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Self-Duality of ALE Ricci-Flat 4-Manifolds and Positive Mass Theorem

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Introduction

In [An, Na, BKN] we have developed the theory of the Hausdorff convergence of Einstein 4-manifolds. Then the importance of ALE Ricci-flat 4-manifolds has appealed to us. The phenomenon is very similar to the case of Yang-Mills connections. When the curvatures of Einstein metrics concentrate around a point, an ALE Ricci-flat 4-manifold bubbles off from there (see §2). This corresponds to the phenomenon that a Yang-Mills connection on S^4 bubbles off from the points at which curvatures of Yang-Mills connections concentrate [Uh]. The classification of Yang-Mills connections on S^4 [AHS, Do2] has been essential in applications of the gauge theory to the differential topology ([Do1, FS]), so that it is very plausible that we think the classification of ALE Ricci-flat 4-manifolds is also important. In view of Kronheimer's classification of such spaces under the additional assumption that the spaces are hyperkähler ([Kr]) (see §1), we pose the following question. *Are there Ricci-flat ALE* 4*-manifolds other than quotients of hyperkähler space* ?

This question is related to physics. For ALE manifolds we can define the *mass* (see §3). If the space is Ricci-flat, the mass vanishes. Some physicists conjecture that the space is self-dual in this case [EGH]. This is a part of *the generalized positive action conjecture.* Although counterexamples of this conjecture are recently constructed by LeBrun [Le], we can show that the conjecture holds under certain topological assumptions (Theorem 3.3).

In this paper we shall make a survey of results on ALE Ricci-flat 4-manifolds and give a partial answer to the above question. In particular we shall prove that a spin Ricci-flat ALE 4-manifold with $\Gamma \subset SU(2)$ has a hyperkähler structure (Theorem 3.3). We shall give some other

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intrinsic characterizations of hyperkähler ALE 4-manifolds (Proposition 4.2) using characteristic numbers.

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§1. ALE gravitational instantons

After the discovery of Yang-Mills instantons on $\mathbb{R}^4$, physicists sought a corresponding object in gravity. Such an object, called "gravitational instanton", was first found by Eguchi and Hanson [EH]. This space is a noncompact (complete) Riemannian 4-manifold which satisfies the Einstein equation and we now call it the "Eguchi-Hanson space". It has an end which looks like the quotient space $\mathbb{R}^4/\{\pm 1\}$. Afterwards Gibbons and Hawking [GH] found a family of Riemannian manifolds (called "multi-Eguchi-Hanson space") which also satisfy the Einstein equation and each space has an end resembled $\mathbb{R}^4/\Gamma$ where Γ is a cyclic group. Then they introduced the following definition to describe this type of ends.

Definition 1.1. An n-dimensional complete Riemannian manifold (M, g) is said to be *asymptotically locally Euclidean* (ALE) of order $\tau > 0$, if there exists a compact subset $K \subset M$ such that $M \setminus K$ has coordinates at infinity; namely there are $R > 0$, a finite group $\Gamma \subset O(n)$ acting freely on $\mathbb{R}^n \setminus B_R(0)$, and a C^∞-diffeomorphism $\mathcal{X}\colon M \setminus K \to (\mathbb{R}^n \setminus \overline{B_R(0)})/\Gamma$ such that $\varphi = \mathcal{X}^{-1} \circ \operatorname{proj}$ satisfies (where proj is the natural projection of $\mathbb{R}^n$ to $\mathbb{R}^n/\Gamma$)

$$(\varphi^* g)_{ij}(x) = \delta_{ij} + a_{ij}(x) \qquad \text{for } x \in \mathbb{R}^n \setminus \overline{B_R(0)},$$

where $|\partial^p a_{ij}(x)| = O(|x|^{-\tau - p})$ for all $p \geqq 0$.

The Einstein manifolds which were found by physicists have a hyperkähler structure which is defined as follows.

