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Harmonic Functions with Growth Conditions on a Manifold of Asymptotically Nonnegative Curvature II

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§0. Introduction

According to a theorem due to Greene-Wu [13], a complete connected noncompact Riemannian manifold M abounds harmonic functions so that M can be imbedded properly into some Euclidean space by them. However various problems on harmonic functions on M with specific conditions (e.g., boundedness, positivity, L^p integrability, etc.) arise in connection with the geometry of M and in fact they have been investigated by many authors (cf. e.g., [11: Section 11], [23], [29: Section 4,6.4] and the references therein). In the previous paper [21], we have discussed bounded or positive harmonic functions on a manifold of asymptotically nonnegative curvature (which will be defined later), and extended all of the results by Li-Tam [24;25] to such manifolds. The purpose of the present paper is to study harmonic functions with finite growth on a manifold of asymptotically nonnegative curvature and then to verify the results stated in [21] without proofs. To state the main results of the paper, we need some definitions. For a harmonic function h on a complete connected noncompact Riemannian manifold M, we denote by $m_x(h,t)$ the maximum of $|\, h \,|$ on the metric sphere $S_t(x)$ around a point x with radius t. In this note, h is said to be *of finite growth,* if $\limsup_{t\to\infty} m_x(h,t)/t^p$ is finite for some constant $p > 0$. After Abresch [1], we call M *a manifold of asymptotically nonnegative curvature,* if the sectional curvature K_M of M satisfies:

(H.1) $$K_M \geq -K \circ r,$$

where r denotes the distance to a fixed point, say o, of M and $k(t)$ is a nonnegative, monotone nonincreasing continuous function on $[0,\infty)$ such that *the integral* $\int^\infty tk(t)dt$ *is finite.* In [19], we have constructed

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a metric space $M(\infty)$ associated with a manifold M of asymptotically nonnegative curvature. Let us here explain it briefly (see [19] for details). We say two rays σ and γ of M *equivalent* if $\mathrm{dis}_M(\sigma(t), \gamma(t))/t$, goes to zero as $t \to \infty$. Define a distance δ_∞ on the equivalence classes by $\delta_\infty([\sigma], [\gamma]) := \lim_{t\to\infty} d_t(\sigma \cap S_t(o), \gamma \cap S_t(o))/t$, where d_t stands for the inner (or intrinsic) distance on $S_t(o)$ induced from the distance $\mathrm{dis}_M(,)$ on M. Then we have a metric space $M(\infty)$ of the equivalence classes of rays with distance δ_∞ which is independent of the choice of the fixed point o and to which a family of scaled metric spheres $\{\frac{1}{t}S_t(o)\}$ converges with respect to the Hausdorff distance as t goes to infinity. We note that the complement $M - B_R(o)$ of a metric ball $B_R(o)$ centered at o with large radius R is homeomorphic to $S_R(o) \times (R, \infty)$. For simplicity, we call a connected component of $M - B_R(o)$ (for large R) *an end* δ *of* M. We write $M_\delta(\infty)$ for the connected component of $M(\infty)$ corresponding to δ, so that $\{\frac{1}{t}\ S_t(o) \cap \delta\}$ converges to $M_\delta(\infty)$ with respect to the Hausdorff distance as $t \to \infty$, and then $M_\delta(\infty)$ turns out to be a compact inner metric space. Since $\mathrm{Vol}_{m-1}(S_t(o) \cap \delta)/t^{m-1}$ $(m := \dim M)$ tends to a nonnegative constant as $t \to \infty$, let us denote the limit by $\mathrm{Vol}\,(M_\delta(\infty))$.

In Euclidean space $\mathbf{R}^m$, the harmonic functions of finite growth (harmonic polynomials) form an important subclass which is closely connected to the eigenfunctions of the unit sphere $S^{m-1}(1)$ $(= \mathbf{R}^m(\infty))$. Moreover if we equip $\mathbf{R}^m$ with a complete metric g which is written in the polar coordinates (r, θ) as $g = dr^2 + r^{2\alpha} d\theta^2$ $(0 \le \alpha < 1)$ for large r, then $(\mathbf{R}^m, g)$ admits no nonconstant harmonic functions of finite growth. In this case, $(\mathbf{R}^m, g)(\infty)$ consists of only one point. We are interested in relationships (if any) between the space of harmonic functions of finite growth on a manifold M of asymptotically nonnegative curvature and the geometry of $M(\infty)$. At this stage, we have rather satisfactory results for the case of dim $M = 2$ and for the case that the sectional curvature of M decays rapidly and the metric balls of M have maximal volume growth (see [3], [4] and the references therein), but for cases without such conditions, little is known. In this paper, we shall prove the following

Theorem A. *Let M be a manifold of asymptotically nonnegative curvature. Suppose that M has one end, i.e., $M(\infty)$ is connected. Then:*

(i) *For a nonconstant harmonic function h on M, one has*

$$\liminf_{t\to\infty} \frac{\log m(h,t)}{\log t} \ge \log\left[\frac{(\exp c(m)\,\mathrm{diam}(M(\infty)) + 1}{(\exp c(m)\,\mathrm{diam}(M(\infty)) - 1}\right] > 0,$$

where $c(m)$ is a positive constant depending only on $m :=$ dim M. *In* *particular, M has no nonconstant harmonic functions of finite growth if $M(\infty)$ consists of only one point.*

(ii) *Suppose that $m = 2$ and $\operatorname{diam}(M(\infty)) > 0$. Then for a nonconstant harmonic function h of finite growth, $\log m(h,t)/\log t$ converges to a constant, say $\operatorname{ord}(h)$, as $t \to \infty$, and $\operatorname{ord}(h)$ is given by $\operatorname{ord}(h) = n\pi/\operatorname{diam}(M(\infty))$ for some positive integer n. Moreover the dimension of the space of harmonic functions h with $\operatorname{ord}(h) \leq n\pi/\operatorname{diam}(M(\infty))$ is equal to $2n+1$.*

It is conjectual that for a manifold of asymptotically nonnegative curvature, the space $\mathcal{H}_p$ of harmonic functions h with $\limsup_{t\to\infty} m(h,t)/t^p < +\infty$ would be of finite dimension for any $p > 0$. In Section 3, we shall show a result related to this question. We remark that Kazdan [23] shows an example of a complete, noncompact Riemannian manifold such that it possesses no nonconstant positive harmonic functions, but the dimension of $\mathcal{H}_p$ is infinite for any $p > 0$. The sectional curvature of his example behaves like $-1/r^2 \log r$ for large r.

