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Hecke Algebra Representations of Braid Groups and Classical Yang-Baxter Equations

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Dedicated to Professor Itiro Tamura on his 60th birthday

Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\boldsymbol{C}$ and let c be the Casimir element. Motivated by the study of rational solutions of the classical Yang-Baxter equations due to Belavin and Drinfel'd [BD], we shall construct a flat connection over

$$X_n=\{(z_1, \cdots, z_n) \in \boldsymbol{C}^n;\ z_i \neq z_j \ \text{ if } i \neq j\}.$$

Let $\rho_i: \mathfrak{g} \to \operatorname{End}(V_i)$, $1 \leq i \leq n$, be irreducible representations. Putting $\Omega = 2^{-1}(\Delta c - c\otimes 1 - 1\otimes c) \in U(\mathfrak{g})\otimes U(\mathfrak{g})$, we define $\Omega_{ij} \in \operatorname{End}(V_1\otimes\cdots\otimes V_n)$ by $\Omega_{ij}=(\rho_i\otimes\rho_j)(\Omega)$. For a complex number λ we consider the connection ω defined by

$$\sum_{1\leq i<j\leq n} \lambda\Omega_{ij} d\log(z_i-z_j).$$

The integrability of this connection follows from the following relations, which we shall call the infinitesimal pure braid relations.

$$(0.1)\qquad \begin{aligned} &[\Omega_{ij}, \Omega_{ik}+\Omega_{jk}]=[\Omega_{ij}+\Omega_{ik}, \Omega_{jk}]=0 \qquad \text{for } i<j<k,\\ &[\Omega_{ij}, \Omega_{kl}]=0 \qquad \text{for distinct } i, j, k, l. \end{aligned}$$

Thus as the monodromy of our connection we obtain a linear representation of the fundamental group of X_n, which is the pure braid group on n strings. If all the representation ρ_i are the same, the above construction gives a linear representation of the braid group depending on a parameter λ.

In the preceding paper [K3], we have shown that in the case $\mathfrak{g} = \mathfrak{sl}(2, \boldsymbol{C})$ and its two dimensional irreducible representation, the linear representation of the braid group obtained in the above manner is the so called Pimsner-Popa-Temperley-Lieb representation appearing in works of Jones [J]. The main theme of this note is to generalize this result to the case $\mathfrak{g}=\mathfrak{sl}(m+1, \boldsymbol{C})$ and its $m+1$ dimensional natural representation. We shall obtain Hecke algebra representations of the braid group corresponding to the Young diagrams of depth $\leq m+1$.

The above type of connections appear in the system of differential equations satisfied by n-point functions in two dimensional conformal field theory [BPZ] [TK]. In [TK] Tsuchiya and Kanie constructed a model of the two dimensional conformal field theory on $\boldsymbol{P}^1$ by means of the affine Lie algebra $A_1^{(1)}$. What is remarkable is that they obtained the irreducible unitarizable Hecke algebra representations corresponding to the Young diagrams of depth ≤ 2 due to Wenzl [W] as the monodromy.

This note is organized in the following way. Section 1 consists of a review of basic facts on braid groups and Hecke algebras. The infinite dimensional Lie algebra defined by the relations (0.1) appeared in works of Aomoto [A1], Chen [C2], Sullivan [S] etc. (see also [G] [H] [K1] [K2] [M]). In Section 2 we briefly report this aspect. In Section 3 we recall works of Belavin-Drinfel'd on the classical Yang-Baxter equations. Our main result is proved in Section 4.

§ 1. Braid groups and Hecke algebras

Let $X_n=\{(z_1, \cdots, z_n) \in \boldsymbol{C}^n; z_i \neq z_j\}$. The symmetric group S_n acts on X_n by $(z_1, \cdots, z_n)\cdot g=(z_{g(1)}, \cdots, z_{g(n)})$, $g \in S_n$. We denote by Y_n the quotient space X_n/S_n. The *Artin's braid group on n strings* is by definition the fundamental group of Y_n. The fundamental group of X_n is called the *pure braid group on n strings*, which we shall denote by P_n. We choose a base point $x_0=(1, 2, \cdots, n) \in X_n$ and we denote by p: $X_n \to Y_n$ the natural projection. We have an exact sequence:

$$1 \longrightarrow P_n \longrightarrow B_n \longrightarrow S_n \longrightarrow 1.$$

Let σ_j, $1 \leq j \leq n-1$, be the element of $\pi_1(Y_n, p(x_0))$ corresponding to the path in X_n given by

$$f(t)=(1, \cdots, j-1, f_j(t), f_{j+1}(t), j+2, \cdots, n), \qquad 0 \leq t \leq 1,$$

where $f_j(t)=j+t-\sqrt{t^2-t}$, $f_{j+1}(t)=j+1-t+\sqrt{t^2-t}$, as in the following picture:

