Advanced Studies in Pure Mathematics 17, 1989
Algebraic Number Theory — in honor of K. Iwasawa
pp. 289–308

Arrangements of Hyperplanes, Higher Braid Groups and Higher Bruhat Orders

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Dedicated to Kenkichi Iwasawa on his seventieth birthday

Introduction

Let z_i be coordinate functions on $\boldsymbol{C}^n$. Consider the arrangement of hyperplanes $D_{ij}: z_i - z_j = 0$ in $\boldsymbol{C}^n$ and let $U = \boldsymbol{C}^n - \bigcup D_{ij}$ be its complement. The fundamental group of U is called the (colored) braid group. This group and the topology of U has been studied in many papers.

The family $\{D_{ij}\}$ is a special example of hyperplane arrangements which we call discriminantal ones. This article is devoted to the study of the topological and combinatorial properties of the discriminantal arrangements.

Among the vast literature on arrangements of hyperplanes we can mention Cartier's Bourbaki report [1] and an important paper [5]. Recently their study was stimulated by the theory of multidimensional hypergeometric functions (cf. [8–10]) and certain models of quantum and statistic physics (see [6], [7] and the bibliography therein).

In section 1 of this paper we recall some results on the hyperplane arrangements, define discriminantal arrangements (considered previously in [6], [7] and [10]), define higher braid groups and calculate their nilpotent completions.

In section 2 we introduce posets $B(n, k)$. Their definition is motivated by combinatorics of the discriminantal arrangements. The poset $B(n, 1)$ is essentially the symmetric group S_n with its weak Bruhat order. We prove some fundamental properties of $B(n, k)$ including the higher analogs of the Coxeter relations.

The results of section 2 were previously announced in [6], [7].

Actually, the construction of section 2 defines on S_n a canonical structure of $(n-1)$-category, whose "1-coskelton" is the category associated to the weak Bruhat order. This $(n-1)$-category is introduced in the section 3. Its structure is closely related to the combinatorial structure of the convex closure of a general orbit of S_n in $\boldsymbol{R}^n$.

Received October 16, 1987.

In conclusion we would like to thank V. A. Hinich, who explained to us the notion of n-category used in section 3, and O. V. Ogievetsky who helped us to prove the main theorem of section 2.

§ 1. Discriminantal arrangements and higher braid groups

1. Preliminaries on hyperplane arrangements. Let F be a field, $D_1, \cdots, D_n$ a finite family of affine hyperplanes in F^m. The main combinatorial invariant of this arrangement is a ranked poset Z, consisting of affine subspaces $D_{i_1} \cap \cdots \cap D_{i_s}$ ordered by inverse inclusion and endowed with codimension as a rank function. Let $U = F^m - \bigcup D_i$. Then for $F = \boldsymbol{C}$ the following topological invariants of U are determined by Z.

a. *The nilpotent completion of* $\pi_1(U)$. Let G be a group, $G = \Gamma_0 G \supset \Gamma_1 G = [G, G] \supset \cdots \supset \Gamma_{j+1} G = [\Gamma_j G, G] \supset \cdots$ its lower central series. Using the central extension

$$1 \longrightarrow \Gamma_j G / \Gamma_{j+1} G \longrightarrow G / \Gamma_{j+1} G \longrightarrow G / \Gamma_j G \longrightarrow 1$$

one can inductively define a Lie $\boldsymbol{Q}$-algebra $[G/\Gamma_j G] \otimes \boldsymbol{Q}$ and the projective limit of these algebras $[G]^{\wedge}_{\boldsymbol{Q}}$ which we shall call the nilpotent completion of G.

For an arrangement of hyperplanes $D = (D_1, \cdots, D_n)$ denote by $L(D)$ the graded Lie $\boldsymbol{Q}$-algebra generated by degree 1 elements $h_1, \cdots, h_n$ satisfying the following relations:

$$R_S : [h_i, \sum_{j \in S} h_j] = 0, \qquad \forall i \in S$$

for all subsets $S \subset \{1, \cdots, n\}$ such that

$$\operatorname{codim} \bigcap_{j \in S} D_j = 2, \quad \operatorname{codim} \bigcap_{j \in S'} D_j > 2 \qquad \text{if } S' \text{ strictly contains } S.$$

Let $L(D)^{\wedge}$ be the completion of $L(D)$ with respect to the lower central series.

1.1. Proposition ([2], [3]). *For* $U = \boldsymbol{C}^m - \bigcup D_j$ *we have*

$$[\pi_1(U)]^{\wedge}_{\boldsymbol{Q}} \cong L(D)^{\wedge}. \qquad \blacksquare$$

b. *Cohomology of* U. Consider a grassmannian $\boldsymbol{Q}$-algebra Λ generated by symbols e_i, $i = 1, \cdots, n$. If $S = (i_1, \cdots, i_k)$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$ put $e_S = e_{i_1} \cdots e_{i_k}$. Define a $\boldsymbol{Q}$-linear differential ∂ in Λ by $\partial 1 = 0$, $\partial e_i = 1$ and $\partial(e_S e_T) = \partial e_S \cdot e_T + (-1)^{|S|} e_S \partial e_T$. Clearly $\partial^2 = 0$.

We say that S is dependent if $\operatorname{codim} \bigcap_{i \in S} D_i > |S|$.

Let $f_i = 0$ be an equation of $D_i \subset \boldsymbol{C}^m$. In 1.2, 1.3 and 1.4 below we assume that $\bigcap_{i=1}^n D_i \neq \varnothing$.

1.2. Proposition ([5]). *Denote by Δ the ideal in Λ generated by $\{\partial e_S \mid S$ dependent$\}$. Then the map*

$$\text{the de Rham class of } \frac{1}{2\pi i} d\log f_i \mapsto e_i \bmod \Delta$$

defines an isomorphism of graded rings $H^*(U, \boldsymbol{Q}) \cong \Lambda/\Delta$. ■

c. *The Betti numbers of U.* Define the Möbius function of the poset Z by

$$\mu(\boldsymbol{C}^m)=1, \qquad \mu(X)=-\sum_{\substack{Y\supset X\\ Y\in Z}} \mu(Y).$$

1.3. Proposition ([5]). *We have*

$$\dim H^i(U, \boldsymbol{Q}) = \Big|\sum_{\substack{X\in Z\\ \operatorname{codim} X=i}} \mu(X)\Big|.$$ ■

d. *Real arrangements.* Assume that all $D_i \subset \boldsymbol{C}^m$ are complexifications of real hyperplanes $D_i^{\boldsymbol{R}} \subset \boldsymbol{R}^m$. Put $u=|\pi_0(\boldsymbol{R}^m - \bigcup D_i^{\boldsymbol{R}})|$.

1.4. Proposition ([11]). *We have*

$$u=\sum_{i\geqslant 0} \dim H^i(U, \boldsymbol{Q}) = \sum_{i\geqslant 0} \Big|\sum_{\substack{X\in Z\\ \operatorname{codim} X=i}} \mu(X)\Big|.$$ ■

2. Discriminantal arrangements. Let now $H_1^0, \cdots, H_n^0 \subset F^k$ be a family of affine hyperplanes in general position. This means in particular that $\operatorname{codim}(H_{i_1}^0 \cap \cdots \cap H_{i_a}^0)=a$ for all $1\leqslant i_1 < \cdots < i_a \leqslant n$ (we agree that $\operatorname{codim} Y>k$ in F^k means that $Y=\varnothing$).