In case of a complete, connected noncompact Riemannian manifold M with nonnegative Ricci curvature, a theorem due to Cheng [8] says that for a harmonic function h on M, any point x of M, and every $t > 0$, $|dh|(x) \leq c(m)\, m_x(h,t)/t$, where $c(m)$ is a constant depending only on $m = \dim M$, and hence h must be constant if h is of sublinear growth, i.e., $\liminf_{t\to\infty} m(h,t)/t = 0$ (see also [29: Section 6.4]). Moreover the Cheeger-Gromoll splitting theorem [6] asserts that M as above contains a distance minimizing geodesic $\sigma : \mathbf{R} \to M$ (which is called *a line* of M) if and only if M splits isometrically into $\mathbf{R} \times M'$. The latter condition is obviously equivalent to saying that M admits a nonconstant totally geodesic function (i.e., a function of vanishing second derivatives). Motivated by these results, we are led to ask whether a nonconstant harmonic function h of linear growth (i.e., $\limsup_{t\to\infty} m(h,t)/t < +\infty$) on such M would be totally geodesic (or equivalently a nonzero d-closed harmonic 1-form on such M with bounded length would be parallel). It is easy to see that the above question is affirmative in case of $\dim M = 2$. In fact, since the Gaussian curvature is nonnegative, $|\omega|^2$ satisfies: $\triangle|\omega|^2 \geq 2|\nabla\omega|^2 \geq 0$. This implies that $|\omega|^2$ is a bounded subharmonic function on M, so that $|\omega|^2$ must be constant, because M possesses no nonconstant bounded subharmonic functions. Thus ω must be parallel and moreover M is flat. In this paper, we shall answer the above question under stronger conditions. Actually we prove the following

Theorem B. *Let M be a complete, connected noncompact Rie-*

mannian manifold of nonnegative sectional curvature: $K_M \geq 0$. *Suppose that* K_M *decays in quadratic order, i.e.,*

$$\text{(H.2)} \qquad K_M \leq \frac{c}{r^2}$$

for some positive constant c, where r stands for the distance to a fixed point of M. Then a nonzero d-closed harmonic 1-form on M with bounded length must be parallel. In particular, if M admits a nonconstant harmonic function h of linear growth, then h is totally geodesic and M splits isometrically into $\mathbf{R}\times M'$ *along the gradient of h.*

Theorem A and Theorem B are, respectively, proved in Section 1 and Section 2. In Section 3, other related results are given.

The author would like to thank Prof. H. Wu for drawing his attention to the lecture [30] in which some open problems related to this paper were proposed.

§1. Proof of Theorem A

We shall begin with proving the first assertion of Theorem A. Let h be a nonconstant harmonic function on M. Set $\overline{m}(h,t) := \max\{h(x) : x \in S_t\}$ and $\underline{m}(h,t) := \min\{h(x) : x \in S_t\}$, where S_t denotes the metric sphere around a fixed point o of M with radius t. Since M has only one end, S_t is connected for large t. Hence for large t, we can take two points p_t and q_t of S_t such that $h(p_t) = \overline{m}(h,t)$ and $h(q_t) = \underline{m}(h,t)$, and then join q_t to p_t by an arc-length parametrized Lipschitz curve $\tau_t : [0, a_t] \to S_t$ whose length a_t is equal to the inner distance $d_t(p_t, q_t)$ between p_t and q_t in S_t. Let us fix here a positive integer n which is greater than $\mathrm{diam}(M(\infty))$ and let $p_{t,i} := \tau_t(ia_t/3n)$ $(i = 0, 1, \cdots, 3n)$. Then we observe that

$$\text{(1.1)} \qquad \begin{aligned} &\limsup_{t\to\infty} \frac{a_t}{t} \leq \mathrm{diam}(M(\infty)) \\ &\limsup_{t\to\infty} \frac{1}{t}\ \mathrm{dis}_M(p_{t,i}, p_{t,i+1}) \leq \frac{\mathrm{diam}(M(\infty))}{3n} < \frac{1}{3}. \end{aligned}$$

Since $\overline{m}(h,t)$ is monotone increasing, $\overline{m}(h, 3t/2) - h$ is a positive harmonic function on the metric ball $B_{t/2}(p_{t,i})$ around $p_{t,i}$ with radius $t/2$ (t is assumed to be sufficiently large). Applying a theorem due to Cheng-Yau [9: Theorem 6] to $\overline{m}(h, 3t/2) - h$, we have

$$\overline{m}(h, \frac{3}{2}t) - h(p_{t,i+1}) \leq \exp\{c_m(1 + t\sqrt{k(\tfrac{t}{2})}\,)\tfrac{a_t}{3nt}\}\{\overline{m}(h, \tfrac{3}{2}) - h(p_{t,i})\}$$

where $k(t)$ is as in (H.1) and c_m is a constant depending only on $m :=$ dim M. Note here that $t\sqrt{k(t/2)}$ goes to zero as $t \to \infty$ (cf. [1: p.667]). This implies that

$$(1.2)\ \overline{m}(h, \frac{3}{2}t) - \underline{m}(h,t) \leq \exp\{c_m(1+t\sqrt{k(\tfrac{t}{2})}\,)\tfrac{a_t}{t}\}\{\overline{m}(h, \tfrac{3}{2}t) - \overline{m}(h,t)\}.$$

Moreover since $\underline{m}(h,t)$ is monotone decreasing, $h - \underline{m}(h, \frac{3}{2}t)$ is a positive harmonic function on $B_{t/2}(p_{t,i})$. Hence by the same reason as above, we have

$$(1.3)\ \overline{m}(h,t) - \underline{m}(h, \frac{3}{2}t) \leq \exp\{c_m(1+t\sqrt{k(\tfrac{t}{2})}\,)\tfrac{a_t}{t}\}\{\underline{m}(h,t) - \underline{m}(h, \tfrac{3}{2}t)\}.$$

If we set $\mu(t) := \overline{m}(h,t) - \underline{m}(h,t)$, then it follows from (1.2) and (1.3) that

$$\mu(\frac{3}{2}t) + \mu(t) \leq \exp\{c_m(1 + t\sqrt{k(\tfrac{t}{2})}\,)\ \tfrac{a_t}{t}\}\{\mu(\tfrac{3}{2}t) - \mu(t)\},$$

which shows

$$(1.4)\qquad \mu(t) \leq \frac{\exp\{c_m(1 + t\sqrt{k(t/2)}\,)\ a_t/t\}\ -\ 1}{\exp\{c_m(1 + t\sqrt{k(t/2)}\,)\ a_t/t\}\ +\ 1}\,\mu(\frac{3}{2}t).$$

Thus it turns out from (1.1), (1.4) and the standard iteration argument that

$$\liminf_{t\to\infty} \frac{\log\ \mu(t)}{\log\ t} > \log\left[\frac{\exp\{c_m\,\mathrm{diam}(M(\infty))\}\ +\ 1}{\exp\{c_m\,\mathrm{diam}(M(\infty))\}\ -\ 1}\right].$$

This proves the first assertion of Theorem A.