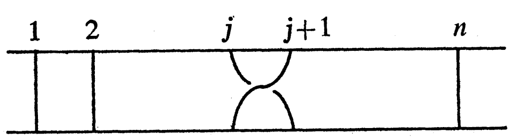

Fig. 1.1

We know by E. Artin that the braid group B_n admits a presentation with generators σ_j, $1 \leq j \leq n-1$, and defining relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad 1 \leq i \leq n-2, \tag{1.2}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if } |i-j| \geq 2. \tag{1.3}$$

For $n \in N$ and $q \in C^*$, we denote by $H_n(q)$ the algebra over C with generators $1, g_1, \cdots, g_{n-1}$ with relations:

$$(g_i+1)(g_i-q)=0, \qquad 1 \leq i \leq n-1, \tag{1.4}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \tag{1.5}$$

$$g_i g_j = g_j g_i \qquad \text{if } |i-j| \geq 2. \tag{1.6}$$

The algebra $H_n(q)$ is called the *Hecke algebra* (or *Iwahori algebra*) of the symmetric group S_n. The original definition for any Coxeter system is due to [IM]. We observe that $H_n(1)$ is nothing but the group algebra of the symmetric group S_n. As is explained in [J], if q is not a root of unity, the algebra $H_n(q)$ is semisimple and the simple $H_n(q)$ modules are in one-to-one correspondence with the Young diagrams associated with S_n. For a more precise statement and the explicit form of irreducible representations of $H_n(q)$ for each Young diagram, see Wenzl's thesis [W].

By means of the correspondence $\sigma_i \to g_i$ we obtain an algebra homomorphism $\pi: C[B_n] \to H_q(n)$. Linear representations of the braid group B_n factoring through the above homomorphism π are called *Hecke algebra representations*. For our purpose it is convenient to take $e_i = (q-g_i)/(1+q)$, $1 \leq i \leq n-1$, as generators of $H_n(q)$. We see that e_i satisfy the following relations:

$$e_i^2 = e_i \tag{1.7}$$

$$e_i e_{i+1} e_i - \tau e_i = e_{i+1} e_i e_{i+1} - \tau e_{i+1}, \qquad \tau = q/(1+q)^2 \tag{1.8}$$

$$e_i e_j = e_j e_i \qquad \text{if } |i-j| \geq 2. \tag{1.9}$$

§ 2. Infinitesimal pure braid relations

We start with a matrix valued 1-form

$$\omega = \sum_{1 \leq i < j \leq n} \Omega_{ij} d \log (z_i - z_j)$$

defined over X_n. Here Ω_{ij}, $1 \leq i < j \leq n$, are $m \times m$ matrices with C coefficients. Let E be a trivial complex vector bundle of rank m over X_n

with a global frame $(e_1, \cdots, e_m)$. Putting $\omega=(\omega_{ij})_{1\leq i\leq j\leq m}$, we define a connection $\nabla: \mathcal{O}(E)\to\Omega^1\otimes\mathcal{O}(E)$ by

$$\nabla(e_i)=-\sum_{j=1}^{m}\omega_{ji}\otimes e_j, \qquad 1\leq i\leq m.$$

Here we denote by Ω^1 the sheaf of the holomorphic 1-forms on X_n. Then the horizontal sections of our connection turn out to be the solutions of the total differential equation

$$dy=\omega y, \qquad y={}^t(y_1, \cdots, y_m).$$

Lemma 2.1. *The connection ∇ is integrable if and only if the following conditions are satisfied.*

$$[\Omega_{ij}, \Omega_{ik}+\Omega_{jk}]=[\Omega_{ij}+\Omega_{ik}, \Omega_{jk}]=0 \text{ for } i<j<k, \tag{2.2}$$

$$[\Omega_{ij}, \Omega_{kl}]=0 \qquad \text{for distinct } i, j, k, l. \tag{2.3}$$

Proof. The curvature form of the connection ∇ is given by $d\omega+\omega\wedge\omega$. Since ω is closed the integrability condition is equivalent to

$$\omega\wedge\omega=0. \tag{2.4}$$

The relations among $\varphi_{ij}=d\log(z_i-z_j)$, $1\leq i<j\leq n$, are generated by

$$\varphi_{ij}\wedge\varphi_{jk}+\varphi_{jk}\wedge\varphi_{ik}+\varphi_{ik}\wedge\varphi_{ij}=0$$

for $i<j<k$ (see for example [Ar] [OS]). Hence the relations (2.2) and (2.3) are equivalent to (2.4). This completes the proof.

We call the relations (2.2) and (2.3) the *infinitesimal pure braid relations*. In the following we shall assume these conditions. Let Y be the matrix consisting of m linearly independent solutions of $dy=\omega y$. For $\gamma\in\pi_1(X_n)$, the result of the analytic continuation γ^*Y can be written in the form

$$\gamma^*Y=Y\cdot\theta(\gamma) \quad \text{with} \quad \theta(\gamma)\in GL(m, \mathbf{C}).$$

This gives the monodromy representation of our connection

$$\theta\colon P_n\longrightarrow GL(m, \mathbf{C}).$$

An expression of this linear representation is given by the iterated integrals of the connection form due to K. T. Chen [C1] [C2]. First we recall the definition.

