Advanced Studies in Pure Mathematics 16, 1988
Conformal Field Theory and Solvable Lattice Models
pp. 123–170

Lectures on Conformal Field Theory

Vl. S. Dotsenko

Lecture 1.

In statistical physics, in the theory of critical phenomena at the second order phase transition points, the global scaling symmetry has been known and extensively used for many years. This led to the renormalization group approach of Wilson to the critical phenomena.

In the scaling theory, which is the theory at the second order phase transition point, a given physical system is classified by a set of basic operators $\{\phi_i(x)\}$, having anomalous dimensions $\{\Delta_i\}$. This is the spectrum of the theory. Under scaling transformations of the space

$$x \longrightarrow \lambda x \tag{1.1}$$

(λ being a constant parameter) the basic operators transform like:

$$\phi_i(x) \longrightarrow \lambda^{\Delta_i}\phi_i(\lambda x). \tag{1.2}$$

It means that if we made the transformation (1.2) then the correlation functions for the basic operators

$$\langle \phi_1(x_1)\phi_2(x_2)\cdots\rangle \tag{1.3}$$

will not change. In particular, this fixes uniquely the 2-point function

$$\langle \phi_i(x_1)\phi_i(x_2)\rangle = \frac{\text{const}}{|x_1-x_2|^{2\Delta_i}} \tag{1.4}$$

(the normalizing constant is arbitrary), so, the anomalous dimensions Δ_i can be found, if the two-point functions are known. By anomalous dimension we mean that it is different from canonical dimensions for free, noninteracting fields. The general situation for the critical phenomena is that the basic fields (operators) will be strongly interacting. And it is reflected in the fact that their dimensions, defined by two-point functions like (1.4), will be anomalous.

Received November 1, 1986.
*) Lectures delivered at RIMS, Kyoto University, October 4–6, 1986.

There are various approximate techniques, numerical and analytical, to calculate critical indices (powers of nonanalytic singularities in free energy, magnetization, on approaching the critical point) for various statistical systems. These are expressed through dimensions of basic operators $\{\Delta_i\}$ by standard scaling relations. Some (many of them in fact) 2D statistical systems can be solved exactly, and so the spectrum of anomalous dimensions $\{\Delta_i\}$ can be found exactly.

The scaling transformation above (1.2) is global, $\lambda=\text{const}$, and the corresponding scaling symmetry of critical phenomena has been well established. The idea that the critical point theories should also be invariant by local scalings like (1.2), in which λ depends on space point x, was put forward by Polyakov [1]. This is a conformal symmetry of critical phenomena. He analyzed consequences of such a symmetry and found that two-point functions for basic operators should be subject to orthogonality condition:

$$\langle\phi_i(x)\phi_j(0)\rangle\propto\frac{\delta_{ij}}{|x|^{2\Delta_i}} \tag{1.5}$$

so that basic operators are orthogonal, in this sense. Also conformal symmetry fixes the 3-point functions like:

$$\langle\phi_1\phi_2\phi_3\rangle=\frac{\text{const}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}. \tag{1.6}$$

The higher point functions are not fixed, but obey certain constraints.

These were the consequences of the conformal symmetry, valid for critical theories in any number of dimensions. The reason that only a finite number of constraints are obtained lies in the fact that in general the conformal group has only a finite number of parameters.

The situation is drastically different in 2D theories. In 2D the conformal transformation, on an infinite plane, is achieved by any analytic transformation of the space points.

$$\begin{gathered} z\longrightarrow\tilde{z}=f(z)\\ \bar{z}\longrightarrow\bar{\tilde{z}}=\overline{f(z)}\\ z=x_1+ix_2,\quad \bar{z}=x_1-ix_2. \end{gathered} \tag{1.7}$$

We will use complex coordinates z, $\bar{z}$ for 2D theory just because they are more convenient.

So, in 2D the group of local rescalings, like (1.2), can be extended to an infinite dimensional group of analytic transformations. Correspondingly, the basic operators transform as:

(1.8)
$$\phi_i(z,\bar{z}) \longrightarrow (\lambda(x))^{\Delta_i}\phi_i(f(z),\overline{f(z)})$$
$$\lambda(x) = \left|\frac{df(z)}{dz}\right|.$$

The conformal theory *for 2D systems*, the theory based on assumption of the symmetry (1.7), (1.8), has been developed rather recently by Belavin-Polyakov-Zamolodchikov [2]. Because the symmetry is infinite dimensional, this makes it possible to fully fix the theories: to get full description of the spectrum of operators, values of their anomalous (conformal) dimensions, to calculate multipoint correlation functions, and to derive the operator algebra coefficients.

So, we start by outlining the basics of this theory.

Basic fields of the theory are field theory operators

$$\{\phi_{\Delta,\bar{\Delta}}(z,\bar{z})\} \tag{1.9}$$

which transform under conformal deformations of the space $(z, \bar{z})$ as (cf. (1.2)):

(1.10)
$$z \longrightarrow \tilde{z} = f(z)$$
$$\bar{z} \longrightarrow \tilde{\bar{z}} = \overline{f(z)}$$

$$\phi_{\Delta,\bar{\Delta}}(z,\bar{z}) \longrightarrow (f'(z))^{\Delta}(\overline{f'(z)})^{\bar{\Delta}}\phi_{\Delta,\bar{\Delta}}(f(z),\overline{f(z)}). \tag{1.11}$$

The deformations of the space (1.10) does correspond to local scaling transformations because the metric element

$$(dx^{\mu})^2 \sim dzd\bar{z} \tag{1.12}$$

gets just rescaled under (1.10):

$$d\tilde{z}d\tilde{\bar{z}} = |f'(z)|^2 dzd\bar{z}. \tag{1.13}$$

An infinitesimal form of the transformation (1.10), (1.11) will be:

(1.14)
$$z \longrightarrow \tilde{z} = z + \alpha(z)$$
$$\bar{z} \longrightarrow \tilde{\bar{z}} = \bar{z} + \overline{\alpha(z)}$$

(1.15)
$$\phi_{\Delta\bar{\Delta}}(z,\bar{z}) \longrightarrow \tilde{\phi}(\tilde{z},\tilde{\bar{z}}) = \phi(z,\bar{z}) + [\alpha(z)\partial_z + \alpha'(z)\Delta + \overline{\alpha(z)}\partial_{\bar{z}} + \overline{\alpha'(z)}\bar{\Delta}]\phi(z,\bar{z}).$$

We can consider infinitesimal conformal transformations by assuming that the theory is formulated in a finite region of $(z, \bar{z})$ plane around the origin $(z=0)$, or in Fig. 1, and assuming that the analytic function $\alpha(z)$ in (1.14),

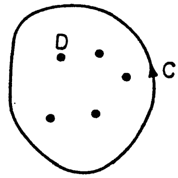

Fig. 1

(1.15) has all its singularities outside this region. In this way, $\alpha(z)$ can be infinitesimally small inside the domain D.

The full information about the quantum field theory of operators $\phi_i(z, \bar{z})$ (1.9) is contained in the multipoint correlation functions

$$(1.16) \qquad \langle \phi_1(z, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\cdots\rangle.$$

So, our main objective will be to find the way to calculate these functions.

First, we should add one more operator, to the space of basic operators (1.9). This is the stress-energy tensor, which is generated by the conformal transformations of the functions (1.16). Suppose the function (1.16) has a representation as a functional integral, with some action. And suppose we make a conformal transformation of fields under the Functional Integral (FI). Because variation of fields under FI should not change its value, we get, in a standard way, the Ward Identity (WI) in the form:

$$(1.17) \quad \frac{1}{2\pi i}\oint_C d\xi[\alpha(\xi)\langle T_{zz}(\xi, \bar{\xi})\phi_1\phi_2\cdots\rangle + \overline{\alpha(\xi)}\langle T_{\bar{z}z}(\xi, \bar{\xi})\phi_1\phi_2\cdots\rangle$$
$$-\frac{1}{2\pi i}\oint_C d\bar{\xi}[\overline{\alpha(\xi)}\langle T_{\bar{z}\bar{z}}(\xi, \bar{\xi})\phi_1\phi_2\cdots\rangle + \alpha(\xi)\langle T_{z\bar{z}}(\xi, \bar{\xi})\phi_1\phi_2\cdots\rangle$$
$$=\sum_i [\alpha(z_i)\partial_i + \alpha'(z_i)\Delta_i + \overline{\alpha(z_i)}\bar{\partial}_i + \overline{\alpha'(z_i)}\bar{\Delta}_i]\langle\phi_1\phi_2\cdots\rangle.$$

The factors $1/2\pi i$ in l.h.s. above are related with the use of complex variables; it is the usual contour integral. The r.h.s. of (1.17) comes from variations of fields ϕ_i of the function (1.16). The l.h.s. comes from variation of the action and of the boundary of the system (Fig. 1) under the conformal transformation (1.14). In this way, the operator

$$(1.18) \qquad T_{\mu\nu} = \{T_{zz}, T_{z\bar{z}}, T_{\bar{z}z}, T_{zz}\}$$

comes into the theory. The other way to say is that $T_{\mu\nu}$ defines conformal transformation of correlation functions, by eq. (1.17), if one chooses (1.17) as a starting point. If the functional integral representation was assumed as a starting point, then we can say that conformal invariance of the theory shows itself in the fact that only the *boundary* term is being produced in the l.h.s. of (1.17).

Starting with (1.17) we shall derive now the local WIs, instead of the integral form above. It will allow us to describe the full spectrum of operators of the conformal theory.

First we remark that if we choose

$$\alpha(z)=a=\text{const} \tag{1.19}$$

and

$$\alpha(z)=a(z-z_0) \tag{1.20}$$

then these two special cases of $\alpha(z)$ will give us, from (1.17), the conservation laws:

$$\partial_{\bar{z}}\langle T_{zz}(z\bar{z})\phi_1\phi_2\cdots\rangle=0 \tag{1.21}$$

$$\langle T_{\bar{z}z}(z\bar{z})\phi_1\phi_2\cdots\rangle=0 \tag{1.22}$$

$$z\neq z_i.$$

The first equation tells us that T_{zz} correlation functions does not depend on $\bar{z}$, which means, because the correlation function is general, that T_{zz} is an analytic field operator

$$T_{zz}=T_{zz}(z). \tag{1.23}$$

Analogously, from (1.22) we find that the cross-component $T_{z\bar{z}}$ (which is $\mathrm{Tr}(T_{\mu\nu})$) vanishes:

$$T_{\bar{z}z}(z,\bar{z})=0. \tag{1.24}$$

We shall use this partial information to reduce the WI (1.17). Analyticity of T_{zz}, and vanishing of $T_{z\bar{z}}$ means that z and $\bar{z}$ parts of WI (1.17) decouple. We can consider just one part of it, say z-part, suppressing for the time being the $\bar{z}$ dependence of correlation functions. So, we arrive at:

$$\frac{1}{2\pi i}\oint_C d\xi\alpha(\xi)\langle T_{zz}(\xi)\phi_1\phi_2\cdots\rangle=\sum_i(\alpha(z_i)\partial_i+\alpha'(z_i)\Delta_i)\langle\phi_1\phi_2\cdots\rangle. \tag{1.25}$$

Reduction to the 1D relation, instead of 2D one, is crucial: $(z,\bar{z})\to z$, $2\mathrm{D}\to 1\mathrm{D}$. It leads to integrability of the theory.

We can now replace the sum in r.h.s. by contour integrals, (as in Fig. 2), and then, deforming them to the integral over the boundary contour C we find $(T(z)=T_{zz}(z))$:

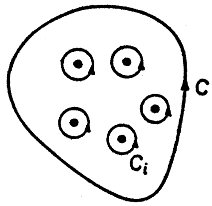

Fig. 2

$$(1.26)\quad \frac{1}{2\pi i}\oint_C dz\alpha(z)\langle T(z)\phi_1\phi\cdots\rangle$$
$$=\sum_i \frac{1}{2\pi i}\oint_C dz\alpha(z)\left(\frac{\Delta_i}{(z-z_i)^2}+\frac{1}{z-z_i}\partial_i\right)\langle\phi_1\phi_2\cdots\rangle$$
$$=\frac{1}{2\pi i}\oint_C dz\alpha(z)\sum_i\left(\frac{\Delta_i}{(z-z_i)^2}+\frac{1}{z-z_i}\partial_i\right)\langle\phi_1\phi_2\cdots\rangle.$$

Because the parameter field $\alpha(z)$ is a general function on z, the integrals can be lifted, leading to the local WI, the conformal WI itself:

$$(1.27)\quad \langle T(z)\phi_1\phi_2\cdots\rangle=\sum_i\left(\frac{\Delta_i}{(z-z_i)^2}+\frac{1}{z-z_i}\partial_i\right)\langle\phi_1\phi_2\cdots\rangle.$$

This is the basic relation of the conformal field theories. Everything follows from it, for the theories which have only conformal, and no additional symmetries.

Let us further analyze the operator content of the theory. We already have the basic operators $\{\phi_i\}$, and also the stress-energy operator $T(z)$. New operators come about from the product $T(z)\phi_1(z_1)$:

$$(1.28)\quad T(z)\phi_1(z_1)=\frac{\Delta_1}{(z-z_1)^2}\phi_1(z_1)+\frac{1}{z-z_1}\partial_1\phi_1(z_1)$$
$$+\phi_1^{(-2)}(z_1)+(z-z_1)\phi_1^{(-3)}(z_1)+\cdots.$$

In the above, we assumed that $T\phi_1$ is placed inside some general correlation function, with other operators at other points, and we take the limit $z\to z_1$. Then the first two singular terms in r.h.s. of (1.28) follow from the WI (1.27). But we will have a whole series in $(z-z_1)$. The expansion coefficients of this series are defined as correlation functions of new operators

$$(1.29)\quad \{\phi_1^{(-2)}(z_1),\ \ \phi_1^{(-3)}(z_1),\ \cdots\}.$$

It is in this sense that we have the operator product expansion (OPE) (1.28). We define the operators (1.29) as a result of applying the operators

L_{-n} to the original operator ϕ_1:

(1.30) $$\phi^{(-n)}(z_1)=L_{-n}(z_1)\phi(z_1).$$

This corresponds to the formal expansion of $T(z)$ around a point z_1:

(1.31) $$T(z)=\sum_{n=-\infty}^{+\infty}\frac{L_n(z_1)}{(z-z_1)^{n+2}}.$$

The action of operators $L_n(z_1)$ on $\phi_1(z_1)$ is defined by the OPE (1.28). So we have:

(1.32) $$\begin{aligned}&L_n(z_1)\phi(z_1)=0 \qquad (n\geq 1)\\&L_0(z_1)\phi(z_1)=\Delta\phi(z_1)\\&L_{-1}(z_1)\phi(z_1)=\partial_1\phi_1(z_1).\end{aligned}$$

Operators $\phi^{(-n)}=L_{-n}\phi$, $n=2, 3, \cdots$, are new ones. They were generated from the product $T\phi_1$:

(1.33) $$T\phi_\Delta\longrightarrow\{\phi_\Delta^{(-n)}=L_{-n}\phi_\Delta\colon n\geq 2\}.$$

Next, we can consider the product $TT\phi$. It will generate a new family of operators:

(1.34) $$T(z)T(z')\phi_\Delta(z_1)\longrightarrow\{\phi_\Delta^{(-n_1,-n_2)}=L_{-n_1}L_{-n_2}\phi_\Delta(z_1)\}$$

and so on. In this way we get a full spectrum of operators of the conformal theory:

There is first a set of basic operators $\{\phi_{\Delta_i}\}$ (primary fields [2]) with conformal transformation properties (1.8). In statistical systems these are "observables" like spin operator (order parameters, in general) and energy operator. Their conformal dimensions Δ define correspondingly magnetic and thermal critical indices. And then, there is an infinite family of operators $\phi_\Delta^{(-n_1,-n_2,\cdots)}$ growing above each basic one, being produced by L_{-n}-components of $T(z)$:

(1.35) $$\phi_\Delta^{(-n_1,-n_2\cdots-n_K)}=L_{-n_1}L_{-n_2}\cdots L_{-n_K}\phi_\Delta(z_1)$$

(1.36) $$\begin{aligned}&L_0\phi_\Delta^{(-n_1,-n_2\cdots-n_K)}=(\Delta+N)\phi_\Delta^{(-n_1,\cdots-n_K)}\\&N=\sum n_i.\end{aligned}$$

They can be classified into "levels", according to their scaling dimensions-the eigenvalues of L_0, see Fig. 3. The conformal transformation properties of the derivative operators (1.35) are different from those of the basic ones.