Let us now prove the second assertion of Theorem A. Since M has finite total curvature: $\int_M K_M\ \mathrm{dvol}(g_M) < +\infty$ (cf. [20:Proposition 4.1]), we can apply some of the results by Finn [12] and Huber [15;16] to our manifold M. In fact, it follows from [15] that the end of M is conformally equivalent to the end of $\mathbf{C}$, to be precise, there is a conformal diffeomorphism $\Psi : M - K \to \mathbf{C} - D_R$ from the complement $M - K$ of a compact set K onto the one of a disk $D_R := \{z \in \mathbf{C} : |z| \leq R\}$. Through the conformal diffeomorphism Ψ, we identify $M - K$ with $\mathbf{C} - D_R$ which has the metric $G := \Psi_* g_M = e^{2u} dz d\bar{z}$. Without loss of generality, we may assume that G defines a complete metric on $\mathbf{C}$ with finite total curvature: $\int_{\mathbf{C}} K_G\ \mathrm{dvol}(G) < +\infty$. Denote here by ρ the distance in $\mathbf{C}$ to the origin with respect to G. Then applying Theorems 11 and 13 in

[12] and Théorème 1 in [16] to $(\mathbf{C}, G)$, we get

$$
\begin{aligned}
(1.5) \qquad \lim_{x\in M\to\infty} \frac{\log\ r(x)}{\log\ |\Psi(x)|} &= \lim_{z\in\mathbf{C}\to\infty} \frac{\log\ \rho(z)}{\log\ |z|} \\
&= 1 - \frac{1}{2\pi}\int_{\mathbf{C}} K_G\ \mathrm{dvol}(G).
\end{aligned}
$$

We note that

$$
\begin{aligned}
1 - \frac{1}{2\pi}\int_{\mathbf{C}} K_G\ \mathrm{dvol}(G) &= \lim_{t\to\infty} \frac{\mathrm{Length}(S_t)^2}{4\pi\ \mathrm{Area}(B_t)} \\
&= \lim_{t\to\infty} \frac{\mathrm{Area}(B_t)}{\pi t^2} \\
(1.6) \qquad &= \lim_{t\to\infty} \frac{\mathrm{Length}(S_t)}{2\pi t} \\
&= \frac{1}{\pi}\ \mathrm{diam}(M(\infty)) \\
&= \chi(M) - \frac{1}{2\pi}\int_M K_M\ \mathrm{dvol}(g_M)
\end{aligned}
$$

(cf. [20: Proposition 4.1], [26]). Let h be a nonconstant harmonic function on M. Since the flux of the restriction of h to $M-K$ $(=\mathbf{C}-D_R)$ vanishes, there exists a harmonic function H on $\mathbf{C}$ such that $|H-h|$ is bounded on $\mathbf{C}-D_R$ (cf. [2: Chap.III]). Hence if h is of finite growth, then we have by (1.5) and (1.6)

$$
(1.7) \qquad \mathrm{ord}(h) = \lim_{x\in M\to\infty} \frac{\log\ |h(x)|}{\log\ r(x)} = \frac{n\pi}{\mathrm{diam}(M(\infty))},
$$

where $n := \lim_{|z|\to\infty} \log|H(z)|/\log|z| \in \{1,2,\cdots\}$. Moreover, for any harmonic function f on $M-K$ the flux of which vanishes, there exists a harmonic function F on M such that $|F-f|$ is bounded on $M-K$ (cf. [2: Chap.III]). Thus it follows from (1.7) that the dimension of harmonic functions h with $\mathrm{ord}(h) \le n\pi/\mathrm{diam}(M(\infty))$ is equal to $2n+1$. This completes the proof of the second assertion of Theorem A. //

Remark. As we have seen in the above proof for Theorem A(ii), the same assertion holds for a complete Riemannian manifold of dimension 2 with finite total curvature and one end, if we replace $\mathrm{diam}(M(\infty))$ in the theorem with $\lim_{t\to\infty} \mathrm{Length}(S_t)^2/(4\ \mathrm{Area}(B_t))$ $(= \lim_{t\to\infty} \mathrm{Area}(B_t)/t^2 = \lim_{t\to\infty} \mathrm{Length}(S_t)/2t = \chi(M) - \frac{1}{2\pi}\int_M K_M)$.

Let us now conclude this section with a corollary and a remark on it.

Corollary. *Let M be a complete connected noncompact Riemannian manifold such that the sectional curvature is bounded from below by $c/r^2 \log r$ outside a compact set, where c is a positive constant and r is the distance to a fixed point of M. Then M has no nonconstant harmonic functions of finite growth, if M has only one end.*

Proof. This follows immediately from Theorem A(i), because $M(\infty)$ consists of only one point (cf. [19: Proposition 5.2]).

Remark. In the above corollary, if M has more than one end, then M may admit nonconstant bounded harmonic functions. Actually, it is easy to construct such manifolds.

§2. Proof of Theorem B

The purpose of this section is to show Theorem B. To begin with, we shall prove the following

Lemma 2.1. *Let N be a complete connected Riemannian manifold of nonnegative sectional curvature. Let h be a nonconstant harmonic function on the Riemannian product $\mathbf{R} \times N$ with $\sup |dh| < +\infty$, and let t be the projection : $\mathbf{R} \times N \to \mathbf{R}$. Then $\langle dt, dh \rangle$ is constant on $\mathbf{R} \times N$ and the restriction of h to $\{t\} \times N$ is harmonic on $\{t\} \times N$. In particular, if N is compact, then $h = ct$ for some constant c.*

Proof. Since $\langle dt, dh \rangle$ is a bounded harmonic function on $\mathbf{R} \times N$, $\langle dt, dh \rangle$ must be constant (cf. Yau [31]), so that, in particular, the derivative of $\langle dt, dh \rangle$ in the direction of grad t vanishes identically. This shows that the restriction of h to $\{t\} \times N$ is harmonic. This completes the proof of Lemma 2.1. //