Let $U(n, k)$ be the manifold of arrangements $H_1, \cdots, H_n$ enjoying two properties: a) H_i is parallel to H_i^0 for all $1\leqslant i\leqslant n$; b) $H_1, \cdots, H_n$ are in general position.

Clearly, $U(n, k)$ is a subset of the space F^n of all parallel transports of H_i^0. Moreover, $F^n - U(n, k)$ is a union of hyperplanes in F^n which we will now describe.

Denote by $C(n, a)$ the set of subsets in $(1, \cdots, n)$ of cardinality a. For $K \in C(n, a)$ put

(1) $\quad D_K=$the set of $(H_1, \cdots, H_n)$ in F^n such that $\bigcap_{i\in K} H_i \neq \varnothing$.

Clearly $D_K=F^n$ if $|K|\leqslant k$ and

(2) $\qquad \operatorname{codim} D_K=|K|-k \qquad \text{for } K\geqslant k+1.$

In particular, D_J for $J \in C(n, k+1)$ are pairwise distinct hyperplanes in F^n. One easily sees (cf. Proposition 4 below) that

$$U(n, k) = F^n - \sum_{J \in C(n, k+1)} D_J. \tag{3}$$

We shall call the set of hyperplanes (D_J) a *discriminantal arrangement* in F^n. Strictly speaking, it depends on $(H_1^0, \cdots, H_n^0)$ and not only on n, k. However, we shall be concerned mostly with its combinatorial invariants which are constant on an open Zariski dense subset of all n-arrangements in F^n. Stating properties of such invariants we shall tacitly assume that our arrangements (H_i^0) are general in this sense. Note that the discriminantal arrangement is almost never general!

3. Definition. The higher braid group $T(n, k)$ is the fundamental group $\pi_1(U(n, k))$ (for $F = \boldsymbol{C}$). ■

If $k=1$ we get the ordinary braid group.

Remark. It would be natural to consider also the manifold $V(n, k)$ of arbitrary families of n hyperplanes in $\boldsymbol{C}^k$ in general position. It coincides with $U(n, k)$ for $k=1$ but leads to a different generalization of the braid group for $k>1$. Unfortunately, $V(n, k)$ is not in general a complement of hyperplanes.

In the next Proposition we state some properties of the poset $Z(n, k)$ generated by hyperplanes D_J. Unfortunately we were unable to obtain its complete combinatorial description.

4. Proposition. a) *The map* $K \mapsto D_K$ (*cf.* (1)) *defines an injection of the lattice of subsets* $\coprod_{k+1 \leqslant a \leqslant n} C(n, a)$ *ordered by inverse inclusion, with operation* $\wedge = \cup$ *and rank function* $a-k$ *into the poset* (*geometric lattice* [5]) $Z(n, k)$.

b) *The codimension one elements in* $Z(n, k)$ *are* D_J *for all* $J \in C(n, k+1)$. *They are pairwise distinct.*

c) *The codimension two elements in* $Z(n, k)$ *are* D_K *for* $K \in C(n, k+2)$ *and* $D_{J_1} \cap D_{J_2}$ *for* $J_i \in C(n, k+1)$, $|J_1 \cup J_2| \geqslant k+3$. *They are pairwise distinct. Moreover,* $D_K \subset D_J$ *for* $J \in C(n, k+1)$ *iff* $J \supset K$, *and* $D_{J_1} \cap D_{J_2} \subset D_J$ *iff either* $J = J_1$, *or* $J = J_2$.

d) $Z(n, k)$ *has a unique minimal element of codimension* $n-k$, *namely* $D_{(1, \cdots, n)}$.

Proof. Everything follows from two simple remarks. First, for arbitrary subsets K_1, K_2 with $|K_i| \geqslant k+1$ we have $K_1 \subset K_2 \Leftrightarrow D_{K_1} \supset D_{K_2}$. Second, if $|J| = k+1$ and $D_J \supset D_{K_1} \cap \cdots \cap D_{K_a}$ then $J \subset \bigcup_{i=1}^{a} K_i$.

In order to prove the second assertion suppose that $J \not\subset \bigcup K_i$. Choose $j \in J - \bigcup K_i$. Then the condition $(H_1, \cdots, H_n) \in D_{K_1} \cap \cdots \cap D_{K_a}$ implies no restrictions on the position of H_j: the other hyperplanes being fixed, this one can be freely moved. Contrariwise, the condition $(H_1, \cdots, H_n) \in D_J$ fixes the position of H_j unambiguously once the position of all H_i, $i \in J - \{j\}$ is known since H_j must pass through the point $\bigcap_{i \in J - \{j\}} H_i$. Therefore in this case $D_J \not\supset D_{K_1} \cap \cdots \cap D_{K_a}$.

Let now $a = |K| \geqslant k+1$. Then we can find $J_1, \cdots, J_{a-k}$ with $|J_i| = k+1$, $|J_{i+1} - \bigcup_{b=1}^{i} J_b| = 1$ and $K = \bigcup J_i$. Therefore $D_K = \bigcap D_{J_i}$ and $\operatorname{codim} D_K = a - k$. Then rest of our assertions follow from this and two remarks stated in the beginning.

The minimal element of $Z(n, k)$ consists of all arrangements intersecting in a point (if $n \geqslant k+1$). This point can be chosen arbitrarily in F^k. ■

5. Theorem. *The nilpotent completion $[T(n, k)]_Q^{\wedge}$ of the higher braid group is generated by elements h_J, $J \in C(n, k+1)$ subjected to the relations*

$$[h_{J_1}, h_{J_2}] = 0, \qquad \text{if } |J_1 \cup J_2| \geqslant k+3, \tag{4}$$

$$[h_J, \sum_{I \subset K} h_I] = 0, \qquad \text{if } K \in C(n, k+2),\ J \subset K. \tag{5}$$ ■

This follows immediately from Propositions 1.1 and 4 a), b).

6. Higher braid groups of real arrangements. If the initial arrangement $(H_1^0, \cdots, H_n^0)$ is the complexification of a real one, then its discriminantal arrangement also is the complexification of a real one. In this case one can use the results of Randell [4] to compute the fundamental group $T(n, k)$ itself. We get the following information.

For each pair $a = (J_1, J_2)$, $J_i \in C(n, k+1)$, $|J_1 \cup J_2| \geqslant k+3$, introduce generators $\alpha_1(a)$, $\alpha_2(a)$. For each set $K \in C(n, k+2)$ introduce generators $\alpha_1(K), \cdots, \alpha_{k+2}(K)$. Then $\pi_1(U(n, k))$ is isomorphic to the group $P(n, k)$ generated by these elements subjected to the relations

$$\alpha_1(a)\alpha_2(a) = \alpha_2(a)\alpha_1(a), \tag{4'}$$

$$\begin{aligned} \alpha_1(K)\alpha_2(K)\cdots\alpha_{k+2}(K) &= \alpha_{k+2}(K)\alpha_1(K)\alpha_2(K)\cdots\alpha_{k+1}(K) \\ &= \cdots = \alpha_{k+2}(K)\alpha_{k+1}(K)\cdots\alpha_1(K), \end{aligned} \tag{5'}$$

(6) some equalities among $\alpha_i(a)$, $\alpha_j(K)$.

The relations (4′) and (5′) correspond to (4) and (5) respectively. However, if one uses Randell's prescriptions, the relations (6) cannot be written in pure combinatorial terms, since they require taking into account some inequalities, i.e. certain characteristics of the real part of the picture.

7. Topology of $U(n, k)$ for $n \leqslant k+3$. If $n=k+1$, we have clearly $U(k+1, k)=\boldsymbol{C}^{k+1}-\boldsymbol{C}^{k}$ and $T(k+1, k)=\boldsymbol{Z}$.