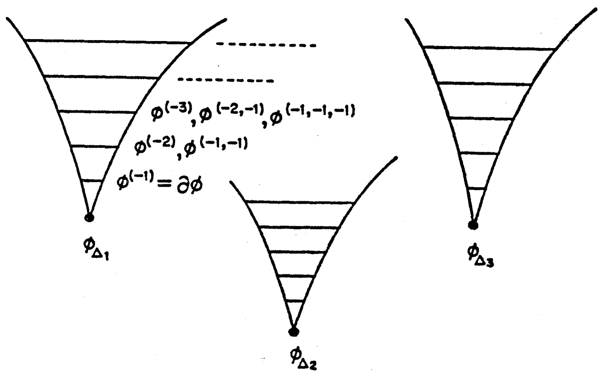

Fig. 3

They are not conformal covariant operators in the sense the basic operators $\{\phi_\Delta\}$ are; see (1.8). The simplest way to see it is to observe that they do not satisfy the condition (cf. (1.32))

$$L_n\phi = 0, \qquad n > 0. \tag{1.36$'$}$$

This followed, for basic operators, from WI (1.27), and, originally, from the transformation property (1.8).

To discuss further the properties of the derivative operators $\phi^{(-n_1 \cdots -n_k)}$, we obviously need the commutation relations for L_n's. So, let us derive them now.

We have

$$T(\xi)\phi(z) = \sum_n \frac{1}{(\xi - z)^{n+2}} L_n\phi(z). \tag{1.37}$$

It implies that the action of the operator $L_n\phi$ can be given in the form of the contour integral:

$$L_n\phi(z) = \frac{1}{2\pi i}\oint_{C_2} d\xi(\xi - z)^{n+1} T(\xi)\phi(z). \tag{1.38}$$

We remark again that the operators are assumed to be inside some correlation functions. So, we are basically dealing with functions, depending on other points that are being suppressed. In this sense all the operator relations should be assumed.

We shall now use the contour integral form (1.38) for $L_n\phi$ to derive the commutation relations $[L_{n_1}, L_{n_2}]$.

Applying successively two operators L_m and L_n, we get, according to (1.38):

$$(1.39)\quad L_n L_m \phi(z) = \frac{1}{(2\pi i)^2} \oint_{C_2} d\xi_2 \oint_{C_1} d\xi_1 (\xi_2 - z)^{n+1} (\xi_1 - z)^{m+1} T(\xi_2) T(\xi_1) \phi(z)$$

Applying these operators in an alternative order, we get:

$$(1.40)\quad L_m L_n \phi(z) = \frac{1}{(2\pi i)^2} \oint_{C_1} d\xi_1 \oint_{C_2} d\xi_2 (\xi_1 - z)^{m+1} (\xi_2 - z)^{n+1} T(\xi_1) T(\xi_2) \phi(z)$$

Subtracting them one from the other we find:

$$(1.41)\quad [L_n, L_m]\phi(z) = \oint_{C_z} d\xi_1 (\xi_1 - z)^{m+1} \oint_{C_{\xi_1}} d\xi_2 (\xi_2 - z)^{n+1} T(\xi_2) T(\xi_1) \phi(z).$$

The contours of integrations are shown in Fig. 4. To take the integral over ξ_2 we have to fix first the OPE for $T(\xi_2)T(\xi_1)$.

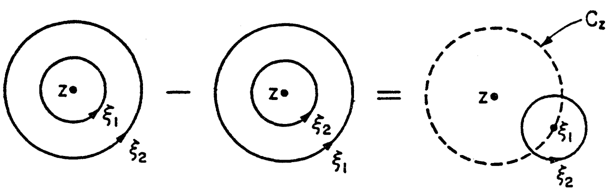

Fig. 4

This is found from the WI involving two T-operators:

$$(1.42)\quad \begin{aligned} &\langle T(z)T(z')\phi_1\phi_2\cdots\rangle \\ &= \frac{c/2}{(z-z')^4}\langle\phi_1\phi_2\cdots\rangle + \left(\frac{2}{(z-z')^2} + \frac{1}{z-z'}\partial_{z'}\right)\langle T(z')\phi_1\phi_2\cdots\rangle \\ &\quad + \sum_i \left(\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i}\partial_i\right)\langle T(z')\phi_1\phi_2\cdots\rangle. \end{aligned}$$

The first term is supplied by the two-point function

$$(1.43)\qquad \langle T(z)T(z')\rangle = \frac{c/2}{(z-z')^4}.$$

So, if, in our theory, we have a nonvanishing 2-point function for T's (1.43), then we have the first term in (1.42). Its other manifestation is in the transformation properties of $T(z)$. We have found the WI (1.27) starting with the function

$$\langle\phi_1\phi_2\cdots\rangle$$

and performing conformal transformations for it. In an analogous way, we get (1.42) if we start with

$$\langle T(z')\phi_1\phi_2\cdots\rangle$$

and use, in addition the following transformations for $T(z')$:

$$\partial T(z)=(2\alpha'(z)+\alpha(z)\partial_z)T(z)+\frac{c}{12}\alpha'''(z). \tag{1.44}$$

The last piece will produce the c-term in WI (1.42). Now, from (1.42) we get

$$T(\xi_2)T(\xi_1)=\frac{c/2}{(\xi_2-\xi_1)^4}+\frac{2}{(\xi_2-\xi_1)^2}T(\xi_1)+\frac{1}{\xi_2-\xi_1}\partial_{\xi_1}T(\xi_1) \\ +\text{Reg. terms.} \tag{1.45}$$

We can now take the ξ_2 integral in (1.41). Simple manipulations lead to the following expression for the r.h.s. of (1.41):

$$\begin{aligned} &\frac{1}{2\pi i}\oint_{C_z} d\xi_1(\xi_1-z)^{m+1}\frac{c}{12}n(n^2-1)(\xi_1-z)^{n-2}\phi(z) \\ &\quad+(n-m)\frac{1}{2\pi i}\oint_{C_z} d\xi_1(\xi_1-z)^{n+m+1}T(\xi_1)\phi(z) \\ &\quad=\frac{c}{12}n(n^2-1)\delta_{n,-m}\phi(z)+(n-m)L_{n+m}\phi(z). \end{aligned} \tag{1.46}$$

So, finally, we get the commutation of L_n's in the form:

$$[L_n, L_m]=(n-m)L_{n+m}+\frac{c}{12}n(n^2-1)\delta_{n,-m}. \tag{1.47}$$

This is the Virasoro algebra [3, 4]. We can now study the properties of the derivative operators (1.35). In particular, it is easy to check that they do not satisfy in general the conformal covariance condition (1.36′) (cf. (1.32) for basic operators ϕ_Δ). But first, let us remark on the role of L_n's as an infinite set of generators of the conformal transformations. In fact, another way to interprete WI (1.17) is that it defines conformal transformations of operators $(\phi_1, \phi_2\cdots)$ inside the correlation function. So, we have

$$\delta\langle\phi_1\phi_2\cdots\rangle|_{\alpha(z)}=\oint_C d\xi\alpha(\xi)\langle T(\xi)\phi_1\phi_2\cdots\rangle. \tag{1.48}$$

If we expand $\alpha(z)$ in a series:

(1.49)
$$\alpha(z)=\sum_{n=0}^{\infty} a_n z^{n+1}$$

we get then, according to the definition (1.38):

(1.50)
$$\begin{aligned}\delta\langle\phi_1\phi_2\cdots\rangle&=\sum_n a_n \oint_C d\xi\xi^{n+1}\langle T(\xi)\phi_1\phi_2\cdots\rangle\\&=\sum_n a_n\langle L_n\phi_1\phi_2\cdots\rangle.\end{aligned}$$

The operator L_n here is defined with respect to the point $z=0$, cf. (1.38), and it acts on all the operators inside the correlation function because the contour C encircles them all. It defines, according to (1.50), the variations of operators in the l.h.s.

Lecture 2.

We have already described the full spectrum of operators in the conformal field theory. These are the basic operators $\{\phi_{\Delta_i}\}$, and the derivative operators (1.35), which grow above each basic one, being produced by the components of $T(z)$ (see (1.31)) which are the Virasoro algebra operators $\{L_n\}$, (1.47), see also Fig. 3. The operator $T(z)$ itself is classified in this family as the second level operator in the family of the identity operator

(2.1)
$$\phi_{\Delta=0}(z)=I=\text{const.}$$

In fact, according to (1.38)

(2.2)
$$I^{(-2)}=\frac{1}{2\pi i}\oint_{C_z} d\xi(\xi-z)^{-1}T(\xi)I=T(z).$$

Also, it can be checked that no new type of operators are produced in OPE of two basic operators, like

(2.3)
$$\phi_{\Delta_1}(z_1)\phi_{\Delta_2}(z_2).$$

Such a product is expanded again in basic operators, and their conformal followers – operators (1.35). (We shall derive the OPE for products of basic operators (operator algebra coefficient in the Lecture 4, and also in the Appendix).

So the space of operators described above does, in fact, represent the full spectrum of operators of the conformal invariant theories. Also, we knew that the Virasoro algebra operators $\{L_n\}$ correspond to an infinite set of generators of the conformal transformations. Our strategy is: having an infinite parameter symmetry (parameters a_n in (1.50)), and so, an infinite set of operators L_n, related to this symmetry, we want to use them to fully fix the theory. We mean, by this, to define all the correlation functions. The way we proceed is a sort of "bootstrap" program: having many symmetries, we impose as many constraints on the theory as we can define consistently, with ultimate aim to define a theory in which everything is fixed – can be calculated.

Proceeding in this way with conformal invariant theories, we shall actually find not just some very special, but all the consistent conformal theories (in a given class which we study – the theories with no additional, except conformal, symmetries).

So, we have infinitely many operators $\{L_n\}$, and we are going to make constraint equations out of them, which we impose on basic operators of the theory $\{\phi_\Delta\}$.

Operators $L_n\phi_\Delta$ with $n>0$ just vanish, see (1.32). The family of operators $L_{-n_1}\cdots L_{-n_k}\phi$ can be classified into levels, see (1.35), (1.36). On the first level we have an operator $L_{-1}\phi=\partial_z\phi$. The constraint like

$$L_{-1}\phi_\Delta=0 \tag{2.4}$$

would lead just to an identity operator $I=\phi_{\Delta=0}$. We prefer to remark on it later, but consider now the second level, where we have two operators

$$L_{-1}^2\phi_\Delta, \qquad L_{-2}\phi_\Delta. \tag{2.5}$$

We form the constraint equation on ϕ_Δ as:

$$L_{-2}\phi+aL_{-1}^2\phi=0. \tag{2.6}$$

It should be noted, that equations like (2.6) should be formed out of operators belonging to the same level (having same scaling dimension). Otherwise we would break even the scaling symmetry.

Next requirement is that equations like (2.6) should be conformally covariant: we have to ensure that the operator in the l.h.s. is a conformal covariant operator, like the basic operators ϕ_Δ, with transformation properties (1.8), and only in that case we can make it vanish consistently, without breaking symmetries (this is analogous to e.g. the gravitation theory, which is generally covariant; then the equations, which are invented, should be generally covariant).

The requirement that (2.6) is conformal covariant is equivalent to the requirement that the l.h.s. is annihilated by L_n's, $n>0$, see (1.32). So we have:

(2.7) $$L_{+1}(L_{-2}\phi+aL_{-1}^2\phi)=0$$

(2.8) $$L_{+2}(L_{-2}\phi+aL_{-1}^2\phi)=0.$$

Conditions with higher L_n's ($n=3, 4\cdots$) follow from the above by Virasoro algebra. By simple algebra, using (1.47), one gets from (2.7)

(2.9) $$a=-\frac{3}{2(2\Delta+1)}$$

and then from (2.8)

(2.10) $$c=\frac{2\Delta(5-8\Delta)}{2\Delta+1}$$

where c is the constant in the Virasoro algebra (1.47) – its central charge. Suppose it is fixed, for a given theory, then equation (2.10) defines the dimension Δ. For the operator ϕ_Δ with this value of Δ we can impose consistently the constraint equation

(2.11) $$L_{-2}\phi_\Delta-\frac{3}{2(2\Delta+1)}L_{-1}^2\phi_\Delta=0.$$

(We can remark now, that analogous requirements, applied to the first level operator (2.4) lead to $\Delta=0$, $\phi_\Delta=I$).

So, we take the operator ϕ_Δ, subject to (2.11), into our theory. We demonstrate next, that the equation (2.11) implies that the correlation functions of this particular operator, with other operators, satisfy linear differential equations. This will be by direct use of the constraints like (2.11).

