{K} \to \infty$ and $\Omega \to 0$, so that we find $|\Delta| = \frac{1}{4}$ corresponding to $\delta_0 = \frac{3}{2}K = -\delta_1$, $\delta_2 = 0$. In the other extreme case $\Delta = 0$, the equation reduces to

$$(3.51) \qquad \qquad (e^{\frac{3}{2}K_c} - 1)(e^{\frac{3}{2}\overline{K}_c} - 1) = 3,$$

which is the critical point in the 3-state Potts model. It is not difficult to show that solutions only exist in the intervals

$$(3.52) \qquad |\Delta - \ell| \leq \frac{1}{4}, \quad \ell = \text{integer}, \quad K_c \leq K < \infty.$$

The literature mostly studied the case $K = \overline{K}$, $\overline{\Delta} = 0$, for which the integrable case is a line starting at $T = 0$, $\Delta = \pm\frac{1}{4}$, ending up at the critical Potts point $\exp(\frac{3}{2}K) = \sqrt{3} + 1$, $\Delta = 0$, see Fig. 9. These are

precisely the end points of the wetting line of [89], and we find the same agreement with [89] for the diagonal case $\Delta = \overline{\Delta}$.

However, a precise analysis of the small T-expansion does not give the same leading corrections as in [89], both for the row and for the diagonal case. We do not precisely know what we are to conclude. If the result of [89] is accurate, then we have a very special line close to the wetting line. On the other hand it is very tempting to believe that we have the exact formula for the wetting line in which case the approximation of [89] is reasonable but not as good as in the two-state Ising case. To be precise, for the diagonal case both the authors of [89] and we find

$$(3.53) \qquad \Delta_w(T) = \frac{1}{4} - \frac{3}{2\pi}\arcsin(\frac{k_B T \delta}{3J}) + \cdots,$$

where [89] has $\delta = \log\frac{3}{2}$ compared to our $\delta = \log 2$. For the row case they have

$$(3.54) \qquad \delta = 3z^2 + \cdots,$$

or

$$(3.55) \qquad \Delta_w(T) = \frac{1}{4} - \frac{3k_B T z^2}{2\pi J} + \cdots,$$

with

$$(3.56) \qquad z \equiv \exp(-3J/2k_B T),$$

whereas we have

$$(3.57) \qquad \delta = 2z + \cdots$$

or

$$(3.58) \qquad \Delta_w(T) = \frac{1}{4} - \frac{k_B T z}{\pi J} + \cdots.$$

So there seems to be an interesting question of either to explain the nature of the exact-solution manifold or to improve upon the approximation made in [89].

The physics of the two-dimensional model is very different from the physics of the quantum spin chain Hamiltonian as considered by Howes, Kadanoff and Den Nijs [34], where there is a wetting line going through their "Lifshitz point". In their model we find commuting transfer matrices both on the selfdual (or critical) line and on the wetting line where

Onsager's operator algebra [15] of A's and G's even holds, see also the next section. But the Boltzmann weights are necessarily complex. As we have said before, we have in general physical cases either real positive Boltzmann weights in the classical two-dimensional model or a Hermitian associated spin chain Hamiltonian, but not both, leading to very different physics.

§4. Integrable quantum spin chain

It is well-known that with a lattice model as found in the previous section corresponds a Hamiltonian limit of a quantum spin chain [50,51]. More precisely, the diagonal-to-diagonal transfer matrix of our model behaves as

$$(4.1)\qquad T_{pq} = 1 + \mathrm{const}\cdot\text{``}(q-p)\text{''}\cdot \mathrm{H} + \cdots, \quad \text{for } q \to p.$$

Explicitly, we find [1–5]

$$(4.2)\qquad \mathrm{H} = \sum_{j=1}^{L}\sum_{n=1}^{N-1}\big[\overline{\alpha}_n(X_j)^n + \alpha_n(Z_j Z_{j+1}^\dagger)^n\big],$$

where

$$(4.3\mathrm{a})\qquad X_j \equiv I_N \otimes I_N \otimes \cdots \otimes X \otimes \cdots \otimes I_N,$$

$$(4.3\mathrm{b})\qquad Z_j \equiv I_N \otimes I_N \otimes \cdots \otimes Z \otimes \cdots \otimes I_N,$$

with I_N the $N \times N$ identity matrix and at the jth place inserted the $N \times N$ generalizations of the Pauli matrices

$$(4.4\mathrm{a})\qquad X = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

$$(4.4\mathrm{b})\qquad Z = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \omega & 0 & \dots & 0 & 0 \\ 0 & 0 & \omega^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \omega^{N-2} & 0 \\ 0 & 0 & 0 & \dots & 0 & \omega^{N-1} \end{pmatrix}.$$