(2.5) **Definition.** Let X be a smooth manifold and let ω_i, $1 \leq i \leq r$, be matrix valued 1-forms on X. For a path γ: $[0, 1] \rightarrow X$ we put $\gamma^* \omega_i = A_i(t)dt$ and we define the *iterated integral* by

$$\int_\gamma \omega_1 \omega_2 \cdots \omega_r = \int_0^1 A_1(t_1) \int_0^{t_1} A_2(t_2) \cdots \int_0^{t_{r-1}} A_r(t_r) dt_r \cdots dt_1$$

Lemma 2.6 (K.T. Chen). *Let us suppose that the connection ∇ is integrable. Then the monodromy representation θ is given by*

$$\theta(\gamma) = \mathbf{1} + \int_\gamma \omega + \int_\gamma \omega\omega + \int_\gamma \omega\omega\omega + \cdots.$$

Let $\mathbf{C}\langle\langle X_{ij} \rangle\rangle$ denote the non-commutative formal power series ring with indeterminates X_{ij}, $1 \leq i < j \leq n$. Let I be the two-sided ideal of $\mathbf{C}\langle\langle X_{ij} \rangle\rangle$ generated by

(2.7) $\qquad [X_{ij}, X_{ik} + X_{jk}], \quad [X_{ij} + X_{ik}, X_{jk}], \quad i < j < k$

(2.8) $\qquad [X_{ij}, X_{kl}], \quad i, j, k, l$ distinct.

We denote by A the quotient algebra $\mathbf{C}\langle\langle X_{ij} \rangle\rangle / I$. Following K.T. Chen, we introduce the *universal integrable* 1-*form* defined by

$$\tilde{\omega} = \sum_{1 \leq i < j \leq n} X_{ij} \otimes d \log (z_i - z_j).$$

As a universal expression of Lemma 2.6, we get a homomorphism $\tilde{\theta}$: $P_n \rightarrow A$. The monodromy θ is obtained by substituting $X_{ij} = \Omega_{ij}$.

For a group G we denote by $\mathbf{C}[G]$ its group algebra over $\mathbf{C}$. Let ε: $\mathbf{C}[G] \rightarrow \mathbf{C}$ denote the augmentation homomorphism and we put $IG = \operatorname{Ker} \varepsilon$. Let $\mathbf{C}[G]^\wedge$ be the completion of $\mathbf{C}[G]$ with respect to the topology defined by $\{I^k G\}_{k \geq 1}$, where $I^k G$ signifies the k-th power of IG. Let j: $G \rightarrow \mathbf{C}[G]^\wedge$ be the natural homomorphism. The following statement is a special case of the theorems discussed in [A1], [H], [K1] and [K2]. We use the formulation due to R. Hain.

Theorem 2.9. *We have an isomorphism of complete Hopf algebras $\mathbf{C}[P_n]^\wedge \cong A$ such that the following diagram is commutative.*

$$\begin{array}{ccc} & \xrightarrow{\ j\ } & \mathbf{C}[P_n]^\wedge \\ P_n & & \downarrow \wr \\ & \xrightarrow[\ \tilde{\theta}\ ]{} & A \end{array}$$

Another description of the above algebra A can be given by means of the *reduced bar construction* on the logarithmic forms. Let $R^{\cdot}$ be the $\boldsymbol{C}$ subalgebra of the algebra of the smooth differential forms on X_n generated by

$$\varphi_{ij}=d\log(z_i-z_j), \qquad 1\leq i<j\leq n.$$

Let us introduce the double complex $\oplus\mathscr{B}^{s,t}$ define by

$$\mathscr{B}^{-s,t}=(\otimes^s R^{\cdot})^t.$$

A typical element of $\mathscr{B}^{-s,t}$ will be denoted by $[a_1|a_2|\cdots|a_s]$, where $a_j\in R^{\cdot}$. The differential $d\colon \mathscr{B}^{-s,t}\to\mathscr{B}^{-s+1,t}$ is defined by

$$d[a_1|\cdots|a_s]=\sum_{i=1}^{s-1}(-1)^{i+1}[Ja_1|\cdots|Ja_i\wedge a_{i+1}|\cdots|a_s],$$

where $Ja=(-1)^{\deg a}a$. The reduced bar construction is by definition the associated total complex $\mathscr{B}(R^{\cdot})$ with the diagonal

$$\Delta\colon \mathscr{B}(R^{\cdot})\longrightarrow\mathscr{B}(R^{\cdot})\otimes\mathscr{B}(R^{\cdot})$$

given by

$$\Delta[a_1|\cdots|a_s]=\sum_{i=0}^{s}[a_1|\cdots|a_i]\otimes[a_{i+1}|\cdots|a_s].$$

It follows from a result of K.T. Chen [C2] that we have an isomorphism of Hopf algebras:

$$\mathrm{Hom}\,(H^0(\mathscr{B}(R^{\cdot})),\boldsymbol{C})\cong A$$

(see also [A1] [H]).