Let us consider the correlation function

(2.12) $$\langle\phi_\Delta(z)\phi_1\phi_2\cdots\rangle$$

and insert the operator in (2.11) into it. So that, we will have

(2.13) $$\langle(L_{-2}(z)\phi_\Delta(z))\phi_1\phi_2\cdots\rangle-\frac{3}{2(2\Delta+1)}\langle L_{-1}^2(z)\phi_\Delta(z)\phi_1\phi_2\cdots\rangle=0.$$

The operator $L_{-1}^2(z)$, applied to $\phi_\Delta(z)$, is a derivative operator, see (1.32). Let us transform also the operator L_{-2} in the first term into a derivative operator. $(L_{-2}\phi_\Delta)$ is defined by a contour integral in (1.38), so that we have:

$$\langle (L_{-2}\phi_\Delta(z)\phi_1\phi_2\cdots\rangle = \frac{1}{2\pi i}\oint_{C_z} d\xi(\xi-z)^{-1}\langle T(\xi)\phi_\Delta(z)\phi_1\phi_2\cdots\rangle. \tag{2.14}$$

Next we can shift the contour C_z away from the point z, by expanding it, so that finally we get the configuration of contour in Fig. 5. The contour

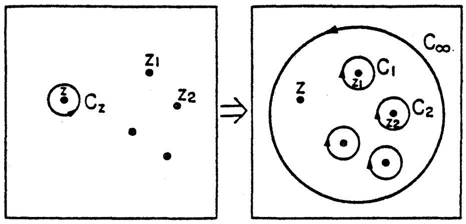

Fig. 5

C_∞ can be shifted to infinity, and its contribution vanishes. The integrals along the contours C_1, C_2, $\cdots$ can be calculated by using the OPE (1.28). So, we get:

$$\begin{aligned} &\langle L_{-2}\phi(z)\phi_1\phi_2\cdots\rangle \\ &= \sum_i \frac{1}{2\pi i}\oint_{C_1} d\xi(\xi-z)^{-1}\left(\frac{\Delta_i}{(\xi-z_i)^2}+\frac{1}{\xi-z_j}\partial_i\right)\langle\phi_\Delta(z)\phi_1\phi_2\cdots\rangle \\ &= \sum_i\left(\frac{\Delta_i}{(z_i-z)^2}+\frac{1}{z-z_i}\partial_i\right)\langle\phi_\Delta(z)\phi_1\phi_2\cdots\rangle. \end{aligned} \tag{2.15}$$

We put this result into (2.13) and find the differential equation

$$\begin{aligned} &\frac{3}{2(2\Delta+1)}\partial_z^2\langle\phi_\Delta(z)\phi_1\phi_2\cdots\rangle \\ &= \sum_{i=1,2\cdots}\left(\frac{\Delta_i}{(z-z_i)^2}+\frac{1}{z-z_i}\partial_i\right)\langle\phi_\Delta(z)\phi_1\phi_2\cdots\rangle. \end{aligned} \tag{2.16}$$

As an exercise, one can do similar things with the third level constraint equation:

$$L_{-3}\phi_\Delta + aL_{-2}L_{-1}\phi_\Delta + bL_{-1}^3\phi_\Delta = 0. \tag{2.17}$$

The operator ϕ_Δ, subjected to (2.17), is a new one, different from that selected above by the second level constraint (2.11). Conformal covariance equations

$$L_{-1}(L_{-3}\phi_\Delta + aL_{-2}L_{-1}\phi_\Delta + bL_{-1}^3\phi_\Delta) = 0 \tag{2.18}$$

(2.19) $$L_2(L_{-3}\phi_\Delta+aL_{-2}L_{-1}\phi_\Delta+bL_{-1}^3\phi_\Delta)=0$$

result into

(2.20) $$a=-\frac{2}{\Delta}, \qquad b=\frac{1}{\Delta(1+\Delta)}.$$

The second equation gives the relation between c and Δ, different from (2.10). So, we get another operator, with another Δ (for fixed c), and one can derive, from (2.17), a third order differential equation for the correlation functions with this particular operator.

One can proceed level by level in this way, selecting special operators for our theory. But the problem can also be formulated in general terms. Consider the set of operators, belonging to the same level N:

(2.21) $$\{\phi^{\{-n_i\}}=L_{-n_1}L_{-n_2}\cdots L_{-n_k}\phi_\Delta\}, \qquad \sum n_i=N.$$

Constraints like (2.6), (2.17) imply that not all the operators in (2.21) are independent. There is (we require it) a special linear combination which vanishes:

(2.22) $$\chi_{(N)}=\sum b_{\{-n_i\}}\phi_\Delta^{\{-n_i\}}=0.$$

In the linear space of operators (2.21) we can intrcduce a scalar (internal) product, generated by the Virasoro algebra, as

(2.23) $$\langle(\phi_\Delta L_{+n_k}\cdots L_{+n_2}L_{n_1})(L_{-m_1}L_{-m_2}\cdots L_{-m_l}\phi_\Delta)\rangle=\hat{M}^{(N)}_{\{n\}\{m\}}$$
$$\sum n_i=\sum m_i=N.$$

It is calculated by commuting $\{L_{+n_i}\}$ through $\{L_{-m_j}\}$ and applying them eventually to ϕ_Δ [4]. In this way, doing it for all the operators in (2.21), we shall get a matrix of norms $\hat{M}^{(N)}$, defined by (2.23). The condition of having a degeneracy on the Nth level, i.e. having a conformal covariant linear combination (2.22), (with the transformation properties of basic operators), which vanishes, is equivalent to vanishing of the determinant of the matrix $\hat{M}^{(N)}$ [4]:

(2.24) $$\mathrm{Det}\,\hat{M}^{(N)}(\Delta, c)=0.$$

This equation relates Δ and c, and is the general form of the relation like (2.10) for the second level operators. The expression for the determinant in (2.24) has been given by Kac, and proved by Feigin and Fuchs [4]. Its zeros give the following discrete set of values for the conformal dimensions Δ (for fixed $c<1$):

(2.25) $$\Delta_{n,m}=\frac{(\alpha_-n+\alpha_+m)^2-(\alpha_-+\alpha_+)^2}{4}$$

Here

(2.26) $$\alpha_\pm=\alpha_0\pm\sqrt{\alpha_0^2+1}$$

(2.27) $$c=1-24\alpha_0^2.$$

The corresponding values for the second level will be $\Delta_{1,2}$ and $\Delta_{2,1}$, which are two solutions of the eq. (2.10). In general, the numbers n, m, which characterize the structure of zeros of the determinant (2.14), are related to the level number N as

(2.28) $$N=n\times m.$$

So, n, m are integer numbers on which N can be factorized. E.g., for level $N=4$ we will have three solutions $\Delta_{1,4}$, $\Delta_{4,1}$, $\Delta_{2,2}$.

Finally, we get a discrete set of basic conformal operators with their dimensions given by the Kac formula (2.25):

(2.29) $$\{\phi_{(n,m)}, \Delta_{(n,m)}\}.$$

The correlation functions for these operators satisfy Nth order linear differential equations, derived from (2.22). So, they can, in principle, be calculated.

What is important is that the theory with operators $\phi_{(n,m)}$ (2.29) closes. It means that if we started with, say, four operators taken from the family (2.29), and calculated the 4-point correlation function for them

(2.30) $$\langle\phi_1\phi_2\phi_3\phi_4\rangle$$

then this function will have the singularities, when $z_1-z_2\to 0$, in the form:

(2.31) $$\sim(z_1-z_2)^{\gamma_{12}^p}=(z_1-z_2)^{-\Delta_1-\Delta_2+\Delta_p}.$$

The exponents of these singularities are identified with the dimensions of new operators $\{\phi_p\}$, coming from the operator product expansion of $\phi_1\phi_2$, for $z_1\to z_2$:

(2.32) $$\phi_1(z_1)\phi_2(z_2)\sim\sum_p\frac{C_{12}^p}{(z_1-z_2)^{\Delta_1+\Delta_2-\Delta_p}}\phi_p(z_2)+\cdots.$$

In this way, looking at the singularities of the function (2.30), we shall find that there are new operators, $\{\phi_p\}$, present in the theory. The closure of the family of operators (2.29) means that these new operators, produced from (2.30), belong to this same family. This can be checked by studying the characteristic equation for the differential equations. It determines the values of the exponents in the singularities (2.31). One finds that, if Δ_1, Δ_2 belong to $\{\Delta_{n,m}\}$, (2.29), then $\{\Delta_p\}$ also belong to this set. Operator algebra (expansions like (2.32)) shall be discussed in greater detail in the Lecture 4. Now, concluding this Lecture, we shall describe the relation of the conformal theories selected above, made of operators $\{\phi_{n,m}\}$, with various statistical systems, at their phase transition points.

Conformal theories, selected above, is a one parameter family, because we still have a free parameter in the Kac formula (2.25). This is the central charge c of the Virasoro algebra ($c<1$), or, equivalently, α_+^2, related to c by equations (2.26), (2.27). It happens that still more special conformal theories result, when the parameter α_+^2 is a rational number, $\alpha_+^2 = p/q$ (p, q are relatively prime integers). It will be shown in Lecture 4 that in such cases a finite family of operators $\{\phi_{n,m}\}$ decouple from the rest, and make a closed theory. They form a $(q-1)\times(p-1)$ table, with $1\leq n\leq q-1$, $1\leq m\leq p-1$. Such theories are called "minimal" in [2]. Let us remark that there is a "doubling" of operators $\{\phi_{n,m}\}$ in such theories, as the next examples of Ising and Z_3 models show. In relation to physical systems, we presume that there is just one basic operator with a given conformal dimension, like the energy operator in the Ising Model, but the conformal table contain its two representatives, operators $\phi_{1,2}$, $\phi_{3,1}$ in the example below. So, it should be consistent to treat $\phi_{1,2}$ and $\phi_{3,1}$ as the same thing, which we will check in Lecture 4, when discussing the operator algebra for them. In short, doubling of operators inside finite tables is a feature of the technique, not of the physics (for some more remarks on this see [2, 5]). The table of operators for the simplest nontrivial theory of this type, corresponding to $\alpha_+^2 = 4/3$, is shown in Fig. 6. It has been identified with the Ising Model [2] at its critical point.

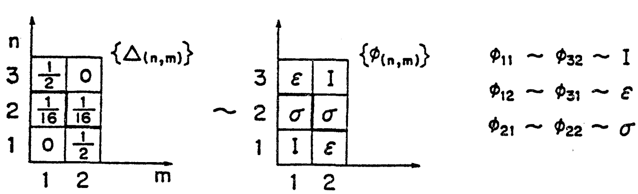

Fig. 6

Next was identified with the Z_3 Potts model. It is defined in the same way as the Ising Model, with the partition function

$$Z(\beta)=\sum_{\{\sigma\}} e^{\beta\Sigma(\bar{\sigma}_x\sigma_{x'}+\sigma_x\bar{\sigma}_{x'})} \tag{2.33}$$

($\{\sigma_x\}$ are spins, attached to sites $\{x\}$ of a 2D lattice; the sum in the exponent goes over the nearest neighbour sites), but the spin variable σ takes three values, instead of two for the Ising Model. They can be assumed to be roots of unity: $\sigma=\exp(2\pi ik/3)$, $k=0, 1, 2$. The indices of this model were known from the solution by Baxter of the model of hard hexagon [12], which is another lattice model, but has same symmetries as the Z_3 model, and so is presumed to be the same at the critical point (belongs to the same universality class).

Conformal theory for this model corresponds to $\alpha_+^2=6/5$, and has a table of operators shown in Fig. 7 [5].

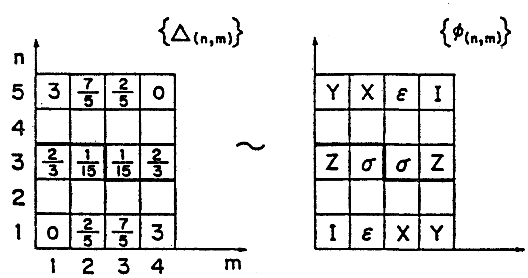

Fig. 7

Q-component Potts models [13] (these are lattice statistical models with spin σ taking Q different values, and withinteraction of nearest neighbours in the form $\delta_{\sigma,\sigma'}$) can be defined (through the high temperature expansion) also for continuous values of Q [14, 15]. For $Q\leq 4$ they are known to have the second order phase transition [16], and its critical exponents has also been found, using the Coulomb gas representation as continuous functions of $Q\leq 4$, see [17] and references there. This one-parameter series of models, corresponding to continuous $Q\leq 4$, have been identified with conformal theories for $c\leq 1$, so that in the conformal theory classification the energy operator is $\phi_{1,2}$. The spin operator belongs to the conformal set of operators only for the discrete series of the values of the Q-parameter, corresponding, in the conformal theory, to $\alpha_+^2=(p+1)/p$, $p=3, 5, 7\cdots$ [5, 6].

Another series of statistical models, the so called tricritical Potts models, $Q\leq 4$ [17], has also been identified with $c\leq 1$ conformal theories, with the difference that now the operator $\phi_{2,1}$ plays the role of the energy operator [11, 6]. The spin operator belongs to the conformal set for values $\alpha_+^2=(p+1)/p$, $p=4, 6, 8\cdots$. The first two of these are tricritical Ising, and tricritical Z_3 Potts models.

One more one parameter model, considered in the lattice statistics, is the $O(n)$ model with continuous $n \leq 2$ [17]. Its indices, known from the Coulomb gas representation, are reproduced in the conformal theory, with now $\phi_{3,1}$ being the energy operator. Spin operator is the same as in the Potts models series [5, 6]. Again $n \leq 2$ is mapped onto $c \leq 1$.

One of the series of critical indices in the recently discovered exactly solvable RSOS lattice models has been identified in [18] with the main series of minimal conformal theories, corresponding to $\alpha_+^2 = (p+1)/p$, $p = 3, 4, 5 \cdots$, the order parameters being the operators $\{\phi_{n,n}\}$. Another series of critical indices in these models have recently been reproduced in the conformal theory, generated by parafermionic currents [7].

One more series of parafermionic conformal theories has been found in [19]. Still the family of most recently found exactly solvable lattice statistical models [20, 21, 22], corresponding to higher spin SOS hierarchy [22], is again bigger than that provided by the conformal theory. Basically the only two existent and different approaches of getting exact analytic solutions for nontrivial, strongly interacting field theory systems, seem nicely stimulate one another, finding the common ground, for the present, in the domain of critical indices.

Lecture 3.

Starting with this lecture, we shall describe an alternative representation for the conformal field theory, which has been outlined in the previous two lectures. This representation happens to be very effective for the calculation of multipoint correlation functions for basic operators $\{\phi_\Delta\}$ of the conformal theory. In the first part of this lecture we shall follow closely the paper [6].

The basic operators will be represented by vertex operators exponentials of free fields:

$$\phi_{\Delta,\bar{\Delta}} \sim V_\alpha(z, \bar{z}) = e^{i\alpha\varphi(z,\bar{z})}. \tag{3.1}$$

Here $\varphi(z, \bar{z})$ is a free field with the action

$$A[\varphi] \sim \int dz d\bar{z} (\partial_z \varphi \partial_{\bar{z}} \varphi) \sim \int d_x^2 (\partial_\mu \varphi)^2 \tag{3.2}$$

α-parameter in (3.1) is to be related to the scaling dimension $\Delta + \bar{\Delta}$ of the field $\phi_{\Delta,\bar{\Delta}}(z, \bar{z})$.

The correlation functions of vertex operators V_α are evaluated as:

$$\langle V_\alpha(z,\bar z)V_{-\alpha}(z',\bar z')\rangle=\frac{\int D\varphi\, e^{-A[\varphi]}e^{i\alpha\varphi(z,\bar z)}\bar{e}^{i\alpha\varphi(z,\bar z)'}}{\int D\varphi\, e^{-A[\varphi]}} \tag{3.3}$$

$$=\exp\left\{-\frac{\alpha^2}{2}(2\langle\varphi^2\rangle-2\langle\varphi(z,\bar z)\varphi(z',\bar z')\rangle\right\}$$

$$=\exp\left\{-4\alpha^2\left(ln\frac{R}{a}-ln\frac{R}{|z-z'|}\right)\right\}$$

$$=\left|\frac{a}{z-z'}\right|^{4\alpha^2}\sim\frac{1}{|z-z'|^{4\alpha^2}}.$$

Here we have used

$$\langle\varphi(z,\bar z)\varphi(z',\bar z')\rangle=4\log\frac{R}{|z-z'|}. \tag{3.4}$$

a is a cut-off scale at small distances; R is the size of the system-cut-off scale on large distances, which is well known to be necessary for quantization of a free scalar field in 2D.