Definition 1.2. A *hyperkähler structure* on a Riemannian manifold (M, g) is a set of three almost complex structures (I, J, K) such that

$$g(Iv, Iw) = g(Jv, Jw) = g(Kv, Kw) = g(v, w) \quad \text{for } v,\, w \in TM, \tag{1.3}$$

$$\nabla I = \nabla J = \nabla K = 0, \tag{1.4}$$

$$IJ = -JI = K. \tag{1.5}$$

Multi-Eguchi-Hanson spaces were also found by Hitchin [Hi] who obtained them by the Penrose's twistor construction. His approach had a close relationship with the deformation theory of rational double points. A quotient singularity $\mathbb{C}^2/\Gamma$ with a finite subgroup $\Gamma \subset SU(2)$ is called a rational double point and studied from several points of view by many mathematicians. Then Hitchin conjectured the existence of ALE gravitational instantons for other finite subgroups $\Gamma \subset SU(2)$ which correspond to the root systems and are classified as follows:

A_n : $\Gamma = \left\{\begin{bmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{bmatrix} : k = 0, \cdots, n\right\} \cong \mathbb{Z}_{n+1}$ where ζ is a primitive $(n+1)$-th root of unity,
D_n : $\Gamma = \mathbb{D}^*_{n-2}$ the binary dihedral group of order $4(n-2)$,
E_6 : $\Gamma = \mathbb{T}^*$ the binary tetrahedral group,
E_7 : $\Gamma = \mathbb{O}^*$ the binary octahedral group,
E_8 : $\Gamma = \mathbb{I}^*$ the binary icosahedral group.

This conjecture was proved affirmatively by Kronheimer [Kr].

Theorem 1.6 [Kr]. *Let Γ be a finite subgroup of $SU(2)$ and $\pi: M \to \mathbb{C}^2/\Gamma$ the minimal resolution of the quotient space $\mathbb{C}^2/\Gamma$ as a complex variety. Suppose that three cohomology classes α_I, α_J, $\alpha_K \in H^2(M;\mathbb{R})$ satisfy the non-degeneracy condition*

for each $\Sigma \in H_2(M;\mathbb{Z})$ with $\Sigma \cdot \Sigma = -2$ there exists $A \in \{I, J, K\}$ with $\alpha_A(\Sigma) \neq 0$.

Then there exists an ALE Riemannian metric g on M of order 4 together with a hyperkähler structure (I, J, K) for which the cohomology class of the Kähler form $[\omega_A]$ determined by the complex structure A is given by α_A for all $A \in I, J, K$. Conversely every hyperkähler ALE 4-manifold of order 4 can be obtained as above.

The exceptional set E of the minimal resolution $M \to \mathbb{C}^2/\Gamma$ decomposes to a union of $\mathbb{CP}^1$

$$E = \Sigma_1 + \Sigma_2 + \cdots + \Sigma_n,$$

and the intersection matrix $(\Sigma_i \cdot \Sigma_j)$ is the negative of the Cartan matrix associated to the root system. In this way the set $\{\Sigma_1, \cdots, \Sigma_n\}$ can be identified with the set of simple roots. On the other hand the set of the classes $\{[\Sigma_1], \cdots, [\Sigma_n]\}$ gives a basis of the second homology group $H_2(M;\mathbb{Z})$. Hence there is an isomorphism between $H_2(M;\mathbb{Z})$ and the weight lattice L. Moreover the cohomology group $H^2(M;\mathbb{R})$ is isomorphic to the Cartan subalgebra and the set $\{\Sigma \in H_2(M;\mathbb{Z}) \mid \Sigma \cdot \Sigma = -2\}$ are identified with the set of roots.

§2. Hausdorff convergence of Einstein manifolds

In [Gr] Gromov introduced a distance (called Hausdorff distance) on the class of all metric spaces. There he proved the precompactness of the set of consisting of the isometry classes of Riemannian manifolds whose Ricci curvatures are bounded from below by -1 (for the definition and some properties of the Hausdorff distance the reader should read Fukaya's survey in this volume). Using Hausdorff distance, we consider the following class of Einstein 4-manifolds.