$U(k+2, k)$ is a complement of a union of $k+1$ hyperplanes in $\boldsymbol{C}^{k+2}$ passing through a common axis $\boldsymbol{C}^{k}$. We leave to the reader a calculation of its cohomology and fundamental group. Proposition 4 suffices also for calculation of the topology of $U(k+3, k)$.

The following table describes the structure of $Z(k+3, k)$. We use the following notation: $(ij)=D_J$, $J=(1, \cdots, k+3)-(i, j)$; $(i)=D_K$, $K=(1, \cdots, k+3)-(i)$; $(ij, lm)=D_{J_1} \cap D_{J_2}$, $(i, j) \cap (l, m)=\varnothing$.

codim	Inclusion diagram	Number of elements
3	$\mathbf{1}=\boldsymbol{C}^{k}$	1
2	$(i)\cdots$ $\quad$ $(ij, lm)\cdots$	$(k+3)+\frac{1}{8}k(k+1)(k+2)(k+3)$
1	$(ij)\cdots$	$\frac{1}{2}(k+2)(k+3)$
0	$\mathbf{0}=\boldsymbol{C}^{k+3}$	1

Hence, by induction on codim

$$\mu(\mathbf{0})=1;\ \mu((ij))=-1;\ \mu((i))=-\sum_{j \neq i} \mu((ij))-1=k+1;$$

$$\mu\ ((ij, lm))=-\mu((ij))-\mu((lm))-\mu(\mathbf{0})=1;$$

$$\mu(\mathbf{1})=-(k+1)(k+3)-\tfrac{1}{8}k(k+1)(k+2)(k+3)+\tfrac{1}{2}(k+2)(k+3)-1$$
$$=-\tfrac{1}{8}k(k+3)(k^2+3k+6)-1.$$

Put $h^i=\dim H^i(U(k+3, k))$. We have $h^0=1$, $h^i=0$ for $i>3$ and, using Proposition 1.4

$$h^1=\tfrac{1}{2}(k+2)(k+3); \qquad h^2=\tfrac{1}{8}(k+1)(k+3)(k^2+2k+8);$$
$$h^3=\tfrac{1}{8}k(k+3)(k^2+3k+6)+1.$$

The sum of these Betti numbers equals to the number of isotopy classes of arrangements of $k+3$ ordered hyperplanes in general position in $\boldsymbol{R}^k$. It is 62 for $k=2$, 140 for $k=3$.

§ 2. Higher Bruhat orders

1. Notation. Let $(I, <)$ be a finite totally ordered set. Denote by $C(I, k)$ the set of all subsets of I of cardinality k.

Usually I will be $(1, \cdots, n)=\underline{n}$ or $(2, \cdots, n)=\underline{n}-(1)$. There is a lexicographic total order on $C(I, k)$: if $J=(j_1, j_2, \cdots, j_k), j_i<j_{i+1}$, and $J'=(j'_1, j'_2, \cdots, j'_k), j'_i<j'_{i+1}$, then $J \leqslant J'$ means that either $j_1<j'_1$, or $j_1=j'_1$ but $j_2<j'_2$ etc.

For $K \in C(I, k+1)$, we shall call a *K-packet* the set $P(K)=\{J \mid J \in C(I, k), J \subset K\}$. If $K=(i_1, \cdots, i_{k+1}), i_j<i_{j+1}$, then $P(K)$ consists of the sets $K_a^\wedge=K-(i_a)$, $a=1, \cdots, k+1$. We have lexicographically $K_{k+1}^\wedge<K_k^\wedge<\cdots<K_1^\wedge$.

Below we shall consider various total orders on $C(I, k)$ which we shall denote ρ, ρ', σ etc. The notation $\rho=J_1 J_2 \cdots J_N$, $N=\binom{n}{k}$ means that $J_i \rho J_j$ for all $i<j$.

2. Definition. a. A total order on $C(I, k)$ is called *admissible* if on each packet it induces either a lexicographic order or the inverse lexicographic one.

We denote by $A(I, k)$ the set of all admissible total orders on $C(I, k)$.

b. Two total orders $\rho, \rho' \in A(I, k)$ are called *elementarily equivalent*, if they differ by an interchange of two neighbours which do not belong to a common packet.

We denote by $B(I, k)$ the quotient of $A(I, k)$ by the corresponding equivalence relation. The natural projection is $\pi: A(I, k) \rightarrow B(I, k)$.

c. An *inversion* in the order $\rho \in A(I, k)$ is an element $K \in C(I, k+1)$ such that ρ induces on $P(K)$ the antilexicographic order. We denote by $\operatorname{Inv}(\rho) \subset C(I, k+1)$ the set of all inversions in ρ and by $\operatorname{inv}(\rho)$ its cardinality. Clearly, for $r=\pi(\rho)=\pi(\rho')$ we can set $\operatorname{Inv}(r)=\operatorname{Inv}(\rho)=\operatorname{Inv}(\rho')$. ∎

Remarks. 1) The lexicographic order $\rho_{\min}$ and the antilexicographic order $\rho_{\max}$ are clearly admissible. We have $\operatorname{Inv}(\rho_{\min})=\varnothing$, $\operatorname{Inv}(\rho_{\max})=C(I, k+1)$. In general, if ρ is admissible, then the inverse order ρ^t also is admissible, and we have $\operatorname{Inv}(\rho^t)=C(I, k+1)-\operatorname{Inv}(\rho)$. If ρ and ρ' are elementarily equivalent then the same is true for ρ^t, ρ'^t, hence t acts upon $B(I, k)$.

2) Suppose that for some $K \in C(I, k+1)$ members of the packet $P(K)$ form a *chain* with respect to an admissible order ρ, i.e. any element of $C(I, k)$ lying between two elements of $P(K)$ belongs to $P(K)$. Define $p_K(\rho)$ as an order in which this chain is reversed while all the rest elements conserve their positions. Evidently $p_K(\rho)$ is admissible, and

$$\mathrm{Inv}\,(p_K(\rho))=\begin{cases}\mathrm{Inv}\,(\rho)\cup\{K\}, & \text{if } K\notin \mathrm{Inv}\,(\rho),\\ \mathrm{Inv}\,(\rho)-\{K\}, & \text{if } K\in \mathrm{Inv}\,(\rho).\end{cases}$$

One easily sees that if ρ and ρ' are elementarily equivalent and if $p_K(\rho)$, $p_K(\rho')$ are both defined then they also are elementarily equivalent.

For $r\in B(I,k)$ we put

$$N(r)=\{K\in C(I,k+1)\,|\,P(K) \text{ forms a chain for some } \rho_0\in r\},$$
$$p_K(r)=\pi(p_K(\rho_0)) \text{ for such } K\in N(r) \text{ and } \rho_0.$$

Clearly, $N(r^t)=N(r)$.

3) Let $f\colon I\to J$ be a strictly increasing map of totally ordered finite sets. It induces maps $f_*\colon C(I,k)\to C(J,k)$ which are strictly increasing with respect to lexicographic orders and map packets into packets. Therefore each admissible order on $C(J,k)$ induces an admissible order on $C(I,k)$, whence we get a map $f^*\colon A(J,k)\to A(I,k)$. It is compatible with elementary equivalencies so that we have a map $f^{\cdot}\colon B(J,k)\to B(I,k)$. In this way our constructions are functorial.

