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On Zeta Functions Associated with the Exceptional Lie Group of Type E_6

Masakazu Muro

Dedicated to Prof. Ichiro Satake on his sixtieth birthday

Abstract. We define a zeta function associated with the exceptional Lie group of type E_6 and compute their functional equations and residues as an application of microlocal analysis.

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Introduction

In 1974, Sato and Shintani [Sm-Sh] proposed a method to construct a new type of zeta functions associated with a "prehomogeneous" action of a linear algebraic group on a finite dimensional vector space. Such zeta functions are called *zeta functions associated with prehomogeneous vector spaces* and some interesting examples have been calculated by Sato-Shintani [Sm-Sh] and by Shintani [Sh].

In particular, Shintani [Sh] analyzed precisely the zeta functions associated with the vector space of symmetric matrices as a typical example of such zeta functions. They are called *Shintani's zeta function.* The reason why Shintani's zeta function has called attention is that some special values of the zeta function defined for positive definite symmetric matrices has close connection with the dimension of Siegel cusp forms through Selberg's trace formula. Though Shintani did not give the special values of his zeta functions explicitly, he calculated the functional equations completely and computed some residues in terms of some special values of the zeta functions of low rank. Similar zeta functions may be defined for other spaces of the same type, for example, zeta functions associated with complex Hermitian matrices or quaternion Hermitian matrices. See [Sm-Sh].

In our paper we shall construct a zeta function of the same type for modular forms on an exceptional symmetric domain. It is a zeta function associated with the prehomogeneous vector space $(\boldsymbol{G}_{\mathrm{C}}, \boldsymbol{X}_{\mathrm{C}})$. In our case, the reductive group $\boldsymbol{G}_{\mathrm{C}} := \boldsymbol{GL}_1 \times \boldsymbol{E}_6$ acts on the space $\boldsymbol{X}_{\mathrm{C}}$ of 3×3 Hermitian matrices over the complex Cayley algebra $\mathfrak{C}_{\mathrm{C}}$. Our purpose of this paper is to analyze invariant tempered distributions on $\boldsymbol{X}_{\mathrm{C}}$ by microlocal analysis. As its consequence, we shall calculate the functional equations and residues of the zeta functions explicitly. The calculation of functional equations is reduced to the computations of the Fourier transforms of tempered distributions obtained as a complex power of a relatively invariant polynomial and the calculation of the residues are reduced to the computation of Fourier transforms of tempered distributions which are given as invariant measures on low dimensional orbits. These facts have already been pointed out in [Sm-Sh]. After all, the key problem is explicit calculations of Fourier transforms of relatively invariant distributions. Microlocal analysis presents us a strong tool to solve this problem.

In Chapter I, we will sketch the procedure of the calculations, which we explained in [Mr1] precisely. In Chapter II, the reader will find how to reduce the calculations of functional equations and residues to the explicit computations of Fourier transforms under suitable convergence conditions. More precise calculations for some specific examples will appear in a future paper.

We close the introduction by giving some comments to relevant papers. Satake and Faraut [Sk-Fa] carried out the Fourier transform of the complex powers of relative invariants on some prehomogeneous vector spaces including our case by making use of the analysis on Jordan algebra. In [Mr 1] the author did similar calculations by using microlocal analysis in the cases of the space of symmetric matrices, complex Hermitian matrices and quaternion Hermitian matrices. On the other hand, F. Sato [Sf 1–3] has obtained a generalization of Sato-Shintani's construction of zeta functions, but it is unknown that microlocal analysis works well for F. Sato's examples.

Notations. We denote by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$, the set of natural numbers, the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For a vector space X, $GL(X)$ means the group of invertible linear endomorphisms of X.

Chapter I. Microlocal Analysis of Invariant Hyperfunctions with respect to the Group E_6

In order to introduce zeta functions associated with the exceptional Lie group of type E_6 and to carry out calculations of their functional equations and residues, we need to analyze invariant hyperfunctions under the group action of the exceptional Lie group of type E_6. In particular, computing Fourier transforms of invariant distributions is the most important. Our final goal of this chapter is to give explicit formulas of Fourier transforms of some type of invariant distributions with respect to the group of type E_6, which are given in § 7 and § 8. Our tool is microlocal analysis mainly developed by Sato and Kashiwara.

§ 1. A realization of the exceptional Lie group of type E_6

We begin with the definition of complex Cayley algebra. Let $\mathbb{H}_{\mathbb{C}}$ be the complex quaternion algebra, which is uniquely determined up to isomorphisms. We define the *complex Cayley algebra* $\mathfrak{C}_{\mathbb{C}}$ to be an $\mathbb{H}_{\mathbb{C}}$-free module of rank two whose basis is $\{1, e\}$; the multiplication low of the algebra $\mathfrak{C}_{\mathbb{C}}$ is given by:

$$(1.1) \qquad (q+re)\cdot(s+te) := (qs-\bar{t}r)+(tq+r\bar{s})e,$$

where q, r, s and t are elements in $\mathbb{H}_{\mathbb{C}}$ and $\bar{t}$ and $\bar{s}$ stand for the conjugate quaternions of t and s, respectively. Then $\mathfrak{C}_{\mathbb{C}}$ is an 8-dimensional non-associative algebra defined over $\mathbb{C}$. For an element $a := q+re \in \mathfrak{C}_{\mathbb{C}}$, we define the *conjugate* Cayley number $\bar{a}$ by $\bar{a} := \bar{q} - re$. We may easily prove that $\overline{a \cdot b} = \bar{b} \cdot \bar{a}$ for any $a, b \in \mathfrak{C}_{\mathbf{C}}$. We put:

$$N(a) := a \cdot \bar{a} \quad \text{and} \quad T(a) := a + \bar{a}, \tag{1.2}$$

and call them the *norm* and the *trace* of $a \in \mathfrak{C}_{\mathbf{C}}$, respectively. The norm and the trace of a are complex numbers.

We put $\boldsymbol{X}_{\mathbf{C}} := \mathbf{Her}_3(\mathfrak{C}_{\mathbf{C}})$, where $\mathbf{Her}_3(\mathfrak{C}_{\mathbf{C}})$ is the set of 3×3 Cayley Hermitian matrices, i.e., the set of 3×3 matrices x whose entries are in $\mathfrak{C}_{\mathbf{C}}$ and which satisfy ${}^t\bar{x} = x$. Then $\boldsymbol{X}_{\mathbf{C}}$ is a 27-dimensional complex vector space. For an element

$$x := \begin{bmatrix} \xi_1, & x_3, & \bar{x}_2 \\ \bar{x}_3, & \xi_2, & x_1 \\ x_2, & \bar{x}_1, & \xi_3 \end{bmatrix} \qquad \text{in } \boldsymbol{X}_{\mathbf{C}},$$

we put:

$$\begin{aligned} \det(x) &:= \xi_1\xi_2\xi_3 - \xi_1 N(x_1) - \xi_2 N(x_2) - \xi_3 N(x_3) + T((x_3 \cdot x_1) \cdot x_2), \\ \mathrm{tr}(x) &:= \xi_1 + \xi_2 + \xi_3, \end{aligned} \tag{1.3}$$

and call them the *determinant* and the *trace* of x, respectively. Let $\boldsymbol{G}_{\mathbf{C}}^1$ be the subgroup of the complex general linear group $\boldsymbol{GL}(\boldsymbol{X}_{\mathbf{C}})$ consisting of elements which keep the polynomial $\det(x)$ invariant. Then $\boldsymbol{G}_{\mathbf{C}}^1$ is a closed linear algebraic subgroup of $\boldsymbol{GL}(\boldsymbol{X}_{\mathbf{C}})$, which is isomorphic to the complex exceptional Lie group of type $\boldsymbol{E}_6$. We denote it by $\boldsymbol{E}_{6\mathbf{C}}$. The group $\boldsymbol{G}_{\mathbf{C}}^1$ is a 78-dimensional complex Lie group.