In an analogous way, the 4-point function will be found in the form:

$$\langle V_{\alpha_1}(z_1,\bar z_1)V_{\alpha_2}(z_2,\bar z_2)V_{\alpha_3}(z_3,\bar z_3)V_{\alpha_4}(z_4,\bar z_4)\rangle\sim\prod_{i<j}|z_i-z_j|^{4\alpha_i\alpha_j}. \tag{3.5}$$

The parameters $\{\alpha_i\}$ here are subject to the condition

$$\sum\alpha_i=0, \tag{3.6}$$

otherwise the large distance scale R (see (3.4)) will appear explicitly in (3.5), and in the limit $(R/a)\to\infty$ correlation function (3.5) will vanish.

The stress energy tensor of the free field theory (3.2) is

$$T_{zz}=-\frac{1}{4}\partial_z\varphi\partial_z\varphi. \tag{3.7}$$

Above outlined is a specific case of a conformal field theory. Again, the relations

$$\partial_{\bar z}\langle T_{zz}(z,\bar z)V_{\alpha_1}(z,\bar z_1)V_{\alpha_2}(z,\bar z_2)\cdots\rangle=0 \tag{3.8}$$

$$\langle T_{z\bar z}(z,\bar z)V_{\alpha_1}(z_1,\bar z_1)V_{\alpha_2}(z,\bar z_2)\cdots\rangle=0 \tag{3.9}$$

hold, and we can study just (z, z_i) dependence of the correlation functions, suppressing, for the time being, the variables $(\bar z, \bar z_i)$ (cf. Lecture 1). Then, the 2-point, and 4-point functions (3.3), (3.5) take the form:

(3.10) $$\langle V_\alpha(z)\, V_{-\alpha}(z')\rangle \sim \frac{1}{(z-z')^{2\alpha^2}}$$

(3.11) $$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) V_{\alpha_4}(z_4)\rangle = \prod_{i<j} (z_i - z_j)^{2\alpha_i \alpha_j}$$
$$\sum \alpha_i = 0.$$

It follows from (3.10) that the vertex operator $V_\alpha(z)$ has a conformal dimension

(3.12) $$\Delta_\alpha = \alpha^2.$$

We remark also on the "normal ordering" regularization for the product of fields $\varphi(z)$ placed at the same point. It has the form:

(3.13) $$:\partial_z\varphi(z)\partial_z\varphi(z): = \lim_{z,z'\to(z+z')/2} \{\partial_z\varphi(z)\partial_{z'}\varphi(z') - \langle\partial_z\varphi(z)\partial_{z'}\varphi(z')\rangle$$

(3.14) $$:e^{i\alpha\varphi(z)}: \sim e^{i\alpha\varphi(z)} + \frac{\alpha^2}{2}\langle\varphi^2\rangle \sim \frac{1}{(a)^{\alpha^2}} \exp(i\alpha\varphi(z)),$$

which means that we cancell out the infinite constants which one would get for the products of fields $\varphi(z)$, taken at the same point z, with the functional integral quantization (3.3).

So, in fact, we have the following expressions for the vertex operator $V_\alpha(z)$ and the stress energy tensor $T(z)$:

(3.15) $$V_\alpha(z) = :e^{i\alpha\varphi(z)}: \sim \frac{1}{(a)^{\alpha^2}} e^{i\alpha\varphi}$$

(3.16) $$T(z) = -\frac{1}{4} : \partial_z\varphi(z)\partial_z\varphi(z):.$$

The normalization of the stress energy tensor in (3.16) is chosen so that the OPE for TV_α would have the standard form (cf. Lecture 1):

(3.17) $$T(z)V_\alpha(z') = \frac{\Delta_\alpha}{(z-z')^2} V_\alpha(z') + \frac{1}{(z-z')}\partial_{z'}V_\alpha(z') + \text{Reg. terms.}$$

This expansion can be derived directly, using the expressions (3.15), (3.16) (see also below).

As was mentioned, the above is a special case of a conformal theory. Normalization of $T(z)$ is fixed by (3.17), and so, the central charge c of the Virasoro algebra (see Lecture 1) can be found from the 2-point function

$\langle T(z)T(z')\rangle$. Using (3.16) and (3.4) one finds

(3.18) $$\langle T(z)T(z')\rangle = \frac{1/2}{(z-z')^4}.$$

So, this is the $c=1$ conformal theory.

Extension to $c<1$ values can be acheived by modifying the quantization prescription. Let us calculate the correlation function for vertex operators using the prescription:

(3.19) $$\begin{aligned}&\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\cdots V_{\alpha_k}(z_k)\rangle\\ &\quad=\lim_{R\to\infty}\{R^{8\alpha_0^2}\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\cdots V_{\alpha_k}(z_k)V_{-2\alpha_0}(R)\rangle_{(0)}\}.\end{aligned}$$

The correlator in the r.h.s. $\langle\cdots\rangle_{(0)}$ is calculated in the standard way. With the prescription (3.19) nonvanishing will be correlators, satisfying the condition:

(3.20) $$\sum_{i=1}^{\kappa}\alpha_i = 2\alpha_0.$$

The nonvanishing 2-point function, made of vertex operators, will be

(3.21) $$\langle V_\alpha(z)V_{2\alpha_0-\alpha}(z')\rangle \sim \frac{1}{(z-z')^{2\alpha(\alpha-2\alpha_0)}}.$$

In conformal invariant theory there is an orthogonality condition (see Lecture 1):

(3.22) $$\langle \phi_{\Delta_i}(z)\phi_{\Delta_j}(z')\rangle = \frac{\delta_{ij}}{(z-z')^{2\Delta_i}}$$

which means that nonvanishing 2-point functions might be only those for operators having the same conformal dimension. So, from (3.21), we get:

(3.23) $$\Delta_\alpha = \Delta_{2\alpha_0-\alpha} = \alpha^2 - 2\alpha\alpha_0$$

—conformal dimensions of vertex operators get modified by α_0 parameter in (3.19).

Next we have to ensure that the OPE (3.17) holds. For this, we modify the stress-energy tensor (3.16) like:

(3.24) $$T(z) = -\frac{1}{4} : \partial_z\varphi\partial_z\varphi : + A\partial_z^2\varphi$$

and calculate the OPE for TV_α:

(3.25) $$T(z)V_\alpha(z')=:\left(-\frac{1}{4}\partial_z\varphi\partial_z\varphi+A\partial_z^2\varphi(z)\right)::e^{i\alpha\varphi(z')}:$$

$$=\frac{\alpha^2+2i\alpha A}{(z-z')^2}:e^{i\alpha\varphi(z')}:+\frac{i\alpha}{z-z'}:\partial_z\varphi(z)e^{i\alpha\varphi(z')}:$$

$$+:\left(-\frac{1}{4}\partial_z\varphi(z)\partial_z\varphi_{(z)}+A\partial_z^2\varphi(z)\right)e^{i\alpha\varphi(z')}:$$

$$=\frac{\alpha^2+2i\alpha A}{(z-z')^2}V_\alpha(z)+\frac{1}{z-z'}\partial_{z'}V_\alpha(z')+\text{Reg. terms.}$$

Comparing this expansion with (3.17), we find that modified stress-energy tensor should have the form:

(3.26) $$T(z)=-\frac{1}{4}:\partial_z\varphi\partial_z\varphi:+i\alpha_0\partial_z^2\varphi.$$

One could also check that, accordingly, the conformal trransformation of the field $\varphi(z)$ has to be modified as:

(3.27) $$z\longrightarrow f(z)$$
$$\varphi(z)\longrightarrow\varphi(f(z))-2_{i\alpha_0}\log f'(z).$$

With this prescrpition, the vertex operator $V_\alpha(z)$ (3.15) will transform in a standard way

(3.28) $$V_\alpha(z)\longrightarrow(f(z))^{\Delta_\alpha}V_\alpha(f(z))$$

(one has to take into account the transformation of the cut-off scale a in (3.15)).

The central charge of the Virasoro algebra for the modified theory can again be form the 2-point function

(3.29) $$\langle T(z)T(z')\rangle=\frac{c/2}{(z-z')^4}.$$

Using (3.26) and (3.4) one finds:

(3.30) $$c=1-24\alpha_0^2.$$

So, we achieved an extension to the values $c<1$.

We shall start now building the multipoint correlation functions, using the representation for the conformal theory given above. The 3-point functions are trivial, same as 2-point ones (see also Lecture 1). So, we start by defining the 4-point functions. In particular, let us define the 4-

point function in which all four operators have same conformal dimension. We intend to define later, using the function provided by vertex operators representation, the actual correlation function of the conformal theory:

$$\langle\phi_{\Delta,\bar{\Delta}}(z_1,\bar{z}_1)\phi_{\Delta,\bar{\Delta}}(z_2,\bar{z}_2)\phi_{\Delta,\bar{\Delta}}(z_3,\bar{z}_3)\phi_{\Delta,\bar{\Delta}}(z_4,\bar{z}_4)\rangle \tag{3.31}$$

On general grounds, we should expect the function (3.31) with all four operators the same should not vanish. Let us try, on the other hand find the representative function, for (3.31), among the functions provided by vertex operators $V_\alpha(z)$. It should be built of vertex operators having the same conformal dimension. We have the following combinations available:

$$\langle V_\alpha V_\alpha V_{2\alpha_0-\alpha} V_{2\alpha_0-\alpha}\rangle, \qquad \sum\alpha_i=4\alpha_0 \tag{3.32}$$

$$\langle V_\alpha V_\alpha V_\alpha V_{2\alpha_0-\alpha}\rangle, \qquad \sum\alpha_i=2\alpha_0+2\alpha \tag{3.33}$$

$$\langle V_\alpha V_\alpha V_\alpha V_\alpha\rangle, \qquad \sum\alpha_i=4\alpha. \tag{3.34}$$

So far none of the functions above satisfy our condition (3.20). So they vanish, if defined like in (3.19), unless α is fixed and special. First remark should be that the theory with $\alpha_0\neq 0$ $(c<1)$ does not have, in general, multipoint correlation functions in a trivial form like (3.11), (3.5).

There is a way to compensate the "charges" $\{\alpha_i\}$ in the functions above, by using the contour operators

$$\oint_C dzJ(z) \tag{3.35}$$

where $J(z)$ is a conformal operator, with $\Delta=1$. There are two such operators among $\{V_\alpha\}$:

$$\begin{aligned}&\Delta_\alpha=\alpha^2+2\alpha_0\alpha=1\longrightarrow\alpha_\pm=\alpha_0\pm\sqrt{\alpha_0^2+1},\\&J_\pm(z)=V_{\alpha_\pm}(z).\end{aligned} \tag{3.36}$$

Conformal transformation of $J(z)$ in (3.35) has the form:

$$\begin{aligned}&z\longrightarrow z+\varepsilon(z)\\&\delta J=(\varepsilon\partial_z+\varepsilon'\Delta_J)J=(\varepsilon\partial_z+\varepsilon')J\\&\quad=\partial_z(\varepsilon(z)J(z)).\end{aligned} \tag{3.37}$$

So, if the contour C in (3.35) is closed, then inserting operators (3.35) inside the correlation functions like in (3.32–3.34) will not effect the conformal transformation properties. On the other hand, the balance of "charges" $\{\alpha_i\}$ will be effected. Since our aim is to construct functions with given conformal transformation properties, we can use freely the operators (3.35), to match the balance (3.20).

The function (3.32) can not be saved from vanishing in this way, but the functions (3.33), (3.43) can. Later, in Lecture 4, after we get the operator algebra coefficients, we shall give the arguments why special operators, provided by the function (3.34) are unacceptable. These special operators, which it provides generate, by the operator algebra, inconsistent theory.

The only choice is the function (3.33). Let us examine it. We can put inside some number of contour operators (3.35), leading to the function:

$$\left\langle V_\alpha V_\alpha V_\alpha V_{2\alpha_0-\alpha} \oint_{C_1} J_- \cdots \oint_{C_n} J_- \oint_{S_1} J_+ \cdots \oint_{S_m} J_+ \right\rangle. \tag{3.38}$$

To have it nonvanishing, under the definition of the average (3.19), we should have the balance:

$$2\alpha_0 + 2\alpha + n\alpha_- + m\alpha_+ = 2\alpha_0. \tag{3.39}$$

This condition can be met, but only if α is *quantized*, takes a discrete set of values

$$\alpha = -\frac{n}{2}\alpha_- - \frac{m}{2}\alpha_+. \tag{3.40}$$

It will be convenient to shift the numbers n, m by a unit, and write the above quantization condition as

$$\alpha_{n,m} = +\frac{1-n}{2}\alpha_- + \frac{1-m}{2}\alpha_+. \tag{3.41}$$

First of all, let us remark that the corresponding conformal dimensions (see (3.23))

$$\Delta_{n,m} = \alpha_{n,m}^2 - 2\alpha_{n,m}\alpha_0 = \frac{(\alpha_- n + \alpha_+ m)^2 + (\alpha_- - \alpha_+)^2}{4} \tag{3.42}$$

are same as those provided by the Kac determinant, see Lecture 2. So, by imposing a natural condition that the 4-point function (3.31) should not vanish, we found that only the operators corresponding to degenerate Verma modules of the Virasoro algebra are allowed. These are the operators of the conformal theory, selected in Lecture 2, by following the conformal bootstrap program. (We remark again that we study the minimal conformal theory. In theories having additional symmetries, like Z_N conformal theory generated by paraformionic currents [7], the 4-point functions like (3.31) may well vanish, and the above arguments are not applicable).

Conformal dimensions being fixed by (3.42), we arrive at the following functions, represented by analytic integrals:

$$(3.43) \qquad \oint_{C_1} du_1 \cdots \oint_{C_{n-1}} du_{n-1} \oint_{S_1} dv_1 \cdots \oint_{S_{m-1}} dv_{m-1}$$
$$\langle V_{\alpha_{nm}}(z_1) V_{\alpha_{nm}}(z_2) V_{\alpha_{nm}}(z_3) V_{2\alpha_0 - \alpha_{nm}}(z_4) J_-(u_1) \cdots J_-(u_{n-1}) J_+(v_1) \cdots J_+(v_{m-1})$$

Above is the Feigin-Fuchs integral representation for the conformal functions in the degenerate conformal theory [8].

Now we shall proceed to construct the actual, physical correlation functions, out of the conformal functions provided by (3.43). First, let us examine the space of independent integrals like (3.43). Let us consider the simplest nontrivial case. It is provided by the function:

$$(3.44) \qquad \langle \phi_{nm} \phi_{12} \phi_{12} \phi_{nm} \rangle \sim \text{(corresponds to)}$$
$$\sim \oint_C \langle V_{nm}(0) V_{12}(z) V_{12}(1) V_{\overline{nm}}(\infty) J_+(v) \rangle dv$$
$$= z^{2\alpha_{12}\alpha_{nm}} (1-z)^{2\alpha_{12}^0} I_C(z);$$

$$(3.45) \qquad I_S(z) = \oint_S dv^{2\alpha_+\alpha_{nm}} (v-1)^{2\alpha_+\alpha_{12}} (v-z)^{2\alpha_+\alpha_{12}}.$$

Here $V_{nm} \equiv V_{\alpha_{nm}}$, $V_{\overline{nm}} = V_{2\alpha_0 - \alpha_{nm}}$. We have fixed three points of the correlation function, out of four, at specific values, corresponding to the standard choice, using the projective invariance of correlation functions. The function for the general values of $\{z_i\}$ will have an additional factor like

$$(3.46) \qquad \prod (z_i - z_j)^{\gamma_{ij}}$$

and the variable z in the above will become

$$(3.47) \qquad z = \frac{z_{12} z_{34}}{z_{13} z_{24}}.$$

These can easily be reproduced, see e.g. [6]. The nontrivial piece of correlation functions is the integral $I_S(z)$ (3.45) above.