4) For $k=1$ we have $C(I,1)\simeq I$, $A(I,1)\simeq$ the set of all total orderings of I, i.e. the symmetric group of permutations of I, $B(I,1)=A(I,1)$; $p_K(\rho)$ is obtained from ρ by transposing two neighbours forming K. (Case $k=n-2$ is described below, cf. Lemma 7).

Thus the following theorem which is the main result of this section is an extension to the case $k\geqslant 2$ of principal properties of the weak Bruhat order on the symmetric group (see e.g. [12]).

3. Theorem. a) *The relation*

$$r\leqslant r'\Longleftrightarrow\exists K_i\in C(n,k+1),\ K_i\in N(p_{K_{i-1}}\cdots p_{K_1}(r))-\mathrm{Inv}\,(p_{K_{i-1}}\cdots p_{K_1}(r)),$$
$$r'=p_{K_m}\cdots p_{K_1}(r)$$

is a partial order on $B(n,k)$.

b) *This partial order defines on* $B(n,k)$ *the structure of a ranked poset, with rank function inv, a unique minimal element* $r_{\min}=\pi(\rho_{\min})$ *and a unique maximal element* $r_{\max}=\pi(\rho_{\max})$.

c) *The map*

$$\{r_{\min}<p_{K_1}(r_{\min})<\cdots<p_{K_M}\cdots p_{K_1}(r_{\min})\}\longmapsto\rho=K_1\cdots K_M$$

defines a bijection

$$\{\textit{the set of maxmial chains in } B(n,k)\}\xrightarrow{\sim}A(n,k+1).$$

d) *Every element* $r\in B(n,k)$ *is uniquely defined by the set* Inv (r). ■

Before we start proving this theorem we digress to give a geometric motivation of our combinatorial notions in terms of the discriminantal hyperplane arrangements.

Choose a real hyperplane arrangement $H_1^0, \cdots, H_n^0 \subset \boldsymbol{R}^{k-1}$. The components of the corresponding discriminantal arrangement $D_J \subset \boldsymbol{R}^n$ are numbered by $J \in C(n, k)$ (cf. (1)). Below we shall consider only the real part of this picture, so that D_J divides $\boldsymbol{R}^n$ into two parts.

Choose in $\boldsymbol{R}^n$ a generic plane and orient it. The arrangement D_J intersects it in a family of lines. The intersection points of these lines are in a bijection with the set

$$C(n, k+1) \cup \text{ the set of unordered pairs } J_1, J_2 \in C(n, k)$$
$$\text{such that } |J_1 \cup J_2| \geqslant k+2.$$

This follows from section 1, Proposition 4.

Draw in our plane P a closed path in the positive direction intersecting each line $D_J \cap P$ twice and containing inside it all intersection points of lines. Then starting with some initial point we shall intersect hyperplanes D_J in some order $D_{J_1}, \cdots, D_{J_R}$, $R = \binom{n}{k}$ and then again $D_{J_1}, \cdots, D_{J_R}$.

One can show that if the initial point and the numbering of H_i^0 are properly chosen, the order $J_1, \cdots, J_R$ will be the lexicographic one.

Now we shall fix the initial point and the endpoint of the first half of our path (after intersecting each D_J once) and deform the path in such a way that at each moment t the deformed path Γ_t(intersects at most one intersection point of lines $P \cap D_{J_i}$. At these critical moments the intersection order of J's will enjoy the following changes (a two-dimensional picture will make it evident to the reader).

When Γ_t intersects $P \cap D_{J_i} \cap D_{J_j}$, $|J_i \cup J_j| \geqslant k+2$, then J_i and J_j which were neighbours just before the critical moment change places.

When Γ_t intersects $P \cap D_K$ then the members of the packet $P(K)$ which formed a chain just before the critical moment become intersected in reverse order.

In this way we get elementary equivalencies and inversions.

Now we return to the proof of the Theorem 3 which requires a number of lemmas. The rest of this section is devoted to this proof.

4. Lemma. *In each class $r \in B(n, k)$ there exists an order $\rho \in A(n, k)$ such that all elements of $C(n, k)$ containing* 1 *form a chain with respect to this order.*

Proof. We shall say that elements $J, J' \in C(n, k)$ commute if they do not belong to a packet, i.e. if $|J \cup J'| \geqslant k+2$ (cf. (4), section 1).

Let ρ be an order from $A(n, k)$, S a subset of $C(n, k)$. We shall denote by $\bar{S}$ the minimal chain containing S. Choose an element $J \in \bar{S}-S$. We shall say that J can be pushed out to the left (resp. to the right) from S, if J commutes with all $J' \in S$ such that $J'\rho J$ (resp. $J\rho J'$).

Now take an arbitrary class $r \in B(n, k)$ represented by an order ρ. Let S consist of all elements $J_1\rho J_2 \cdots \rho J_N$, $N=\binom{n-1}{k-1}$, containing 1.

We affirm that any element $J \in \bar{S}-S$ can be pushed out from S either to the left or to the right (or both). In effect, let $J=(j_1, \cdots, j_k)$, $K=\{1\} \cup J$. Any element of S not commuting with J must be of the form $(1, j_1, \cdots, \hat{j}_p, \cdots, j_k)=K^{\wedge}_{p+1}$ for some p, $1 \leqslant p \leqslant k$. Moreover, $J=K^{\wedge}_1$. Since ρ is admissible, J must be either the maximal or the minimal element of the packet $\{K^{\wedge}_i\}$ with respect to ρ. Therefore J can be pushed out of S.

Let now J', J'' be such a couple of elements of $\bar{S}-S$ that $J'\rho J''$, J' cannot be pushed out to the left and J'' cannot be pushed out to the right. We affirm that in this case J' and J'' must commute. In effect, if they do not commute, we have $J'=K^{\wedge}_p$, $J''=K^{\wedge}_q$ for some $K \in C(n, k+1)$, where $K=(i_1 \cdots, i_{k+1})$, $1<i_1$, $1 \leqslant p, q \leqslant k+1$. By assumption, all sets $\{1\} \cup K^{\wedge}_{s,p}=(1, i_1, \cdots, \hat{i}_s, \cdots, \hat{i}_p, \cdots, i_{k+1})$, $1 \leqslant s \leqslant k+1$, $s \neq p$, lie to the left of J' with respect to ρ. In particular, for $J=\{1\} \cup K^{\wedge}_{p,q}$ we have $J\rho J'\rho J''$. Since J clearly does not commute with J'', J'' cannot be pushed out of S to the left. But by assumption it cannot be pushed out to the right either which contradicts our previous result.

Now we shall apply elementary equivalencies to ρ trying to push out all elements of $\bar{S}-S$ so that in the end S forms a chain. If all elements of $\bar{S}-S$ can be pushed out to the left then it suffices to interchange them in turn with all elements of S lying to the left of them. Otherwise let $J \in \bar{S}-S$ be the maximal of all elements that cannot be pushed out to the left. Then it can be pushed out to the right from S by changing places with all its right neighbours, belonging to S or not. The proof concludes by induction on the number of elements which cannot be pushed out to the left. ■

An order ρ whose existence is asserted in Lemma 4 will be called a *good* one. We shall write it in the standard form

$$\rho=J'_1 \cdots J'_a;\ J_1 \cdots J_N;\ J'_{a+1} \cdots J'_M, \tag{7}$$

where $1 \in J_i$, $1 \notin J'_i$, $N=\binom{n-1}{k-1}$, $M=\binom{n-1}{k}$. Put $J_i=\{1\} \cup L_i$, $\rho'=L_1 \cdots L_N \in A(\underline{n}-\{1\}, k-1)$, $\sigma=J'_1 \cdots J'_a J'_{a+1} \cdots J'_M \in A(\underline{n}-\{1\}, k)$. Then we can rewrite (7) in the form

(8) $$\rho=\rho_1 1*\rho'\rho_2, \qquad \sigma=\rho_1\rho_2.$$

We shall call ρ_1, ρ_2, $1*\rho'$ the left, right and middle part of ρ respectively. Left or right part may be empty.