We put $\boldsymbol{G}_{\mathbf{C}} := \boldsymbol{GL}_1(\mathbb{C}) \times \boldsymbol{E}_{6\mathbf{C}}$. We define a representation ρ of $\boldsymbol{G}_{\mathbf{C}}$ into $\boldsymbol{GL}(\boldsymbol{X}_{\mathbf{C}})$ by:

$$\rho(g) \colon x \longmapsto g_1(g_2 \cdot x), \tag{1.4}$$

with $g := (g_1, g_2) \in \boldsymbol{G}_{\mathbf{C}}$ and $x \in \boldsymbol{X}_{\mathbf{C}}$. We may naturally regard $\boldsymbol{G}_{\mathbf{C}}$ as a subgroup of $\boldsymbol{GL}(\boldsymbol{X}_{\mathbf{C}})$ by the inclusion map ρ. On the other hand, when we put:

$$\langle x, y \rangle := \mathrm{tr}(x \cdot y), \tag{1.5}$$

it gives a non-degenerate complex bilinear form on $\boldsymbol{X}_{\mathbf{C}}$ and hence it is an inner product on $\boldsymbol{X}_{\mathbf{C}}$. We may identify $\boldsymbol{X}_{\mathbf{C}}$ with its dual vector space $\boldsymbol{X}_{\mathbf{C}}^*$ through the inner product (1.5). The contragredient representation ρ^* on $\boldsymbol{X}_{\mathbf{C}}$ with respect to (1.5) is defined naturally and $\boldsymbol{G}_{\mathbf{C}}$ is embedded in $\boldsymbol{GL}(\boldsymbol{X}_{\mathbf{C}})$ by ρ^*. The images of $\boldsymbol{G}_{\mathbf{C}}^1$ by ρ and ρ^* are the same. The orbit structure of $\boldsymbol{X}_{\mathbf{C}}$ with respect to the group action of $\rho(\boldsymbol{G}_{\mathbf{C}})$ and $\rho^*(\boldsymbol{G}_{\mathbf{C}})$ is the same, that is to say, the orbital decompositions of $\boldsymbol{X}_{\mathrm{C}}$ by the group $\rho(\boldsymbol{G}_{\mathrm{C}})$ and $\rho^*(\boldsymbol{G}_{\mathrm{C}})$ are the same.

We put $P(X) := \det(x)$. Then $P(X)$ is an irreducible polynomial of degree 3 on $\boldsymbol{X}_{\mathrm{C}}$. Apparently we have:

$$P(\rho(g)x) = \chi(g)P(x),$$

for all $g := (g_1, g_2) \in \boldsymbol{G}$ where $\chi(g) = g_1^3$. Namely $P(x)$ is an irreducible relatively invariant polynomial corresponding to the character $\chi(g)$ on $\boldsymbol{X}_{\mathrm{C}}$ under the action $\rho(g)$. For the contragredient action ρ^*, we have $P(\rho^*(g) \cdot x) = \chi(g)^{-1}P(x)$, which means that $P(x)$ is a relatively invariant polynomial corresponding to the character $\chi(g)^{-1}$ under the contragredient action $\rho^*(g)$. We put:

$$\boldsymbol{S}_{\mathrm{C}} := \{x \in \boldsymbol{X}_{\mathrm{C}};\, P(x) = 0\}, \tag{1.6}$$

and call it the *singular set* of $\boldsymbol{X}_{\mathrm{C}}$.

Proposition 1.1. 1) *The triplets* $(\boldsymbol{G}_{\mathrm{C}}, \rho, \boldsymbol{X}_{\mathrm{C}})$ *and* $(\boldsymbol{G}_{\mathrm{C}}, \rho^*, \boldsymbol{X}_{\mathrm{C}})$ *are regular irreducible prehomogeneous vector spaces with an irreducible relatively invariant polynomial* $P(x) = \det(x)$.

2) *The vector space* $\boldsymbol{X}_{\mathrm{C}}$ *decomposes into four orbits; the open orbit is unique and it is* $\boldsymbol{X}_{\mathrm{C}} - \boldsymbol{S}_{\mathrm{C}}$; *the singular set* $\boldsymbol{S}_{\mathrm{C}}$ *decomposes into the following three orbits*:

$$\begin{aligned}
&\boldsymbol{S}_{1\mathrm{C}} := \rho(\boldsymbol{G}_{\mathrm{C}}) \cdot \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} = \rho^*(\boldsymbol{G}_{\mathrm{C}}) \cdot \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \\
&\boldsymbol{S}_{2\mathrm{C}} := \rho(\boldsymbol{G}_{\mathrm{C}}) \cdot \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} = \rho^*(\boldsymbol{G}_{\mathrm{C}}) \cdot \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \\
&\boldsymbol{S}_{3\mathrm{C}} := \{0\}.
\end{aligned} \tag{1.7}$$

3) *The orbits in* (1.7) *are not only* $\boldsymbol{G}_{\mathrm{C}}$*-orbits but also* $\boldsymbol{G}_{\mathrm{C}}^1$*-orbits.*

Proof. The claims 1) and 2) were proved in Sato-Kimura [Sm-Ki] Proposition 47 and Kimura [Ki] Proposition 6-2, respectively. For 3), we may easily check the validity by computing the dimension of the $\boldsymbol{G}_{\mathrm{C}}^1$-orbits generated by

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}.$$

q.e.d.

§ 2. An invariant holonomic system $\mathfrak{M}_s$ and its holonomy diagram

In this section we shall give a holonomic system $\mathfrak{M}_s$ having the solution $P(x)^s$ and give the complex holonomy diagram of it. For the details about theoretical aspect of microlocal analysis, see [Sm-Ka-Ki-Os], and for the technical aspect of it, see [Ki] Proposition 6-2.

Let $\mathscr{G}_{\mathrm{C}}$ be the Lie algebra of $\boldsymbol{G}_{\mathrm{C}}$ and let $d\rho$ and $\delta\chi$ be infinitesimal representations of ρ and χ, respectively. Consider the following system of linear differential equations with one unknown function $u(x)$:

$$(2.1)\qquad \mathfrak{M}_s\colon \left(\left\langle d\rho(A)x, \frac{\partial}{\partial x}\right\rangle - s\delta\chi(A)\right)u(x)=0, \qquad \text{for all } A\in\mathscr{G}_{\mathrm{C}}.$$

on $\boldsymbol{X}_{\mathrm{C}}$. Here, s is a complex number, $\langle\ ,\ \rangle$ is the canonical bilinear form on $\boldsymbol{X}_{\mathrm{C}}\times\boldsymbol{X}_{\mathrm{C}}^*$, and $\partial/\partial x$ is the first order homogeneous differential operator corresponding to the dual basis.

We see that $\mathfrak{M}_s$ is a holonomic system, i.e., the characteristic variety $\mathrm{ch}(\mathfrak{M}_s)$ is a Lagrangian subvariety in $T^*\boldsymbol{X}_{\mathrm{C}}$, as a consequence of the following proposition:

Proposition 2.1. *The characteristic variety* $\mathrm{ch}(\mathfrak{M}_s)$ *is contained in* $\bigcup_{i=1}^n \overline{T^*_{\boldsymbol{S}_{i\mathrm{C}}}\boldsymbol{X}_{\mathrm{C}}}$ *where* $T^*_{\boldsymbol{S}_{i\mathrm{C}}}\boldsymbol{X}_{\mathrm{C}}$ *is the conormal bundle of* $\boldsymbol{S}_{i\mathrm{C}}$ *in* $\boldsymbol{X}_{\mathrm{C}}$. *Here the upper bar means the Zariski closure of the variety.*

For the proof, see the argument in Kashiwara [Ka 1] p. 116 Example 1 or Muro [Mr 1].

