

## On Eisenstein's Copy of the *Disquisitiones*

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*To Kenkichi Iwasawa as a token of friendship*

In 1979, Professor B. Artmann, of the *Technische Hochschule*, Darmstadt, having become aware of my deep interest in Eisenstein, very kindly informed me of an interesting discovery he had made some fifteen years earlier, while browsing in the library of the Mathematical Institute of Giessen University. As he explained, he had casually taken down from the shelf a bound volume bearing the title “GAUSS, *Disquisitiones Arithmeticae*”. This turned out to have been Eisenstein's own copy, not of the original edition of the *Disquisitiones*, but of its French translation, *Recherches Arithmétiques*, published in Paris in 1807 ([1]). The volume was interfoliated; the first blank page bore Eisenstein's name, and the others bore copious annotations in his hand. Bound with the *Recherches* were two of Dirichlet's most famous papers ([2a, b]), perhaps a gift from Dirichlet, whose lectures Eisenstein had attended as early as 1840.

Of course, after this discovery, the book was put under lock and key; but Professor Artmann indicated that the librarian, Frau Helga Bertram, might be willing to let me have it for consultation for a limited time. Naturally I wrote Frau Bertram at once, and, through her kindness, the precious volume arrived promptly in Princeton.

As to its previous history, Frau Bertram told me of the results of her investigations. It has been one of a number of volumes that had belonged to the mathematician Eugen Netto and had come to the Mathematical Institute after his death; Netto had been a professor at Giessen from 1888 to 1913 and had retired there as emeritus. The previous history cannot be ascertained; but it is of interest to note that Netto's teacher at the Friedrich-Werdersche Gymnasium in Berlin had been the same renowned K. Schellbach who had been Eisenstein's teacher there many years before. Thus one may fancy that the book came into Schellbach's hands after Eisenstein's premature death in 1852, and passed from there into those of Netto, who must have seemed a fit recipient for this valuable gift.

Inserted in the volume are a visiting card of Eisenstein's, inscribed "Gotthold Eisenstein, Stud. math.", with some formulas, and some loose sheets, among them an affectionate short letter from his father, also bearing mathematical remarks and formulas. The blank page preceding the title bears the heading:

*Gotthold Eisenstein*

gekauft von Dr. Michaelis 1842 für 2 th. 13½ sgr.  
eingebunden in Sept. 1843 für 25½ sgr.

Thus Eisenstein had bought the book when he was in the final year of the Gymnasium, shortly before leaving for England and Ireland in the summer of 1842, and had the book bound soon after coming back to Berlin. For a young man in Eisenstein's straitened circumstances, this purchase may have been a costly one. As to why he acquired the French translation and not the original Latin edition, it must be simply that the latter had long been out of print and was unavailable at the time; surely Eisenstein could have read Gauss's Latin as easily as his translator's French.

Still on the same page, Eisenstein had inscribed in large capital letters Gauss's famous motto;

PAUCA *sed* MATURA

and a less known one, that he also attributes to Gauss:

*Procreare jucundum, sed parturire molestum*

(i.e.: "to beget is pleasant, but to give birth is painful"). Further, he has this comment on Gauss (his own, or possibly a quotation)

"Gauss, grand géomètre que la postérité placera à côté de Fermat non seulement pour ses admirables découvertes dans la théorie des nombres, mais aussi pour le peu d'empressement qu'il met à faire paraître ses travaux."

As to the annotations that fill up many of the intercalary pages inserted by the bookbinder, they are of varied value and interest. Some are in French, others in German. They begin with a page copied from a paper by Jacobi ([3]); there follows, on five pages, an elementary treatment of the Gaussian ring  $\mathbb{Z}[i]$ , under the title "*Théorie fondamentale des entiers complexes*", obviously the work of a raw but highly talented beginner. Others are drafts for papers published by Eisenstein; this

includes his 1844 paper on cubic forms ([4], pp. 1–5) and his two proofs for the quadratic reciprocity law ([4], pp. 114–116 and pp. 164–166), the latter on a loose sheet, and the former bearing the title “*Eigener Beweis des Fundamentaltheorems*” (concerning which cf. his comments in [4], pp. 477–478). The most extensive texts are those facing Gauss’s theory of genera, of the group of classes of binary forms, and his treatment of ternary quadratic forms; they include results on the number of representations of integers by the form  $x^2 + y^2 + z^2$  and its relation to class-numbers; noteworthy are the observation that “the most remarkable ternary form is  $x^2 - yz$ ”, and a statement facing Gauss’s title “Digression sur les formes ternaires”, that “Im Herbste 1844 habe ich mich selbst mit den ternären, and auch mit den quatern. usw. Formen beschäftigt und habe allgemeine Untersuchungen darüber angestellt, die namentlich die Anzahl der Klassen betreffen”. I confess that I have not analysed those annotations in detail; so far as I have examined them, they would seem to add little of importance to Eisenstein’s record, while throwing an interesting light on the maturing of his genius during the early years of his career.