We remark now, that the contour S in (3.45) need not be closed, in fact. It can be taken, e.g., from point $v=0$ to point $v=1$, *if* the integral is convergent at these points. In the following, we shall always assume that such integrals are convergent in the end points. For particular values of parameters, like $a = 2\alpha_+\alpha_{12}$, $b = 2\alpha_+\alpha_{nm}$ in (3.45), when the integral is not convergent, we assume the definition of the integral by analytic continuation in a, b.

Next, we find that there are two linearly independent integrals like (3.45). They may be chosen, e.g., like in Fig. 8. Accordingly we have two independent functions

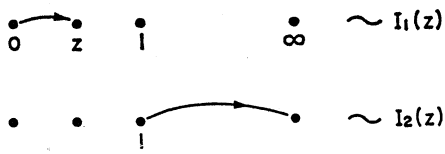

Fig. 8

(3.48) $$I_1(a, b, c; z) = \int_1^\infty dv v^a (v-1)^b (v-z)^c$$
$$= \frac{\Gamma(-a-b-c-1)\Gamma(b+1)}{\Gamma(-a-c)} F(-c, -a-b-c-1, -a-c; z)$$

(3.49) $$I_2(a, b, c; z) = (z)^{1+a+c} \int_0^1 dv v^a (1-v)^c (1-zv)^b$$
$$= z^{1+a+c} \frac{\Gamma(1+a)\Gamma(1+c)}{\Gamma(2+a+c)} F(-b, 1+a, 2+a+c; z).$$

Here $a = 2\alpha_+\alpha_{nm}$, $b = c = 2\alpha_+\alpha_{12}$; the functions F are hypergeometric functions. The two functions $I_1(z)$, $I_2(z)$ above provide two independent solutions to the 2nd order differential equation, which is corresponding to the operator $\phi_{12}(z)$ in (3.44), according to the theory described in Lecture 2.

Next, we still consider two other examples of conformal functions containing higher order operators.

The independent sets of integrals corresponding to the correlation functions

(3.50) $$\langle \phi_{nm}\phi_{13}\phi_{13}\phi_{nm} \rangle$$

and

(3.51) $$\langle \phi_{nm}\phi_{22}\phi_{22}\phi_{nm} \rangle$$

are shown in Figs. 6, 7. In particular, the first of integrals in Fig. 7 has the following analytic form:

(3.52) $$I(z) = \underset{(C)}{\int_1^\infty} du \underset{(S)}{\int_1^\infty} dv (u)^{a'} (u-1)^{b'} (u-z)^{c'} \cdot (v)^a (v-1)^b (v-z)^c (u-v)^f$$

Here

$$a'=2\alpha_{nm}\alpha_-,\ a=2\alpha_{nm}\alpha_+,\ c'=b'=2\alpha_{22}\alpha_-,\ c=b=2\alpha_{22}\alpha_+,\ f=2\alpha_+\alpha_-=-2.$$

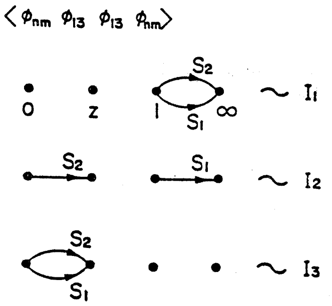

Fig. 9

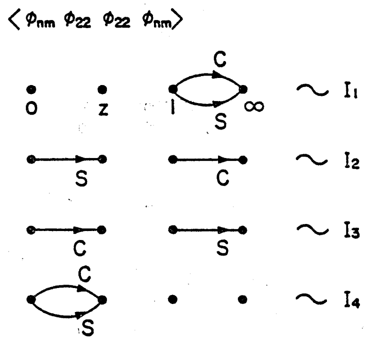

Fig. 10

The four functions in Fig. 10 provide four independent solutions to the fourth order differential equation, corresponding to the operator ϕ_{22} (see Lecture 2).

The generalization is now obvious.

Having established an independent set of conformal functions, provided by the integrals above, our next task is to construct the physical correlation functions out of them.

We note, that these conformal functions are not defined uniquely on a complex plane. As functions of z, they have 0, 1 (and ∞) as singular points. It is easy to check that when the integral $I_i(z)$ is continued analytically around the point 0, or the point 1, as in Fig. 11, along the contours g_0 or g_1, the integral will transform linearly as:

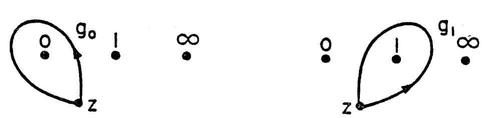

Fig. 11

$$I_i(z)\xrightarrow{g_0}I_i^{(g_0)}(z)=(g_0)_{ij}I_j(z) \tag{3.53}$$

$$I_i(z)\xrightarrow{g_1}I_i^{(g_1)}(z)=(g_1)_{ij}I_j(z). \tag{3.54}$$

These are the monodromy transformations of functions $\{I_i(z)\}$, which form a full set of Fuchs-type differential equations. In the minimal conformal theory that we describe, the basic operators $\{\phi_{\Delta,\bar{\Delta}}(z,\bar{z})\}$ have no spin, which means $\Delta=\bar{\Delta}$, $\Delta+\bar{\Delta}$ being their scaling dimension. They are operators like spin, or energy operator, in statistical systems. Obviously, the correlation functions for such operators should be uniquely defined on a 2D plane. Also, we have to remember now, that they depend both on z and $\bar{z}$ variables. It is easy to convince oneself that physical correlation functions for the basic operators $\{\phi_{\Delta,\bar{\Delta}}\}$ should be quadratic forms of functions $I_i(z)$, $\overline{I_j(z)}$:

$$G(z,\bar{z})\equiv\langle\phi_1(0,0)\phi_2(z,\bar{z})\phi_3(1,1)\phi_4(\infty,\infty)\rangle \sim\sum X_{ij}\,I_i(z)\overline{I_j(z)}. \tag{3.55)}$$

Next, we notice that our choice of basic functions $\{I_i(z)\}$ corresponds to the monodromy matrix $(g_0)_{ij}$ being diagonal—see the examples above—(3.48), (3.49), (4.52), and Figs. 5, 6, 7. This means that the quadratic form (3.55) should be diagonal, which ensures g_0-invariance:

$$G(z,\bar{z})=\sum X_i I_i(z)\overline{I_i(z)}. \tag{3.56}$$

The coefficients X_i of this form are determined by the requirement that (3.56) be also invariant to g_1 transformations. So, one has to study the monodromy transformations of the analytic functions $I_i(z)$, and then determine the coefficients $\{X_i\}$, to have $G(z,\bar{z})$ invariant.

This route was followed in [6, 9], where 4-point correlation functions has been calculated for all the basic operators of the minimal conformal theory.

Here, instead, we shall give another way of defining the physical correlation functions. In a way, it is more straightforward, and can directly be generalized to the case of correlation functions with the number of points greater than four.

Explicitly invariant form for the correlation functions is provided by the 2D integrals over the whole 2D plane:

$$\begin{aligned}&\langle\phi_{nm}(z_1\bar{z}_1)\phi_{kl}(z_2\bar{z}_2)\phi_{kl}(z_3\bar{z}_3)\phi_{nm}(z_4\bar{z}_4)\rangle\\ &\sim\int d^2u_1\cdots\int d^2u_{k-1}\int d^2v_1\cdots\int d^2v_{l-1}\\ &\times\langle V_{nm}(z_1\bar{z}_1)V_{kl}(z_2\bar{z}_2)V_{kl}(z_3\bar{z}_3)V_{\overline{nm}}(z_4\bar{z}_4)\\ &\times J_-(u_1\bar{u}_1)\cdots J_-(u_{k-1},\bar{u}_{k-1})J_+(v_1\bar{v}_1)\cdots J_+(v_{l-1},\bar{v}_{l-1})\rangle.\end{aligned} \tag{3.57}$$

Here it is assumed that $n\times m>k\times l$, so that the number of required J_-, J_+ operators is determined by the lower order operator, $\phi_{k,l}$.

To demonstrate the technique of reducing the formal 2D integrals like (3.57) to quadratic forms of analytic functions like (3.56), we shall consider for simplicity the case of a single integral; the generalization to multiple integrals being straightforward.

So, we shall consider the integral:

$$G(z,\bar{z})\sim\square(a,b,c;z,\bar{z})=\int d^2u|u|^{2a}|u-z|^{2c}|u-1|^b. \tag{3.58}$$

But first we take even more simple case, the integral

$$\square_0\equiv\square(z=0)=\int d^2u|u|^{2a}|u-1|^{2b}. \tag{3.59}$$

We are going to define the value of this integral by reducing it to the product of two 1D integrals.

We have

$$\square_0(a,b)=\int_{-\infty}^{+\infty}du_1\int_{-\infty}^{+\infty}du_2(u_1^2+u_2^2)^a((u_1-1)^2+u_2^2)^b. \tag{3.60}$$

We continue analytically the contour of u_2-integration into a u_2-complex plane, so that the contour is shifted close the imaginary axis, see Fig. 9.

$$u_2\longrightarrow ie^{-2i\varepsilon}u_2\approx i(1-2i\varepsilon)u_2. \tag{3.61}$$

Here ε is a vanishingly small positive number. The integral (3.60) takes a form:

$$\begin{aligned}\square_0(a,b)&=i\int_{-\infty}^{+\infty}du_1\int_{-\infty}^{+\infty}du_2\,(u_1^2-u_2^2e^{-4i\varepsilon})^a((u_1-1)^2-u_2^2e^{-4i\varepsilon})^b\\&=\frac{i}{2}\int_{-\infty}^{+\infty}du_+[u_+-i\varepsilon(u_+-u_-)]^a[u_+-1-i\varepsilon(u_+-u_-)]^b\\&\quad\times\int_{-\infty}^{+\infty}du_-[u_-+i\varepsilon(u_+-u_-)]^a[u_--1+i\varepsilon(u_+-u_-)]^b\end{aligned} \tag{3.62}$$

where $u_\pm=u_1\pm u_2$. The integral $\square_0(a,b)$ get basically factorized on u_+ and u_- integrals, up to the ε-terms, which define the way we go around the sigular points. In particular, suppose $u_+\in(-\infty,0)$, then the contour of u_- integration goes below the singular points $u_-=0$ and $u_-=1$. In fact:

$$u_-=0,\qquad i\varepsilon(u_+-u_-)=i\varepsilon u_+.$$

Because u_+ is negative, this small imaginary piece will be negative, which means that the u_- contour goes below the point $u_-=0$. In the same way, we find that the u_- contour goes also below $u_-=1$, for $u_+\in(-\infty,0)$.

Next, it is easy to check that for $u_+\in(0,1)$, the u_- contour will go

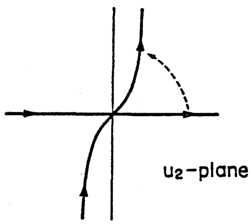

Fig. 12

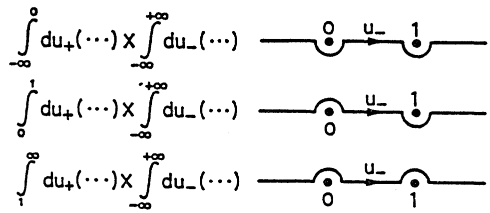

Fig. 13

above the point $u_-=0$, but below still the point $u_-=1$, and so on. The integral $\square_0$ gets expanded into a sum of three products of u_+ and u_- integrals, which are shown in Fig. 13. The first and their integrals vanish, because the u_- contours can be deformed to infinity, for these cases, after which it vanishes, because the integrals are assumed to be convergent. For

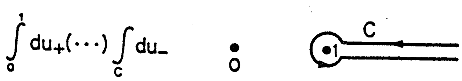

Fig. 14

the second integral, we deform the contour as in Fig. 14, after which it reduces to:

$$
\begin{aligned}
(3.63)\quad & \int_0^1 du_+(u_+)^a(1-u^+)^b \times \sin(\pi b)\cdot \int_1^\infty du_-(u_-)^a(u_- -1)^b \\
&= \int_0^1 du_+(u_+)^a(1-u_+)^b \cdot \sin(\pi b)\cdot \int_0^1 du_-(u_-)^{-2-a-b}(1-u_-)^b \\
&= \sin \pi b \,\frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)} \times \frac{\Gamma(-1-a-b)\Gamma(1+b)}{\Gamma(-a)}.
\end{aligned}
$$

Now we can turn back to the integral $\square(a, b, c; z, \bar{z})$ in (3.58), which has one additional singular point, $u=z$.

It can be reduced to the product of 1D integrals in precisely the same way—the corresponding pictures for nonvanishing pieces of the integral $\square(a, b, c; z, \bar{z})$ are shown in Fig. 15. To make it more symmetric, the u_+ integral over the $(z, 1)$ interval, and u_- integral over the $(-\infty, 0)$ interval can be expressed by linear combinations of integrals, corresponding to our canonical choice, the integrals over the intervals $(0, 1)$ and $(1, \infty)$. It requires a simple manipulation with contours (see also ([6]). One more remark is that the two products of integrals in Fig. 15 have to be summed, with a relative phase factor, to make the integral $\square(a, b, c; z, \bar{z})$. This phase factor can be defined by keeping u_- variable fixed in, say, the $(\infty, 0)$

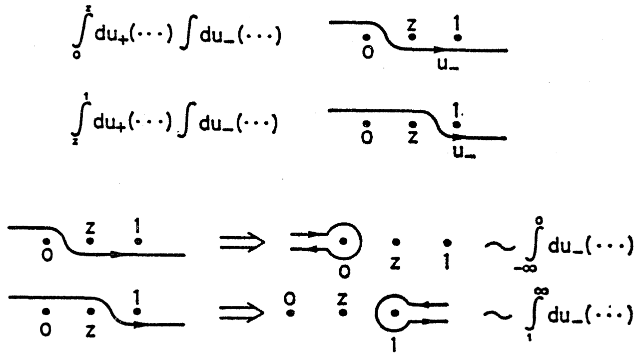

Fig. 15

interval, and continuing the u_+ integrand according to the small imaginary pieces in it, like in the integral (3.62).

In this way, we get the following result for the function $G(z, \bar{z})$, (3.58):

$$G(z, \bar{z}) = \frac{s(a)s(c)}{s(a+c)} |I_1(z)|^2 + \frac{s(b)s(a+b+c)}{s(a+c)} |I_2(z)|^2 \tag{3.64}$$

where $s(a) = \sin(\pi a)$, and

$$I_1(z) = \int_0^z du u^a (z-u)^c (1-u)^b \tag{3.65}$$

$$I_2(z) = \int_1^\infty du u^a (u-z)^c (u-1)^b. \tag{3.66}$$

In the same way one can proceed in the general case, e.g. for the function (3.57), and find a reduction of the corresponding 2D integral in the form:

$$\langle \phi_1(0, 0) \phi_2(z, \bar{z}) \phi_3(1, 1) \phi_4(\infty, \infty) \rangle \sim G(z, \bar{z}) = \sum_p X_p |I_p(z)|^2. \tag{3.67}$$

More important, one can define in this way also the multipoint correlation functions, with the number of points greater than four. The generalization of the technique should be straightforward.