In [K3] we have shown the following result by proving that the universal monodromy $\tilde{\theta}\colon P_n\to A$ is injective. We put

$$\gamma_{ij}=\sigma_i\sigma_{i+1}\cdots\sigma_{j-1}\sigma_j^2\sigma_{j-1}^{-1}\cdots\sigma_i^{-1}, \qquad 1\leq i<j\leq n.$$

Theorem 2.10. *Let θ; $P_n\to GL(m,\boldsymbol{C})$ be a linear representation such that each $\|\theta(\gamma_{ij})-\mathbf{1}\|$ is sufficiently small for $1\leq i<j\leq n$. Then there exist constant matrices Ω_{ij}, $1\leq i<j\leq n$, close to $\mathbf{0}$, satisfying the infinitesimal pure braid relations, such that the monodromy of the connection $\omega=\sum_{1\leq i<j\leq n}\Omega_{ij}d\log(z_i-z_j)$ is the given θ.*

The above theorem may be considered as a version of the Riemann-Hilbert correspondence. The object of the next section is to give an explicit form of our connection in the case of Hecke algebra representations.

§ 3. Review of classical Yang-Baxter equations

In this section we briefly review rational solutions of the classical Yang-Baxter equations associated with simple Lie algebras and we discuss a relation with the infinitesimal pure braid relations. A general reference of the classical Yang-Baxter equations is [BD].

Let V be a finite dimensional complex vector space. By the *classical Yang-Baxter equation* we mean the following functional equation for a matrix valued meromorphic function $r(u) \in \mathrm{End}(V\otimes V)$ of $u \in \boldsymbol{C}$:

$$(3.1) \qquad [r_{12}(u), r_{13}(u+v)]+[r_{12}(u), r_{23}(v)]+[r_{13}(u+v), r_{23}(v)]=0$$

for any $u, v \in \boldsymbol{C}$. Here $r_{ij}(u) \in \mathrm{End}(V_1\otimes V_2\otimes V_3)$, $V_1=V_2=V_3=V$, signifies the matrix $r(u)$ on the space $V_i\otimes V_j$, acting as identity on the third space; e.g., $r_{12}(u)=r(u)\otimes \mathbf{1}_V$, $r_{23}(u)=\mathbf{1}_V\otimes r(u)$. Since the equation (3.1) is written in terms of bracket products, it makes sense for a $\mathfrak{g}\otimes\mathfrak{g}$-valued function $r(u)$, with an abstract Lie algebra $\mathfrak{g}$. To each such $\mathfrak{g}\otimes\mathfrak{g}$-valued solution $r(u)$, we may associate a solution of (3.1) $(\rho\otimes\rho)(r(u)) \in \mathrm{End}(V\otimes V)$, by specifying an irreducible representation ρ: $\mathfrak{g}\to\mathrm{End}(V)$. In the case $\mathfrak{g}$ is a simple Lie algebra over $\boldsymbol{C}$, solutions of the classical Yang-Baxter equation have been classified by Belavin-Drinfel'd (see [BD] for a precise statement). In particular, we know the following rational solution.

Proposition 3.2 (Belavin-Drinfel'd [BD]). *Let* $\mathfrak{g}$ *be a simple Lie algebra over* $\boldsymbol{C}$ *and let* $\{I_\alpha\}$ *be an orthonormal basis of* $\mathfrak{g}$ *with respect to the Cartan-Killing form. We put* $\Omega=\sum_\alpha I_\alpha\otimes I_\alpha \in \mathfrak{g}\otimes\mathfrak{g}$. *Then* $r(u)=\Omega/u$ *is a solution of the classical Yang-Baxter equation.*

Proof. We denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Let $c \in U(\mathfrak{g})$ be the Casimir element defined by $c=\sum_\alpha I_\alpha\cdot I_\alpha$. Let Δ: $U(\mathfrak{g})\to U(\mathfrak{g})\otimes U(\mathfrak{g})$ be the diagonal homomorphism defined by $\Delta(x)=x\otimes 1+1\otimes x$ for $x \in \mathfrak{g}$. Then Ω can be written as

$$2\Omega=\Delta(c)-c\otimes 1-1\otimes c.$$

Let us recall the well-known fact that the Casimir element c lies in the center of $U(\mathfrak{g})$ (see for example [Hu]). It follows from the above expression for Ω that

$$[\Delta(x), \Omega]=0 \qquad \text{for any } x \in \mathfrak{g}.$$

In particular, we have

$$(3.3) \qquad [\Omega_{12}+\Omega_{13}, \Omega_{23}]=[\Omega_{12}, \Omega_{13}+\Omega_{23}]=0.$$

Here Ω_{ij} is defined by $r_{ij}(u)=\Omega_{ij}/u$. By an elementary computation we check that the equation 3.3 signifies that $r(u)$ is a solution of the classical Yang-Baxter equation.