What chiefly makes the volume noteworthy is the content of the last page, whose purpose is to give a proof of the functional equation for the  $L$ -function modulo 4:

$$L(s) = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-s} = 1 - 3^{-s} + 5^{-s} - 7^{-s} + \dots$$

in the range  $0 < s < 1$ . This will now be given in full, with a translation and a commentary.

| Text                           | Translation                             |
|--------------------------------|---|
| Nach Dirichlet auf seite 9 ist | According to Dirichlet, page 9, one has |

|     |   |
|-----|---|
| (1) | $\int_0^{\infty} e^{\sigma\psi} \psi^{\alpha-1} d\psi = \frac{\Gamma(q)}{(\pm\sigma)^q} e^{\pm q\pi i/2}$ |
|-----|---|

wo  $0 < q < 1$  u. das obere oder untere Zeichen gilt jenachdem  $\sigma$  pos. oder neg. Hieraus, wenn  $\alpha$  zwischen 0 u.  $2\pi$  liegt, wobei also  $\alpha + 2\sigma\pi$  pos. für  $\sigma = 0, 1, 2, \dots$ ,  $\alpha + 2\sigma\pi$  neg. für  $\sigma = -1, -2$  etc.

where  $0 < q < 1$  and the upper or lower sign is valid according as  $\sigma$  is positive or negative. From this, when  $\alpha$  is between 0 and  $2\pi$ , so that  $\alpha + 2\sigma\pi$  is positive for  $\sigma = 0, 1, 2, \dots$ , and negative for  $\sigma = -1, -2, \dots$ , one gets

$$(2) \quad \sum_{\sigma=-\infty}^{\sigma=\infty} \int_0^{\infty} e^{(\alpha+2\sigma\pi)\psi i} f(\sigma) \psi^{q-1} d\psi = \sum_{\sigma=-\infty}^{\sigma=\infty} \int_0^{\infty} f(\sigma) e^{\alpha\psi i} \psi^{q-1} d\psi e^{2\sigma\pi i \psi}$$

$$= \Gamma(q) e^{+(q\pi/2)} \sum_{\sigma=0}^{\sigma=\infty} \frac{f(\sigma)}{(\alpha+2\pi\sigma)^q} + \Gamma(q) e^{-(q\pi/2)} \sum_{\sigma=1}^{\sigma=\infty} \frac{f(-\sigma)}{(2\pi\sigma-\alpha)^q}.$$

Wenn z.B.  $f(\sigma) = (-1)^\sigma$ , so hat man | If e.g.  $f(\sigma) = (-1)^\sigma$ , one has

$$(3) \quad \sum_{\sigma=-k}^{\sigma=k} (-1)^\sigma e^{2\sigma\pi i \psi} = 1 - e^{2\pi i \psi} + e^{4\pi i \psi} \dots + (-1)^k e^{2k\pi i \psi}$$

$$- e^{-2\pi i \psi} + e^{-4\pi i \psi} \dots + (-1)^k e^{-2k\pi i \psi} = \text{etc.}$$

Allgemein  $f(\sigma) = e^{2\beta\pi i \sigma}$ , wo  $\beta$  reell u.  $<1$  anzunehmen ( $\beta$  positiv) | Generally take  $f(\sigma) = e^{2\beta\pi i \sigma}$ , where  $\beta$  is to be taken real and  $<1$  (and  $>0$ )

$$(4) \quad e^{(\alpha+2\sigma\pi)\psi i} f(\sigma) = e^{\alpha\psi i} e^{2\pi i \sigma(\psi+\beta)},$$

sei  $\psi + \beta = \varphi$ , so ist  $\psi = \varphi - \beta$ , u.  $\varphi$  geht von  $\beta$  bis  $\infty$ , also hat man zu finden | here, putting  $\psi + \beta = \varphi$ , one has  $\psi = \varphi - \beta$ , and  $\varphi$  goes from  $\beta$  to  $\infty$ , so that one has to calculate