The equation (3.67) gives the general analytic structure of the 4-point correlation functions, for general basic operators, in the minimal conformal theories.

Using the knowledge of 4-point functions, we shall derive, in the next Lecture, the operator algebra for the basic operators, and we shall discuss some of its properties.

Lecture 4.

In the previous Lecture we have derived the 4-point functions in the form (3.67). Explicit expressions for the coefficients $\{X_p\}$ can be found in [9].

To analyze the operator algebra we should better use the normalized function $\{F_p(z)\}$ which are related to the integrals $I_p(z)$ as follows:

$$I_p(z) = N_p \cdot F_p(z) \tag{4.1}$$
$$F_p(z \to 0) \approx (z)^{\gamma_p}(1 + \beta_1 z + \beta_2 z^2 + \cdots).$$

Then we will have the 4-point functions in the form:

$$\langle \phi_s(0)\phi_n(z, \bar{z})\phi_m(1)\phi_k(\infty)\rangle \sim \sum_p A^p_{snmk}|F_p(z)|^2 \tag{4.2}$$

(dependence of $F_p(z)$ on (s, n, m, k) is being suppressed, in the same way as are the previous expressions with the integrals $I_p(z)$).

For $z \to 0$, we shall find:

$$\langle \phi_s(0)\phi_n(z, \bar{z})\phi_m(1)\phi_k(\infty)\rangle \sim \sum_p \frac{A^p_{snmk}}{|z|^{\Delta_s + \Delta_n - \Delta_p}}(1 + 0(z)). \tag{4.3}$$

(From now on, we are using Δ for the full scaling dimension of operators, i.e. $\Delta_s + \bar{\Delta}_s = 2\Delta_s \to \Delta_s$). Only singular (nonanalytic) terms are shown in the above equation. We remind that, for general values of the points z_1, z_2, z_3, z_4, the variable z will be replaced by $(z_{12}z_{34})/(z_{13}z_{24})$, and also an additional factor

$$\prod_{i,j} |z_{ij}|^{\gamma_{ij}}, \qquad |z_{ij}| = |z_i - z_j|$$

will have to be supplied into (4.2), (4.3). Yet, the nonanalytic singularities for $z_{12} \sim z \to 0$ are shown correctly by the eq. (4.3). This will be enough to make connection with the operator algebra expansion for the operators $\phi_s\phi_n$, $\phi_m\phi_k$.

We remark next, that the representation (4.2) corresponds directly to the operator algebra (OA) expansion for pairs of operators (ϕ_s, ϕ_n) and (ϕ_m, ϕ_k). In fact, suppose we consider the above 4-point function for general values of z_1, z_2, z_3, z_4:

$$\langle \phi_s(z_1\bar{z}_1)\phi_n(z_2\bar{z}_2)\phi_m(z_3\bar{z}_3)\phi_k(z_4\bar{z}_4)\rangle. \tag{4.4}$$

We can find its singularities, for $|z_{12}| \sim |z_{34}| \to 0$ (which means just $|z_{12}| \sim |z_{34}| \ll |z_{24}|$) by using the OA expansions:

$$\phi_s(z_1\bar{z}_1)\phi_n(z_2\bar{z}_2) \approx \sum_p \frac{D^p_{sn}}{|z_1-z_2|^{\Delta_s+\Delta_n-\Delta_p}}(\phi_p(z_2\bar{z}_2)+\cdots). \tag{4.5}$$

Again, only leading singular terms are shown; not shown are the subleading operators $\phi_p^{(-n_1,-n_2,\cdots,-n_k)}(z_2,\bar{z}_2)$, which appear at the positive integer powers of (z_1-z_2) in the brackets of (4.4)—see the Appendix. We have the expansion like (4.5) also for the operators (ϕ_m, ϕ_k) in (4.4):

$$\phi_m(z_3\bar{z}_3)\phi_k(z_4\bar{z}_4) \approx \sum_q \frac{D^q_{mk}}{|z_3-z_4|^{\Delta_m+\Delta_k-\Delta_q}}(\phi_q(z_4\bar{z}_4)+\cdots). \tag{4.6}$$

We can use the OA expansions (4.5) and (4.6) in the correlation function (4.4), and find, for $|z_{12}|\sim|z_{34}|\ll|z_{24}|$

$$\begin{aligned}&\langle\phi_s(1)\phi_n(2)\phi_m(3)\phi_k(4)\rangle \\ &\approx \sum_p \frac{D^p_{sn}D^p_{mk}}{|z_{12}|^{\Delta_s+\Delta_n-\Delta_p}|z_{34}|^{\Delta_m+\Delta_k-\Delta_p}}\cdot\frac{1}{|z_{24}|^{2\Delta_p}}\times(1+\cdots).\end{aligned} \tag{4.7}$$

We assumed here that all the two-point functions are normalized as:

$$\langle\phi_p(z\bar{z})\phi_p(z'\bar{z}')\rangle = \frac{1}{|z-z'|^{2\Delta_p}}. \tag{4.8}$$

For the projective group fixed values of points $z_1=0$, $z_2=z$, $z_3=1$, $z_4\to\infty$, as in the function (4.2), (4.3) above, we have to watch only for the variable z_{12}, which is z. With this remark, we can compare the expansions (4.3) and (4.7), and find that they are the same thing. Which means next, that

$$A^p_{snmk} = D^p_{sn}D^p_{mk}. \tag{4.9}$$

So, the structure coefficients of 4-point functions, A^p_{snmk} in (4.2), should factorize into a product of OA coefficients in (4.5), (4.6).

We remark again here that the function in (4.2) is presented by a sum over the intermediate operators in the (s, n) "channel", see also Fig. 16.

$$\langle\phi_s\,\phi_n\,\phi_m\,\phi_k\rangle = \sum_p$$

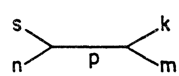

Fig. 16

Alternatively, by choosing a different basic set of basic functions $F_p(z)$ (different choice of the integrals $I_p(z)$ in (3.67), (3.56), see also Figs. 8, 9, 10), we would have got the correlation function $\langle\phi_s\phi_n\phi_m\phi_k\rangle$ as a sum over the intermediate operators in the n, m channel, see Fig. 17. "Duality"

$$\langle \phi_s \phi_n \phi_m \phi_k \rangle = \sum_q$$

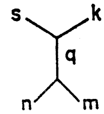

Fig. 17

relation, which requires that the sums in Fig. 16 and in Fig. 17 give the same function [2], is automatic in our approach. In fact, the duality relation is equivalent to our requirement, which we discussed in Lecture 3, that the correlation functions should be monodromy invariant, see also [9].

We turn back to deriving the OAE coefficients, $\{D^p_{sn}\}$, (4.5). We have found, that they can be derived from the structure constants of 4-point functions, see (4.9). It is more convenient to get them from symmetric 4-point functions, like

$$\langle \phi_s \phi_n \phi_n \phi_s \rangle = \sum A^p_{snn\,s} |F_p(z)|^2 \tag{4.10}$$

for which we will have

$$A_{snn\,s} = (D^p_{sn})^2. \tag{4.11}$$

We still make one more remark, before going to discuss the properties of the OA. At the end of Lecture 3, we have described how the physical correlation functions can be derived directly from explicitly monodromy invariant 2D integrals, with vertex operators $\{V_{n,m}(z, \bar{z})\}$, see eq. (3.57) and the discussion that follows. We remark now that the vertex operators representation in general, described in Lecture 3, has an asymmetry involved. In particular, the 4-point functions in this representation involve one vertex operator which is different from the rest, see (3.57) and also (3.43), (3.44). This asymmetry shows itself in the structure constants of the 4-point functions, let us define them as $\tilde{A}^p_{snn\bar{s}}$, with one index staying for two, which we derive from (3.57). In particular, instead of (4.11), they factorize as

$$A^p_{snn\bar{s}} = C^p_{sn} C^{\bar{p}}_{n\bar{s}} \tag{4.12}$$

with C^p_{sn}, $C^{\bar{p}}_{n\bar{s}}$ having different form, see [9]. This asymmetry is easily cured. The fact is that any 4-point function $\langle \phi_s \phi_n \phi_m \phi_k \rangle$ calculated starting with vertex operators $V_s V_n V_m V_{\bar{k}}$ will have an overall numerical factor, dependent on indices $s, n, m, \bar{k}$, which is asymmetric. Apart from this, it has all the symmetries of the function $\langle \phi_s \phi_n \phi_m \phi_k \rangle$. This means that, after calculating a particular correlation function, starting with vertex operators representation, undoing the 2D integrals, as in Lecture 3, the final result, expression like (3.67), still have to be normalized properly, after which the function becomes symmetric. Normalizing factors are defined, basically, by the requirement that two-point functions are normalized as (4.8) which means $D^I_{nn}=1$, I being an identity operator. This is worked out in detail in [10].

Finally, using the techniques described above, the OAE coefficients $\{D^p_{sn}\}$ has been found in the following form [10]:

$$(D^{(p'p)}_{(n'n),(s's)})^2=(C^{(p'p)}_{(n'n)(s's)})^2 a_{n'n} a_{s's}(a_{p'p})^{-1} \tag{4.13}$$

$$a_{n'n}=\prod_{i=1}^{n'-1}\prod_{j=1}^{n-1}\frac{(1+i-\rho(1+j))^2}{(i-j\rho)^2} \tag{4.14}$$

$$\times\prod_{i=1}^{n'-1}\frac{\Gamma(i\rho')\Gamma(2-\rho'(1+i))}{\Gamma(1-\rho')\Gamma(-1+\rho'(1+i))}\cdot\sum_{j=1}^{n-1}\frac{\Gamma(j\rho)\Gamma(2+\rho(1+j))}{\Gamma 1-(j\rho)\Gamma(-1+\rho(1+j))}$$

$$C^{(p'p)}_{(n'n)(s's)}=\rho^{4(l-1)(k-1)}\cdot\prod_{i=1}^{l-1}\prod_{j=1}^{k-1}\frac{1}{(i-\rho j)^2}\cdot\prod_{i=1}^{l-1}\frac{\Gamma(i\rho)}{\Gamma(1-i\rho')}\prod_{j=1}^{k=1}\frac{\Gamma(\rho j)}{\Gamma(1-\rho j)} \tag{4.15}$$

$$\times\prod_{i=0}^{l-2}\prod_{j=0}^{k-2}\frac{1}{\begin{array}{c}((s'-1-i)-\rho(s-1-j))^2((n'-1-i)-\rho(n-1-j))^2\\ \times((p'+l+i)-\rho(p+1+j))^2\end{array}}$$

$$\times\prod_{i=0}^{l-2}\frac{\begin{array}{c}\Gamma(1-\rho'(s'-1-i)+(s'-1))1-\rho'(n'-1-i)\\ +(n-1))\Gamma(1+\rho'(p'+1+i)-(p+1))\end{array}}{\begin{array}{c}\Gamma(\rho'(s'-1-i)-(s-1))\Gamma(\rho'(n'-1-i)-(n-1))\\ \times\Gamma(-\rho'(p'+1+i)+(p+1))\end{array}}$$

$$\times\prod_{j=0}^{k-2}\frac{\begin{array}{c}\Gamma(1-\rho(s-1-j)+(s'-1))\Gamma(1-\rho(n-1-j)\\ +(n'-1))\Gamma(1+\rho(p+1+j)-(p'+1))\end{array}}{\begin{array}{c}\Gamma(\rho(s-1-j)-(s'-1))\Gamma(\rho(n-1-j)\\ -(n'-1))-\Gamma(\rho(p+1+j)+(p'+1))\end{array}}.$$

Here

$$l=\frac{s'+n'-p'+1}{2},\quad k=\frac{s+n-p+1}{2};$$

$$\rho=\alpha_+^2,\qquad \rho'=\rho^{-1}=\alpha_-^2. \tag{4.16}$$

The operators $\{\phi_{1,n}\}$ (the same as opetators $\phi_{n,1}$) make a subalgebra of their own. We shall give coefficients for this subalgebra, because they are simpler, and in many cases it is sufficient to analyze this subalgebra:

$$(D^p_{sn})^2=(C^p_{sn})^2 a_s a_n (a_p)^{-1} \tag{4.17}$$

$$a_n=\prod_{i=1}^{n-1}\frac{\Gamma(i\rho)\Gamma(2-\rho(1+i))}{\Gamma(1-i\rho)\Gamma(-1+\rho(1+i))} \tag{4.18}$$

(4.19)
$$C_{sn}^{p}=\prod_{i=1}^{k-1}\frac{\Gamma(i\rho)}{\Gamma(1-i\rho)}\prod_{i=0}^{k-2}\frac{\Gamma(1-\rho(s-1-i))\Gamma(1-\rho(n-1-i))\times\Gamma(-1+\rho(p+1+i))}{\Gamma(\rho(s-1-i))\Gamma(\rho(n-1-i))\times\Gamma(2-\rho(p+1+i))}$$

Here

(4.20)
$$k=\frac{s+n-p+1}{2}.$$

Now we shall discuss some properties of the OA.

1. First remark is that for the k-product in (4.19) the minimal value is $(k)_{\min}=1$, for which the products are just replaced by 1, and which corresponds, according to (4.20), to

(4.21)
$$p_{\max}=s+n-1.$$

The maximal value $(k)_{\max}$ is defined by the first appearance of the pole, in Γ's, in the subproduct

(4.22)
$$\prod_{i=0}^{k-2}\frac{1}{\Gamma(\rho(s-1-i))\Gamma(\rho(n-1-i))}$$

of (4.19). It means, that for $k=k_{\max}+1$, one of Γ's in (4.22) gets a pole, and the coefficient D_{sn}^{p} will vanish. This defines $p_{\min}$ as:

(4.23)
$$p_{\min}=|s-n|+1.$$

So, p varies between (4.21) and (4.23) with steps $\Delta_p=2$. Generalization to the algebra with coefficients (4.13) is straightforward. This corresponds to the general properties of the OA, as derived from the differential equations, which are described in [2].

2. Next property is the closure of the OA by a finite number of

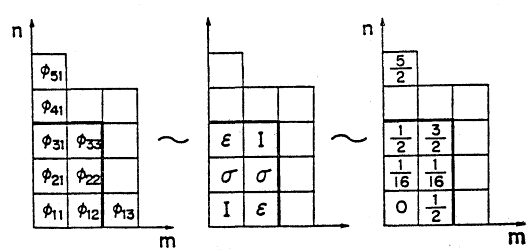

Fig. 18

operators for rational values of $\alpha_+^2=\rho$. Let's consider the case of $\rho=4/3$, which is Ising model, see Fig. 18 (cf. Fig. 6). Consider the OAE for the product of two operators $\phi_{1,2}=\varepsilon$. According to the rules (4.21), (4.23) described above, we have:

$$\phi_{1,2}\phi_{1,2}\sim D^1_{2,2}\phi_{1,1}+D^3_{2,2}\phi_{1,3} \tag{4.24}$$

(this symbolic form stands for the OAE (4.5), with scaling factors dropped). In (4.24) $D^1_{2,2}=1$ as it should, but it happens that for $\rho=4/3$, $D^3_{2,2}$ vanishes. This is due to the pole in one of Γ's in the expression for $a_{13}\equiv a_3$ (4.18). So that a_3 diverges, and $D^3_{2,2}$ vanishes according to (4.17). So, the operator $\phi_{1,3}$ decouples.