5. Lemma. a) $\mathrm{Inv}(\rho')=\{J'_1, \cdots, J'_a\}=$ *the set of elements of the left part of* ρ; $1*\mathrm{Inv}(\rho')=\{\{1\}\cup J'_1, \cdots, \{1\}\cup J'_a\}=$ *the set of inversions of* ρ, *containing* 1.

b) *Any packet from* $C(n, k)$ *either is disjoint with the middle part of* ρ, *or* k *of its members lie in the middle part. In the latter case the exceptional member lies in the left part of* ρ, *iff the whole packet belongs to* $\mathrm{Inv}(\rho)$.

Corollary. *Left, right and middle parts of* ρ *as sets depend only on* $r=\pi(\rho)$.

Proof. Let $K\in C(n, k+1)$. If $1\notin K$ then $P(K)$ is disjoint with the middle part of ρ. If $1\in K$ then only $K_1^\wedge$ does not contain 1. If ρ induces on $P(K)$ the lexicographic order then $K_1^\wedge$ belongs to the right part, otherwise to the left one. The rest of the packet belongs to the middle part. If we delete 1 from them we obtain a packet in $C(\underline{n}-\{1\}, k-1)$, which belongs to $\mathrm{Inv}(\rho')$ exactly when $K_1^\wedge$ is in the left part of ρ. ■

6. Lemma. *Let* $\rho\in A(n, k)$, $r=\pi(\rho)$, $K\in C(n, k+1)$, $\bar{P}$ *is the minimal* ρ*-chain containing the packet* $P=P(K)$. *Then the following properties are equivalent.*

a) $K\in N(r)$, *i.e.* P *forms a chain with respect to an appropriate representative of* r.

b) *Every element* $L\in\bar{P}-P$ *can be pushed out either to the left or to the right.*

Proof. We shall consider only the case $K\notin \mathrm{Inv}(r)$; applying it to r^t we shall get the rest. Implication a)⇒b) results from the following observation. Suppose that P forms a chain with respect to an order $\rho'\in r$. Then in the series of elementary equivalencies connecting ρ to ρ' every element $L\in\bar{P}-P$ must change places with all elements of P lying to the same side of L where it is eventually pushed out.

We shall prove b)⇒a) by induction on $n+k$. The first case $n=k=1$ is trivial.

First note that it suffices to prove b)⇒a) for an arbitrary representative ρ_0 of r. In effect, if b) is true for some ρ then it is true for any ρ' elementarily equivalent to ρ.

Take for ρ_0 a good order (7). Consider two cases.

1) $1 \in K$, i.e. $K=(1, i_2, \cdots, i_{k+1})$. Then $\{K_{k+1}, \cdots, K_2\} \subset \{J_1, \cdots, J_N\}$ and since the induced order on P is the lexicographic one, we have $K_1^{\wedge}=(i_2, \cdots, i_{k+1})=J'_j$ for some $j \geqslant a+1$. By the inductive assumption for $(n-1, k-1)$ it follows from b) that the order ρ_0 can be changed to an equivalent one in such a way that $\{K_{k+1}, \cdots, K_2\}$ form a chain. We shall assume that this is true for ρ_0.

Suppose first that $j > a+1$. We affirm that in this case $J'_{a+1}, \cdots, J'_{j-1}$ commute with J'_j so that one can put J'_j to the $(a+1)$-th place.

In effect, if J'_t does not commute with $J'_j = K_1$ for some $a+1 \leqslant t \leqslant j-1$, then for some $s \geqslant 2$ we have $i_s \notin J'_t$. But $1 \notin J'_t$, hence J'_t cannot be pushed out of P neither to the left nor to the right, which contradicts our assumption.

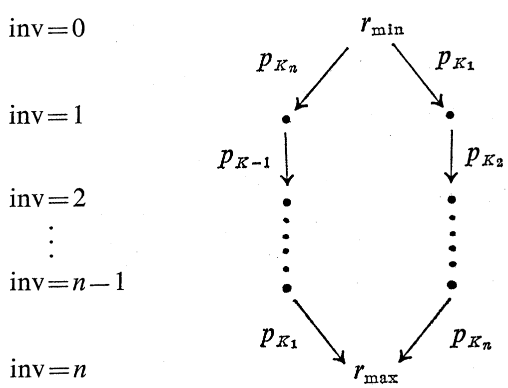
Let now $K_1^{\wedge} = J'_{a+1}$. Then among all sets $J_1, \cdots, J_N$ only the elements of P do not commute with $K_1^{\wedge}$. But P already forms a chain and $K_1^{\wedge}$ can be moved to the left to extend this chain.

2) $1 \notin K$. We shall show that in this case all members of P belong simultaneously either to the left or to the right part of ρ_0. Therefore we can apply the inductive assumption for the case $(n-1, k)$.

In effect, suppose that $\{K_{k+1}, \cdots, K_{t+1}\} \subset \{J'_1, \cdots, J'_a\}$ and $\{K_t, \cdots, K_1\} \subset \{J'_{a+1}, \cdots, J'_M\}$. Let $1 \leqslant p \leqslant t$, $t+1 \leqslant q \leqslant k+1$. Then $K^{\wedge}_{p,q} \cup \{1\} \in \{J_1, \cdots, J_N\}$ does not commute neither with $K_p^{\wedge}$ nor with $K_q^{\wedge}$ and therefore cannot be pushed out of P, contrary to our assumption. ■

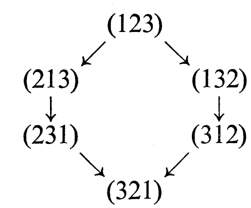
7. Lemma. a) $A(n, n-1) = B(n, n-1) = \{K_n \cdots K_1, K_1 \cdots K_n\}$ *where* $K=(1, \cdots, n)$.

b) *The poset* $B(n, n-2)$ *is described by the following picture*:



Proof. The first statement is evident.

No less evident is the structure of $B(3, 1)$:



Now we shall prove by induction the following statement which is a reformulation of the second part of our Lemma.

$(B)_{n-1}$. For any $r \in B(n-1, n-3)$ we have either $\mathrm{Inv}(r)=\{K_1^\wedge, \cdots, K_i^\wedge\}$ or $\mathrm{Inv}(r)=\{K_{n-1}^\wedge, \cdots, K_{n-i}^\wedge\}$ for an appropriate i, where $K=\{1, \cdots, n-1\}$. In the first case we have $N(r)=\{K_i^\wedge, K_{i+1}^\wedge\}$, in the second $N(r)=\{K_{n-i}^\wedge, K_{n-i-1}^\wedge\}$ with exception of border effects: for $i=0$, where $r=r_{\min}$, and for $i=n-1$, where $r=r_{\max}$, we have $N(r)=\{K_1^\wedge, K_{n-1}^\wedge\}$.

In order to deduce $(B)_n$ from $(B)_{n-1}$ take an arbitrary $r \in B(n, n-2)$ with a good representative (7). Then $M=n-1$ and since $\cup J_j'=\underline{n}-\{1\}$, sets J_j' form a packet. Consider three possibilities.