From the definition, each conormal bundles $T^*_{\boldsymbol{S}_{i\mathrm{C}}}\boldsymbol{X}_{\mathrm{C}}$ is an irreducible Lagrangian subvariety in $T^*\boldsymbol{X}_{\mathrm{C}}$. Therefore $\mathfrak{M}_s$ is a holonomic system on $\boldsymbol{X}_{\mathrm{C}}$. We denote by $\Lambda_{i\mathrm{C}}$ the Zariski closure $\overline{T^*_{\boldsymbol{S}_{i\mathrm{C}}}\boldsymbol{X}_{\mathrm{C}}}$. From the argument in Kimura [Ki] § 6, $\mathfrak{M}_s$ is a simple holonomic system on each $\Lambda_{i\mathrm{C}}$ and the order of $\mathfrak{M}_s$ on $\Lambda_{i\mathrm{C}}$ is $-is-i(4i-3)/2$. We may look upon the complex cotangent bundle $T^*\boldsymbol{X}_{\mathrm{C}}$ of $\boldsymbol{X}_{\mathrm{C}}$ as the product $\boldsymbol{X}_{\mathrm{C}}\times\boldsymbol{X}_{\mathrm{C}}^*$. The group $\boldsymbol{G}_{\mathrm{C}}$ acts on $T^*\boldsymbol{X}_{\mathrm{C}}\simeq\boldsymbol{X}_{\mathrm{C}}\times\boldsymbol{X}_{\mathrm{C}}^*\colon g\cdot(x,y)=(\rho(g)\cdot x, \rho^*(g)\cdot y)$ with $g\in\boldsymbol{G}_{\mathrm{C}}$ and $(x,y)\in\boldsymbol{X}_{\mathrm{C}}\times\boldsymbol{X}_{\mathrm{C}}^*$. It is easily proved that $T^*_{\boldsymbol{S}_{i\mathrm{C}}}\boldsymbol{X}_{\mathrm{C}}$ is a $\boldsymbol{G}_{\mathrm{C}}$-stable subset and so is the closure $\Lambda_{i\mathrm{C}}=\overline{T^*_{\boldsymbol{S}_{i\mathrm{C}}}\boldsymbol{X}_{\mathrm{C}}}$. Each $\Lambda_{i\mathrm{C}}$ decomposes into finitely many $\boldsymbol{G}_{\mathrm{C}}$-orbits. Namely:

Proposition 2.2. *The* $\boldsymbol{G}_{\mathrm{C}}$*-stable Lagrangian subvariety* $\Lambda_{i\mathrm{C}}$ *has the* $\boldsymbol{G}_{\mathrm{C}}$*-orbital decomposition*:

$$(2.2)\qquad \coprod_{\substack{p\geq i\\ q\geq 3-i}} \boldsymbol{G}_{\mathrm{C}}\cdot\left(\begin{bmatrix} I_{3-p} & \\ & 0_p\end{bmatrix}, \begin{bmatrix} 0_q & \\ & I_{3-q}\end{bmatrix}\right)=\Lambda_{i\mathrm{C}}.$$

Here I_i *means* $i\times i$ *identity matrix.*

This proposition can be easily proved by the direct calculation so we omit the proof. We denote by $\Sigma_{p,q\mathbf{C}}$ the $G_{\mathbf{C}}$-orbit

$$G_{\mathbf{C}}\cdot\left(\begin{bmatrix} I_{3-p} & \\ & 0_p\end{bmatrix}, \begin{bmatrix} 0_q & \\ & I_{3-q}\end{bmatrix}\right) \qquad \text{for each } 0\leq p,\ q\leq 3.$$

Finally we give the complex holonomy diagram of $\mathfrak{M}_s$.

Proposition 2.3. 1) *The Lagrangian subvarieties $\Lambda_{i\mathbf{C}}$ and $\Lambda_{j\mathbf{C}}$ have an intersection of codimension one if $|i-j|=1$. Otherwise there are no intersections of codimension one. The $G_{\mathbf{C}}$-orbit $\Sigma_{i+1,3-i\mathbf{C}}$ is an open dense orbit in $\Lambda_{i\mathbf{C}}\cap\Lambda_{i+1\mathbf{C}}$ $(i=0, 1, 2)$.*

2) *For each $p\in\Sigma_{i+1,3-i}$, we have*

$$(T(\Lambda_{i\mathbf{C}}\cap\Lambda_{i+1\mathbf{C}}))_p=(T\Lambda_{i\mathbf{C}})_p\cap(T\Lambda_{i+1\mathbf{C}})_p$$

where $(TA)_p$ means the tangent space of A at p. Namely $\Lambda_{i\mathbf{C}}$ and $\Lambda_{i+1\mathbf{C}}$ have a regular intersection at each $p\in\Sigma_{i+1,3-i\mathbf{C}}$.

3) *The complex holonomy diagram of $\mathfrak{M}_s$ is*

$$(2.3)\qquad \begin{array}{ccccccc} 0 & & -s-\frac{1}{2} & & -2s-5 & & -3s-\frac{27}{2} \\ \circ & \longrightarrow & \circ & \longrightarrow & \circ & \longrightarrow & \circ \\ \Lambda_{0\mathbf{C}} & (s+1) & \Lambda_{1\mathbf{C}} & (s+5) & \Lambda_{2\mathbf{C}} & (s+9) & \Lambda_{3\mathbf{C}} \end{array}\ .$$

Each $\circ$ stands for an irreducible Lagrangian subvariety and the line $\longrightarrow$ means an intersection of codimension one. This implies that the b-function is:

$$(2.4)\qquad b(s):=(s+1)(s+5)(s+9).$$

Namely we have $P(\partial/\partial x)P(x)^{s+1}=\text{const. } b(s)\cdot P(x)^s$.

For the proof see [Ki].

§ 3. Real forms of the prehomogeneous vector spaces and real holonomy diagrams of $\mathfrak{M}_s$

In this section, we shall determine all the real forms of the prehomogeneous vector space $(G_{\mathbf{C}}, \rho, X_{\mathbf{C}}):=(GL_1(\mathbb{C})\times E_{6\mathbf{C}}, \rho, \mathbf{Her}_3(\mathfrak{C}_{\mathbf{C}}))$ and give their orbit structure. Let $(G_{\mathbf{C}}, \rho, X_{\mathbf{C}})$ be a given complex prehomogeneous vector space. For a real form $G_{\mathbf{R}}$ of the complex Lie group $G_{\mathbf{C}}$ and a real form $X_{\mathbf{R}}$ of the complex vector space $X_{\mathbf{C}}$, we say that $(G_{\mathbf{R}}, \rho, X_{\mathbf{R}})$ is a *real form* of the complex prehomogeneous vector space $(G_{\mathbf{C}}, \rho, X_{\mathbf{C}})$ if $\rho(G_{\mathbf{R}})\subset GL(X_{\mathbf{R}})\cap\rho(G_{\mathbf{C}})$.

We begin with determining all the possible real forms of the complex quaternion algebra. As well known, there are following two real forms of the complex quaternion algebras $\mathbb{H}_{\mathbb{C}}$ and they are all:

(3.1) 1) (quaternion division algebra)

$\mathbb{H}_{\mathbb{R}}^{d}$:= a four dimensional real vector space generated by $\{1, u, v, w\}$ endowed with the structure of algebra defined over $\mathbb{R}$; the multiplication low is given by $w=uv=-vu$ and $u^2=v^2=-1$.

2) (quaternion split algebra)

$\mathbb{H}_{\mathbb{R}}^{s}$:= a four dimensional real vector space generated by $\{1, u, v, w\}$ endowed with the structure of algebra defined over $\mathbb{R}$; the multiplication low is given by $w=uv=-vu$ and $u^2=v^2=+1$.

For any element $p=\alpha+\beta u+\gamma v+\delta w$ in $\mathbb{H}_{\mathbb{R}}^{d}$ or in $\mathbb{H}_{\mathbb{R}}^{s}$, the quaternion conjugate $\bar{p}$ is defined to be $\bar{p}=\alpha-\beta u-\gamma v-\delta w$. The complexifications of $\mathbb{H}_{\mathbb{R}}^{d}$ and $\mathbb{H}_{\mathbb{R}}^{s}$ are naturally isomorphic to $\mathbb{H}_{\mathbb{C}}$.