We observe that the relation (3.3) corresponds to the infinitesimal pure braid relations in the case $n=3$ (see 2.1). This leads us to the following construction. Let $\rho_i: \mathfrak{g}\to\mathrm{End}(V_i)$, $1\leq i\leq n$, be a family of irreducible representations of $\mathfrak{g}$. We define $\Omega_{ij}\in\mathrm{End}(V_1\otimes\cdots\otimes V_n)$ by $\Omega_{ij}=(\rho_i\otimes\rho_j)(\Omega)$, $1\leq i<j\leq n$. Here ρ_i signifies the representation ρ_i on V_i, acting as identity on the other factors. Then we have

Lemma 3.4. *Let* $\Omega_{ij}\in\mathrm{End}(V_1\otimes\cdots\otimes V_n)$ *be the matrices defined above. Then these satisfy the infinitesimal pure braid relations.*

Proof. The relations (2.2) follows from (3.3). The other relations are clear from the construction.

Thus we obtain an integrable connection

$$\omega=\sum_{1\leq i<j\leq n}\Omega_{ij}\,d\log(z_i-z_j).$$

In the following, we shall consider the connection $\lambda\omega$ for a complex number λ. As the monodromy we obtain a one-parameter family of linear representations

$$\theta_\lambda\colon P_n\longrightarrow GL(V_1\otimes\cdots\otimes V_n).$$

Our problem is to give an explicit form of the above representation as a function of λ. By means of the expression of the monodromy using the Chen's iterated integrals (Lemma 2.6), this function is entire with respect to λ. Incidently, this type of connection appears naturally as the differential equations satisfied by the n-point functions in the two dimensional conformal field theory with gauge symmetry (see [TK]).

§ 4. The case $\mathfrak{g}=\mathfrak{sl}(m+1, C)$

In the case $\mathfrak{g}=\mathfrak{sl}(2, C)$ and all $\rho_i: \mathfrak{g}\to\mathrm{End}(V_i)$ are two dimensional irreducible representations, the one-parameter family of the monodromy representations constructed in the previous section was determined in [K3]. This representation factors through the Hecke algebra and is known as the Pimsner-Popa-Temperley-Lieb representation. In this section we generalize this result to the case $m>1$.

Let $\mathfrak{g}=\mathfrak{sl}(m+1, C)$ and we denote by $c\in U(\mathfrak{g})$ the Casimir element. Let V be an $m+1$ dimensional complex vector space and we fix a natural action of $\mathfrak{g}$ on V. As in the previous section we put

$$\Omega=2^{-1}(\Delta c-c\otimes 1-1\otimes c)$$

and we define $\Omega_{ij}\in \mathrm{End}(V^{\otimes n})$, $1\leq i<j\leq n$, by means of the above natural representation.

To describe the associated monodromy representation θ_λ, we introduce the following Hecke algebra representation. We put

$$\mathscr{F}_n=\bigotimes_{i=1}^{n} M_{m+1}(\boldsymbol{C}).$$

Let e be the element of $\mathscr{F}_n$ defined by

$$\begin{aligned}e=\{&(1+q)^{-1}(\sum_{i<j} e_{ii}\otimes e_{jj})+(1+q)^{-1}q(\sum_{i>j} e_{ii}\otimes e_{jj})\\&+(1+q)^{-1}\sqrt{q}\,(\sum_{i\neq j} e_{ij}\otimes e_{ji})\}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1},\end{aligned}$$

where $q\in \boldsymbol{C}^*$ and e_{ij} are matrix units for $M_{m+1}(\boldsymbol{C})$. We define a shifting endomorphism of $\mathscr{F}_n$ by

$$\sigma\cdot(x_1\otimes x_2\otimes\cdots\otimes x_n)=x_n\otimes x_1\otimes\cdots\otimes x_{n-1}.$$

Putting $e_i=\sigma^{i-1}(e)$, we observe that e_i satisfy the relations (1.7), (1.8) and (1.9). Hence the correspondence $\sigma_i\to q-(1+q)e_i$ defines a Hecke algebra representation of the braid group. This representation appeared in works by M. Jimbo [Ji]. The case $m=1$ is the Pimsner-Popa-Temperley-Lieb representation (see [J]). Our main result is the following.

Theorem 4.1. *As the monodromy of the connection*

$$\omega=\sum_{1\leq i<j\leq n}\lambda\Omega_{ij}\,d\log(z_i-z_j),\qquad \lambda\in\boldsymbol{C}$$

defined above by means of $\mathfrak{g}=\mathfrak{sl}(m+1,\boldsymbol{C})$ *and its* $m+1$ *dimensional natural representation, we obtain a linear representation of the braid group* B_n *equivalent to the representation given by the correspondence*

$$\sigma_i\longrightarrow q^{-\mu}\{q-(1+q)e_i\},\qquad 1\leq i\leq n-1,$$

where $q=\exp(2\pi i\lambda)$ *and* $\mu=2^{-1}(m+1)^{-1}(m+2)$.

Proof. We devide the proof into several claims.

claim 1: The monodromy representation θ_λ: $P_n\to GL(V^{\otimes n})$ can be extended to a linear representation of B_n.