Furthermore, we have

$$(4.5) \qquad \alpha_k = \frac{e^{i(2k-N)\phi/N}}{\sin(\pi k/N)}, \qquad \overline{\alpha}_k = \lambda \frac{e^{i(2k-N)\overline{\phi}/N}}{\sin(\pi k/N)},$$

$$(4.6) \qquad \cos\phi = \lambda \cos\overline{\phi},$$

and

$$(4.7) \qquad e^{2i\phi/N} = \omega^{\frac{1}{2}} \frac{a_p c_p}{b_p d_p}, \qquad e^{2i\overline{\phi}/N} = \omega^{\frac{1}{2}} \frac{a_p d_p}{b_p c_p}, \qquad \lambda \equiv k'.$$

The Hamiltonian (4.2) is Hermitian for real ϕ and $\overline{\phi}$.

Of particular interest is the "superintegrable" point defined by

$$(4.8) \qquad \phi = \overline{\phi} = \frac{\pi}{2},$$

or

$$(4.9) \qquad a_p c_p = b_p d_p, \qquad a_p d_p = b_p c_p.$$

For the superintegrable point, the Hamiltonian (4.2), rewritten as

$$(4.10) \qquad \mathrm{H} = A_0 - \lambda A_1,$$

where A_0 contains the α-terms and A_1 the $\overline{\alpha}$-terms, satisfies the Dolan-Grady condition [110]

$$(4.11) \qquad [A_0, [A_0, [A_0, A_1]]] = \mathrm{const}\, [A_0, A_1].$$

In fact, von Gehlen and Rittenberg [35] used this very criterion to construct spin chain Hamiltonians for the superintegrable case. One can prove that this condition means that one can introduce operators A_n and G_n, satisfying the Onsager algebra [15,4]

$$(4.12) \qquad \begin{aligned} [A_j, A_k] &= 4G_{j-k}, \\ [G_m, A_l] &= 2A_{l+m} - 2A_{l-m}, \\ [G_j, G_k] &= 0. \end{aligned}$$

We find the same algebra as in the Ising model, except for the boundary condition $A_{j+2L} = A_j$ which does not hold now. But we do not have an Ising model, but a kind of higher-spin representation. There are many sectors within which the behavior of the spectrum is additive. This behavior does only hold at the superintegrable point.

Numerical studies of the "ground-state sector", which is also discussed in the talks of Baxter and McCoy [10,11], have shown that the eigenvalues of the superintegrable Hamiltonian are given by

$$(4.13) \qquad E_L = -N \sum_{j=1}^{m} \sqrt{1 + \lambda^2 - \lambda a_j}$$
$$- \{2Q + [(N-1)L - 2Q - Nm](1+\lambda)\},$$

where

$$(4.14) \qquad m \equiv [\frac{(N-1)L - Q}{N}],$$

with $[\cdots]$ denoting integer part, L the length of the spin chain, Q the eigenvalue of the spin-shift operator

$$(4.15) \qquad \prod_{j=1}^{L} X_j = \omega^Q,$$

and a_j being m constants, independent of λ, for some given m.

We can now exploit the theorem that, if

$$(4.16) \qquad \lim_{L\to\infty} \frac{E_L}{L} = \sum_{n=0}^{\infty} c_n \lambda^n$$

and

$$(4.17) \qquad \frac{E_L}{L} = \sum_{n=0}^{\infty} c_{n,L} \lambda^n,$$

then

$$(4.18) \qquad c_{n,L} = c_n \quad \text{for } n < L.$$

A heuristic proof of this fact is the following: In the perturbation expansion in powers of λ for a chain with cyclic boundary conditions, you do not see the fact that the chain is finite until you have enough powers to bring you around the system, involving L interactions $j \to j+1$.