According to the real form of $\mathbb{H}_{\mathbb{C}}$, there are two types of real forms of the complex Cayley algebra: one is the division Cayley algebra $\mathfrak{C}_{\mathbb{R}}^{d}$ and the other is the split Cayley algebra $\mathfrak{C}_{\mathbb{R}}^{s}$, which are defined to be a $\mathbb{H}_{\mathbb{R}}^{d}$- and $\mathbb{H}_{\mathbb{R}}^{s}$- free modules of rank two generated by $\{1, e\}$, respectively. The multiplication law is given by $(q+re)(s+te):=(qs-\bar{t}r)+(tq+r\bar{s})e$ for two elements $q+re$ and $s+te$ in $\mathfrak{C}_{\mathbb{R}}^{d}$ or $\mathfrak{C}_{\mathbb{R}}^{s}$. The trace $T(a)$ and the norm $N(a)$ of $a \in \mathfrak{C}_{\mathbb{R}}^{d}$ or $\mathfrak{C}_{\mathbb{R}}^{s}$ are defined in the same manner as in the case of the complex Cayley algebra $\mathfrak{C}_{\mathbb{C}}$. The complexifications $\mathfrak{C}_{\mathbb{R}}^{d}\otimes_{\mathbb{R}}\mathbb{C}$ and $\mathfrak{C}_{\mathbb{R}}^{s}\otimes_{\mathbb{R}}\mathbb{C}$ are naturally isomorphic to $\mathfrak{C}_{\mathbb{C}}$.

According to the real forms of $\mathfrak{C}_{\mathbb{C}}$, there are two real forms of the complex vector space $\boldsymbol{X}_{\mathbb{C}}=\mathbf{Her}_3(\mathfrak{C}_{\mathbb{C}})$. They are

(3.2) 1) the division case: $\boldsymbol{X}_{\mathbb{R}} := \mathbf{Her}_3(\mathfrak{C}_{\mathbb{R}}^{d})$

and

2) the split case: $\boldsymbol{X}_{\mathbb{R}} := \mathbf{Her}_3(\mathfrak{C}_{\mathbb{R}}^{s})$.

The *determinant* $\det(x)$ and the *trace* $\mathrm{tr}(x)$ of an element x in $\boldsymbol{X}_{\mathbb{R}}$ are defined in the same way as the complex case. By definition, they coincide with the restrictions of the polynomials $\det(x)$ and $\mathrm{tr}(x)$ on $\boldsymbol{X}_{\mathbb{C}}$ to $\boldsymbol{X}_{\mathbb{R}}$. Putting $\boldsymbol{G}_{\mathbb{R}}^{1} := \boldsymbol{G}_{\mathbb{C}}^{1} \cap \boldsymbol{GL}(\boldsymbol{X}_{\mathbb{R}})$, $\boldsymbol{G}_{\mathbb{R}}^{1}$ is a real form of the complex Lie group $\boldsymbol{G}_{\mathbb{C}}^{1}$ from the definition. Then $\boldsymbol{G}_{\mathbb{R}}^{+} := \boldsymbol{GL}_1(\mathbb{R})^{+} \times \boldsymbol{G}_{\mathbb{R}}^{1}$ is a real form of $\boldsymbol{G}_{\mathbb{C}}$ and $(\boldsymbol{G}_{\mathbb{R}}^{+}, \rho, \boldsymbol{X}_{\mathbb{R}})$ is a real form of the complex prehomogeneous vector space $(\boldsymbol{G}_{\mathbb{C}}, \rho, \boldsymbol{X}_{\mathbb{C}})$. Here $\boldsymbol{GL}_1(\mathbb{R})^{+}$ is the connected component of $\boldsymbol{GL}_1(\mathbb{R})$ containing the identity. Thus we have two real forms of $(\boldsymbol{G}_{\mathbb{C}}, \rho, \boldsymbol{X}_{\mathbb{C}})$.

Proposition 3.1. *All possible real forms of* $(\boldsymbol{G}_{\mathbb{C}}, \rho, \boldsymbol{X}_{\mathbb{C}})$ *are the above two.*

Proof. As proved in Rubenthaler [Ru], there are only two possible real forms of the complex prehomogeneous vector space $(\boldsymbol{G}_{\mathbf{C}}, \rho, \boldsymbol{X}_{\mathbf{C}})$. The two real forms given in (3.2) are actually real forms. When $\boldsymbol{X}_{\mathbf{R}}=\mathbf{Her}_3(\mathfrak{C}^d_{\mathbf{R}})$, the group $\boldsymbol{G}^1_{\mathbf{R}}$ is the real exceptional Lie group of type $\boldsymbol{E}_6$ whose signature of the killing form is -26, which is denoted by $\boldsymbol{E}_6^{(-26)}$. When $\boldsymbol{X}_{\mathbf{R}}=\mathbf{Her}_3(\mathfrak{C}^s_{\mathbf{R}})$, $\boldsymbol{G}^1_{\mathbf{R}}$ is the real form of signature 6, which is denoted by $\boldsymbol{E}_6^{(6)}$. Thus the above two real forms in (3.2) are different to each other and they are all possible real forms. q.e.d.

After all, the following two real forms are obtained.

(3.3) 1) The division case:
$\boldsymbol{G}^+_{\mathbf{R}} := \boldsymbol{GL}_1(\mathbb{R})^+ \times \boldsymbol{E}_6^{(-26)}$ and $\boldsymbol{X}_{\mathbf{R}} := \mathbf{Her}_3(\mathfrak{C}^d_{\mathbf{R}})$.

2) The split case:
$\boldsymbol{G}^+_{\mathbf{R}} := \boldsymbol{GL}_1(\mathbb{R})^+ \times \boldsymbol{E}_6^{(6)}$ and $\boldsymbol{X}_{\mathbf{R}} := \mathbf{Her}_3(\mathfrak{C}^s_{\mathbf{R}})$.

The restriction of the inner product in (1.5) to $\boldsymbol{X}_{\mathbf{R}}$ is a real valued inner product on the real vector space $\boldsymbol{X}_{\mathbf{R}}$. The contragredient representation ρ^* of $\boldsymbol{G}_{\mathbf{R}}$ gives a real form $(\boldsymbol{G}_{\mathbf{R}}, \rho^*, \boldsymbol{X}_{\mathbf{R}})$ of $(\boldsymbol{G}_{\mathbf{C}}, \rho^*, \boldsymbol{X}_{\mathbf{C}})$. We denote by $\boldsymbol{G}^+_{\mathbf{R}}$ the connected component of $\boldsymbol{G}_{\mathbf{R}}$ containing the identity element. The character $\chi(g)$ is real-valued and positive on $\boldsymbol{G}^+_{\mathbf{R}}$ because we see by a direct computation that the restriction of $P(x)$ on $\boldsymbol{X}_{\mathbf{R}}$ is a polynomial with real coefficients in each real form.

For a fixed real form $(\boldsymbol{G}^+_{\mathbf{R}}, \rho, \boldsymbol{X}_{\mathbf{R}})$, we denote $\boldsymbol{S}_{\mathbf{R}} := \boldsymbol{S}_{\mathbf{C}} \cap \boldsymbol{X}_{\mathbf{R}}$ and call it the *real singular set*. The set $\boldsymbol{X}_{\mathbf{R}}$ has the same orbital decomposition under the action $\rho(\boldsymbol{G}^+_{\mathbf{R}})$ and $\rho^*(\boldsymbol{G}^+_{\mathbf{R}})$. Namely:

Proposition 3.2. 1) (*The division case*): *When $\boldsymbol{X}_{\mathbf{R}}=\mathbf{Her}_3(\mathfrak{C}^d_{\mathbf{R}})$, each real locus of the $\boldsymbol{G}_{\mathbf{C}}$-orbits in $\boldsymbol{X}_{\mathbf{C}}$ decomposes into the following connected components which are $\boldsymbol{G}^+_{\mathbf{R}}$-orbits*:

i) $\boldsymbol{X}_{\mathbf{R}} - \boldsymbol{S}_{\mathbf{R}} = \coprod_{i=0}^{3} \boldsymbol{O}_i$ (*disjoint union*), *where* $\boldsymbol{O}_i := \rho(\boldsymbol{G}^+_{\mathbf{R}})\cdot I_3^{(i)} = \rho^*(\boldsymbol{G}^+_{\mathbf{R}})\cdot I_3^{(i)}$ *with* $I_3^{(i)} := \begin{bmatrix} -I_i & \\ & I_{3-i} \end{bmatrix}$ $(i=0, 1, 2, 3)$.