Let us start with the trivial vector bundle E over X_n with fiber $V^{\otimes n}$ equipped with the connection form ω. The symmetric group S_n acts diagonally on E by

$$((z_1, \cdots, z_n), x_1\otimes x_2\otimes\cdots\otimes x_n)\cdot g=((z_{g(1)}, \cdots, z_{g(n)}), x_{g(1)}\otimes\cdots\otimes x_{g(n)})$$

for $g \in S_n$, $(z_1, \cdots, z_n) \in X_n$, $x_1\otimes\cdots\otimes x_n \in V^{\otimes n}$. Let E' be the quotient bundle over Y_n. Since the connection form ω is invariant by the above action of the symmetric group S_n, this defines a flat connection on E'. The associated monodromy gives a linear representation of B_n, which is an extension of θ_λ. We denote by φ_λ the linear representation of B_n obtained in the above way by replacing the connection form ω by

$$\omega+\sum_{1\le i<j\le n} 2\mu\cdot\mathbf{1}\, d\log(z_i-z_j). \tag{4.2}$$

claim 2. The linear representation φ_λ obtained above is a Hecke algebra representation with $q=\exp(2\pi i\lambda)$.

First we look at the action of the Casimir element on $V\otimes V$. Let $V\otimes V=W^+\oplus W^-$ be the decomposition into the symmetric part W^+ and the antisymmetric part W^-. Then a computation shows that the Casimir element c acts as $(m+1)^{-1}(2m^2+6m)\times\mathrm{id}$ on W^+ and as $(m+1)^{-1}(2m^2+2m-4)$ on W^- respectively. Hence $\Omega=2^{-1}(\Delta c-c\otimes 1-1\otimes c)$ acts as $(m+1)^{-1}m$ on W^+ and as $-(m+1)^{-1}(m+2)$ on W^-.

Now we are going to compute the residue of the connection of E' associated with φ_λ introduced at the end of the proof of claim 1. We denote by H_{ij} the hyperplane in C^n defined by $z_i=z_j$. Let us choose $x_0 \in H_{12}$ in such a way that x_0 is not a singular point of the union of the hyperplanes H_{ij}. Let D be a small disc transversal to H_{12} at x_0 with a local coordinate $t=z_1-z_2$, the connection from (4.2) is written locally on D as

$$(A_{-1}t^{-1}+\sum_{j\ge 0} A_j t^j)dt \tag{4.3}$$

with $A_{-1}=\lambda\Omega_{12}+2\mu\cdot\mathbf{1}$, $A_j \in \mathrm{End}(V\otimes\cdots\otimes V), j\ge 0$. Since our connection is invariant by the action of S_n, (4.3) is invariant, in particular, by the transposition (1, 2). Hence we have $A_0=A_2=A_4=\cdots=0$. Let p: $C^n\to C^n/S_n$ denote the natural projection. Outside $\bigcup_{1\le i<j\le n}H_{ij}$, p induces an unramified covering $X_n\to Y_n$. Let Δ denote the discriminant set defined by $p(\bigcup H_{ij})$. We may take a coordinate around x_0, $(y_1, \cdots, y_n)$ with $t=y_1$, and a coordinate around $p(x_0)$, $(w_1, \cdots, w_n)$, so that Δ is locally defined by $w_1=0$, and that projection p is locally given by

$$p: (y_1, \cdots, y_n)\longrightarrow(y_1^2, y_2, \cdots, y_n).$$

We put $\xi=t^2$. In a small disc D' transverse to Δ at $p(x_0)$, the connection (4.3) can be written in the form:

$$2^{-1}(A_{-1}\xi^{-1}+\sum_{j\geq 1} A_{2j+1}\xi^{2j+1})d\xi.$$

In partucular, it has residue $Z=2^{-1}(\lambda\Omega_{12}+2\mu\cdot\mathbf{1})$ at 0. It follows that the local monodromy of the trivial bundle over $D'-p(x_0)$ with the connection (4.3) is given by $e^{2\pi i Z}$ (see [D]). Hence the monodromy of E' restricted to $D'-p(x_0)$ is given by $Pe^{2\pi i Z}$, where $P \in GL(V\otimes\cdots\otimes V)$ is the permutation matrix defined by $P(x_1\otimes x_2\otimes x_3\otimes\cdots)=x_2\otimes x_1\otimes x_3\otimes\cdots$. The monodromy matrix $\varphi_\lambda(\sigma_1)$ is a conjugate of $Pe^{2\pi i Z}$, hence it has q and -1 as eigenvalues. We can apply the same argument to $\varphi_\lambda(\sigma_j)$, $j\geq 2$. Moreover the matrices $\varphi_\lambda(\sigma_j)$ are semisimple. Hence by putting $g_j=\varphi_\lambda(\sigma_j)$, we have the relations:

$$(g_j+1)(g_j-q)=0.$$

The other relations (1.5) and (1.6) follow from the fact that φ_λ of a linear representation of the braid group. Thus we have proved claim 2.

The proof of our main theorem is completed in the following way. Let us compare φ_λ with the linear representation given by

$$\sigma_i \longrightarrow q-(1+q)e_i.$$

Each representation is a Hecke algebra representation and is the permutation representation if $q=0$. Hence we conclude that they are equivalent if q is close to 1. Moreover we know by Lemma 2.6 that the monodromy φ_λ is an entire function with respect to λ. Thus by an analytic continuation we have proved our theorem.