Lemma 2.2. *Let M be a complete, connected noncompact Riemannian manifold of nonnegative sectional curvature. Suppose M admits a nonconstant harmonic function h which satisfies:*

$$|dh|(x) \longrightarrow c_1, \tag{2.1}$$

$$r(x)\, |\nabla dh|(x) \longrightarrow 0 \tag{2.2}$$

as $x \in M$ goes to infinity, where c_1 is a positive constant and $r(x)$ denotes as usual the distance to a fixed point of M. Then the second *derivative ∇dh of h vanishes identically and moreover M splits isometrically into $\mathbf{R} \times M'$ along the gradient vector ∇h of h.*

Proof. According to the splitting theorem by Toponogov [27], M has one end (namely, M is connected at infinity) or M is isometric to $\mathbf{R} \times M'$, where M' is compact. If the latter case occurs, then Lemma 2.2 is obvious (cf. Lemma 2.1). Hence in what follows, we assume that M has one end, and further that c_1 is equal to 1 for simplicity. Define a vector field Λ on the open set $U := \{x \in M : \nabla h(x) \neq 0\}$ by $\Lambda := \nabla h / |\nabla h|^2$, and for a point $x \in U$, denote by $\lambda_x(t)$ $(-\infty \leq \underline{\tau}_x < t < \bar{\tau}_x \leq +\infty)$ the maximal integral curve of Λ such that $\lambda_x(0) = x$. Then by (2.1), it is not hard to see that for some point $x \in U$, the integral curve $\lambda_x(t)$ is defined for all t and the length is bounded away from zero. We fix such a point x. Now we claim first that

$$\lim_{t \to \pm\infty} \frac{1}{|t|} \operatorname{dis}_M(x, \lambda_x(t)) = 1. \tag{2.3}$$

In fact, let $\sigma_t : [0, a_t] \to M$ be a distance minimizing geodesic joining $x = \sigma_t(0)$ with $\lambda_x(t) = \sigma_t(a_t)$ $(a_t := \operatorname{dis}_M(x, \lambda_x(t)))$. Consider the case: $t > 0$. Then we have

$$\begin{aligned} t &= h(\lambda_x(t)) - h(x) = h(\sigma_t(a_t)) - h(\sigma_t(0)) \\ &= \int_0^{a_t} < \nabla h, \dot{\sigma}_t(s) > ds < a_t, \end{aligned}$$

since $|\nabla h|^2$ is subharmonic (i.e., $\triangle |\nabla h|^2 = 2|\nabla dh|^2 + 2\operatorname{Ric}_M(\nabla h, \nabla h) \geq 0$) and so $|\nabla h| < \sup |\nabla h| = 1$. On the other hand, we get

$$\begin{aligned} a_t &\leq \text{the length of } \lambda_{x|[0,t]} \\ &= \int_0^t \frac{1}{|\nabla h|(\lambda_x(s))} \, ds. \end{aligned}$$

Therefore we have

$$\begin{aligned} 1 \leq & \liminf_{t \to \infty} \frac{a_t}{t} \leq \limsup_{t \to \infty} \frac{a_t}{t} \leq \\ & \limsup_{t \to \infty} \frac{1}{t} \int_0^t \frac{1}{|\nabla h|(\lambda_x(s))} \, ds \leq \limsup_{t \to \infty} \frac{1}{|\nabla h|(\lambda_x(t))} = 1. \end{aligned}$$

Thus we have shown (2.3) in case: $t > 0$. The same argument can be applied to the case: $t < 0$.

Let us next claim

$$\lim_{t \to \infty} \frac{1}{t} \operatorname{dis}_M(\lambda_x(t), \lambda_x(-t)) = 2. \tag{2.4}$$

In fact, let $\eta_t : [0, b_t] \to M$ be a distance minimizing geodesic joining $\eta_t(0) = \lambda_x(-t)$ with $\eta_t(b_t) = \lambda_x(t)$. Then by (2.3), we have

$$(2.5)\quad \limsup_{t\to\infty} \frac{b_t}{t} \le \limsup_{t\to\infty} \frac{1}{t}\{\mathrm{dis}_M(x, \lambda_x(t)) + \mathrm{dis}_M(x, \lambda_x(-t))\} = 2.$$

On the other hand, if $\mathrm{dis}_M(x, \eta_t([0, b_t]))/t = \mathrm{dis}_M(x, \eta_t(c_t))/t$ tends to zero as $t \to +\infty$, then we have

$$(2.6)\quad \begin{aligned} \liminf_{t\to+\infty} \frac{b_t}{t} \ge& \liminf_{t\to+\infty} \frac{1}{t}\ \{\mathrm{dis}_M(x, \lambda_x(t)) - \mathrm{dis}_M(x, \eta_t(c_t))\} + \\ & \liminf_{t\to+\infty} \frac{1}{t}\ \{\mathrm{dis}_M(x, \lambda_x(-t)) - \mathrm{dis}_M(x, \eta_t(c_t))\} \\ =& 2. \end{aligned}$$

Moreover if $\mathrm{dis}_M(x, \eta_{t(i)}(c_{t(i)}))/t(i) > d > 0$ for some divergent sequence $\{t(i)\}$ and a positive constant d, then by the assumption (2.2), we have

$$(2.7)\quad |\nabla dh(\dot\eta_{t(i)}(s), \dot\eta_{t(i)}(s))| \le \frac{\delta(dt(i))}{dt(i)} \quad (0 \le s \le b_{t(i)}),$$

where $\delta(u)$ goes to zero as $u \to +\infty$. Hence we get

$$\begin{aligned} 2 &= \frac{1}{t(i)} \int_0^{b_{t(i)}} \frac{d}{ds}\ h(\eta_{t(i)}(s))\ ds \\ &= \frac{1}{t(i)} \left(\int_0^{b_t(i)} \int_0^s \nabla dh(\dot\eta_{t(i)}(u), \dot\eta_{t(i)}(u))\ duds + b_{t(i)}\langle \nabla h, \dot\eta_{t(i)}(0)\rangle \right) \\ &\le \frac{\delta(dt(i))}{2d} \left(\frac{b_{t(i)}}{t(i)}\right)^2 + \left(\frac{b_{t(i)}}{t(i)}\right) \quad \text{(by (2.7) and} |\nabla h| < 1). \end{aligned}$$

This shows that

$$(2.8)\quad \liminf_{t(i)\to+\infty} \frac{b_{t(i)}}{t(i)} \ge 2.$$

Thus (2.4) follows from (2.5), (2.6) and (2.8).