ii) $\boldsymbol{S}_{1\mathbf{R}} := \boldsymbol{S}_{1\mathbf{C}} \cap \boldsymbol{X}_{\mathbf{R}} = \boldsymbol{S}_1^0 \amalg \boldsymbol{S}_1^1 \amalg \boldsymbol{S}_1^2$ (*disjoint union*), *where* $\boldsymbol{S}_1^i := \rho(\boldsymbol{G}^+_{\mathbf{R}})\cdot I_2^{(i)} = \rho^*(\boldsymbol{G}^+_{\mathbf{R}})\cdot I_2^{(i)}$ *with* $I_2^{(i)} := \begin{bmatrix} -I_i & & \\ & I_{2-i} & \\ & & 0 \end{bmatrix}$ $(i=0, 1, 2)$.

iii) $\boldsymbol{S}_{2\mathbf{R}} := \boldsymbol{S}_{2\mathbf{C}} \cap \boldsymbol{X}_{\mathbf{R}} = \boldsymbol{S}_2^0 \amalg \boldsymbol{S}_2^1$ (*disjoint union*), *where* $\boldsymbol{S}_2^i := \rho(\boldsymbol{G}^+_{\mathbf{R}})\cdot I_1^{(i)} = \rho^*(\boldsymbol{G}^+_{\mathbf{R}})\cdot I_1^{(i)}$ *with* $I_1^{(i)} := \begin{bmatrix} (-1)^i & \\ & 0_2 \end{bmatrix}$ $(i=0, 1)$.

vi) $\boldsymbol{S}_{3\mathbf{R}} := \boldsymbol{S}_{3\mathbf{C}} \cap \boldsymbol{X}_{\mathbf{R}} = \boldsymbol{S}_3^0 := \{0\}$ (*the origin*).

2) *The split case*: *When $\boldsymbol{X}_{\mathbf{C}}=\mathbf{Her}_3(\mathfrak{C}^s_{\mathbf{R}})$, each real locus of the $\boldsymbol{G}_{\mathbf{C}}$-orbits in $\boldsymbol{X}_{\mathbf{C}}$ decomposes into the following connected components which are $\boldsymbol{G}^+_{\mathbf{R}}$-orbits.*

i) $X_{\mathbf{R}}-S_{\mathbf{R}}=O_{+}\amalg O_{-}$ *with* $O_{+}:=\{x\in X_{\mathbf{R}};\det(x)>0\}$ *and* $O_{-}:=\{x\in X_{\mathbf{R}};\det(x)<0\}$.

ii) $S_{i\mathbf{R}}:=S_{i\mathbf{C}}\cap X_{\mathbf{R}}\ (i=1,2,3)$ *is a single* $G_{\mathbf{R}}^{+}$*-orbit. We denote it by* S_i^0.

3) *In both of the cases* 1) *and* 2), *the* $G_{\mathbf{R}}^{+}$*-orbits in the real singular set* $S_{\mathbf{R}}:=S_{\mathbf{C}}\cap X_{\mathbf{R}}$ *are* $G_{\mathbf{R}}^{1}$*-orbits.*

For the proof of this proposition we have only to carry out similar computation as we did in the complex case (see [Ki]). We omit the proof because it is complicated.

We conclude this section by calculating the real holonomy diagrams of $\mathfrak{M}_s$, which indicates the configuration of the real Lagrangian subvarieties appearing in $\mathrm{ch}(\mathfrak{M}_s)_{\mathbf{R}}:=\mathrm{ch}(\mathfrak{M}_s)\cap T^*X_{\mathbf{R}}$. For the precise definition of a real holonomy diagram, see [Ka-Mi], [Mr 1], p. 414 or [Mr 3]. Each real locus $\Lambda_{i\mathbf{R}}:=\Lambda_{i\mathbf{C}}\cap T^*X_{\mathbf{R}}$ of $\Lambda_{i\mathbf{C}}$ is a real Lagrangian subvariety. Indeed, we may easily check that $\Lambda_{i\mathbf{R}}:=\overline{T^*_{S_{i\mathbf{R}}}X_{\mathbf{R}}}$ where $S_{i\mathbf{R}}:=S_{i\mathbf{C}}\cap X_{\mathbf{R}}$. For each irreducible component $\Lambda_{i\mathbf{C}}$ of $\mathrm{ch}(\mathfrak{M}_s)$, we denote by $\Lambda_{i\mathbf{C}}^{\circ}$ the subset of $\Lambda_{i\mathbf{C}}$ consisting of points at which $\mathrm{ch}(\mathfrak{M}_s)$ is non-singular, i.e., if $\xi\in\Lambda_{i\mathbf{C}}$, then there exists a neighborhood U of ξ such that $U\cap\mathrm{ch}(\mathfrak{M}_s)=U\cap\Lambda_{i\mathbf{C}}$ and it is a non-singular variety. Then $\Lambda_{i\mathbf{C}}^{\circ}$ is an open dense non-singular subset of $\Lambda_{i\mathbf{C}}$. In our case, we see that $\Lambda_{i\mathbf{C}}^{\circ}=\Sigma_{i,3-i}$. The real locus $\Lambda_{i\mathbf{R}}^{\circ}:=\Lambda_{i\mathbf{C}}^{\circ}\cap T^*X_{\mathbf{R}}$ decomposes into finitely many connected components, each of which is a $G_{\mathbf{R}}^{+}$-orbit. Explicit calculation is easy and the result is the following.

(3.5) 1) The division case: $X_{\mathbf{R}}=\mathbf{Her}_3(\mathfrak{C}_{\mathbf{R}}^{d})$.

$$\Lambda_{i\mathbf{R}}^{\circ}:=\coprod_{\substack{0\le p\le 3-i\\0\le q\le i}}\Lambda_i^{p,q}\qquad(i=0,1,2,3)$$

where $\Lambda_i^{p,q}:=G_{\mathbf{R}}^{+}\cdot\left(\begin{bmatrix}I_{3-i}^{(p)}&\\&0_i\end{bmatrix},\begin{bmatrix}0_{3-i}&\\&I_i^{(q)}\end{bmatrix}\right)$,

with $I_j^{(k)}:=\begin{bmatrix}I_k&\\&-I_{j-k}\end{bmatrix}\in M_j(\mathbf{R})$.

2) The split case: $X_{\mathbf{R}}=\mathbf{Her}_3(\mathfrak{C}_{\mathbf{R}}^{s})$.

$$\Lambda_{i\mathbf{R}}^{\circ}:=\coprod_{\varepsilon=\pm}\Lambda_i^{\varepsilon}\qquad(i=0,1,2,3)$$

where $\Lambda_i^{\varepsilon}:=G_{\mathbf{R}}^{+}\cdot\left(\begin{bmatrix}I_2&\\&\varepsilon\end{bmatrix},\begin{bmatrix}0_3\end{bmatrix}\right)\qquad(i=0)$ or

$$\Lambda_i^{\varepsilon}:=G_{\mathbf{R}}^{+}\cdot\left(\begin{bmatrix}I_{3-i}&\\&0_i\end{bmatrix},\begin{bmatrix}0_{3-i}&&\\&\varepsilon&\\&&I_{i-1}\end{bmatrix}\right)\qquad(i\ge 1).$$

Next we consider the real locus of the codimension one intersection $\Sigma_{i+1,3-i\mathbf{C}}$ between $\Lambda_{i\mathbf{C}}$ and $\Lambda_{i+1\mathbf{C}}$: $\Sigma_{i+1,3-i\mathbf{C}}\cap T^*X_{\mathbf{R}}$ $(i=0,1,2)$. It is a regular intersection of real codimension one between $\Lambda_{i\mathrm{R}}$ and $\Lambda_{i+1\mathrm{R}}$. Each connected component of $\Sigma_{i+1,3-i\mathrm{R}}$ is a G_{R}^{+}-orbit. We have the following decomposition of $\Sigma_{i+1,3-i\mathrm{R}}$ into G_{R}^{+}-orbits in each case.