Let us suppose that q is not a root of unity. We denote by $\rho(d_1, d_2, \cdots, d_k)$ the Hecke algebra representation of B_n corresponding to the Young diagram of type $(d_1, \cdots, d_k)$, $d_1\geq d_2\geq\cdots\geq d_k$, $d_1+d_2+\cdots+d_k=n$ (see [W] for its explicit form). As a corollary to our main theorem, we have the following.

Corollary 4.4. *If q is not a root of unity, our monodromy representation of B_n associated with $\mathfrak{sl}(m+1, \mathbf{C})$ is a direct sum of $\rho(d_1, \cdots, d_k)$, $k\leq m+1$, $d_1\geq\cdots\geq d_k$, $d_1+\cdots d_k=n$, with the multiplicity*

$$\prod_{i<j}(l_i-l_j)/m!\,(m-1)!\cdots 1!, \quad \textit{where} \quad l_i=d_i+m-i+1.$$

Remark 4.5. If $m=1$, let us recall that e_i appearing in the definition of the PPTL representation satisfy the relations:

$$(4.6) \qquad e_i^2 = e_i$$

$$(4.7) \qquad e_i e_{i\pm 1} e_i = \beta^{-1} e_i, \qquad \beta = 2 + q + q^{-1}$$

$$(4.8) \qquad e_i e_j = e_j e_i \qquad \text{if } |i-j| \geq 2$$

Let $A_{\beta,n}$ denote the abstract $\boldsymbol{C}$ algebra generated by $1, e_1, \cdots, e_{n-1}$ with the relations (4.6), (4.7), (4.8). In the case $\beta \geq 4$, this is a kind of the *Jones algebra* (see [J]). We have shown that our monodromy representation factors through the natural homomorphisms:

$$\boldsymbol{C}[B_n] \longrightarrow H_n(q) \longrightarrow A_{\beta,n}$$

where the second homomorphism is defined by the correspondence

$$g_i \longrightarrow q-(1+q)e_i.$$

Remark 4.6. The monodromy representation of B_n corresponding to the Young diagram of type $(n-1, 1)$ is up to sign the so called *reduced Burau representation* (see [B]). The associated system of differential equation has as the solution a simple integral representation. We shall explain this briefly (see [A2] [K3]).

We start with the natural projection $\pi: X_{n+1} \to X_n$. Let us fix a complex number λ such that λ and $n\lambda$ are not integers. Let $\mathscr{L}$ be the local system over X_{n+1} associated with the representation defined by sending γ_{ij} to $e^{-2\pi i\lambda}$. The main object of this paragraph is to study the local system over X_n defined by $\boldsymbol{R}^1\pi_*\mathscr{L}^*$. Here $\mathscr{L}^*$ stands for the dual local system of $\mathscr{L}$. Let $i: Z \hookrightarrow X_{n+1}$ denote the fiber of π over $(1, \cdots, n)$.

We put $\Phi = (\zeta - z_1)^\lambda \cdots (\zeta - z_n)^\lambda$, where $\zeta = z_{n+1}$. for $w \in H_1(Z, i^*\mathscr{L})$, let us consider the integral

$$F_i(z_1, \cdots, z_n) = \int_w \Phi \, d\log(\zeta - z_i), \qquad 1 \leq i \leq n$$

which are considered as multi-valued functions on X_n.

Let w_j, $1 \leq j \leq n-1$, be a basis of $H_1(Z, i^*\mathscr{L})$ chosen as the segment $]j, j+1[$. Let us consider

$$Y = \begin{pmatrix} \int_{w_1} \Phi \, d\log(\zeta - z_1), & \cdots, & \int_{w_{n-1}} \Phi \, d\log(\zeta - z_1), & 1 \\ \vdots & & \vdots & \vdots \\ \int_{w_1} \Phi \, d\log(\zeta - z_n), & \cdots, & \int_{w_{n-1}} \Phi \, d\log(\zeta - z_n), & 1 \end{pmatrix}$$

The following $n\times n$ matrix J_{ij}, $1\leq i<j\leq n$, is called the *Jordan-Pochhammer matrix.*

$$J_{ij}=\begin{bmatrix} \ddots & & & & \\ & \lambda & \cdots & -\lambda & \\ & \vdots & & \vdots & \\ & -\lambda & \cdots & \lambda & \\ & & & & \ddots \end{bmatrix}\begin{matrix} \\ (i \\ \\ (j \\ \\ \end{matrix}$$

Here all the other components are zero. Then the matrix Y is a fundamental solution of the total differential equation

$$dy=\sum_{1\leq i<j\leq n} J_{ij}\, d\log(z_i-z_j)\cdot y.$$

The 1-form $\sum_{1\leq i<j\leq n} J_{ij}\, d\log(z_i-z_j)$ defines an integrable connection on the trivial bundle $X_n\times \boldsymbol{C}^n$. Let the symmetric group S_n act diagonally on this bundle via the permutation of coordinates. The above connection is invariant by this action, hence it defines a local system over $Y_n=X_n/S_n$, which is the direct sum of a local system of rank $n-1$, say $\mathscr{J}_n$, and a rank 1 trivial local system. We call $\mathscr{J}_n$ the *Jordan-Pochhammer system.* Let β_j be matrices with $\boldsymbol{Z}[t, t^{-1}]$ coefficients defined by

$$\beta_j=\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 1-t & 1 & & \\ & & & t & 0 & & \\ & & & & & 1 & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{bmatrix}(j$$