1) $a=0$. From lemma 5 it follows that $\mathrm{Inv}(\rho')=\varnothing$ (in notation (8)). By inductive assumption, ρ' is equivalent to the lexicographic order on $C(\underline{n}-\{1\}, k-1)$ and $N(\pi(\rho'))=\{(\underline{n}-\{1\})_1^\wedge, (\underline{n}-\{1\})_{n-1}^\wedge\}$. Therefore

$$\underline{n}_1^\wedge \in N(r) \subset \{\underline{n}_2^\wedge, \underline{n}_n^\wedge, \underline{n}_1^\wedge\},$$

where the first two sets in the right-hand side emerge after adding $\{1\}$ to the members of ρ' and the third one corresponds to the packet (J_j'). This packet is ordered by ρ either lexicographically or antilexicographically. In the first case $r=r_{\min}$, the packet $P(\underline{n}_n^\wedge)$ can be moved to form a chain in an order representing r, but the packet $P(\underline{n}_2^\wedge)$ cannot be moved in this way since its member $(\underline{n}_2^\wedge)_1^\wedge=(\underline{n}-\{1\})_1^\wedge$ is among $(J_1, \cdots, J_N)$ and cannot be put to the right part. Hence $N(\pi(\rho))=\{\underline{n}_1^\wedge, \underline{n}_n^\wedge\}$ in accord with $(B)_n$.

In the second case $r=p_{\underline{n}-\{1\}}(r_{\min})$ and for similar reasons one can form a chain from the packet $P(\underline{n}_2^\wedge)$ but not from the packet $\underline{n}_n^\wedge$ since $(\underline{n}_n^\wedge)_1^\wedge=(\underline{n}-\{1\})_{n-1}^\wedge$ is ρ-maximal. Therefore $N(r)=\{\underline{n}_1^\wedge, \underline{n}_2^\wedge\}$ as should be by $(B)_n$.

2) $0<a<n-1$. Since here $\mathrm{Inv}(\rho')=\{J_1', \cdots, J_a'\}$, ρ' is neither minimal, nor maximal one. And since the packet $\{J_i'\}$ does not belong to one part of ρ, it is not contained in $N(r)$ (cf. the end of the proof of Lemma 6).

By inductive assumption $(B)_{n-1}$, one of two cases can occur:

$$\mathrm{Inv}(\rho')=\{(\underline{n}-\{1\})_1^\wedge, \cdots, (\underline{n}-\{1\})_a^\wedge\},\ N(\pi(\rho'))=\{(\underline{n}-\{1\})_a^\wedge, (\underline{n}-\{1\})_{a+1}^\wedge\}$$

or

$$\operatorname{Inv}(\rho')=\{(\underline{n}-\{1\})^{\wedge}_{n-1}, \cdots, (\underline{n}-\{1\})^{\wedge}_{n-a}\},$$
$$N(\pi(\rho'))=\{(\underline{n}-\{1\})^{\wedge}_{n-a}, (\underline{n}-\{1\})^{\wedge}_{n-a-1}\}.$$

To get $\operatorname{Inv}(r)$, $N(r)$ one must add $\{1\}$ and include the packet $P(\underline{n}-\{1\})$ into $\operatorname{Inv}(r)$ if necessary. This should be done iff $J'_b=(\underline{n}-\{1\})^{\wedge}_b$ for $1\leqslant b \leqslant a$, i.e. in the first case. Hence we have respectively

$$\operatorname{Inv}(r)=\{\underline{n}^{\wedge}_1, \underline{n}^{\wedge}_2, \cdots, \underline{n}^{\wedge}_{a+1}\}, \qquad N(r)=\{\underline{n}^{\wedge}_{a+1}, \underline{n}^{\wedge}_{a+2}\}$$

or

$$\operatorname{Inv}(r)=\{\underline{n}^{\wedge}_n, \cdots, \underline{n}^{\wedge}_{n-a+1}\}, \qquad N(r)=\{\underline{n}^{\wedge}_{n-a+1}, \underline{n}^{\wedge}_{n-a}\}$$

in accordance with $(B)_n$.

3) $a=n-1$. This case is treated similarly to $a=0$. ■

8. Lemma. *The following properties of $r \in B(n, k)$ are equivalent*:

a) *r is a maximal (resp. a minimal) element of $B(n, k)$.*

b) *$r=\pi(\rho_{\max})$ (resp. $\pi(\rho_{\min})$).*

c) *$\operatorname{Inv}(r)=C(n, k+1)$ (resp. $\operatorname{Inv}(r)=\varnothing$).*

Let $r_{\max}=p_{K_R}\cdots p_{K_1}(r_{\min})$, $R=\binom{n}{k+1}$. Then $K_1\cdots K_R \in A(n, k+1)$.

Corollary. *Any two elements of $A(n, k)$ are connected by a series of elementary equivalencies and operations p_K.*

Proof. Clearly b)⇒c)⇒a). We shall show that a)⇒b). In order to do that we shall prove by induction on $n+k$ the conjunction of two statements: if $\operatorname{Inv}(r)\neq\varnothing$, r is not minimal; otherwise r contains the lexicographical order. The first case is evident. The inductive step depends on the value of a in a good representative ρ of r (see (7), (8)).

$a=0$. Here $\operatorname{Inv}(\rho')=\varnothing$. Hence if $\operatorname{Inv}(r)\neq\varnothing$, all inversions of ρ lie in $\rho_2 \in A(\underline{n}-\{1\}, k)$. By inductive assumption then ρ_2 is not minimal, therefore r is not minimal. If $\operatorname{Inv}(r)=\varnothing$, both ρ' and ρ_2 are equivalent to lexicographical orders, hence r also is.

$a>0$. In this case $\operatorname{Inv}(r)\neq\varnothing$. Let J' be the maximal element of ρ_1 and $K=\{1\}\cup J'$. We shall show that $K\in N(r)$. Since $K\in \operatorname{Inv}(r)$, it follows that $p_K(r)<r$, hence r is not minimal.

First we prove that $J'\in N(\rho')$, i.e. that $K^{\wedge}_2, \cdots, K^{\wedge}_{k+1}$ can be gathered to form a chain in $1*\rho'$. In order to show this we shall apply Lemma 6 and check that any element L lying between two elements of this packet (but not belonging to it) can be pushed out. Assume the contrary and let L lie between $K_p^\wedge$ and $K_q^\wedge$ not commuting with them. Then $L = K_p^\wedge \cap K_q^\wedge \cup \{l\}$ for some $l \in L$. Therefore the set $M = K \cup L$ contains only $k+2$ elements and the situation which we want to exclude must be realizable already in $A(k+2, k)$. But in this situation we would have that the maximal element of ρ_1 is not contained in $N(r)$ while this is impossible by Lemma 7.

Thus we may assume that $K_2^\wedge, \cdots, K_{k+1}^\wedge$ form a chain in $1 * \rho'$. Clearly $K_1^\wedge$ can be moved to this chain since all sets lying in between commute with $K_1^\wedge$.

We have established that a), b), c) are equivalent. In order to prove the last statement we must choose among $K_1, \cdots, K_R$ all members of a packet $P(L)$, $L \in C(n, k+2)$, and to show that the order induced on them is either lexicographic one, or inverse. But if we shall look in all our constructions only at elements and subsets of L, we shall reduce our task to that in $B(k+2, k)$ which is dealt with in Lemma 7.

Up to now we have proved statements a) and b) of Theorem 3. We can now conclude.