(3.6) 1) The division case: $X_{\mathrm{R}} = \mathbf{Her}_3(\mathfrak{C}_{\mathrm{R}}^{d})$

$$\Sigma_{i+1,3-i\mathrm{R}} = \coprod_{\substack{0\le p\le 2-i\\ 0\le q\le i}} \Sigma_i^{p,q} \qquad (i=0,1,2)$$

where $\Sigma_i^{p,q}$ is a G_{R}^{+}-orbit generated by

$$\left(\begin{bmatrix} I_{2-i}^{(p)} & \\ & 0_{i+1}\end{bmatrix}, \begin{bmatrix} 0_{3-i} & \\ & I_i^{(q)}\end{bmatrix}\right).$$

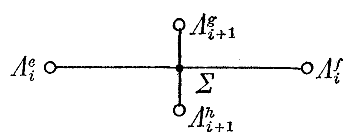
2) The split case: $X_{\mathrm{R}} = \mathbf{Her}_3(\mathfrak{C}_{\mathrm{R}}^{s})$.

$\Sigma_{i+1,3-i\mathrm{R}}$ is a single G_{R}^{+}-orbit generated by

$$\left(\begin{bmatrix} I_{2-i} & \\ & 0_{i+1}\end{bmatrix}, \begin{bmatrix} 0_{3-i} & \\ & I_i\end{bmatrix}\right),$$

which is denoted by Σ_i $(i=0, 1, 2)$.

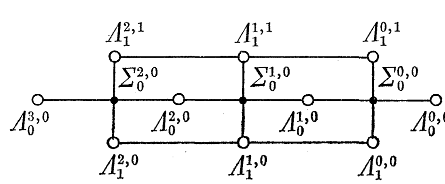
Let Σ be a connected component of $\Sigma_{i+1,3-i\mathrm{R}}$. Take a point (x, y) in Σ. Then there exist two connected components Λ_i^e and Λ_i^f in $\Lambda_{i\mathrm{R}}^{\circ}$, and, Λ_{i+1}^g and Λ_{i+1}^h in $\Lambda_{i+1\mathrm{R}}^{\circ}$ such that $\Lambda_{i\mathrm{R}} - \Sigma = \Lambda_i^e \amalg \Lambda_i^f$ and $\Lambda_{i+1\mathrm{R}} - \Sigma = \Lambda_{i+1}^g \amalg \Lambda_{i+1}^h$. Such Λ_i^e, Λ_i^f, Λ_{i+1}^g and Λ_{i+1}^h do not depend on the choice of the point (x, ξ) in Σ. In other words, $\Lambda_i^e \cup \Lambda_i^f \cup \Sigma$ and $\Lambda_{i+1}^g \cup \Lambda_{i+1}^h \cup \Sigma$ coincide with $\Lambda_{i\mathrm{R}}$ and $\Lambda_{i+1\mathrm{R}}$ in a neighborhood of (x, y), respectively and they have the intersection Σ. We denote this situation by the following diagram.



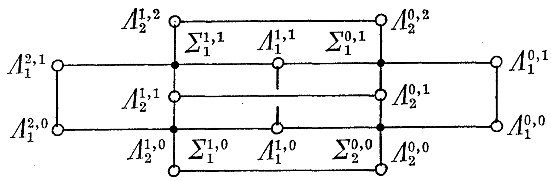
The real holonomy diagram between $\Lambda_{i\mathrm{R}}$ and $\Lambda_{i+1\mathrm{R}}$ is a diagram obtained by drawing all the intersections of codimension one between $\Lambda_{i\mathrm{R}}$ and $\Lambda_{i+1\mathrm{R}}$. In our case we have the following real holonomy diagrams.

(3.7) The division case: $X_{\mathrm{R}} = \mathbf{Her}_3(\mathfrak{C}_{\mathrm{R}}^{d})$,

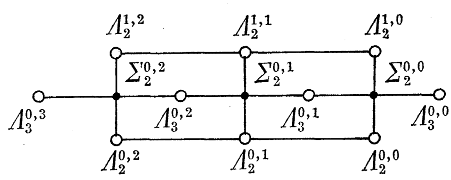
1) The intersection of $\Lambda_{0\mathrm{R}}$ and $\Lambda_{1\mathrm{R}}$.



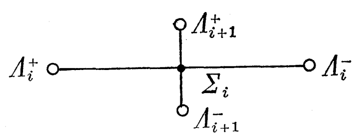
2) The intersection of $\Lambda_{1\mathrm{R}}$ and $\Lambda_{2\mathrm{R}}$.



3) The intersection of $\Lambda_{2\mathrm{R}}$ and $\Lambda_{3\mathrm{R}}$.



(3.8) The split case: $\boldsymbol{X}_{\mathrm{R}}=\mathbf{Her}_3(\mathfrak{C}^s_{\mathrm{R}})$,
The intersection of $\Lambda_{i\mathrm{R}}$ and $\Lambda_{i+1\mathrm{R}}$:



§ 4. Local zeta functions as a solution of the holonomic system $\mathfrak{M}_s$

Let $\mathfrak{M}_s$ be the holonomic system defined in (2.1) and let $(\boldsymbol{G}_{\mathrm{R}}^{+}, \rho, \boldsymbol{X}_{\mathrm{R}})$ be a real form of the complex prehomogeneous vector space $(\boldsymbol{G}_{\mathrm{C}}, \rho, \boldsymbol{X}_{\mathrm{C}})$. For a hyperfunction $f(x)$ on $\boldsymbol{X}_{\mathrm{R}}$, we say that $f(x)$ is a χ^s-*invariant* hyperfunction if it satisfies $f(\rho(g)\cdot x)=\chi(g)^s\cdot f(x)$ for all $g\in \boldsymbol{G}_{\mathrm{R}}^{+}$. Note the following proposition.

Proposition 4.1. *If $f(x)$ is a hyperfunction solution of $\mathfrak{M}_s$, then it is a χ^s-invariant tempered distribution. Conversely, if $f(x)$ is a χ^s-invariant hyperfunction, it is a tempered distribution and is a solution to $\mathfrak{M}_s$.*

Proof. Since $\mathfrak{M}_s$ is a simple holonomic system, $\mathfrak{M}_s$ is a completely regular holonomic system, hence any solution of $\mathfrak{M}_s$ is a tempered distribution on $X_{\mathbf{R}}$. It is easily checked that a solution $f(x)$ of $\mathfrak{M}_s$ is χ^s-invariant if it is a solution to $\mathfrak{M}_s$ in a more general situation. See for example Proposition 2.3 in [Mr 2]. q.e.d.

This proposition enables us to treat any χ^s-invariant hyperfunction as a solution of the holonomic system $\mathfrak{M}_s$.

Let $\coprod_{i\in I_0} O_i = X_{\mathbf{R}} - S_{\mathbf{R}}$ be the connected component decomposition of $X_{\mathbf{R}} - S_{\mathbf{R}}$ obtained in Proposition 3.2. Here the index set I_0 is $\{0, 1, 2, 3\}$ in the division case and $\{+, -\}$ in the split case. We put:

$$|P(x)|_i^s := \begin{cases} |P(x)|^s & x \in O_i \\ 0 & x \notin O_i \end{cases} \tag{4.1}$$

with $s \in \mathbb{C}$. When the real part of s is sufficiently large, $|P(x)|_i^s$ is a continuous function. It can be continued with respect to s to the whole complex plane $\mathbb{C}$ as a tempered distribution with a meromorphic parameter $s \in \mathbb{C}$. The function defined by (4.1) stands for the extended one as a distribution with a meromorphic parameter in the whole complex plane. The possible poles of $|P(x)|_i^s$ is contained in the set of *critical points*: $\mathrm{Cri}\,(P(x)^s) := \{s \in ; b(s+k)=0 \text{ with } k \in \mathbb{N} \cup \{0\}\}$. Following Sato-Shintani [Sm-Sh], we call $|P(x)|_i^s$ a *local zeta function.* For a fixed complex number $\lambda \in \mathbb{C}$, $|P(x)|_i^\lambda$ is a χ^λ-invariant tempered distribution if $\lambda \notin \mathrm{Cri}\,(P(x)^s)$. Moreover let λ be a point in $\mathbb{C}$ and let $a_i(s)$ $(i \in I_0)$ be holomorphic functions defined near $s=\lambda$ such that $\sum_{i\in I_0} a_i(s)\cdot|P(x)|_i^s$ is holomorphic at $s=\lambda$. Then

$$\sum_{i\in I_0} a_i(s)\cdot|P(x)|_i^s|_{s=\lambda}, \tag{4.2}$$

is a χ^λ-invariant tempered distribution. Conversely we have the following proposition.

