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An Application of Kähler-Einstein Metrics to Singularities of Plane Curves

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Throughout this note, we use the same terminology as [K1] and [K-N-S].

It is a classical problem which kind of singularities a plane curve of degree d can admit. Several authors studied this problem. Among them, Hirzebruch [H], Ivinskis [I] and Yoshihara [Yh] obtained some partial answers under the restriction on the degree d of the given curve. They needed the restriction on d because they used the covering method and the Miyaoka-Yau inequality [M] for surfaces with isolated quotient singularities.

In this note we apply a new differential geometric method to this problem and estimate the maximum number of certain class of singularities of a curve of degree d. Namely, we consider the given curve as a formal branch curve and use the canonical Kähler-Einstein metric on log-canonical normal surfaces with branch loci ([K1], [K-N-S]). Then a general Miyaoka-Yau inequality ([K1], [K-N-S]) gives an estimate for the number of the singularities. Since the Kähler-Einstein metric exists with arbitrarily prescribed formal branch indices, our method has an advantage that we need no covering trick and so need no assumption on d.

Recently, Sakai [S] studies this problem systematically and compares the effectiveness of the estimate obtained by our method and those obtained by other various methods.

Let C be a plane curve of degree d and $C = C_1 + \cdots + C_r$ the decomposition into irreducible components. Let d_i denote the degree of C_i. To each curve C_i, we assign a formal branch index b_i ($2 \leq b_i \leq \infty$ for all i) and make a Q-divisor

$$D = \sum_{i=1}^{r}(1 - \frac{1}{b_i})C_i. \tag{1}$$

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We suppose that b_i's are chosen so that the normal surface pair $(P_2(C), D)$ has at worst log-canonical singularities. write LTS (resp. LCS) for log-terminal singularities (resp. log-canonical singularities which are not log-terminal). We further assume bigness of $K + D$:

$$(2) \qquad \sum_{i=1}^{r}(1 - \frac{1}{b_i}) > 3.$$

Then (1) and (2) imply that $(P_2(C), D)$ is a log-canonical normal surface. It follows from [K1, Section 3, Theorem 1] that there exists a canonical Kähler-Einstein metric ω on

$$X = P_2(C) - LCS(P_2(C), D) - \cup_{b_i=\infty} C_i.$$

A possibly non-compact surface X with the canonical metric is essentially an orbifold with "cusps". Then we integrate the Chern forms $\gamma_1(\gamma)^2$ and $\gamma_2(\omega)$ and compute the correction terms coming from the singularities. We have

$$(3) \qquad \int \gamma_1(\omega)^2 = \left(-3 + \sum_{i=1}^{r}(1 - \frac{1}{b_i})d_i\right)^2$$

and

$$(4) \quad \int \gamma_2(\omega) = \Big\{ e(P_2(C) - \Lambda) - \sum_{b_i<\infty}(1 - \frac{1}{b_i})(e(C_i - \Lambda) - \sharp(C_i \cap T)) - \sum_{p\in T}(1 - \frac{1}{|\Gamma|}) \Big\}$$

where

$$\Lambda = \cup_{b_i=\infty} C_i \cup \mathrm{LCS}(P_2(C), D), \qquad T = \mathrm{LTS}(P_2(C), D).$$

The Kähler-Einstein property of ω implies that if we write the Chern forms in terms of the quadratic polynomials of the components of the curvature tensor, then

$$(5) \qquad \gamma_1(\omega)^2 \leq 3\gamma_2(\omega)$$

holds pointwise. This may be called the pointwise Miyaoka-Yau inequality. Combining (3), (4) and (5), we have

Theorem. *Let $(P_2(C), D)$ be as above. Then we have*

$$(6) \quad (-3+\sum_{i=1}^{r}(1-\frac{1}{b_i})d_i)^2 \leq 3\{e(P_2(C)-\Lambda) - \sum_{b_i<\infty}(1-\frac{1}{b_i})(e(C_i-\Lambda)-\sharp(C_i\cap T)) - \sum_{p\in T}(1-\frac{1}{|\Gamma|})\}.$$

Note that we have no assumption on d other than the inequality (2). To state the following Corollary, we need the following notations:

$$C(b) := \cup_{b_i=b}C_i, \qquad d(b) := \sum_{b_i=b} d_i,$$

$$\mu_p(b) := \text{the Milnor number of } (C(b), p).$$

Corollary 1. *Let $(P_2(C), D)$ be as above. Then we have*

$$(7) \quad \sum_{2\leq b\leq\infty}\left\{\sum_{p\in \mathrm{Sing}(C)\cap C(b)}(1-\frac{1}{b})(\mu_p(b)-1)\right\} + \sharp(\mathrm{Sing}(C)) - \sum_{p\in LTS(P_2(C),D)}\frac{1}{|\Gamma(p)|} \leq \sum_{2\leq b\leq\infty}(1-\frac{1}{b})d(b)\{(\frac{2}{3}+\frac{1}{3b})d(b)-1\}.$$

Theorem is a direct consequence of a general Miyaoka-Yau type inequality proved in [K1] (see also [K-N-S]). If we use the genus formula (see, for example, [B-K])

$$\sum_{p\in \mathrm{Sing}(C)} \mu_p = d(d-3)+e(C),$$

where μ_p is the Milnor number of the singularity (C, p) and $e(C)$ is the topological Euler number of C, we can rewrite (6) in the form (7). As a convention, $d(b) = 0$ if there is no C_i with $b_i = b$.