We are now in a position to complete the proof of Lemma 2.2. Let $\sigma_t : [0, a_t] \to M$, $\sigma_{-t} : [0.a_{-t}] \to M$, and $\eta_t : [0, b_t] \to M$ be as above. For each (s, u) $(0 \le s \le a_t,\ 0 \le u \le a_{-t})$, let $\triangle_t(s, u)$ be the triangle sketched on $\mathbf{R}^2$ whose edge lengths are s, u, and $\mathrm{dis}_M(\sigma_t(s), \sigma_{-t}(u))$, and denote by $\theta_t(s, u)$ the angle of $\triangle_t(s, u)$ opposite to the edge of length $\mathrm{dis}_M(\sigma_t(s), \sigma_{-t}(u))$. Then by a theorem due to Toponogov [28: Lemma

19], we see that $\theta_t(s,u) \le \theta_t(s',u')$ if $s' \le s$ and $u' \le u$. Note that by (2.4)

$$\lim_{t\to+\infty} \theta_t(a_t, a_{-t}) = \pi.$$

This shows that for any $s, u \in (0,\infty)$, we have

$$\lim_{t\to+\infty} \theta_t(s,u) = \pi. \tag{2.9}$$

If we take a divergent sequence $\{t(i)\}$ such that $\sigma_{t(i)}$ (resp. $\sigma_{-t(i)}$) converges to a ray $\sigma_\infty : [0,\infty) \to M$ (resp., a ray $\sigma_{-\infty} : [0,\infty) \to M$) starting at x, and if we define a curve $\xi : \mathbf{R} \to M$ by $\xi(t) = \sigma_\infty(t)$ for $t \ge 0$ and $\xi(t) = \sigma_{-\infty}(-t)$ for $t \le 0$, then it turns out from (2.9) that ξ is a line, namely, ξ is a distance minimizing geodesic defined on $\mathbf{R}$. Thus it follows from the Toponogov splitting theorem that M is isometric to $\xi(\mathbf{R}) \times M'$. Now it is clear from Lemma 2.1 and the above construction of the line ξ that for some constant c, $h((t,x')) = t + c$ on $M = \xi(\mathbf{R}) \times M'$. This completes the proof of Lemma 2.2.

Finally we need the following

Lemma 2.3. *Let M and ω be as in Theorem* B. *Then $|\omega|(x)$ tends to a constant $c_1 > 0$ and $r(x)|\nabla\omega|(x)$ converges to zero, as $x \in M$ goes to infinity, where $r(x)$ denotes the distance to a fixed point, say o of M.*

Proof. We first observe that $|\omega|^2$ is subharmonic on M, by the Weitzenböck's formula:

$$\triangle|\omega|^2 = 2|\nabla\omega|^2 + 2\ \mathrm{Ric}_M(\omega^\#, \omega^\#) \tag{2.10}$$

($\omega^\# :=$ the dual vector field of ω). Set $m(t) :=$ the maximum of $|\omega|$ on the metric sphere S_t around o with radius t. Then it follows from the maximum principle for subharmonic functions that $m(t)$ is nondecreasing, and hence $m(t)$ converges to a positive constant c_2 as t goes to infinity. For the sake of simplicity, we assume that $c_2 = 1$. Let us here take points $\{x_t\}$ of M such that $x_t \in S_t$ and $|\omega|(x_t)$ converges to 1 as $t \to \infty$. Choosing an orthonormal basis of the tangent space $T_{x_t}M$ of M at each x_t, we identify $T_{x_t}M$ with Euclidean space $\mathbf{R}^m$, and write $\mathbf{B}_R$ for the ball of $\mathbf{R}^m$ around the origin with radius R. Then by the assumption (H.2) in Theorem B, we can fix a sufficienty small constant $a > 0$ so that for each x_t, the restriction Ψ_t of the exponential map $\exp_{x_t} : \mathbf{R}^m\ (= T_{x_t}M) \to M$ to $\mathbf{B}_{at}$ induces a smooth map of maximal rank from $\mathbf{B}_{at}$ onto the metric ball $B_{at}(x_t)$ of M around x_t with radius at. Define a family of Riemannian metrics $\{g_t\}$ on $\mathbf{B}_a$ by

$g_t := \frac{1}{t^2}\Psi_t^* g_M$, where g_M denotes the Riemannian metric on M. Then (H.2) implies that the sectional curvature of g_t is bounded uniformly in t. Hence, choosing a smaller constant a if necessarily and taking harmonic coordinates appropriately around the origin with respect to g_t, we can see that the coefficients of g_t (with respect to the harmonic coordinates) have $C^{1,\alpha}$-Hölder norms $(0 < \alpha < 1)$ and $W^{2,p}$-Sobolev norms bounded uniformly in t (cf. e.g., [14], [20]). Thus we can assert that

(2.11) : for any divergent sequence $\{t(i)\}$, there exists a subsequence $\{t(j)\}$ of $\{t(i)\}$ such that $g_{t(j)}$ converges to $C^{1,\alpha}$ Riemannian metric g_∞ on $\mathbf{B}_a$ in the $C^{1,\alpha}$-norm with respect to the harmonic coordinates. Moreover the coefficients of g_∞ belong to the Sobolev space $W^{2,p}$ $(p \geq 1)$.

Let us now define a family of 1-forms ω_t on $\mathbf{B}_a$ by $\omega_t := \frac{1}{t}\Psi_t^*\omega$. Then ω_t is a d-closed harmonic 1-form such that the length $|\omega_t|$ (with respect to g_t) satisfies: $|\omega_t| < 1$ and $|\omega_t(o)| \to 1$ as $t \to \infty$. Since $\mathbf{B}_a$ is simply connected, there exists a smooth function h_t on $\mathbf{B}_a$ with $\omega_t = dh_t$. Here we may assume that $h_t(o) = 0$. Hence $|h_t|$ is bounded uniformly in t. Moreover since the coefficients of g_t (with respect to the harmonic coordinates) have bounded $C^{1,\alpha}$-norms uniformly in t, it follows from the a priori estimates that the $C^{2,\alpha}$-norms of h_t is bounded uniformly in t. Thus by (2.11), we see that for any divergent sequence $\{t(i)\}$, there exists a subsequence $\{t(j)\}$ such that in the $C^{2,\alpha}$-norm (with respect to the harmonic coordinates), $h_{t(j)}$ converges to a $C^{2,\alpha}$ function h_∞ which is harmonic with respect to g_∞. We put here $\omega_\infty := dh_\infty$. Then the length $|\omega_\infty|$ (with respect to g_∞) satisfies: $|\omega_\infty| \leq 1$ and $|\omega_\infty|(o) = 1$. Since $|\omega_t|^2$ is subharmonic (with respect to g_t), so is $|\omega_\infty|^2$ (with respect to g_∞). Hence applying the maximum principle to $|\omega_\infty|^2$, we see that $|\omega_\infty|$ is constantly equal to 1. Noting that (2.10) holds for each ω_t, and $\omega_{t(j)}$ (resp. $g_{t(j)}$) converges to ω_∞ (resp. g_∞) in the $C^{1,\alpha}$-norm as $t(j) \to \infty$, we have the identity (2.10) for ω_∞ in a weak sense. Namely, for any smooth function η with compact support in $\mathbf{B}_a$,