The correspondence $\sigma_j\rightarrow\beta_j$ defines a linear representation of B_n called the *Burau representation.* Since this representation remains invariant the subspace such that the sum of the coordinates is zero, we get an $n-1$ dimensional representation

$$\psi_n\colon B_n\longrightarrow GL_{n-1}(\boldsymbol{Z}[t, t^{-1}])$$

which we call the *reduced Burau representation.* The monodromy representation of B_n associated with the above Jordan-Pochhammer system is the reduced Burau representation with $t=e^{2\pi i\lambda}$. This type of a system of differential equations was studied extensively by Deligne and Mostow [DM].

References

[A1] K. Aomoto, Fonctions hyperlogarithmiques et groupes de monodromie unipotents, J. Fac. Sci. Tokyo, **25** (1978), 149–156.

[A2] ——, Gauss-Manin connection of integral of difference products, J. Math. Soc. Japan, **39** (1987), 191–208.

[Ar] V. I. Arnold, The cohomology ring of the colored braid group, Mat. Zametki, **5** (1969), 227–231.

[B] J. Birman, Braids, links, and mapping class groups, Ann. Math. Stud., **82** (1974).

[BD] A. A. Belavin and V. G. Drinfiel'd, Solutions of the classical Yang-Baxter equation for simple Lie algebras, Funct. Anal. Appl., **16** (1982), 1–29.

[Bo] N. Bourbaki, Groupes et algèbres de Lie, IV, V, VI, Masson, Paris (1982).

[BPZ] A. A. Belavin, A. N. Polyakov and A. B. Zamolodchikov, Infinite dimensional symmetries in two-dimensional quantum field theory, Nucl. Phys., **B241** (1984), 333–380.

[C1] K. T. Chen, Extensions of C^∞ function algebra by integrals and Malcev completion of π_1, Adv. in Math., **23** (1977), 181–210.

[C2] ——, Iterated path integrals, Bull. Amer. Math. Soc., **83** (1977), 831–879.

[CC] D. Chudnovsky and G. Chudnovsky, Editors, The Riemann problem, complete Integrability, and arithmetic applications, Lect. Notes in Math. **925** (1982), Springer Verlag.

[D] P. Deligne, Equations différentielles à points singuliers réguliers, Lect. Notes in Math. **163** (1970), Springer Verlag.

[DM] P. Deligne and G. D. Mostow, Monodromy of hypergeometric functions and non-lattice integral monodromy, Publ. Math. IHES, **63** (1986), 5–90.

[G] V. Golubeva, On the recovery of Pfaffian systems of Fuchsian type from the generators of the monodromy group, Math. USSR. Izvest. **17** (1981), 227–241.

[H] R. Hain, On a generalization of Hilbert 21st problem, Ann. Scient. Ec. Norm. Sup., **19** (1986), 609–627.

[Hu] J. E. Humphreys, Introduction to Lie algebras and representation theory, GTM 9, Springer Verlag.

[IM] N. Iwahori and H. Matsumoto, On some Bruhat decomposition and the structure of the Hecke ring of p-adic Chevalley groups, Publ. IHES, **25** (1965), 5–48.

[J] V. Jones, Hecke algebra representations of braid groups and link polynomials, to appear in Ann. of Math.

[Ji] M. Jimbo, A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation, Lett. in Math. Phys., **11** (1986), 247–252.

[K1] T. Kohno, On the holonomy Lie algebra and the nilpotent completion of the fundamental group of the complement of hypersurfaces, Nagoya Math. J., **92** (1983), 21–37.

[K2] ——, Série de Poincaré-Koszul associée aux groupes de tresses pures, Invent. Math., **82** (1985), 57–75.

[K3] ——, Linear representations of braid groups and classial Yang-Baxter equations, to appear in Contemp. Math.

[L] I. A. Lappo-Danilevsky, Mémoires sur la théorie des systèmes des équations différentielles linéaires, Chelsea, 1953.

[M] J. Morgan, The algebraic topology of smooth algebraic varieties, Publ. Math. IHES, **48** (1978), 137–204.

[OS] P. Orlik and L. Solomon, Combinatorics and topology of complement of hyperplanes, Invent. Math., **56** (1980), 167–189.

[S] D. Sullivan, Infinitesimal computations in topology, Publ. Math. IHES, **47** (1977), 269–331.

[TK] A. Tsuchiya and Y. Kanie, Vertex operators in conformal field theory on $\boldsymbol{P}^1$ and monodromy representations of braid groups, in this volume, 297–372.

[W] H. Wenzl, Representations of Hecke algebras and subfactors, Thesis, University of Pensylvania (1985).

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