The above form (4.13) and this theorem together give a recurrent algorithm for calculating c_n's and E_L's for increasing values of n and L. For example, the simplest sector is $Q = N - 1$, where m is the smallest. All Q-sectors have the same λ-series, due to the asymptotic degeneracy

of the eigenvalues for $\lambda \leq 1$ as $L \to \infty$. We can start with $L = 2$ for which $m = 0$. Then, from (4.13) and the little theorem, we find

$$(4.19) \qquad -\frac{E_2}{2} = N - 1 + O(\lambda^2).$$

For $L = 3$ we have $m = 1$, so we find

$$(4.20) \qquad -\frac{E_3}{3} = \frac{2N - 3 - \lambda + N\sqrt{1 + \lambda^2 - a_1\lambda}}{3}$$

$$= N - 1 - \frac{\lambda}{3} - \frac{Na_1\lambda}{6} + \cdots.$$

Comparing (4.19) and (4.20) we find

$$(4.21) \qquad a_1 = -\frac{2}{N}.$$

Expanding the so-obtained exact E_3 to one more order we find

$$(4.22) \qquad -\frac{E_3}{3} = (N - 1) + \frac{(N^2 - 1)}{6N}\lambda^2 + O(\lambda^3).$$

So we can continue to play the same game, writing an algebraic computer program and using self-submitting batchjobs. We have found [7] the small λ series coefficients upto λ^{88} for $N = 3$, λ^{62} for $N = 4$, λ^{60} for $N = 5$, and λ^{16} for $N = N$. This involves many selfconsistency checks of the form (4.13). Using a technique involving incomplete zeta functions, we extrapolated these results to $\lambda = 1$, finding, for $N = 3$,

$$(4.23) \qquad e_0 \equiv \lim_{L \to \infty} \frac{E_L}{L} = -2.497025859702117629\cdots,$$

$$\frac{E_L}{L} = e_0 - \sum_{k=0}^{``\infty"} u_k L^{-k-\frac{5}{2}},$$

$$u_0 = 1.11820671, \cdots.$$

These series were exactly evaluated by Baxter [8,10] and his results can be rewritten as

$$(4.24) \qquad \lim_{L \to \infty} \frac{E_L}{L} = -(1 + \lambda) \sum_{k=1}^{N-1} F\left(-\frac{1}{2}, \frac{k}{N}; 1; \frac{4\lambda}{(1 + \lambda)^2}\right),$$

where F is the hypergeometric function. In the selfdual case, $\lambda = k' = 1$, this reduces to

$$(4.25) \qquad \lim_{L \to \infty} \frac{E_L}{L} = -2 \sum_{k=1}^{N-1} \frac{\Gamma(\frac{3}{2} - \frac{k}{N})}{\Gamma(\frac{3}{2})\Gamma(1 - \frac{k}{N})}.$$

This result was also derived by Albertini, McCoy, and one of us [9,11] for the special case of $N = 3$. To be more specific, on the basis of a recent preprint (now published [37]) for the rSOS model, we arrived at a conjecture for a functional equation for the transfer matrix

$$(4.26) \qquad T_{p,q}T_{p,Rq}T_{p,R^2q} = e^{-iP}\{f^L_{p,Rq}f^L_{Rq,p}T_{p,q}$$
$$+ f^L_{p,q}f^L_{q,p}T_{p,R^2q} + f^L_{p,R^5q}f^L_{R^2q,p}T_{p,R^4q}\},$$

which we could verify numerically for chains up to length 7. This result is apparently true in general, not just for the superintegrable case.

The further analysis of this equation taught us that the result (4.25) is not the groundstate energy for sufficiently long chains [9,11], which is discussed in great detail in McCoy's talk [11]. In fact, for some value of λ energy-levels cross which is associated with a transition to an incommensurate state. The behavior is much like the one predicted by Pokrovsky and Talapov [111], see also [34].