9. Proof of 3c). Our map is clearly injective. In view of Corollary to Lemma 8, in order to prove surjectivity it suffices to establish that in each equality

$$r_{\max} = p_{K_R} \cdots p_{K_1}(r_{\min})$$

one can change places the neighbouring $p_{K_i}, p_{K_{i-1}}$ if K_i, K_{i-1} commute, or reverse the order of the members of a packet if they are applied consecutively, without breaking the validity of such an equality. Note that if only the right hand side makes sense it necessarily coincides with $r_{\max}$ in view of Lemma 8c).

Suppose that K_i, K_{i-1} commute. Then $P(K_i) \cap P(K_{i-1}) = \varnothing$. From Lemma 6 it follows that if one can make chain first out of $P(K_{i-1})$ and second out of $P(K_i)$ applying only elementary equivalencies, one can make this also in reverse order.

Finally, let us show that if $p_{L_1^\wedge} \cdots p_{L_{k+2}^\wedge}(r)$ makes sense for some $L \in C(I, k+2)$, then $p_{L_{k+2}^\wedge} \cdots p_{L_1^\wedge}(r)$ also makes sense and gives the same result.

Both operators act nontrivially only upon $C(L, k) \subset C(n, k)$ and coincide in view of Lemma 7. Any element $J \in C(n, k) - C(L, k)$ does not commute with at most three elements of the set $\{L_{p,q}^\wedge\}$: if $|L \cap J| \leqslant k-2$, then J commutes with all $L_{p,q}^\wedge$, and if $|L \cap J| = k-1$ and $L - J = \{l_p, l_q, l_r\}$, then J does not commute with $L_{p,q}^\wedge, L_{p,r}^\wedge, L_{q,r}^\wedge$. From Lemma 7 it follows that in each of the packets $P(L_p)$, $P(L_q)$, $P(L_r)$ the set J lies at the same side of both elements with which it does not commute. Therefore the same will be true for the packet with reverse order. Hence if one of the expressions we work with make sense, so is the other one. ■

10. Proof of 3d). By induction on $n+k$ we shall show that if $\mathrm{Inv}(\rho)=\mathrm{Inv}(\sigma)$ then $\rho\sim\sigma$. Let $\rho=\rho_1 1*\rho'\rho_2$, $\sigma=\sigma_1 1*\sigma'\sigma_2$ From Lemma 5 it follows that ρ_1 and σ_1 coincide as sets and $\mathrm{Inv}(\rho_1\rho_2)=\mathrm{Inv}(\sigma_1\sigma_2)$, $\mathrm{Inv}(\rho')=\mathrm{Inv}(\sigma')$. Hence by inductive assumption $\pi(\rho_1\rho_2)=\pi(\sigma_1\sigma_2)$, $\pi(\rho')=\pi(\sigma')$.

Clearly, it follows that $\pi(\rho)=\pi(\sigma)$. ■

11. Question. In [12], [13] a nice combinatorial description of the set $A(n, 2)$ is given in terms of Young tableaux. Is there a generalization to $A(n, k)$, $k>2$?

§ 3. $(n-1)$-category S_n

1. Definition. An n-spheric set A consists of sets $A_0, A_1, \cdots, A_n$ and maps

$$s_k,\ t_k\colon A_k \rightrightarrows A_{k-1};\quad 1\leqslant k\leqslant n,$$
$$i_k\colon A_k\longrightarrow A_{k+1};\quad 0\leqslant k\leqslant n-1$$

such that

a) $s_{k-1}s_k=s_{k-1}t_k$; $t_{k-1}s_k=t_{k-1}t_k$.

b) $s_{k+1}i_k=t_{k+1}i_k=\mathrm{id}_{A_k}$. ■

We shall sometimes omit the subscript k and write s, t, i. The elements of A_k are called k-cells of A. A k-cell is called degenerate if it is contained in $i(A_{k-1})$.

A geometric realization of the n-spheric set A is the n-dimensional CW-complex in which to a nondegenerate k-cell there corresponds a k-dimensional disk $D(k)$ represented as the union of an open k-ball and two open k'-balls for all $0\leqslant k'<k$. Example:

$D(2)$: 

Let A be an n-spheric set. For $k'<k$ we set

$$s_{kk'}=s_{k'+1}s_{k'+2}\cdots s_k,\ t_{kk'}=t_{k'+1}\cdots t_k\colon A_k\longrightarrow A_{k'},$$
$$i_{k'k}=i_{k-1}\cdots i_{k'+1}i_{k'}\colon A_{k'}\longrightarrow A_k.$$

2. Definition. A (small) n-category is an n-spheric set C endowed with a family of multiplication maps

$$\mu_{pq}: \{(f, g) \in C_p \times C_p \mid s_{pq}(f) = t_{pq}(g)\} \longrightarrow C_p, \qquad 0 \leqslant q < p.$$

We shall denote $\mu_{pq}(f, g)$ by $f \circ_q g$.

These maps must satisfy the following axioms:

(AO) Let $f, g \in C_p$, $q < p$, $s_{pq}(f) = t_{pq}(g)$. Then $s_p(f \circ_q g) = s_p(f) \circ_q s_p(g)$, $t_p(f \circ_q g) = t_p(f) \circ_q t_p(g)$ if $q < p-1$, $s_p(f \circ_q g) = s_p(g)$, $t_p(f \circ_q g) = t_p(f)$ if $q = p-1$.

(Ass 1) $(f \circ_q g) \circ_q h = f \circ_q (g \circ_q h)$.

(Ass 2) Let $p < q; f, f', g, g' \in C_r$ and $t_q(f) = s_q(f')$, $t_q(g) = s_q(g')$, $t_p(f) = s_p(g) (= t_p(f') = s_p(g'))$. Then $(f \circ_q f') \circ_p (g \circ_q g') = (f \circ_p g) \circ_q (f' \circ_p g')$.

(Id) Let $f \in C_p$, $q < p$. Then $f \circ_q i_{qp} s_{pq}(f) = i_{qp} t_{pq}(f) \circ_q f = f$. ∎

Elements of C_p are called p-morphisms, 0-morphisms are called objects.

For $n=1$ we get the usual definition of a category.

Let C be an n-category and $m < n$. Then we can define an m-category $\tau_{\leqslant m} C$ as follows:

$$(\tau_{\leqslant m} C)_p = C_p \qquad \text{for } p < m,$$
$$(\tau_{\leqslant m} C)_m = \operatorname{Coker}(s, t;\, C_{m+1} \rightrightarrows C_m).$$

We shall show now how the construction of section 2 allows us to define for each $n \geqslant 1$ an $(n-1)$-category $\boldsymbol{S}_n$.

Let $I = \{1, 2, \cdots, n\}$, $\mathscr{P} \subset C(I, k)$ an arbitrary subset.

3. Definition (cf. sec. 2, $n^0 2$). a) A total order ρ on $\mathscr{P}$ is called admissible if for each packet P it induces on $P \cap \mathscr{P}$ either the lexicographic order or the reverse one.

We denote by $A(n, k; \mathscr{P})$ the set of all admissible total orders on $\mathscr{P}$.

b) Two total orders $\rho, \rho' \in A(n, k; \mathscr{P})$ are called elementarily equivalent, if they differ by a transposition of two neighbours which do not belong to a common packet.

We denote by $B(n, k; \mathscr{P})$ the quotient of $A(n, k; \mathscr{P})$ by the induced equivalence relation and by π the natural projection.

c) An inversion in the order $\rho \in A(n, k; \mathscr{P})$ is an element $K \in C(I, k+1)$ such that $P(K) \subset \mathscr{P}$ and induces on $P(K)$ the antilexicographic order. We denote by $\operatorname{Inv}(\rho) \subset C(I, k+1)$ the set of all inversions of ρ. Clearly $\operatorname{Inv}(\rho) = \operatorname{Inv}(\rho')$ if $\pi(\rho) = \pi(\rho')$ so that one can define $\operatorname{Inv}(r)$ for any $r \in B(n, k; \mathscr{P})$. As in section 2 we define $p_K(\rho)$.