(2.12)
$$\int g_\infty(d|\omega_\infty|^2, d\eta)\,\mathrm{dvol}(g_\infty) = -2\int \{|\nabla_\infty\omega_\infty|^2 + \mathrm{Ric}_\infty(\omega_\infty^\#, \omega_\infty^\#)\}\eta\ \mathrm{dvol}(g_\infty).$$

Here we have used the fact that g_∞ has the Ricci tensor Ric_∞ in the L^p-sense $(p \geq 1)$ and the Ricci tensor $\mathrm{Ric}_{t(j)}$ of $g_{t(j)}$ converges weakly to Ric_∞ as $t(j) \to \infty$. Since the left-hand side of (2.12) vanishes, we see that $|\nabla_\infty \omega_\infty|^2 + \mathrm{Ric}_\infty(\omega_\infty^\#, \omega_\infty^\#) = 0$ almost everywehre and hence ω_∞ is parallel. Thus we have shown that if we take points $x_t \in S_t$ with $\lim_{t\to\infty} |\omega|(x_t) = 1$, then

$$\begin{aligned} &\max\{1 - |\omega|(x) : x \in B_{at}(x_t)\} \longrightarrow 0, \\ &\max\{r(x)|\nabla\omega|(x) : x \in B_{at}(x_t)\} \longrightarrow 0, \end{aligned} \tag{2.13}$$

as t goes to infinity. Since the diameter of S_t with respect to the inner distance on S_t is bounded by bt for some constant b, (2.13) proves Lemma 2.3. //

We are now in a position to complete the proof of Theorem B. Let M and ω be as in Theorem B, and let $\Pi : \widetilde{M} \to M$ be the universal covering of M. Set $\widetilde{\omega} := \Pi^*\omega$. Then there is a harmonic function h on $\widetilde{M}$ which satisfies: $\widetilde{\omega} = dh$. Therefore if the fundamental group $\pi_1(M)$ of M is finite, then $\widetilde{M}$ also satisfies assumption (H.2), and hence by Lemmas 2.2 and 2.3, ∇dh vanishes identically and $\widetilde{M}$ splits isometrically into $\mathbf{R} \times M'$ along the gradient ∇h of h. Moreover in this case, M' is flat, because the sectional curvature of M decays to zero. We shall now consider the case that $\pi_1(M)$ is infinite. Let Σ be a soul of M (i.e., a compact, totally geodesic and totally convex submanifold of M). Then by Theorem 9.1 in [7], $\widetilde{\Sigma} := \Pi^{-1}(\Sigma)$ splits isometrically into $\mathbf{R}^k \times \widetilde{\Sigma}_o$, where $\widetilde{\Sigma}_o$ is a compact simply connected manifold of nonnegative curvature and furthermore $k \geq 1$, because $\pi_1(M) = \pi_1(\Sigma)$ is infinite. Hence $\widetilde{M}$ is isometric to the Riemannian product $\mathbf{R}^k \times \widetilde{M}_o$ of Euclidean space $\mathbf{R}^k$ and a complete, noncompact simply connected manifold $\widetilde{M}_o$ with nonnegative sectional curvature. We observe here that the sectional curvature of $\widetilde{M}_o$ decays in quadratic order, since $\widetilde{M}_o$ is compact. Now it follows from Lemma 2.1 that the restriction $\widetilde{h}$ of h to $\{o\} \times \widetilde{M}_o$ is constant or it gives a nonconstant harmonic function on $\widetilde{M}_o$, the gradient of which has bounded length. If the former case occurs, then it is clear that h is totally geodesic. When the latter case occurs, we can apply Lemmas 2.2 and 2.3 and show that h is totally geodesic. This completes the proof of Theorem B. //

Corollary. *Let M be as in Theorem* B. *Suppose that the Ricci curvature of M is positive somewhere. Then any d-closed harmonic* 1-*form with bounded length must be zero.*

Proof. This is clear from the above proof of Theorem B. //

§3. Some other results

Let M be a manifold of asymptoticallty nonnegative curvature. In this section, we shall make some observations on the asymptotic behavior of harmonic functions on M with finite growth and then that of the Green function on M, under certain additional conditions. Throughout this section, the dimension m of M is assumed to be greater than two. First we recall the following

Fact 3.1 (cf. [20: Lemma 2.3]). Let M be as above and δ an end of M. Suppose that the sectional curvature K_M of M decays in quadratic order on the end δ, i.e.,

$$K_M \leq \frac{c}{r^2} \quad on \quad \delta, \quad and \tag{3.1}$$

$$\mathcal{V}o\ell(M_\delta(\infty)) > 0 \tag{3.2}$$

where c is a positive constant and r denotes the distance to a fixed point of M. Then :

(i) $M_\delta(\infty)$ is a compact, connected smooth manifold with $C^{1,\alpha}$ Riemannian metric g_∞ $(0 < \alpha < 1)$.

(ii) Fix two positive numbers a, b with $a > b$, and set $A_t(a,b) := \{x \in M : b < r(x)/t < a\}$ for $t > 0$. If t is sufficienty large, then there exists a $C^{2,\alpha}$ diffeomorphism Π_t from $A_t(a,b) \cap \delta$ into the cone $\mathcal{C}(M_\delta(\infty))$ over $M_\delta(\infty)$ (i.e., $\mathcal{C}(M_\delta(\infty)) := (0,\infty) \times_{t^2} M_\delta(\infty)$) which has the following properties: as t goes to infinity, $\Pi_t(A_t(a,b) \cap \delta)$ converges to $(b,a) \times M_\delta(\infty)$ and $\frac{1}{t^2}\Pi_{t*}g_M$ also converges to the metric $dt^2 + t^2 g_\infty$ in $C^{1,\alpha'}$ topology $(0 < \alpha' < \alpha < 1)$. Here g_M stands for the Riemannian metric of M.

Let us now prove the following

Proposition C. *Let M be a manifold of asymptotically nonnegative curvature and δ an end of M. Suppose* (3.1) *and* (3.2) *hold for the end δ. Then if there exists a harmonic function h defined on δ such that* $0 < \limsup_{X \in \delta \to \infty} |h(x)|/r(x)^p < +\infty$ *for some positive constant p, then* $p(p+m-2)$ $(m := \dim\ M \geq 3)$ *is an eigenvalue of $M_\delta(\infty)$. Moreover* $p \geq 1$ *and if $p = 1$, then $M_\delta(\infty)$ is isometric to the $(m-1)$-sphere $S^{m-1}(1)$ of constant curvature* 1.