For one other case, namely $\phi = \overline{\phi} = 0$ and $\lambda = 1$, is the groundstate energy explicitly known. Here the results of [29–33] can be further simplified to

$$(4.27) \qquad \lim_{L \to \infty} \frac{E_L}{L} = -\frac{2}{\pi}(N - \delta_{N,\text{odd}}) + \delta_{N,\text{even}}$$
$$- 2 \sum_{k=1}^{[N/2]} \Big[\operatorname{cosec}\frac{\pi k}{N} - \Big(1 - \frac{2k-1}{N}\Big)\tan\frac{\pi(2k-1)}{2N}\Big],$$

where the Kronecker delta symbols have the usual meaning of being one or zero, depending on whether equality of the indices holds or not.

Finally, we would like to remark that we also have a conjecture for the exact form of the order parameters, namely [7]

$$(4.28) \qquad \langle\sigma_0{}^n\rangle = (1 - k'^2)^{\beta_n}, \quad (k' = \lambda \le 1),$$

where

$$(4.29) \qquad \beta_n = \frac{n(N-n)}{2N^2}, \quad 1 \le n \le N - 1.$$

There is much supporting evidence for this conjecture. For the case $N = 2$, the Ising model, this is the result of Onsager and Yang [40,41]. Also for the eight-vertex model such a form appears [21]. Howes, Kadanoff, and den Nijs [34] found this result up to 16 terms in the $k' = \lambda$ expansion for the $N = 3$ quantum spin chain, and we found this to two nontrivial orders for general N. Henkel and Lacki [112] expanded the

sum $\sum \langle \sigma_0{}^n \rangle$ of the order parameters up to k'^6 and that result agrees. The critical exponent β_n is precisely the one predicted by conformal field theory and the rSOS model, see e.g. [113]. On the basis of Baxter's Z-invariance arguments [21,47] the $\langle \sigma_0{}^n \rangle$ can only depend on the modulus k'. Therefore, the result (4.28) should indeed be an exact result even though we do not yet know how to prove it. There is a lot more to be discovered about our new models and we hope to report further progress in the future.

Acknowledgements

Appendix

In this appendix we shall present a proof of the following theorem which we established in collaboration with Prof. R.J. Baxter [5]:

Theorem. *Let the Boltzmann weights be given by*

(A.1a)
$$\frac{W_{pq}(n)}{W_{pq}(0)} \equiv \prod_{j=1}^{n} \left(\frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j} \right),$$

(A.1b)
$$\frac{\overline{W}_{pq}(n)}{\overline{W}_{pq}(0)} \equiv \prod_{j=1}^{n} \left(\frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j} \right),$$

and

(A.2) $\quad \omega \equiv e^{2\pi i/N}, \qquad W_{pq}(N) = W_{pq}(0), \qquad \overline{W}_{pq}(N) = \overline{W}_{pq}(0).$

Then there exists a number $R \equiv R_{pqr}$ such that the star-triangle equation

$$
\text{(A.3)} \quad \sum_{d=1}^{N} \overline{W}_{qr}(b-d)W_{pr}(a-d)\overline{W}_{pq}(d-c)
$$

$$
= R_{pqr}W_{pq}(a-b)\overline{W}_{pr}(b-c)W_{qr}(a-c)
$$

holds.

Proof. Define the quantities

$$
\text{(A.4a)} \qquad V_{m,n} = V_{m,n}(p,q,r) \equiv \sum_{k=1}^{N} \omega^{nk}W_{pr}(m+k)\overline{W}_{qr}(k),
$$

$$
\text{(A.4b)} \qquad \overline{V}_{m,n} = \overline{V}_{m,n}(p,q,r) \equiv \sum_{k=1}^{N} \omega^{mk}\overline{W}_{pr}(k)W_{qr}(n+k),
$$

satisfying the property

$$
\text{(A.5)} \qquad \overline{V}_{m,n}(p,q,r) = V_{n,m}(q,p,r).
$$

(In the following we shall suppress the arguments of the $V_{m,n}$'s, as we shall assume them to be (p,q,r) throughout without permutations.) Then the star-triangle equation (A.3) becomes

$$
\text{(A.6)} \qquad V_{m,n}\widehat{\overline{W}}_{pq}(n) = R_{pqr}\overline{V}_{n,m}W_{pq}(m),
$$

where

$$
\text{(A.7)} \qquad \widehat{\overline{W}}_{pq}(m) \equiv \sum_{l=1}^{N} \omega^{ml}\overline{W}_{pq}(l).
$$