Definition 2.1. For a positive numbers R, D and $k = \pm 1$, 0 let $\mathcal{E}(k, D, V, R)$ denote the set of isometry classes of compact 4-dimensional Riemannian manifolds (X, g) satisfying the following conditions:

(2.2) $\operatorname{Ric} g = kg$,

(2.3) $\operatorname{diam}(X, g) \leqq D$,

(2.4) $\operatorname{vol}(X, g) \geqq V$,

(2.5) $\displaystyle\int_X |R_g|^2 dV_g \leqq R$,

where R_g denotes the curvature tensor of g.

By the Gromov's precompactness theorem $\mathcal{E}(k, D, V, R)$ is precompact in the set of all compact metric spaces. Its boundary $\partial\mathcal{E}(k, D, V, R)$ is described as follows.

Theorem 2.6 [An, Na, BKN]. *Let (X_i, g_i) $(i = 1, 2, \cdots)$ be a sequence in $\mathcal{E}(k, D, V, R)$. Then there exists a subsequence $\{j\} \subset \{i\}$ such that (X_j, g_j) converges to a compact Einstein orbifold (X_∞, g_∞) in the Hausdorff distance. More precisely, outside a finite set $S = \{x_1, x_2, \cdots, x_n\} \subset X_\infty$ there exists an into diffeomorphism $\phi_j \colon X_\infty \backslash S \to X_j$ such that the metric $\phi_j^* g_j$ converges to g_∞ in the C^∞-topology on each compact subset of $X_\infty \setminus S$.*

This compactness theorem is very similar to Uhlenbeck's compactness theorem for Yang-Mills connections [Uh]. In her results Yang-Mills connections on S^4 bubble off from points where the convergence are broken. In our case ALE Ricci-flat manifolds bubble off from S.

Theorem 2.7 [BKN]. *Let (X_j, g_j), (X_∞, g_∞) and S be as in Theorem 2.6. For each j and $a = 1, \ldots, n$, there exists a point $x_{a,j} \in X_j$ and a positive number r_j such that*

(2.8) *$(X_j, g_j, x_{a,j})$ converges to $(X_\infty, g_\infty, x_a)$ in the pointed Hausdorff distance,*

(2.9) $\displaystyle\lim_{j \to \infty} r_j = \infty$, (2.10) *the rescaled manifold* $(X_j, r_j g_j, x_{a,j})$ *converges to an ALE Ricci flat 4-manifold* (M_a, h_a, y_a) *in the pointed Hausdorff distance.*

Examples of Hausdorff convergence of Einstein 4-manifolds can be found in [Na] or [Ko] (in this volume).

In the course of the proof we obtained the following intrinsic characterization of the existence of coordinates at infinity.

Theorem 2.11 [BKN]. *Let* (M, g) *be a 4-dimensional Ricci-flat manifold which satisfies*

$$\mathrm{vol}(B_t(p)) \geqq Vt^4 \qquad \text{for all } t > 0,$$

$$\int_M |R|^2 dV_g < \infty$$

for some $p \in M$ *and a positive constant* V. *Then* (M, g) *is ALE of order* 4.

We do not have any simply-connected non-hyperkähler examples which satisfy the conditions of Theorem 2.11. We conjecture that there are no such examples. In the following sections we shall give partial answers to this problem.

§3. Positive mass conjecture

Suppose that (M, g) is an n-dimensional ALE manifold of order $\tau > \dfrac{n-2}{2}$. In general relativity the mass of (M, g) is defined by

$$(3.1) \qquad m(M,g) = \lim_{R\to\infty} \int_{|x|=R} (\partial_i(\varphi^* g)_{ij}(x) - \partial_j(\varphi^* g)_{ii}(x))\partial_j \lrcorner\, dx,$$

where φ is as in Definition 1.1. We remark that it is not at all clear that the mass $m(M, g)$ is independent of the choice of coordinates at infinity. Bartnik [Ba] proved it by proving the uniqueness of harmonic coordinates at infinity under the condition $\tau > \dfrac{n-2}{2}$.