There will be many ways in arranging this theorem. Here we consider the simplest case.

Definition. The singularity (C,p) is simple if $(P_2(C), \frac{1}{2}C, p)$ is log-canonical. We say that a simple singularity (C,p) is LTS (resp. LCS) if $(P_2(C), \frac{1}{2}C, p)$ is LTS (resp. LCS).

It is easy to classify simple singularities (C,p). A complete list of them is found in the following tables separately in the cases of LTS and LCS. The double cover of a simple singularity is either a rational double point, a Hilbert modular cusp singularity or a simple elliptic singularity.

Simple Singularities

LTS			
(C,p)	μ_p	$\lvert\Gamma(p)\rvert$	Double Cover
x^2+y^{2a+2}	$2a+1$	$4(a+1)$	A_{2a+1}
$y(x^2+y^{2a+2})$	$2a+4$	$16(a+1)$	D_{2a+4}
x^2+y^{2a+1}	$2a$	$2(2a+1)$	A_{2a}
$y(x^2+y^{2a+1})$	$2a+3$	$8(2a+1)$	D_{2a+3}
x^3+y^4	6	48	E_6
$x(x^2+y^3)$	7	96	E_7
x^3+y^5	8	240	E_8

LCS of simple elliptic type	
(C,p)	μ_p
x^4+y^4	9
$x(x^2+y^4)$	10

LCS of Hilbert modular cusp type	
(C,p)	μ_p
$(x^2+y^2)(x^2+y^{2a+2}),\quad a \geq 1$	$2a+9$
$(x^{2b+2}+y^2)(x^2+y^{2a-2b+2}),\quad 1 \leq b \leq [\frac{a}{2}],\ a \geq 2$	$2a+9$
$(x+y^2)(x^2+y^{2a+4}),\quad a \geq 0$	$2a+10$
$(x^2+y^2)(x^2+y^{2a+3}),\quad a \geq 0$	$2a+10$
$(x^2+y^{2b+2})(x^{2(a-b)+3}+y^2),\quad 1 \leq b \leq a-1,\ a \geq 2$	$2a+10$
$(x+y^2)(x^2+y^{2a+5}),\quad a \geq 0$	$2a+11$
$(x^3+y^2)(x^2+y^{2a+3}),\quad a \geq 0$	$2a+11$
$(x^2+y^{2a+3})(x^{2(a-b)+3}+y^2),\quad 1 \leq b \leq [\frac{a}{2}],\ a \geq 2$	$2a+11$

If we set $D = C(2)$, i.e., $b_i = 2$ for all i in Corollary 1, we have

Corollary 2. *Let C be a plane curve of degree $d \geq 7$ having only simple singularities. Then we have*

$$(8) \qquad \sum_{p \in \mathrm{Sing}(C)} (\mu_p + 1) - \sum_{p \in LTS(P_2(C), \frac{1}{2}C)} \frac{2}{|\Gamma|} \leq \frac{5}{6}d^2 - d.$$

In the case d is even, this was known by considering the double covering and applying Miyaoka's inequality [M] for surfaces with isolated quotient singularities. From Corollary 2, we get an effective estimate of the number of ordinary cusps for irreducible plane curves of degree $d \geq 7$ with at worst ordinary cusps.

Probably we can understand Corollary 2 in the frame work of [B-K-N] (bubbling out of complete noncompact Ricci-flat manifolds). A partial support is this: suppose we are given a degeneration of algebraic surfaces of general type such that the singular member acquires at worst rational double points, Hilbert modular cusp singularities or simple elliptic singularities. Then we can show the convergence of the canonical Kähler-Einstein metric to a complete orbifold Kähler-Einstein metric on the complement of the simple elliptic and ball cusp singularities. The existence of such Kähler-Einstein metric is proved in [K1] (canonical Kähler-Einstein metric on the log-canonical model). The convergence is a consequence of the maximum principle as in [T] (see also Sugiyama's paper in this volume). We now look at one simple elliptic singularity. If this is a triple point, i.e., locally given by the equation $x^3 + y^3 + z^3 = 0$, then probably an affine cubic surface with a complete Ricci-flat Kähler metric [K2] bubbles out.

Ivinskis showed that Corollary 2 holds also for $d = 6$. He takes a double covering and uses Miyaoka's version of generalized Miyaoka-Yau inequality valid for surfaces with $\kappa = 0$. Note that the double cover is a degenerate K3 surface. In [K2], we have presented a geometric picture of the degeneration of Kähler-Einstein K3 surfaces. Ivinskis' result (Miyaoka's inequality for surfaces with zero Kodaira dimension [M]) may probably be understood if, as in [K2, Section 2.3.3, Examples], we prove that the squared L^2-norm of the curvature tensor (representing the Euler number of Einstein K3 surfaces) is conserved provided we take the bubbling out Ricci-flat complete Kähler surfaces into account. Making a firm mathematical footing explaining these phenomena is a problem not yet fully answered.

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