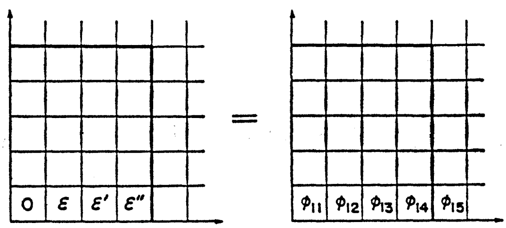

Fig. 19

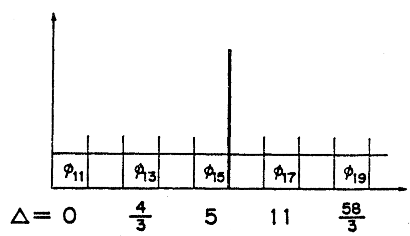

Fig. 20

Same happens in other cases. One more example is the Z_3 table of operators, Fig. 19. For the product of two $\phi_{1,3}$ operators we would expect:

$$\phi_{1,3}\phi_{1,3}\sim D^1_{3,3}\phi_{1,1}+D^3_{3,3}\phi_{1,3}+D^5_{3,3}\phi_{1,5}. \tag{4.25}$$

Here $D^1_{3,3}=1$ according to our normalization; one can check that $D^3_{3,3}$ is finite, but $D^5_{3,3}$ vanishes, because a_5 in (4.17) diverges for Z_3 model value of $\rho=6/5$.

3. Described above are the cases of trivial decouplings of operators outside the table. There are still cases when it is not so obvious. Consider again the IM table of opertators, Fig. 18. For minimal theories, corresponding to rational values of $\rho=\alpha_+^2$, there is a doubling of basic operators inside the table (see Lecture 2). So for IM case, Fig. 18, the operators $\phi_{3,1}$ and $\phi_{1,2}$ should be same. On the other hand, the general rules give the following OAE for the product of two $\phi_{3,1}$:

$$\phi_{3,1}\phi_{3,1}\sim D^1_{3,3}\phi_{1,1}+D^3_{3,3}\phi_{3,1}+D^5_{3,2}\phi_{5,1}. \tag{4.26}$$

We have to compare this with (4.24) and check if in fact the operators $\phi_{3,1}$ and $\phi_{1,2}$ behave like being the same. In (4.24) $\phi_{1,3}$ decouples, so that there is only the identity operator $I=\phi_{1,1}$ in the r.h.s. so, we should expect that in (4.26) the operators $\phi_{3,1}$ and $\phi_{5,1}$ decouple. One finds, on the other hand, that $D^3_{3,3}=0$, but $D^5_{3,3}$ is finite. (The coefficients D^p_{sn} for the $\{\phi_{n,1}\}$ subalgebra are same as for $\{\phi_{1,n}\}$, with the replacement $\rho\to\rho'$. So, for IM case $\rho'=3/4$ in this last analysis). There is apparent inconsistancy. It is resolved by noting that the dimensions of operators $\phi_{3,1}$ and $\phi_{5,1}$ differ by an integer number $\Delta_{5,1}-\Delta_{3,1}=2$. So, we have to look at the subleading operators:

$$\begin{aligned}\phi_{3,1}(z)\phi_{3,1}(0)\sim{}& D^1_{3,3}\phi_{1,1}\frac{1}{|z|^{2\Delta_3}}\\&+\frac{1}{|z|^{\Delta_3}}D^3_{3,3}(\phi_{3,1}+z\beta^{(-1)}\phi^{(-1)}_{3,1}+z^2(\beta^{(-1,-1)}\phi^{(-1,-1)}_{3,1}\\&+\beta^{(-2)}\phi^{(-2)}_{3,1})+z^3(\cdots)+\cdots)\\&+\frac{1}{|z|^{2\Delta_3-\Delta_5}}D^5_{3,3}\phi_{5,1}+\cdots\end{aligned} \tag{4.27}$$

(above, in the brackets, only z part of the expansion is shown; it is to be completed, in an obvious way, by $\bar{z}$ terms, see also the Appendix). What happens is that the coefficient $D^3_{3,3}$ vanishes, but the β-coefficients in the brackets, starting with the z^2 term, diverge for $\rho'=3/4$. To resolve the ambiguity, we take ρ' out of the point $\rho'=3/4$, and take the limit. The product of $D^3_{3,3}$ and β's turns out to be finite, and is equal (the z^2 term in brackets) to $D^5_{3,3}$, but with an opposite sign. The scaling factors also match, because $\Delta_{5,1}-\Delta_{3,1}=2$. So, the effect is that the operator $\phi_{3,1}$ still decouples, as it was going from the start, but it cancels also the $\phi_{5,1}$ piece of the OAE. In the result, both $\phi_{3,1}$ and $\phi_{5,1}$ decouple, as they should, to match the expansion (4.24).

Again, similar things happen in other cases. In Fig. 20 a piece of the table is shown for $\rho=7/6$. For $\phi_{1,5}\phi_{1,5}$ we have:

$$\phi_{1,5}\phi_{1,5}=D^1_{5,5}\phi_{1,1}+D^3_{5,5}\phi_{1,3}+D^5_{5,5}\phi_{1,5}+D^7_{5,5}\phi_{1,7}+D^9_{5,5}\phi_{1,9}. \tag{4.28}$$

One can check that $D^3_{5,5}$ and $D^5_{5,5}$ vanish, $D^7_{5,5}$, $D^9_{5,5}$ stay finite, though the corresponding operators are outside the table. Again, we find that the dimensions differ by integers, $\Delta_{1,9}-\Delta_{1,3}=18$, $\Delta_{1,7}-\Delta_{1,5}=6$, and the same happens: operators $\phi_{1,3}$, $\phi_{1,5}$ decouple, but cancel also $\phi_{1,9}$ and $\phi_{1,7}$ correspondingly.

4. By the above examples we have explained the way how the finite number of operators, belonging to a finite table $(q-1)\times(p-1)$, decouple from the rest, for the parameter $\alpha_+^2=\rho$ being a rational number q/p (p, q are relative prime numbers). Now, we shall argue that only these finite cases can be acceptable as a complete field theory. The reason is that the full set of values of $\Delta_{n,m}$, given by the Kac formula (3.42), involves, as one can check, some values being negative. They are placed near the line, in the (n, m) plane,

$$\alpha_+m+\alpha_-n=\alpha_+m-|\alpha_-|\cdot n=0. \tag{4.29}$$

The corresponding operators are pathological, for the field theory, because their 2-point correlation function grows with distance. So, we wouldn't like to have such operators in our theory. On the other hand, if we started with some "good operators", having Δ's positive, and began calculating their multipoint correlation functions, they will generate, in the intermediate channels (see Figs. 16, 17) some new operators, which we have to add to our set of basic operators, to make the theory complete, and so on. Or, in other words, doing the operator algebra, we shall be getting more and more operators. In this way, unless the OA closes by a finite family of operators, sooner or later we shall get into our theory the pathological operators, with negative dimensions, which are placed near the line (4.29).

So, we have to make the family of the basic operators finite, which means to stick to the rational values of $\alpha_+^2=q/p$. Still, it is readily checked that only the family with $q=p+1$ will have all the Δ's positive inside a

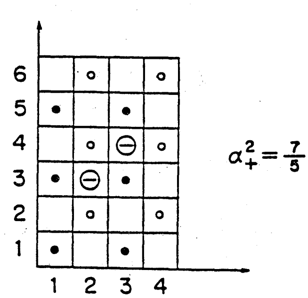

Fig. 21

finite $(q-1)\times(p-1)$ table. For illustration, in Fig. 21 given an example of the table, not in the $p=q+1$ family, having pathological operators in the center.

There is a way, in fact, to avoid the two operators in the center, shown in Fig. 21 by θ. We can pick up the operators, marked by dots and open circles in Fig. 21. They form a closed subalgebra and they do not involve the two operators in the center. Thus, all the operators in this subalgebra have positive dimensions Δ, and so it is acceptable in this respect.

There is still a difference between the theory which we selected above, and the theories in the main family, with $q=p+1$. We turn back to our OA coefficients D^p_{sn}, given by (4.17). We can redefine the normalization of operators in the following way:

$$\phi_s = \sqrt{a'_s}\,\tilde{\phi}_s. \tag{4.30}$$

The operators $\tilde{\phi}_s$ will have the OA as in (4.5) with the coefficients C^p_{sn}:

$$\tilde{\phi}_s\tilde{\phi}_n = \sum_p C^p_{sn}\tilde{\phi}_p \tag{4.31}$$

(scaling factors suppressed) and the two-point functions will become:

$$\langle\tilde{\phi}_s(z)\tilde{\phi}_s(0)\rangle = \frac{(a_s)^{-1}}{|z|^{2\Delta s}}. \tag{4.32}$$

So, $(a_s)^{-1}$ plays the role of a norm of the operator $\tilde{\phi}_s$. One can check using the expression (4.18) that only for the main series of the conformal theories, corresponding to $\alpha_+^2=(p+1/p)$, the norms a_s^{-1} will be positive for all the operators inside the table. If the original normalization of two-point functions was used, it would mean that the OA coefficients D^p_{sn}, given by (4.17), can all be defined to be real and positive, The symmetric 4-point functions in these theories:

$$\langle\phi_s\phi_n\phi_n\phi_s\rangle \sim \sum (D^p_{sn})^2 |F_p(z)|^2 \tag{4.33}$$

are positive defined, for all values of z_1, z_2, z_3, z_4.

It is not difficult to check that the theories of the main series ($\alpha_+^2=q/p$, $q-p\geq 2$) do not have these properties. It means that there will always be some operators inside the table, which have their norms $a_{n,n}^{-1}$ negative.

In our example, in Fig. 21, the operator $\phi_{1,3}$, which belongs to the selected subalgebra of operators with positive Δ's it has the norm a_3^{-1} which is negative (negative also is a_4^{-1}). We can put the norms inside the OA coefficients. Then D^p_{sn}, (4.17), becomes imaginary. In one way or another, the unitarity is lost for all the theories, except for the main series, with

$\alpha_+^2=(p+1)/p$. This agrees with the selection of unitary conformal theories in [11].

For statistical physics applications absence of unitarity may not be a serious trouble. And in fact such theories are considered as lattice statistical models (Potts model with continuous number of components parameter Q: $1\leq Q\leq 4$; $O(n)$ model with continuous n: $-2\leq n\leq 2$; Yang-Lee singularity, for the IM in an imaginary magnetic field). Particular expectation values, and particular correlation functions, are still well defined for these models. It would become a problem only when one attempted to construct the corresponding field theories. One can say that field theory starts with analysis of 4-point correlation functions. Some of them (great majority, among the Potts model series with continuous Q) do not correspond to well defined field theories.

Our last remark, related to OA, is on the special operators, provided by the function (3.34). By arguments that follow the eq. (3.33), we would get from (3.34) the quantized values

$$\alpha_{n,m}=\frac{1-n}{4}\alpha_-+\frac{1-m}{4}\alpha_+ \tag{4.34}$$

instead of the Kac formula (3.42). Special operators, provided by (4.34) are, e.q. $V_{-\alpha_+/4}$, $V_{-3\alpha_+/4}$ and so on. The correlation functions can be formed in the usual way

$$\iiint\langle V_{-\alpha_+/4}(1)V_{-\alpha_+/4}(2)V_{-\alpha_+/2}(3)V_{-\alpha_+/2}(4)J_+J_+J_-\rangle \tag{4.35}$$

(comp. (3.38) or (3.57)). The operator algebra can be analyzed, using these vertex operators.

If we consider a product of two operators, like

$$V_{-\alpha_+/4}(1)V_{-\alpha_+/4}(2) \tag{4.36}$$

then this product is expanded first into the operator, having the net "change parameter" α of the two, i.e.

$$V_{-\alpha_+/2} \tag{4.37}$$

and then the operators with the charge parameter α shifted by J's, which are present in the correlation function (4.35). J's are coupled to the product one by one, and so we get, next, after (4.37), the operators

$$V_{\alpha_+/2},\quad V_{(3/2)\alpha_+},\quad V_{(3/2)\alpha_++\alpha_-} \tag{4.38}$$

(for details of such an analysis of the OA, see [9]). What is essential, is that the identity operator, which is $V_{\alpha=0}$, is missing. So, the OAE for the product (4.36) will not contain the identity operator. As a consequence, the two-point function for such operators will vanish. In our minimal conformal theory, which we consider, and which has no additional symmetries except for the conformal one, we do not allow such operators. Vanishing of the two-point function means that such an operator vanish itself. For this reason, the special operators, provided by (4.34), are unacceptable for our theory.

Concluding Remarks

In these Lectures we have described in full detail the minimal conformal theories. Minimal they are not only in the sence that they are based on a finite number of basic operators [2], this is one aspect, but also because they contain no additional symmetries except for the conformal one. In particular, all the basic operators are Lorentz (Euclid, to be more precise) scalars, and there are no multiplicities in the spectrum of conformal dimensions for them—there is just one basic operator with a given conformal dimension.

For the present, these theories are studied most profoundly. We have got 4-point correlation functions for most general basic operators in such theories, and have found the OA coefficients for all of them. By the technique described in Lecture 3, it should be possible also to calculate the higher-point correlation functions (with number of points greater than four). Some particular OA coefficients for these theories have already been checked, on a particular lattice statistical problem, using a different approach [23, 24].

There are presently developed other conformal theories, which contain additional symmetries. These are minimal super conformal ($N=1$) theories [25], $N=2$ superconformal theories [26], conformal theories based on current algebras (2D Wess-Zumino model) [27], and marked by different new class of 2D field theories—the conformal theories based on parafermionic current algebras [7, 19]. These particular directions of research in the domain of conformal field theories, which is rapidly expanding at present, deserves each separate set of Lectures.

These set of lectures has been sponsered, stimulated and kindly helped by the reasearch workers at RIMS, Kyoto University, to whom I am grateful for their support and generous hospitality. I am especially grateful to E. Date, M. Jimbo, J. Miwa, M. Okado, and also to N. Kawamoto from the Physics Department, Kyoto University.