Physicists have conjectured that when (M, g) is an AE manifold (namely an ALE manifold with $\Gamma = \{e\}$) and has nonnegative scalar curvature, the mass $m(M, g)$ is nonnegative and $m(M, g) = 0$ if and only if (M, g) is isometric to $\mathbb{R}^n$ with the standard metric. This positive mass conjecture was proved by Schoen and Yau [SY1] when $n = 3$ and $\tau \geqq 1$ by the use of minimal surfaces. Shortly afterwards Witten [Wi] gave a different proof under the same assumptions using spinors. The technique of Schoen and Yau can be used to prove the corresponding results when $n \leqq 7$. (Schoen and Yau announced that the positive mass conjecture holds in arbitrary dimensions.) Witten's proof can be directly generalized to higher dimensional spin manifolds [Ba, LP].

For ALE 4-manifolds physicists (see [EGH]) conjectured that when (M, g) has nonnegative scalar curvature, the mass $m(M, g)$ is nonnegative and $m(M, g) = 0$ if and only if (M, g) has self-dual curvature tensor. Recently LeBrun [Le] constructed counter-examples for this conjecture:

Theorem 3.2 [Le]. *The total space of any complex line bundle L over $\mathbb{CP}^1$ with Chern class $c_1 < -2$ carries an ALE Kähler metric with zero scalar curvature of negative mass.*

In coordinates $\varphi: \mathbb{C}^2 \setminus B_a(0) \to L \setminus \mathbb{CP}^1$ $(a > 0)$, the metric is written as

$$g = (1 - \frac{a^2}{r^4})^{-1}(1 + \frac{ka^2}{r^2})^{-1} dr^2 + r^2 \left(\sigma_1^2 + \sigma_2^2 + (1 - \frac{a^2}{r^4})(1 + \frac{ka^2}{r^2})\sigma_3^2 \right),$$

where $k = -c_1$ and σ_1, σ_2, σ_3 is a left-invariant coframe of S^3. The fundamental group Γ of the end is the cyclic group of order $n = -c_2 + 1$ acting on $\mathbb{C}^2$ by

$$\mathbb{C}^2 \ni (z_1, z_2) \mapsto (\zeta z_1, \zeta z_2),$$

where ζ is a primitive n-th root of unity. Dividing by this action, we can remove the apparent singularity at $r = a$, and define a smooth metric on L. Then a direct calculation shows that the scalar curvature vanishes, and the mass is negative. Moreover we have the anti-self-duality of the Weyl tensor.

Recall that the universal covering space of $SO(4)$ is $\mathrm{Spin}(4) = SU(2) \times SU(2) = Sp(1) \times Sp(1)$. The group $Sp(1) \times Sp(1)$ acts on $\mathbb{R}^4$ which is identified with the space of quaternion $\mathbb{H}$ by the multiplication from right and left:

$$Sp(1) \times Sp(1) \ni (q, p) \qquad x \mapsto qxp \quad \text{for } x \in \mathbb{R}^4 = \mathbb{H}.$$

When the ALE manifold (M, g) is spin (this is true for hyperkähler spaces), we suppose that the coordinates at infinity $\varphi: \mathbb{R}^4 \setminus \overline{B_R(0)} \to M \setminus K$ preserve the spin structure. Then Γ has a lift to $\mathrm{Spin}(4)$. If we assume furthermore that Γ is contained in $SU(2)$ and acts on $\mathbb{C}^2$ from the right, we have a satisfactory answer to our problem.

Theorem 3.3. *Suppose that (M, g) is a spin ALE 4-manifold of order $\tau > 1$ with the end S^3/Γ where $\Gamma \subset SU(2)$ acts from the right and* *the scalar curvature S of (M,g) is nonnegative. Then its mass $m(M,g)$ is nonnegative. Moreover $m(M,g) = 0$ if and only if (M,g) admits a hyperkähler structure.*

When Γ acts from the left, the same conclusion holds for the orientation reversing manifold of (M,g).