4. Definition. a) Let $r, r' \in B(n, k; \mathscr{P})$ and $\operatorname{Inv}(r') \supset \operatorname{Inv}(r)$. Set $\mathscr{P}' = \operatorname{Inv}(r') - \operatorname{Inv}(r)$. Call an *arrow* from r to r' an element $f \in B(n, k+1; \mathscr{P}')$ such that if $f \sim K_1 \cdots K_M$, we have $r' = p_{K_1} \cdots p_{K_M}(r)$ (cf.

sec. 2, Theorem 3a)). We shall write simply f; $r \to r'$ and denote by $\mathrm{Ar}(r, r')$ the set of arrows from r to r'.

b) Composition: if $f: r \to r', f \sim K_1 \cdots K_M$; $g: r' \to r''$, $g \sim K'_1 \cdots K'_N$, we put $g \circ f: r \to r''$, $g \circ f \sim K'_1 \cdots K'_N K_1 \cdots K_M$.

c) Concatenation: let $f_0, f_1: r \to r'$; $g_0, g_1: r' \to r''$; $h_0: f_0 \to f_1$, $h_0 \sim K_1 \cdots K_M$; $h_1: g_0 \to g_1$, $h_1 \sim K'_1 \cdots K'_N$. Then $h = h_1 * h_0 : \sim K_1 \cdots K_M K'_1 \cdots K'_N \sim K'_1 \cdots K'_N K_1 \cdots K_M$ is an arrow from $g_0 f_0$ to $g_1 f_1$. ■

5. Now we can define $\boldsymbol{S}_n$. Set $\boldsymbol{S}_{n,0} = S_n$. For $1 \leqslant p \leqslant n-1$, a p-morphism of $\boldsymbol{S}_n$ consists of the following data:

a) A family of subsets $\mathscr{P}_i \subset C(I, i+1)$, $i = 0, 1, \cdots, p$, such that $\mathscr{P}_0$ $C(I, 1) = S_n$.

b) A family of pairs $r_0^i, r_1^i \in B(n, i+1; \mathscr{P}_i)$, $i = 0, 1, \cdots, p-1$, and an element $r^p \in B(n, p+1; \mathscr{P}_p)$ such that

$$r_0^i, r_1^i \in \mathrm{Ar}\,(r_0^{i-1}, r_1^{i-1}),\ i = 1, \cdots, p-1,\ r^p \in \mathrm{Ar}(r_0^{p-1}, r_1^{p-1}).$$

Thus $\mathscr{P}_j = \mathrm{Inv}\,(r_1^{i-1}) - \mathrm{Inv}\,(r_0^{i-1})$ for $i \geqslant 1$.

For $r = (r_0^0, r_1^0; r_0^1, r_1^1; \cdots; r_0^{p-1}, r_1^{p-1}; r^p) \in \boldsymbol{S}_p$ set

$$s(r) = (r_0^0, r_1^0; \cdots; r_0^{p-2}, r_1^{p-2}; r_0^{p-1}),$$
$$t(r) = (r_0^0, r_1^0; \cdots; r_0^{p-2}, r_1^{p-2}; r_1^{p-1}),$$
$$i(r) = (r_0^0, r_1^0; \cdots; r_0^{p-1}, r_1^{p-1}; r^p, r^p; \mathrm{Id}),$$

where Id means the identity, an only element of $B(n, p+2; \varnothing)$.

This data define on $\boldsymbol{S}_n = \{\boldsymbol{S}_{n,0}, \cdots, \boldsymbol{S}_{n,n-1}\}$ a structure of a spheric set. Now we shall describe multiplication.

Let $r, r' \in \boldsymbol{S}_p$, $t_q(r) = s_q(r')$. If $q = p-1$, we have

$$r = (r_0^0, r_1^0; \cdots; r_0^{p-1}, r_1^{p-1}: r^p),$$
$$r' = (r_0^0, r_1^0; \cdots; r_0^{p-2}, r_1^{p-2}; r_1^{p-1}, r_2^{p-1}; r'^p)$$

and we set

$$r' \circ_{p-1} r = (r_0^0, r_1^0; \cdots; r_0^{p-2}, r_1^{p-2}; r_1^{p-1}, r_2^{p-1}; r'^p \circ r^p).$$

If $q < p-1$, we have

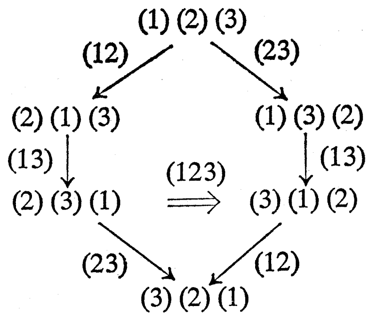
$$r = (r_0^0, r_1^0; \cdots; r_0^{q-1}, r_1^{q-1}; r_0^q, r_1^q; \cdots; r^p),$$
$$r' = (r_0^0, r_1^0; \cdots; r_0^{q-1}, r_1^{q-1}; r_1^q, r_2^q; r_0'^{q+1}, r_1'^{q+1}; \cdots; r'^p)$$

and we set

$$r' \circ_q r = (r_0^0, r_1^0; \cdots; r_0^{q-1}, r_1^{q-1}; r_0^q, r_2^q; r'^{q+1}_0 \circ r_0^{q+1},$$
$$r'^{q+1}_1 \circ r'^{q+1}_1; r'^{q+2}_0 * r'^{q+2}_0, r'^{q+2}_1 * r'^{q+2}_1; \cdots; r'^p * r^p).$$

It is straightforward to check the axioms of the $(n-1)$-category. The truncated category $\tau_{\leqslant 1}\boldsymbol{S}_n$ is a category with objects S_n in which $\mathrm{Hom}(x, y)$ is either empty or consists of one element. The latter occurs iff $x \geqslant y$ with respect to the weak Bruhat order.

6. Example. 2-category $\boldsymbol{S}_3$:



7. Convex hull. Let $x = (x_1, \cdots, x_n)$ be a point with pairwise distinct coordinates, M_n the convex hull of points $(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$, $\sigma \in S_n$. We shall be interested only in combinatorial structure of M_n which does not depend on a choice of x. The $(n-1)$-polytope M_n in a sense may be considered as a "geometric realization" of $\boldsymbol{S}_n$. More precisely, the set of vertices of M_n is (bijective to) $S_n = \boldsymbol{S}_{n,0}$. The set of 1-faces is bijective to the set of indecomposable 1-morphisms of $\boldsymbol{S}_n$ (this is well-known in the theory of the weak Bruhat order). In general, each p-face of M_n is a product $M_{p_1} \times \cdots \times M_{p_k}$, where $\sum (p_i - 1) = p$. We shall call *indecomposable* the faces M_{p+1}. E.g., M_4 has eight indecomposable 2-faces (hexagons M_3) and six decomposable ones (quadrangles $M_2 \times M_2$).

Conjecture. The set of indecomposable p-faces of M_n is naturally bijective to the set of indecomposable p-morphisms of $\boldsymbol{S}_n$.

This is true for $p=0, 1$ and also for $p=n$ (one morphism) and $p=n-1$ (2n morphisms). The faces $M_{p_1} \times \cdots \times M_{p_k}$ correspond to the products of indecomposable morphisms with commuting factors.

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