Appendix

The complete OPE for the product of two basic operator ϕ_1 and ϕ_2 has the following form [2]:

$$\text{(A.1)}\qquad \phi_2(z)\phi_1(0)=\sum_p \frac{C_{21}^p}{(z)^{\Delta_1+\Delta_2-\Delta_p}} \times\{\phi_p(0)+z\beta_p^{(-1)}\phi_p^{(-1)}(0)+z^2(\beta^{(-1,-1)}\phi^{(-1,-1)}(0)+\beta_p^{(-2)}\phi^{(-2)}(0))+\cdots\}$$
$$\equiv\sum_p \frac{C_{21}^p}{(z)^{\Delta_1+\Delta_2-\Delta_p}}\{\sum_{\vec{k}}(z)^{|\vec{k}|}\beta^{\{-\vec{k}\}}\phi_p{}^{\{-\vec{k}\}}(0)\}.$$

Here we use the notation:

$$\text{(A.2)}\qquad |\vec{k}|=k_1+k_2+\cdots,\ \beta^{\{-\vec{k}\}}=\beta^{\{-k_1,-k_2,\cdots\}}.$$

To simplify the notation, dependence of $\{\beta_p^{\{-\vec{k}\}}\}$ on indices 1, 2 of the operators $\phi_1\cdot\phi_2$ are being suppressed. As in many developments for the conformal field theory, in (A.1) only the z-dependence is shown. Actually, for physical operators $\phi(z, \bar{z})$ it will be a double expansion, in powers of z and $\bar{z}$, starting with:

$$\text{(A.3)}\qquad \phi_2(z,\bar{z})\phi_1(0,0)=\sum_p \frac{C_{21}^p}{|z|^{2\Delta_1+2\Delta_2-2\Delta_p}}\{\phi_p(0,0)+z\beta_p^{(-1)}\phi_p^{(-1)}(0,0)$$
$$+\bar{z}\beta_p^{(-\bar{1})}\phi_p^{(-\bar{1})}(0,0)+z\bar{z}\beta_p^{(-1)}\beta_p^{(-\bar{1})}\phi^{(-1)(-\bar{1})}(0,0)$$
$$+z^2\beta_p^{(-2)}\phi_p^{(-2)}(0,0)+\cdots\}$$

where $\phi^{(-\bar{1})}=\bar{L}_{-1}\phi$, and the operators $\{\bar{L}_n\}$ and $\{L_m\}$ commute with one another. Since (A.3) is just a trivial doubling of (A.1), we shall continue by considering only the z-part of it, the expansion (A.1).

Calculation of OA coefficients $\{C_{12}^p\}$ is a dynamical problem. In general, the coefficients C_{12}^p are related to the knowledge of 4-point correlation functions, which is a dynamical problem of a given theory. The way to derive them for minimal conformal theories has been described in Lecture 4.

The coefficients $\beta^{\{-\vec{k}\}}$, on the other hand, present a kinematic problem. They are defined by conformal invariance in narrow sense, without references to its particular quantum field theory realization.

Coefficients $\beta_p^{\{-\vec{k}\}}$ can be found as follows. Consider a product of operators

$$\text{(A.4)}\qquad \phi_2(z)\phi_1(0)$$

and apply an operator L_n, $n \geq 1$, to it:

$$L_n(\phi_1(z)\phi_2(0)) = \frac{1}{2\pi i}\oint_C d\xi \xi^{n+1} T(\xi)\phi_2(z)\phi_1(0). \tag{A.5}$$

The operator L_n is defined with respect to the origin, i.e. $L_n = L_n(0)$, see Lecture 1, and act on both operators, ϕ_1 and ϕ_2, which means that the contour C in (A.5) encircles them both, as in Fig. 22.

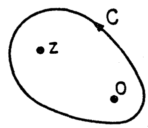

Fig. 22

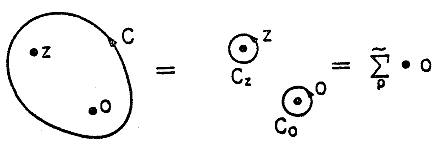

Fig. 23

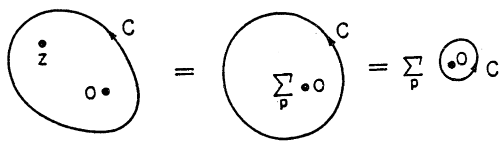

Fig. 24

Then, there are two possibilities. First, we can deform the contour as in Fig. 23, so that the operator L_n gets applied separately to $\phi_2(z)$ and $\phi_1(0)$, which give, as one can check, taking the integral over C_z, C_0 (C_0 contribution vanishes):

$$\begin{aligned} L_n(\phi_2(z)\phi_1(0)) &= (L_n\phi_2(z))\phi_1(0) + \phi_2(z)(L_n\phi_1(0)) \\ &= [(n+1)z^n \Delta_2 + z^{n+1}\partial_z]\phi_2(z)\phi_1(0). \end{aligned} \tag{A.6}$$

And after this, expand the product $\phi_2\phi_1$ as in (A.1), to find:

$$\begin{aligned} &L_n(\phi_2(z)\phi_1(0)) \\ &= [(n+1)z^n\Delta_2 + z^{n+1}\partial_z] \sum_p \frac{C_{21}^p}{z^{\Delta_1+\Delta_2-\Delta_p}} \{\sum_{\{\vec{k}\}} z^{|\vec{k}|} \beta_p^{\{-\vec{k}\}} \phi_p^{\{-\vec{k}\}}(0)\}. \end{aligned} \tag{A.7}$$

For operators L_{+1} and L_{+2} it gives:

(A.8) $$L_{+1}(\phi_2(z)\phi_1(0))$$
$$=\sum_p \frac{C_{21}^p}{(z)^{\Delta_2+\Delta_1-\Delta_p}}\{z(\Delta_2-\Delta_1+\Delta_p)\phi_p(0)+z^2(\Delta_2-\Delta_1+1)\beta^{(-1)}\phi_p^{(-1)}(0)+\cdots\}$$

(A.9) $$L_{+2}(\phi_2(z)\phi_1(0))=\sum_p \frac{C_{21}^p}{(z)^{\Delta_2+\Delta_1-\Delta_p}}\{z^2(2\Delta_2-\Delta_1+\Delta_p)\phi_p(0)+\cdots\}.$$

The other way will be to *first* expand the product $\phi_2\phi_1$ in (A.6) into OPE (A.1), and *then* apply the operator L_n to the terms of the expansion. This gives the following expansion of (A.6):

(A.10) $$L_n(\phi_2(z)\phi_1(0))=L_n\sum_p \frac{C_{21}^p}{(z)^{\Delta_2+\Delta_1-\Delta_p}}\{\sum_{\{\vec{k}\}}\beta^{\{-\vec{k}\}}\phi_p^{\{-\vec{k}\}}(0)\}$$
$$=\sum_p \frac{C_{21}^p}{(z)^{\Delta_2+\Delta_1-\Delta_p}}\{\sum_{\{\vec{k}\}}\beta_p^{\{-\vec{k}\}}(L_n\phi_p^{\{\vec{k}\}}(0))\}.$$

For operators L_{+1} and L_{+2} it gives, as one can check, using the definition $\phi^{\{-\vec{k}\}}=\phi^{\{-k_1,-k_2,\cdots\}}=L_{-k_1}L_{-k_2}\cdots\phi$, and commuting the Virasoro algebra operators L_{+1}, L_{+2} through those of $\phi^{\{-\vec{k}\}}$,

(A.11) $$L_{+1}(\phi_2(z)\phi_1(0))=\sum_p \frac{C_{12}^p}{z^{\Delta_1+\Delta_2-\Delta_p}}$$
$$\times\{0+z\beta^{(-1)}2\Delta_p\phi_p(0)+z^2[\beta^{(-1,-1)}2(2\Delta_p+1)+\beta^{(-2)}3]\phi_p^{(-1)}(0)+\cdots\}$$

(A.12) $$L_{+2}(\phi_2(z)\phi_1(0))=\sum_p \frac{C_{12}^p}{z^{\Delta_1+\Delta_2-\Delta_p}}$$
$$\times\left\{0+0+z^2\left[\beta^{(-1,-1)}6\Delta_p+\beta^{(-2)}\left(4\Delta_p+\frac{c}{2}\right)\right]\phi_p(0)+\cdots\right\}.$$

By comparing (A.8), (A.9) with, correspondingly, (A.11), (A.12), one gets the relations:

(A.13) $$\beta^{(-1)}2\Delta_p=\Delta_2-\Delta_1+\Delta_2$$

(A.14) $$\beta^{(-1,-1)}2(2\Delta_p+1)+3\beta^{(-2)}=(\Delta_2-\Delta_1+\Delta_p+1)\beta^{(-1)}$$

(A.15) $$\beta^{(-1,-1)}6\Delta_p+\beta^{(-2)}\left(4\Delta_p+\frac{c}{2}\right)=(2\Delta_2-\Delta_1+\Delta_p).$$

Equations (A.13–15) determine the coefficients $\beta^{(-1)}$, $\beta^{(-1,-1)}$, $\beta^{(-2)}$. One can proceed further, by comparing higher power terms in (A.8), (A.9) and

(A.11), (A.12). In this way all the coefficients $\beta_p^{\{-\vec{k}\}}$ in (A.1) can, in principle, be found [2]. It is sufficient to consider only the operators L_{+1} and L_{+2}, because the relations found from L_n, $n \geq 3$ in (A.6) will follow from those for the operators L_{+1}, L_{+2}, by the Virasoro algebra.

There is, in principle, another way of finding the coefficients $\beta^{\{-\vec{k}\}}$. They can be derived also from the series eapansion for the functions $F_p(z)$ introduced in Lectures 3 and 4 by multiple integrals, see also [6, 9]. These functions define 4-point correlation functions $\langle\phi_1(0)\phi_2(z)\phi_3(1)\phi_4(\infty)\rangle$, and so the series expansion of $F_p(z)$ and the OPE in (A.1) can obviously be related. The problem, in this approach, is to actually find the series expansion for the functions $F_p(z)$, which involves calculation of multiple integrals, and which is not so straightforward. So far the problem of finding more or less explicit representation for the general coefficient of the series expansion for the functions $F_p(z)$ has not been resolved. One exception, of course, is when there is only one integration involved in the definition of $F_p(z)$, and which is the hypergeometric function. Such examples are given in Lecture 3.

References

[1] A. M. Polyakov, Conformal symmetry of critical fluctuations, JETP Lett., **12** (1970), 381–383.

[2] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov and B. Aleksandr, Infinite conformal symmetry of critical fluctuations in two dimensions, J. Statist. Phys., **34** (1984), no. 5–6, 763–774; Infinite conformal symmetry in two dimensional quantum field theory, Nuclear Phys. B, **241** (1984), no. 2, 333–380.

[3] M. A. Virasoro, Subsidiary conditions and ghosts in dual-resonance models, Phys. Rev. D, **1** (1970), 2933–2936.

[4] V. G. Kac, Contravariant form for infinite-dimensional Lie algebras and superalgebras, Lecture Notes in Phys., **94** (1979), 441–445; B. L. Feigin and D. B. Fuks, Invariant skew-symmetric differential operators on the Line and Verma modules over the Virasoro algebra, Functional Anal. Appl., **16** (1982), no. 2, 114–126.

[5] Vl. S. Dotsenko, Critical behavior and associated conformal algebra of the Z_3 Potts model, J. Statist. Phys., **34** (1984), no. 5–6, 781–791; Nuclear Phys. B, **235** (1984), no. 1 FS11, 54–74.

[6] Vl. S. Dotsenko and V. A. Fateev, Conformal algebra and multipoint correlation functions in 2D statistical models, Nuclear Phys. B, **240**, no. 3, 312–348.

[7] A. B. Zamolodchikov and V. A. Fateev, Nonlocal (parafermion) currents in two-dimensional conformal quantum field theory and self-dual critical points in Z_N-symmetric statistical systems, Soviet Phys. JETP, **62** (1985), no. 2, 215–225.

[8] B. L. Feigin and D. B. Fuks, Preprint (1983).

[9] Vl. S. Dotsenko and V. A. Fateev, Four-point correlation functions and the operator algebra in 2D conformal invariant theories with central charge $C \leq 1$ Nuclear Phys. B, **251** (1985), no. 5–6, 691–734.

[10] ——, Operator algebra of two-dimensional conformal theories with central

charge $C \leq 1$, Phys. Lett. B, **154** (1985), no. 4, 291–295.

[11] D. Friedan, Z. Qiu and S. Shenker, Conformal invariance, unitarity, and critical exponents in two dimensions, Phys. Rev. Lett. **52** (1984), no. 18, 1575–1578.

[12] R. J. Baxter, Hard hexagons: exact solution, J. Phys. A: Math. Gen., **13** (1980), L61–L70.

[13] R. B. Potts, Some generalized order-disorder transformations, Proc. Cambridge Philos. Soc., **48** (1952), 106–109.

[14] H. N. V. Temperley and E. H. Lieb, Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem, Proc. Roy. Soc. London A, **322** (1971), no. 1549, 251–280.

[15] R. J. Baxter, S. B. Kelland and F. Y. Wu, Equivalence of the Potts model or Whitney polynomial with an ice-type model, J. Phys. A: Math. Gen., **9** (1976), no. 3. 397–406.

[16] R. J. Baxter, Potts model at the critical temperature, J. Phys. C: Solid State Phys., **6** (1973), L445–L448.

[17] B. Nienhuis, Critical behavior of two-dimensional spin models and charge asymmery in the Coulomb gas, J. Statist. Phys., **34** (1984), no. 5–6, 731–761.

[18] D. A. Huse, Exact exponents for infinitely many new multicritical points, Phys. Rev. B, **30** (1984), no. 7, 3908–3915.

[19] V. A. Fateev and A. B. Zamolodchikov, Landau Institute preprint, 1986–3; Teoret. Mat. Fiz., to be published.

[20] A. Kuniba, Y. Akutsu and M. Wadati, Exactly solvable IRF models, I — A three-state model, J. Phys. Soc. Japan, **55** (1986), no. 4, 1092–1101; Preprint (1986).

[21] R. J. Baxter and G. E. Andrews, Preprint (1986).

[22] E. Date, M. Jimbo, T. Miwa and M. Okado, Fusion of the eight vertex SOS model, Lett. Math. Phys., **12** (1986), 209–215.

[23] J. Cardy, Univ. California Preprint, **79** (1986).

[24] ——, Phase Transitions and Critical Phenomena, Vol. 11 (Eds. C. Domb and J. L. Lebowiz, Academic, New York 1986), to be published.

[25] M. A. Bershadsky, V. G. Knizhnik and M. G. Teitelman, Superconformal symmetry in two dimensions, Phys. Lett. B, **151** (1985), no. 1, 31–36; D. Friedan, Z. Qiu, S. Shenker and Stephen, Superconformal invariance in two dimensions and the tricritical Ising model, ibid., 37–43.

[26] P. Di Vecchia, J. L. Petersen, M. Yu and H. B. Zheng, Explicit construction of unitary representations of the N=2 superconformal algebra, Phys. Lett., **174** (1986), no. 3, 280–284;

W. Boucher, D. Friedan and A. Kent, Determinant formulae and unitarity for the N=2 superconformal algebras in two dimensions or exact results on string compactification, Phys. Lett. B, **172** (1984), no. 3, 316–222.

[27] V. G. Knizhnik and A. B. Zamolochikov, Current algebra and Wess-Zumino model in two dimensions, Nuclear Phys. B, **247** (1984), 83–103;

V. A. Fateev and A. B. Zamolodchikov, Operator algebra and correlation functions in the two-dimensional SU(2) (times) SU(2) chiral Wess-Zumino model, Soviet J. Nuclear Phys., **43** (1986), no. 4 657–664; Preprint, HE-86–10, (1986).

Landau Institute for Theoretical Physics
The Academy of Sciences of USSR
Kosygin st. 2
117940 Moscow
USSR