Note that there is no summation in eq. (A.6) which appeared first in

[1]. Indeed, we can verify successively that

$$(A.8) \quad V_{a-b,m}\widehat{\overline{W}}_{pq}(m)$$

$$= \sum_{k=1}^{N}\sum_{l=1}^{N} \omega^{m(k+l)} W_{pr}(a-b+k)\overline{W}_{qr}(k)\overline{W}_{pq}(l)$$

$$= \sum_{c=1}^{N}\sum_{d=1}^{N} \omega^{m(b-c)} W_{pr}(a-d)\overline{W}_{qr}(b-d)\overline{W}_{pq}(d-c)$$

$$\overset{(A.3)}{=} \sum_{c=1}^{N} \omega^{m(b-c)} R_{pqr} W_{pq}(a-b)\overline{W}_{pr}(b-c)W_{qr}(a-c)$$

$$= R_{pqr} W_{pq}(a-b)\overline{V}_{m,a-b}.$$

Secondly, we must note that the Fourier transform of

$$(A.9) \qquad \frac{f(n)}{f(0)} = \prod_{j=1}^{n}\left(\frac{\omega x_1 - x_2\omega^j}{x_4 - x_3\omega^j}\right)$$

is

$$(A.10) \qquad \frac{\widehat{f}(n)}{\widehat{f}(0)} = \prod_{j=1}^{n}\left(\frac{x_4 - x_1\omega^j}{x_3 - x_2\omega^j}\right),$$

see also (3.9), so that we find from (A.1b)

$$(A.11) \qquad \frac{\widehat{\overline{W}}_{pq}(n)}{\widehat{\overline{W}}_{pq}(0)} = \prod_{j=1}^{n}\left(\frac{c_p b_q - a_p d_q\omega^j}{b_p c_q - d_p a_q\omega^j}\right).$$

We can now derive two sets of two linear recurrence relations for the $V_{m,n}$'s, both sets being equivalent to the original product formulae (A.1). The star-triangle equation (A.3) follows when we demand consistency of the two sets, noting also the uniqueness of their solutions.

To be more explicit, let us apply

$$\sum_{k=1}^{N} \omega^{nk} W_{pr}(m+k)\cdots$$

to

$$(c_q b_r - b_q c_r\omega^k)\overline{W}_{qr}(k) = (\omega a_q d_r - d_q a_r\omega^k)\overline{W}_{qr}(k-1),$$

which is equivalent to (A.1b). In this way we arrive at

(A.12a) $c_q b_r V_{m,n} - b_q c_r V_{m,n+1}$

$$= a_q d_r \omega^{n+1} V_{m+1,n} - d_q a_r \omega^{n+1} V_{m+1,n+1}.$$

Similarly, applying

$$\sum_{k=1}^{N} \omega^{nk} \overline{W}_{qr}(k) \cdots$$

to

$$(b_p d_r - c_p a_r \omega^{m+k+1}) W_{pr}(m+k+1) = (d_p b_r - a_p c_r \omega^{m+k+1}) W_{pr}(m+k),$$

we get

(A.12b) $d_p b_r V_{m,n} - b_p d_r V_{m+1,n}$

$$= a_p c_r \omega^{m+1} V_{m,n+1} - c_p a_r \omega^{m+1} V_{m+1,n+1}.$$

In the same way we can derive from (A.1) the two relations

(A.13a) $d_q b_r \overline{V}_{m,n} - b_q d_r \overline{V}_{m,n+1}$

$$= a_q c_r \omega^{n+1} \overline{V}_{m+1,n} - c_q a_r \omega^{n+1} \overline{V}_{m+1,n+1},$$

(A.13b) $c_p b_r \overline{V}_{m,n} - b_p c_r \overline{V}_{m+1,n}$

$$= a_p d_r \omega^{m+1} \overline{V}_{m,n+1} - d_p a_r \omega^{m+1} \overline{V}_{m+1,n+1}.$$

At this point it may be noted that the four equations (A.12) and (A.13) are closely related. In fact, from the symmetry relation (A.5) we see that (A.12a) $\leftrightarrow$ (A.13b), (A.12b) $\leftrightarrow$ (A.13a). Under the operation $c \leftrightarrow d$, $p \leftrightarrow q$, $m \leftrightarrow n$ and transposing V, we also have (A.12a) $\leftrightarrow$ (A.12b). Finally, under the duality operation $c \leftrightarrow d$, $V \leftrightarrow \overline{V}$, we have simply (A.12a) $\leftrightarrow$ (A.13a), (A.12b) $\leftrightarrow$ (A.13b).