Proposition 4.2. *Every χ^λ-invariant tempered distribution is written in the form in* (4.2).

The proof of this proposition has been given as a special case of the main theorem in [Mr 3]. A more general result related to Proposition 4.2 is announced in [Mr 5].

§ 5. Real principal symbols of the solutions of $\mathfrak{M}_s$ and their explicit expressions

Let $\mathscr{B}_{X_{\mathbf{R}}}$ be the sheaf of hyperfunctions on $X_{\mathbf{R}}$ and let $\mathscr{C}_{X_{\mathbf{R}}}$ be the sheaf of microfunctions on $T^*X_{\mathbf{R}}$ (and not on $\mathring{T}^*X_{\mathbf{R}} := T^*X_{\mathbf{R}} - X_{\mathbf{R}}$ as usually used). Then there is an isomorphism:

$$sp: \mathscr{B}_{X_{\mathrm{R}}} \xrightarrow{\sim} \pi_*(\mathscr{C}_{X_{\mathrm{R}}}), \tag{5.1}$$

where π is the projection map from T^*X_{R} to X_{R}. Especially, for the restriction sheaf of $\mathscr{C}_{X_{\mathrm{R}}}$ to $T^*X_{\mathrm{R}}-X_{\mathrm{R}}$, we have Sato's fundamental exact sequence:

$$0 \longrightarrow \mathscr{A}_{X_{\mathrm{R}}} \longrightarrow \mathscr{B}_{X_{\mathrm{R}}} \longrightarrow \pi_*(\mathscr{C}_{X_{\mathrm{R}}}|_{T^*X_{\mathrm{R}}-X_{\mathrm{R}}}) \longrightarrow 0, \tag{5.2}$$

where $\mathscr{A}_{X_{\mathrm{R}}}$ is the sheaf of real analytic functions on X_{R}.

Let $f(x)$ be a hyperfunction solution to $\mathfrak{M}_s$ on X_{R}. It is a section of $\mathscr{B}_{X_{\mathrm{R}}}$ on X_{R}. Through the map sp in (5.1), $f(x)$ is regarded as a section of $\mathscr{C}_{X_{\mathrm{R}}}$ on T^*X_{R}, which is denoted by $sp(f(x))$. From Sato's fundamental theorem (Theorem 3.4.3 in [Ka-Kw-Ki]), we see that:

$$\boldsymbol{SS}(f(x)) := \mathrm{supp}\,(sp(f(x))) \subset \mathrm{ch}\,(\mathfrak{M}_s) \cap T^*X_{\mathrm{R}}. \tag{5.3}$$

Let $\Lambda_{i\mathrm{C}}$ be an irreducible component of $\mathrm{ch}\,(\mathfrak{M}_s)$ and $\Lambda_{i\mathrm{R}} := \Lambda_{i\mathrm{C}} \cap T^*X_{\mathrm{R}}$. The open dense subset $\Lambda_{i\mathrm{R}}^{\circ}$ in $\Lambda_{i\mathrm{R}}$ defined in § 3 decomposes into a finite number of connected components. We denote it by $\Lambda_{i\mathrm{R}}^{\circ} := \coprod_{j=1}^{j_i} \Lambda_i^j$: the connected component decomposition of $\Lambda_{i\mathrm{R}}^{\circ}$. We see that two solutions which coincide with each other on every Λ_i^j are the same. Namely:

Proposition 5.1. *Let $f(x)$ and $g(x)$ be two hyperfunction solutions of the holonomic system $\mathfrak{M}_s$ on X_{R}. If $sp(f(x))$ and $sp(g(x))$ coincide on every Λ_i^j ($i=0, 1, 2, 3$ and $j=1, \cdots, j_i$), then $f(x)=g(x)$ as hyperfunctions on X_{R}.*

Proof. The proof of this proposition is almost the same as the proof of Lemma 2.15 and Corollary 2.16 in [Mr 1]. A generalization of this theorem has been given in [Mr 4]. We shall give an outline of the proof.

We may show this proposition by applying an induction on the orbits and the Holmgren's uniqueness theorem. We have seen that $f(x)-g(x)$ is zero on $X_{\mathrm{R}}-S_{\mathrm{R}}$. Let $x \in S_1$. There exists a neighborhood U of x such that $U \cap S_{\mathrm{R}} = U \cap S_{1\mathrm{R}}$. From the Holmgren's uniqueness theorem, we see that $(f-g)(x)=0$ on U by estimating the singular spectrum of $(f-g)$ on $\mathring{T}^*U$. Next we assume that $f-g$ is zero on $X_{\mathrm{R}}-(S_{k\mathrm{R}} \cup \cdots \cup S_{3\mathrm{R}})$ $(2 \leq k \leq 3)$. Then, for any $x \in S_{k\mathrm{R}}$, there exists a neighborhood U of x such that $U \cap (S_{k\mathrm{R}} \cup \cdots \cup S_{3\mathrm{R}}) = U \cap S_{k\mathrm{R}}$. By applying Holmgren's uniqueness theorem, $f-g$ is 0 on U, which means $f-g=0$ on $X_{\mathrm{R}}-(S_{k+1\mathrm{R}} \cup \cdots \cup S_{3\mathrm{R}})$. Therefore by induction we obtain that $f-g$ is zero on X_{R}.

q.e.d.

With a microfunction solution $u(x)$ on Λ_i^j, we may associate a real analytic section of $|\Omega_{\Lambda_{i\mathbf{R}}}|^{\otimes 1/2} \otimes |\Omega_{X_{\mathbf{R}}}|^{\otimes -1/2}$ on Λ_i^j. Here $|\Omega_{\Lambda_{i\mathbf{R}}}|$ and $|\Omega_{X_{\mathbf{R}}}|$ are the sheaf of volume forms on $\Lambda_{i\mathbf{R}}$ and $X_{\mathbf{R}}$, respectively. It is called the *real principal symbol* of the solution $u(x)$ and is denoted by $\sigma_{\Lambda_i^j}(u)$. For the precise definition of real principal symbols, see [Ka-Mi] or [Mr 1], pp. 416–419. Since the holonomic system $\mathfrak{M}_s$ is simple on each $\Lambda_{i\mathbf{C}}$, the dimension of the microfunction solution on each Λ_i^j is one. Any microfunction solution on Λ_i^j is written as a constant multiple of a non-trivial section of the solution space on Λ_i^j. When we denote by $\mathcal{S}ol\,(\mathfrak{M}_s)(\Lambda_i^j)$ (resp. $\mathcal{S}ymbol\,(\mathfrak{M}_s)(\Lambda_i^j)$) the space of microfunction solutions of $\mathfrak{M}_s$ on Λ_i^j (resp. the space of real principal symbols of the solutions on Λ_i^j), $\sigma_{\Lambda_i^j}$ gives a linear isomorphism:

$$\begin{array}{rccc} (5.4) \qquad \sigma_{\Lambda_i^j}: & \mathcal{S}ol\,(\mathfrak{M}_s)(\Lambda_i^j) & \longrightarrow & \mathcal{S}ymbol\,(\mathfrak{M}_s)(\Lambda_i^j) \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & u(x) & \longmapsto & \sigma_{\Lambda_i^j}(u). \end{array}$$

Thus we may consider real principal symbols on Λ_i^j instead of microfunction solutions on Λ_i^j. Proposition 5.1 means that two hyperfunction solutions whose real principal symbols on each Λ_i^j coincide with each other are the same.