Proof of Theorem 3.3. We consider the bundle of self-dual spinors V^+. Since $\Gamma \subset SU(2)$ acting from the right, V^+ is trivial on $M \setminus K$. Hence we can take a constant spinor ψ_0 with respect to this trivialization. We normalize ψ_0 so that $|\psi_0| \to 1$ at infinity. As in [Ba, LP] we can take a spinor ψ which is asymptotic to ψ_0 and satisfies $D\psi = 0$. Then it holds

$$m(M,g) = c\int_M (4|\nabla\psi|^2 + S|\psi|^2)dV_g$$

for some positive constant c depending only on Γ. Since $S \geqq 0$, we have $m(M,g) \geqq 0$. If the mass vanishes, then $\nabla\psi \equiv 0$, and $S \equiv 0$. Since ψ_0 is an arbitrary constant spinor, we can find a basis for the bundles of self-dual spinors V^+ consisting of parallel spinors. This implies (M,g) has a hyperkähler structure. Q.E.D.

Corollary 3.4. *If (M,g) is a spin ALE Ricci flat 4-manifold of order $\tau > 0$ with the end S^3/Γ where $\Gamma \subset SU(2)$ acts from the right, then it has a hyperkähler structure.*

Proof. By Theorem 2.11 (M,g) is ALE of order 4. Hence the mass $m(M,g)$ vanishes. So (M,g) has a hyperkähler structure by Theorem 3.3. Q.E.D.

An example of LeBrun is spin if and only if c_1 is even. But even if c_1 is even, the group Γ is not contained in $SU(2)$ in the coordinates at infinity preserving the spin structure.

§4. Inequalities between characteristic numbers

In this section we give other characterizations of hyperkähler manifold among ALE Ricci-flat 4-manifold. First we derive an equality which corresponds to Hitchin's inequality for compact Einstein 4-manifolds.

Theorem 4.2. *Let (M,g) be an ALE Ricci-flat 4-manifold (M,g) with the end S^3/Γ with Euler number $\chi(M)$ and signature $\tau(M)$. Then*

$$2(\chi(M) - \frac{1}{|\Gamma|}) \geqq 3|\tau(M) + \eta_S(S^3/\Gamma)| \tag{4.3}$$

where $|\Gamma|$ is the order of Γ and $\eta_S(S^3/\Gamma)$ is the eta invariant of the space S^3/Γ for the signature operator. Moreover the equality holds if and only if W^+ or W^- vanishes identically, namely (M,g) or the opposite orientation space of (M,g) is a quotient of an ALE hyperkähler 4-manifold.

Proof. For a Ricci-flat 4-manifold (M,g) the Gauss-Bonnet-Chern form is given by (cf. [Be])

$$\frac{1}{32\pi^2}\sum \varepsilon_{ijkl}\Omega^i_j \wedge \Omega^k_l = \frac{1}{8\pi^2}|W|^2 dV_g,$$

where W is the Weyl tensor of (M,g). Hence by the Gauss-Bonnet theorem for manifolds with boundary we have

$$\begin{aligned} \chi(M) &= \frac{1}{32\pi^2}\int_M \sum \varepsilon_{ijkl}\Omega^i_j \wedge \Omega^k_l + \frac{1}{|\Gamma|} \\ &= \frac{1}{8\pi^2}\int_M |W|^2 dV_g + \frac{1}{|\Gamma|}. \end{aligned} \tag{4.4}$$

The first Pontrjagin form p_1 of (M,g) is given by (cf. [Be])

$$-\frac{1}{8\pi^2}\operatorname{tr}\Omega^2 = \frac{1}{4\pi^2}(|W^+|^2 - |W^-|^2)dV_g,$$

where W^+, W^- are the self-dual and the anti-self-dual Weyl tensors respectively. By the signature theorem for manifolds with boundary [APS] we have

$$\begin{aligned} \tau(M) &= \frac{1}{3}\int_M p_1 - \eta_S(S^3/\Gamma) \\ &= \frac{1}{12\pi^2}\int_M (|W^+|^2 - |W^-|^2)dV_g - \eta_S(S^3/\Gamma). \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5), we have the inequality (4.3) in Theorem 4.2. When the equality holds in (4.3), we have $W^+ = 0$ or $W^- = 0$. Q.E.D.