It is not difficult to show that (A.12), and similarly (A.13), is equivalent to (A.1). In fact, if we define

(A.14) $Y_{m+k,k} \equiv \frac{1}{N} \sum_{n=1}^{N} V_{m,n} \omega^{-nk},$

compare (A.4a), then we see from (A.12) that $Y_{j,k}$ satisfies

(A.15) $\dfrac{Y_{j,k-1}}{Y_{j,k}} = \dfrac{\overline{W}_{qr}(k-1)}{\overline{W}_{qr}(k)}, \qquad \dfrac{Y_{j+1,k}}{Y_{j,k}} = \dfrac{W_{pr}(j+1)}{W_{pr}(j)}.$

Hence,

(A.16) $$Y_{j,k} = \text{const } W_{pr}(j)\overline{W}_{qr}(k)$$

is the complete solution of (A.12). In other words, the solution of (A.12) is unique and given by (A.4a) with the equations (A.1) substituted in it.

It is straightforward to show that the pair (A.12) is equivalent to

(A.17a) $$c_r(d_p b_q - a_p c_q \omega^{m+1})V_{m,n+1}$$
$$- d_r(b_p c_q - d_p a_q \omega^{n+1})V_{m+1,n}$$
$$+ a_r(c_p c_q \omega^{m+1} - d_p d_q \omega^{n+1})V_{m+1,n+1} = 0,$$

(A.17b) $$d_r(b_p d_q - c_p a_q \omega^{m+1})\omega^n V_{m+1,n}$$
$$- c_r(c_p b_q - a_p d_q \omega^{n+1})\omega^m V_{m,n+1}$$
$$+ b_r(c_p c_q \omega^m - d_p d_q \omega^n)V_{m,n} = 0.$$

In the same way, the pair (A.13) is equivalent to

(A.18a) $$d_r(c_p b_q - a_p d_q \omega^{m+1})\overline{V}_{m,n+1}$$
$$- c_r(b_p d_q - c_p a_q \omega^{n+1})\overline{V}_{m+1,n}$$
$$+ a_r(d_p d_q \omega^{m+1} - c_p c_q \omega^{n+1})\overline{V}_{m+1,n+1} = 0,$$

(A.18b) $$c_r(b_p c_q - d_p a_q \omega^{m+1})\omega^n \overline{V}_{m+1,n}$$
$$- d_r(d_p b_q - a_p c_q \omega^{n+1})\omega^m \overline{V}_{m,n+1}$$
$$+ b_r(d_p d_q \omega^m - c_p c_q \omega^n)\overline{V}_{m,n} = 0.$$

We note that in (A.17) and (A.18) the coefficients of c_r and d_r are numerators and denominators of W_{pq} and $\overline{W}_{pq}$, see (A.1). Also, all the coefficients of a_r and b_r are very similar. Therefore, it takes only little algebra to show that (A.18) becomes (A.17) under the substitution

(A.19) $$\overline{V}_{m,n} \to \overline{\overline{V}}_{m,n} \equiv V_{n,m} \frac{\widehat{\overline{W}}_{pq}(m)}{W_{pq}(n)}$$

Hence, from the uniqueness of the solution of (A.18), or (A.13), up to an overall constant factor, say, R, we are forced to conclude

(A.20) $$\overline{\overline{V}}_{m,n} = R\overline{V}_{m,n}$$

for all m and n, that is (A.6), using the definition in (A.19).   Q.E.D.

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Helen Au-Yang
Department of Physics
Oklahoma State University
Stillwater, Oklahoma 74078-0444
U.S.A.

Jacques H.H. Perk
Department of Physics
Oklahoma State University
Stillwater, Oklahoma 74078-0444
U.S.A.