$$(5a) \quad \sum_{\sigma=-\infty}^{\sigma=\infty} \int_{\beta}^{\infty} e^{\alpha(\varphi-\beta)i} (\varphi-\beta)^{q-1} e^{2\pi i \sigma \varphi} d\varphi,$$

welches, wenn  $0 < \beta < 1$ , | which, for  $0 < \beta < 1$ , is

$$(5b) \quad = \frac{e^{\alpha(1-\beta)i}}{(1-\beta)^{1-q}} + \frac{e^{\alpha(2-\beta)i}}{(2-\beta)^{1-q}} + \frac{e^{\alpha(3-\beta)i}}{(3-\beta)^{1-q}} \dots = e^{-\alpha\beta i} \sum_{\sigma=1}^{\sigma=\infty} \frac{e^{\sigma\alpha i}}{(\sigma-\beta)^{1-q}}.$$

Setzt man noch auf beiden Seiten  $2\alpha\pi$  statt  $\alpha$ , so hat man also | Replacing  $\alpha$  by  $2\pi\alpha$  on both sides, one has therefore

$$(6) \quad e^{-\alpha\beta \cdot 2\pi i} \sum_{\sigma=1}^{\sigma=\infty} \frac{e^{\sigma\alpha \cdot 2\pi i}}{(\sigma-\beta)^{1-q}}$$

$$= \frac{\Gamma(q)}{(2\pi)^q} e^{(q\pi/2)i} \sum_{\sigma=0}^{\sigma=\infty} \frac{e^{\sigma\beta \cdot 2\pi i}}{(\sigma+\alpha)^q} + \frac{\Gamma(q)}{(2\pi)^q} e^{-(q\pi/2)i} \sum_{\sigma=1}^{\sigma=\infty} \frac{e^{\sigma\beta \cdot 2\pi i}}{(\sigma-\alpha)^q}$$

wo  $\alpha$  u.  $\beta$  zwei pos. ächte Brüche sind,  $q$  ist ebenfalls ein pos. ächter Bruch. Man kann hier auch statt  $\beta$  schreiben  $-\beta$ , muss dann aber die Summation links statt von  $\sigma=1$  von  $\sigma=0$  anfangen lassen. Für  $\alpha = \frac{1}{2}$  hat man einfacher

where  $\alpha$  and  $\beta$  are both positive and  $<1$ ; so is  $q$ . Here one can also, instead of  $\beta$ , write  $-\beta$ , but the summation must then begin with  $\sigma=0$  instead of  $\sigma=1$ . For  $\alpha = \frac{1}{2}$ , one gets the simpler formulas

$$(7) \quad e^{-\beta\pi i} \sum_{\sigma=1}^{\infty} \frac{(-1)^\sigma}{(\sigma-\beta)^{1-q}} = \frac{\Gamma(q)}{(2\pi)^q} [e^{(q\pi/2)i} + e^{2\beta\pi i - (q\pi/2)i}] \sum_{\sigma=0}^{\infty} \frac{e^{\sigma\beta \cdot 2\pi i}}{(\sigma+\frac{1}{2})^q}$$

$$(8) \quad e^{\beta\pi i} \sum_{\sigma=0}^{\infty} \frac{(-1)^\sigma}{(\sigma+\beta)^{1-q}} = \frac{\Gamma(q)}{(2\pi)^q} [e^{(q\pi/2)i} + e^{-2\beta\pi i - (q\pi/2)i}] \sum_{\sigma=0}^{\infty} \frac{e^{-\sigma\beta \cdot 2\pi i}}{(\sigma+\frac{1}{2})^q}$$

Wenn auch  $\beta = \frac{1}{2}$  | If also  $\beta = \frac{1}{2}$

$$(9) \quad 1 - \frac{1}{3^{1-q}} + \frac{1}{5^{1-q}} - \frac{1}{7^{1-q}} + \dots$$

$$= \frac{2^q \Gamma(q)}{\pi^q} \sin \frac{q\pi}{2} \left[ 1 - \frac{1}{3^q} + \frac{1}{5^q} - \frac{1}{7^q} + \dots \right]$$

Setzt man hier  $1-q$  statt  $q$  u. | If here one replaces  $q$  by  $1-q$   
 multipl. die beiden Formeln, so | and takes the product of both  
 erhält man | formulas, one obtains

$$\Gamma(q)\Gamma(1-q) = \frac{\pi}{\sin q\pi}$$

Scripti 7 April 1849 MDCCCIL

**Commentary**

[N.B. In the above text, Eisenstein did not number the formulas; the numbering has been added for purposes of reference.]