To prove Proposition C, we need the following

Fact 3.2. Let h be a nonconstant harmonic function on the cone $\mathcal{C}(M_\delta(\infty))$ $(= (0,\infty) \times {}_{t^2} M_\delta(\infty))$ over $M_\delta(\infty)$ such that $|h(t,\theta)|/t^p$ is bounded on $\mathcal{C}(M_\delta(\infty))$ for some $p > 0$. Then $\lambda := p(p+m-2)$ is equal to an eigenvalue of $M_\delta(\infty)$ and $h(t,\theta)/t^p$ defines an eigenfunction of $M_\delta(\infty)$ with eigenvalue λ.

Proof. For the convenience of the reader, we shall give a proof of the fact. Let $\phi(s,\theta)$ ($s = \log\, t$) be a function on $\mathbf{R} \times M_\delta(\infty)$ defined by $\phi(s,\theta) := \mathrm{e}^{-ps}\, h(\mathrm{e}^s,\theta)$. Then ϕ satisfies:

$$\frac{\partial^2\phi}{\partial s^2} + (2p+m-2)\,\frac{\partial\phi}{\partial s} + p(p+m-2)\phi + \triangle_\infty\phi = 0,$$

where $\triangle_\infty$ denotes the Laplacian on $M_\delta(\infty)$. Let $\{\mu_i\}_{i=1,2,\ldots} : \mu_1 \le \mu_2 \le \ldots$ be the eigenvalues of $M_\delta(\infty)$ and $\{E_i(\theta)\}_{i=12,\ldots}$ an orthonormal system of eigenfunctions on $M_\delta(\infty)$ corresponding to $\{\mu_i\}$. Set $\phi_i(s) := \int_{M_\delta(\infty)} \phi(s,\theta)E_i(\theta)\,\mathrm{dvol}(g_\infty)$ $(i = 1,2,\ldots)$. Then ϕ_i obeys the following ordinary differential equation on $\mathbf{R}$:

$$\phi_i'' + (2p+m-2)\phi_i' + (p(p+m-2) - \mu_i)\phi_i = 0.$$

Since $|h(s,\theta)|/t^p$ is bounded, so is $|\phi(s,\theta)|$. Hence each ϕ_i is also bounded. Then it turns out that ϕ_i is equal to a constant a_i which is zero unless $\mu_i = p(p+m-2)$, so that $\phi(s,\theta) = \sum_i a_iE_i(\theta)$, where the summation is taken over the indices i's with $\mu_i = p(p+m-2)$. This proves Fact 3.2. //

Proof of Proposition C. Let M, h and p be as in the proposition. Let us first fix a positive integer n and a sufficiently large R for a while, and define a function h_R on $\Pi_R(A_R(n,n^{-1}))$ $(\subset \mathcal{C}(M_\delta(\infty))$ by $h_R := h \circ \Pi_R^{-1}/R^p$, where Π_R and A_R are as in Fact 3.1. Then h_R is harmonic with respect to the metric $\frac{1}{R^2}\Pi_{R*}g_M$. Moreover since $\mu := \limsup_{x\in\delta\to\infty}|h|(x)/r^p(x)$ is finite $|h_R|$ is bounded from above by cn^p for some positive constant c independent of R and n. Thus it follows from Fact 3.1 and the a priori estimates that the $C^{2,\alpha}$-Hölder norm of h_R is bounded uniformly in R. Let us take here a divergence sequence $\{R(i)\}$ such that $\max\{|h(x)| : x \in S_{R(i)} \cap \delta\}/R(i)^p$ converges to $\mu > 0$ as $R(i)$ goes to infinity. Then we can take inductively a subsequence $\{R(n,j)\}$ of $\{R(i)\}$ so that $\{R(n+1,j)\} \subset \{R(n,j)\}$ and as $j \to \infty$, $h_{R(n,j)}$ converges to a harmonic function h_n on $A_\infty(n,n^{-1}) := \{(t,\theta) \in \mathcal{C}(M_\delta(\infty)) : n^{-1} < t < n\}$ in the $C^{2,\alpha}$-Hölder norm. Note that h_n satisfies: $|h_n(t,\theta)| \le ct^p$ on $A_\infty(n,n^{-1})$. Hence if we set $h_\infty := h_n$ on $A_\infty(n,n^{-1})$, then we get a harmonic function h_∞ on $\mathcal{C}(M_\delta(\infty))$ such that $|h_\infty(t,\theta)| \le ct^p$. By the choice of $\{R(i)\}$, we see that h_∞ does not vanish identically. Thus it turns out from Fact 3.2 that $\lambda := p(p+m-2)$ must be an eigenvalue of $M_\delta(\infty)$ and $h_\infty(t,\theta)/t^p$ gives an eigenfunction on $M_\delta(\infty)$ with the eigenvalue λ. Finally the remaining assertion of Proposition C follows from Lemma 3.3 below. //

Lemma 3.3. *The first eigenvalue μ_1 of $M_\delta(\infty)$ is greater than or equal to $m-1$. Moreover if $\mu_1 = m-1$, then $M_\delta(\infty)$ is isometric to the $(m-1)$-sphere $S^{m-1}(1)$ of constant curvature 1.*

Proof. Let $\Pi_t : A_t(a,b) \to \mathcal{C}(M_\delta(\infty))$ be as in Fact 3.1. Set $M_t := \Pi_t^{-1}(\{1\} \times M_\delta(\infty))$. Then we observe that the sectional curvature K_t of M_t satisfies: $1 - \varepsilon_1(t) \le K_t \le 1 + \varepsilon_1(t) + \kappa_\delta$, where $\varepsilon_1(t) > 0$ goes to zero as $t \to \infty$ and $\kappa_\delta := \limsup_{x \in \delta \to \infty} r(x)^2 K_M(x)$. Let $\mu_{t,1}$ be the first eigenvalue of M_t. Then applying the Lichnerowicz' theorem (cf. [10]) to M_t, we see that $\mu_{t,1} \ge (m-1) - \varepsilon_2(t)$, where $\varepsilon_2(t) > 0$ tends to zero as $t \to \infty$. This implies that $\mu_1 \ge (m-1)$. Suppose that $\mu_1 = (m-1)$. Then the diameter of $M_\delta(\infty)$ must take the maximum value π. In fact if the diameter is less than π, then the diameter of M_t is less than $\pi - \varepsilon_3$ for large t and some positive constant ε_3. It follows now from [10] that $\mu_{t,1} \ge (m-1) + \varepsilon_4$ for large t and some positive constant ε_4. This is a contradiction. Thus $M_\delta(\infty)$ has the maximum diameter π, so that the volume of $M_\delta(\infty)$ must be equal to the volume of $S^{m-1}(1)$ (cf. [18: Theorem 4.1] or [5]). Then it turns out from a theorem by Katsuda [22] that the Hausdorff distance between $M_\delta(\infty)$ and $S^{m-1}(1)$ is equal to zero, namely, $M_\delta(\infty)$ is isometric to $S^{m-1}(1)$. This completes the proof of Lemma 3.3. //