The eta invariant of S^3/Γ is calculated as follows ([APS]):

$$\eta_S(S^3/\Gamma) = -\frac{1}{|\Gamma|}\sum_{g\neq 1}\cot\frac{r(g)}{2}\cot\frac{s(g)}{2}$$

where $r(g)$ and $s(g)$ are the rotation numbers corresponding to the action of $g \in \Gamma \subset SO(4)$ at $0 \in \mathbb{R}^4$. A direct calculation shows that for

$\Gamma \subset SU(2)$ we have

$$\eta_S(S^3/\Gamma) = \frac{n(n-1)}{3(n+1)}, \frac{2n^2-8n+9}{6(n-2)}, \frac{49}{36}, \frac{121}{72}, \frac{361}{180}$$

according to whether Γ is of type A_n, D_n, E_6, E_7, E_8. Since every hyperkähler ALE 4-manifold is simply-connected and its intersection form is the negative of the Cartan matrix, we have

$$\chi(M) = -\tau(M) + 1 = n + 1,$$

where n is the size of the Cartan matrix associated with Γ. The equality in Proposition 4.2 for an ALE hyperkähler 4-manifold can also be obtained from this directly.

Now we recall the eta invariant η_D for the Dirac operator. Suppose that an ALE Ricci-flat 4-manifold (M, g) is spin. As in [Kr] we compactify M to an orbifold $\widehat{M}$ by a conformal change of the metric g. We denote by $D: \Gamma(V_+) \to \Gamma(V_-)$ the Dirac operator on $\widehat{M}$. Then the index theorem for orbifolds [Ka] says

$$\text{(4.6)} \qquad \operatorname{Index} D = \frac{1}{96\pi^2} \int_M (|W^-|^2 - |W^+|^2) dV_g - \frac{\eta_D(S^3/\Gamma)}{2}.$$

The eta invariant $\eta_D(S^3/\Gamma)$ can be calculated as [HR]

$$\eta_D(S^3/\Gamma) = -\frac{1}{2|\Gamma|} \sum_{g \neq 1} \csc \frac{r(g)}{2} \csc \frac{s(g)}{2}.$$

For a hyperkähler 4-manifold (M, g) we have

$$\frac{\eta_D(S^3/\Gamma)}{2} = \frac{n(n+2)}{12(n+1)}, \frac{4n^2+4n-7}{48(n-2)}, \frac{167}{288}, \frac{383}{576}, \frac{1079}{1440}$$

according to Γ is of type A_n, D_n, E_6, E_7, E_8. Combining (4.5) and (4.6) we have

$$\text{(4.7)} \qquad \operatorname{Index} D = -\frac{1}{8}(\tau(M) + \eta_S(S^3/\Gamma)) - \frac{\eta_D(S^3/\Gamma)}{2}$$

for a spin ALE 4-manifold (M, g). In fact we have the following (see also [Kr; Lemma 7-2-1)]).

Proposition 4.8. *If (M, g) is a spin ALE 4-manifold of order $\tau > 0$ with zero scalar curvature, then the index of the Dirac operator D on the compactification $\widehat{M}$ equals to* 0.

Proof. Suppose we have a nonzero element of the kernel of the Dirac operator on $\widehat{M}$. Then by the conformal invariance of the Dirac operator, there exists a solution ψ of $D\psi = 0$ which satisfies the decay conditions

$$|\psi| = O(r^{-3}),$$
$$|\nabla\psi| = O(r^{-4}),$$

where $r = dist(o, *)$ for some fixed $o \in M$. The Weitzenböck formula for the Dirac operator says

$$D^*D = \nabla^*\nabla + \frac{1}{8}S,$$

where S is the scalar curvature. Since $S = 0$ on M, we have

$$\nabla^*\nabla\psi = 0$$

By the decay conditions we can apply the Stokes theorem on M to get

$$\begin{aligned} 0 &= \int_M (\nabla^*\nabla\psi, \psi) \\ &= \int_M (\nabla\psi, \nabla\psi). \end{aligned}$$

Thus we have $\nabla\psi = 0$ and hence $\psi = 0$. This is a contradiction. Similarly the adjoint D^* also has a trivial kernel. Q.E.D.