In modern language, formula (1) says that the Fourier transform of the function  $\Phi_0(x)$ , equal to  $x^{q-1}$  for  $x > 0$  and to 0 for  $x < 0$  is given by

$$\Psi_0(y) = \int_0^\infty e^{2\pi i x y} x^{q-1} dx = \frac{\Gamma(q)}{(2\pi|y|)^q} e^{(q\pi i/2) \operatorname{sgn}(y)}$$

Eisenstein's reference for this is to page 9 of his reprint of Dirichlet's paper [2b], bound with the present volume; this corresponds to page 401 of Dirichlet's *Werke* ([2], vol I); according to Dirichlet, the formula in question, or rather a more general one, had been Euler's discovery, and he attributes its proof to Poisson (cf. [2], pp. 378, 386, 397, and [7], vol. 19, p. 221). From (1), (2) follows formally, no question of convergence being raised; its purpose, as well as that of (3) and (4), seems to be to prepare the way for the calculation of (5a). As to the passage from (5a) to (5b), it appears to be no more and no less than an unmotivated application of the so-called "Poisson summation formula" to the function  $\Phi_{\alpha,\beta}(x)$  equal to

$$e^{\alpha(x-\beta)^i}(x-\beta)^{\alpha-1}$$

for  $x > \beta$  and to 0 for  $x < \beta$ . In fact, (5b) is nothing else than

$$\sum_{n \in \mathbf{Z}} \Phi_{\alpha, \beta}(n),$$

and (5a) is

$$\sum_{m \in \mathbf{Z}} \Psi_{\alpha, \beta}(m)$$

where  $\Psi_{\alpha, \beta}$  is the Fourier transform of  $\Phi_{\alpha, \beta}$ . Now, as we have

$$\Phi_{\alpha, \beta}(x) = e^{\alpha(x-\beta)^i} \Phi_0(x-\beta),$$

the value of its Fourier transform  $\Psi_{\alpha, \beta}$  is an easy consequence of that of  $\Psi_0$ ; it is given by

$$\Psi_{\alpha, \beta}(y) = e^{2\pi i \beta y} \Psi_0\left(y + \frac{\alpha}{2\pi}\right),$$

where  $\Psi_0$ , as above, is the Fourier transform of  $\Phi_0$ . Using this value for  $\Psi_{\alpha, \beta}$ , one obtains (6), of which (7), (8) and (9) are special cases: (9) is of course the well-known functional equation for  $L(s)$ , and (7) and (8) could be used to prove the functional equations also for other  $L$ -series.

The date of this proof, 7 April 1849, deserves special attention. In this same year 1849, the functional equation for  $L(s)$ , also in the range  $0 < s < 1$ , was published by Schlömilch, as an "exercise" in *Grunert's Archiv* ([5]); also a full proof for it was published in *Crelle's Journal*, as part of an extensive investigation on definite integrals ([6]), by Malmstén, who mentioned "having seen it somewhere in Euler" (actually in Euler's paper "*Remarques sur un beau rapport*"; cf. [7], vol. 15, pp. 70–90). Malmstén's paper is dated Upsala, May 1846; thus it must have lain with Crelle from 1846 to 1849, and Eisenstein could have seen it, either in manuscript or in print, before April 1849. Conceivably he could also have seen the functional equation for  $L(s)$  in *Grunert's Archiv*, or even in Euler; anyway, in the proof transcribed above, which is surely his own original contribution, he makes no claim to the discovery of the formula in question.

Even more interesting is the fact that April 1849 is the month when Riemann, who had been Eisenstein's student in Berlin, was leaving Berlin for Göttingen; Riemann's biography by Dedekind ([8], p. 544) mentions that he left Berlin shortly after the visit to Berlin of a delegation from the Diet in Frankfurt, which visit took place in the first week of April of that year. Eisenstein's correspondence with Stern shows how close

Eisenstein and Riemann were at that time (cf. [4], vol. II, pp. 804 and 821). It is therefore tempting to imagine that Eisenstein had communicated his proof to Riemann before the latter's departure; this, rather than Malmstén's and Schlömilch's publications, could well have been at the origin of Riemann's famous paper on the zeta-function.

### References

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