Let us now show a proposition on the minimal positive Green function $G(x,y)$ on $M \times M$. According to Li-Tam [24], we call an end δ of M *large* (resp., *small*) if the integral $\int^\infty t V_\delta(t)^{-1} dt$ is finite (resp., infinite), where $V_\delta(t) := \mathrm{Vol}_m(B_t \cap \delta)$. Suppose that M has at least one large end δ. Then based on some of the results in [19] and the arguments in [24;25], we have shown in [21] the following results:

(3.3) There exists a unique positive harmonic function h_δ on M such that $\lim_{x \in \delta \to \infty} h_\delta(x) = 1$ and $\lim_{y \in \delta' \to \infty} h_{\delta'}(y) = 0$ for another large end δ' (if any).

(3.4) There exists a unique minimal positive Green function $G(x,y)$ on $M \times M$ such that

$$G(x,y) \le c(x) \int_{\mathrm{dis}_M(x,y)}^\infty \frac{t}{V_\delta(t)}\, dt$$

for all $y \in \delta - B_{R(x)}$, and $G(x,y) \longrightarrow c(x,\mathcal{D})$ as $y \in \mathcal{D} \longrightarrow +\infty$ for a small end $\mathcal{D}$ (if any). Here the constants $R(x), C(x)$ and $C(x,\mathcal{D})$ are positive constants depending on the quantities in parentheses.

We remark that the value $h_\delta(x)$ of the function h_δ at a point x is equal to the hitting probability of the paths starting at x to the large end δ. Moreover as we mentioned in [21], we see that if $G(x,y) / \int_{\mathrm{dis}_M(x,y)}^{\infty} m^{-1}tV_\delta(t)^{-1}dt$ converges to $h_\delta(x)$ as $y \in \delta$ goes to infinity for *some* x, then this holds for *all* $x \in M$. It is unclear whether the limit should exist and be equal to $h_\delta(x)$ for some x. The following proposition answers this question partially.

Proposition D. *Let M be an m-dimensional manifold of asymptotically nonnegative curvature which has at least one large end δ. Suppose* (3.1) *and* (3.2) *hold for δ. Then for any point x of M, one has*

$$\frac{G(x.y)}{\int_{\mathrm{dis}_M(x,y)} \frac{t}{mV_\delta(t)}dt} \longrightarrow h_\delta(x)$$

as $y \in \delta$ goes to infinity. In particular, in this case, one has

$$G(x,y)\ \mathrm{dis}_M(x,y)^{m-2} \longrightarrow \frac{h_\delta(x)}{(m-2)\,\mathrm{Vol}(M_\delta(\infty))}$$

as $y \in \delta$ goes to infinity.

Proof. We fix a point x of M. We first observe that for some positive constants c_1 and c_2,

$$c_1 \leq G(x,y)\ \mathrm{dis}_M(x,y)^{m-2} \leq c_2 \tag{3.5}$$

on δ. The first inequality is a consequence of the assumption that M has asymptotically nonnegative curvature (cf. [17: Theorem 4.3]) and the second one follows from (3.4). Set $G_R(y) := R^{m-2}G(x,y)$. Then by the same argument as in the proof of Proposition C, we see that given a divergent sequence $\{R(i)\}$, there exists a subsequence $\{R(j)\}$ for which $G_{R(j)}$ converges as $j \to \infty$ to a harmonic function G_∞ on compact sets of the cone $\mathcal{C}(M_\delta(\infty)) = (0,\infty) \times_{t^2} M_\delta(\infty)$ in the $C^{2,\alpha}$ Hörder norm. By (3.5), we have

$$c_1 \leq G_\infty(t,\theta)t^{m-2} \leq c_2$$

for any $(t,\theta) \in \mathcal{C}(M_\delta(\infty))$. Moreover it turns out from the same argument as in Lemma 3.2 that $G_\infty(t,\theta)t^{m-2}$ is in fact a constant, say c_3. Then it is not hard to see that the constant c_3 is given by

$c_3(m-2)\mathcal{V}o\ell(M_\delta(\infty)) = h_\delta(x)$. Thus the constant c_3 is independent of the choice of a divergent sequence $\{R(i)\}$. This shows that

$$G(x,y)\ \mathrm{dis}_M(x,y)^{m-2} \longrightarrow \frac{h_\delta(x)}{(m-2)\mathcal{V}o\ell(M_\delta(\infty))}$$

as $y \in \delta$ goes to infinity. Since

$$\mathrm{dis}_M(x,y)^{m-2} \int_{\mathrm{dis}_M(x,y)}^{\infty} \frac{t}{V_\delta(t)}\, dt \longrightarrow \frac{m}{(m-2)\mathcal{V}o\ell(M_\delta(\infty))}$$

as $y \in \delta$ goes to infinity, we have proven Proposition D. //

Remark. Let M and δ be as in Proposition D. Define a function $F_\delta(y)$ on M by $F_\delta(y) := c_4 G(o,y)^{1/(2-m)}$, where o is a fixed point of M and $c_4 := \left(h_\delta(o)/((m-2)\mathcal{V}o\ell(M_\delta(\infty)))\right)^{1/(m-2)}$. Then we can prove by using the same argument as in the proof of Proposition D that as $y \in \delta$ goes to infinity,

(i) $$\frac{F_\delta(y)}{\mathrm{dis}_M(o,y)} \longrightarrow 1,$$

(ii) $$|\nabla F_\delta|(y) \longrightarrow 1,$$

(iii) $$\left|\frac{1}{2}\ \nabla dF_\delta^2 - g_M\right| \longrightarrow 0,$$

where g_M denotes the Riemannian metric of M. Thus F_δ gives a nice smooth approximation for the distance function $r = \mathrm{dis}_M(o,*)$ on the end δ.

Added in proof. Theorem B does not hold for a complete, noncompact Riemannian manifold of nonnegative Ricci curvature (even if the sectional curvature decays quadratically).

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