Corollary 4.9. *Under the same condition with Proposition* 4.8 *we have*

$$\tau(M) + \eta_S(S^3/\Gamma) + 4\eta_D(S^3/\Gamma) = 0.$$

In particular for a spin ALE Ricci-flat 4-manifold (M, g), the signature $\tau(M)$ is determined by the end S^3/Γ.

References

[An] M. Anderson, Ricci curvature bounds and Einstein metrics on compact manifolds, preprint.

[AHS] M. Atiyah, N. Hitchin and I. Singer, Self-duality in four dimensional Riemannian geometry, Proc. Roy. Soc. London Ser. A, **362** (1978), 425–461.

[APS] M. Atiyah, V. Patodi and I. Singer, Spectral asymmetry and riemannian geometry. I, Math. Proc. Cambridge Philos. Soc., **77** (1975), 43–69; II, Math. Proc. Cambridge Philos. Soc., **78** (1975), 405–432.

[Ba] R. Bartnik, The mass of an asymptotically flat manifold, Comm. Pure and Appl. Math., **39** (1986), 661–693.

[Be] A.L. Besse, "Einstein manifolds", A series of Modern Surveys in Mathematics, Springer-Verlag, Berlin.

[BKN] S. Bando, A. Kasue and H. Nakajima, On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth, Invent. Math., **97**, 313–349.

[Do1] S.K. Donaldson, An application of gauge theory to four dimensional topology, J. Diff. Geom., **18** (1983), 279–315.

[Do2] ———, Instantons and geometric invariant theory, Comm. Math. Phys., **93** (1984), 453–460.

[EH] T. Eguchi and A.J. Hanson, Asymptotically flat self-dual solutions to Euclidean Gravity, Phys. Lett., **237** (1978), 249–251.

[EGH] T. Eguchi, P.B. Gilkey and A.J. Hanson, Gravitation, gauge theories and differential geometry, Physics Reports, **66** (1980), 213-393.

[FS] R. Fintushel and R.J. Stern, Pseudofree orbifolds, Ann. of Math., **122** (1985), 335–364.

[GH] G.W. Gibbons and S.W. Hawking, Gravitational multi-instantons, Phys. Lett., **B 78** (1978), 430–432.

[Gr] M. Gromov, "Structures métrique pour les variété riemanniennes, Redigé par J. Lafontaine et P. Pansu", Textes Math. No. 1, Cedic/ Fernand Nathan, Paris, 1981.

[HR] A.J. Hanson and H. Römer, Gravitational instanton contribution to spin 3/2 axial anormaly, Phys. Lett., **80B** (1978), 58–60.

[Hi] N.J. Hitchin, Linear field equations on self-dual spaces, Proc. Roy. Soc., **A370** (1980), 173–191.

[Ka] T. Kawasaki, The index of elliptic operators over V-manifolds, Nagoya Math. J., **84** (1981), 135–157.

[Ko] R. Kobayashi, Moduli of Einstein Metrics on K3-Surfaces and Mild Degeneration of Kähler-Einstein Metrics, in this volume.

[Kr] P.B. Kronheimer, ALE gravitational instantons, Ph.D. thesis, Univ. of Oxford 1986.

[Le] C. LeBrun, Counter-examples to the generalized positive action conjecture, Comm. Math. Phys, **118** (1988), 591–596.

[LP] J.M. Lee and T.H. Parker, The Yamabe problem, Bull. of the A.M.S., **17** (1987), 37–91.

[PT] T.H. Parker and C.H. Taubes, On Witten's proof of positive energy theorem, Comm. Math. Phys., **84** (1982), 223–238.

[Na] H. Nakajima, Hausdorff convergence of Einstein 4-manifolds, J. Fac. Sci. Univ. Tokyo, **35** (1988), 411–424.

[Uh] K. Uhlenbeck, Connection with L^p bounds on curvature, Comm. Math. Phys., **83** (1982), 31–42.

[Wi] E. Witten, A new proof of the positive energy theorem, Comm. Math. Phys., **80** (1981), 381–402.

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