Next we will see that we may take a canonical basis for the space $\mathcal{S}ymbol\,(\mathfrak{M}_s)(\Lambda_i^j)$. Namely, for a microfunction solution $u(x)$ on Λ_i^j, we may express its principal symbol as:

$$(5.5) \qquad \sigma_{\Lambda_i^j}(u(x)) = c_{\Lambda_i^j}(u) \cdot |P_{\Lambda_{i\mathbf{R}}}|^s \cdot \sqrt{|\omega_{\Lambda_{i\mathbf{R}}}|}/\sqrt{|dx|}$$

with a constant $c_{\Lambda_i^j}(u)$ depending only on u. Here:

$$(5.6) \quad 1) \quad P_{\Lambda_{i\mathbf{R}}} := P \circ \pi / \sigma^{m_{\Lambda_i}}|_{\Lambda_{i\mathbf{R}}^\circ},$$

$$2) \quad \omega_{\Lambda_{i\mathbf{R}}} := \frac{\pi^{-1}(dx) \wedge d\sigma}{\sigma^{\mu_{\Lambda_i}}} \Big/ d\sigma \Big|_{\Lambda_{i\mathbf{R}}^\circ},$$

and; 1) π is the projection map from W to $X_{\mathbf{R}}$ where W is the Zariski closure of the set $\{(x, y) \in X_{\mathbf{R}} \times X_{\mathbf{R}}^*;\, x = s \cdot \mathrm{grad}(\log P(y)),\, y \in X_{\mathbf{R}}^* - S_{\mathbf{R}},\, s \in \mathbb{C}\}$; 2) $\sigma := \langle x, y \rangle / 3$, (a function on W); 3) $dx := d\xi_1 \wedge d\xi_2 \wedge d\xi_3 \wedge dx_1 \wedge dx_2 \wedge dx_3$ where $dx_i = \bigwedge_{j=1}^8 dx_i^j$ with $x_i := (x_i^1 + x_i^2 u + x_i^3 v + x_i^4 w) + (x_i^5 + x_i^6 u + x_i^7 v + x_i^8 w)e \in \mathfrak{C}_{\mathbf{R}}^d$ or $\mathfrak{C}_{\mathbf{R}}^s$ ($x_i^j \in \mathbb{R}$ and $\{1, u, v, w\}$ is the basis of $\mathbb{H}^d$ or $\mathbb{H}^s$ defined in (3.1)); 4) the numbers m_{Λ_i} and μ_{Λ_i} are constants appearing in the expression of the order $-m_{\Lambda_i} s - (\mu_{\Lambda_i}/2)$ on $\Lambda_{i\mathbf{C}}$, namely $m_{\Lambda_i} = i$ and $\mu_{\Lambda_i} = 4i^2 - 3i$. As proved in [Sm-Ka-Ki-Os], $P_{\Lambda_{i\mathbf{R}}}(x, y)$ (resp. $\omega_{\Lambda_{i\mathbf{R}}}(x, y)$) is a real analytic function (resp. a real analytic volume form) on $\Lambda_{i\mathbf{R}}^\circ$ which does not vanish on $\Lambda_{i\mathbf{R}}^\circ$. The above argument is a consequence of fundamental calculations developed in [Sm-Ka-Ki-Os]. For details see [Sm-Ka-Ki-Os].

Let $f(x)$ be a hyperfunction solution of $\mathfrak{M}_s$ on $X_{\mathbf{R}}$. Then the real principal symbol of $sp(f(x))$ on Λ_i^j is written as:

$$\sigma_{\Lambda_i^j}(sp(f))=c_i^j(f)\cdot|P_{\Lambda_{i\mathrm{R}}}|^s\cdot\sqrt{|\omega_{\Lambda_{i\mathrm{R}}}|}/\sqrt{|dx|}, \tag{5.7}$$

with a constant $c_i^j(f)$ depending on f. We call the constant $c_i^j(f)$ the coefficient of $f(x)$ on Λ_i^j. Proposition 5.1 is rewritten in terms of the coefficients of microfunctions.

Proposition 5.2. *For two hyperfunction solutions $f(x)$ and $g(x)$, if the coefficient $c_i^j(sp(f))$ coincides with $c_i^j(sp(g))$ for each Λ_i^j in $\mathrm{ch}(\mathfrak{M})_{\mathrm{R}}$, then $f(x)=g(x)$ as hyperfunctions on X_{R}.*

As a result of this section, a hyperfunction solution f of $\mathfrak{M}_s$ is completely determined by its coefficients $\{c_i^j(f)\}$ over all the connected components of $\mathrm{ch}(\mathfrak{M}_s)_{\mathrm{reg}}:=$the regular locus of $\mathrm{ch}(\mathfrak{M}_s)$. In other words, a hyperfunction solution with a given set of coefficients is unique. However there may exist sets of coefficients to which no hyperfunction solutions correspond. For a given set of coefficients, there are some linear relations between the coefficients in order that it is a set of coefficients of a hyperfunction solution of $\mathfrak{M}_s$. In the next section we shall give their relations.

§ 6. A relation between real principal symbols on two Lagrangian varieties having an intersection of codimension one

We assume the same situation as in § 5. Let $f(x)$ be a hyperfunction solution. Since $\Lambda_{i\mathrm{C}}$ and $\Lambda_{i+1\mathrm{C}}$ have an intersection of codimension one, the microfunction solutions on $\Lambda_{i\mathrm{R}}^{\circ}$ and $\Lambda_{i+1\mathrm{R}}^{\circ}$ have linear relations. It was one of the main results in [Ka-Mi]. In [Mr 1], the author has given such solutions for relatively invariant microfunctions on other prehomogeneous vector spaces closely related to our case. It was presented as linear relation between the coefficients on $\Lambda_{i\mathrm{R}}^{\circ}$ and $\Lambda_{i+1\mathrm{R}}^{\circ}$. In our case, we have the following relations of the coefficients.

Proposition 6.1. *Let $f(x)$ be a hyperfunction solution of $\mathfrak{M}_s$. Then there are following linear relations among the coefficients of f on $\Lambda_{i\mathrm{R}}^{\circ}$ and $\Lambda_{i+1\mathrm{R}}^{\circ}$.*

i) *In the division case*:

$$\begin{bmatrix} c_{\Lambda_{i+1}^{j-1,k+1}}(f) \\ c_{\Lambda_{i+1}^{j-1,k}}(f) \end{bmatrix} = \frac{\Gamma(s+4i+1)}{\sqrt{2\pi}} \times \begin{bmatrix} \exp\left(-\frac{\pi}{2}\sqrt{-1}\,(s+1)\right), & \exp\left(\frac{\pi}{2}\sqrt{-1}\,(s+1)\right) \\ \exp\left(\frac{\pi}{2}\sqrt{-1}\,(s+1)\right), & \exp\left(-\frac{\pi}{2}\sqrt{-1}\,(s+1)\right) \end{bmatrix} \begin{bmatrix} c_{\Lambda_i^{j,k}}(f) \\ c_{\Lambda_i^{j-1,k}}(f) \end{bmatrix}, \tag{6.1}$$

provided that $s \neq -(4i+1), -(4i+1)-1, \cdots$. *For* $s=-(4i+1), -(4i+1)-1, \cdots$, *the inverse matrix of*

$$\frac{\Gamma(s+4i+1)}{\sqrt{2\pi}} \times \begin{bmatrix} \exp\left(-\frac{\pi}{2}\sqrt{-1}(s+1)\right), & \exp\left(\frac{\pi}{2}\sqrt{-1}(s+1)\right) \\ \exp\left(\frac{\pi}{2}\sqrt{-1}(s+1)\right), & \exp\left(-\frac{\pi}{2}\sqrt{-1}(s+1)\right) \end{bmatrix}$$

is *well defined and the relation is given by this inverse matrix.*

ii) *In the split case*:

$$\begin{bmatrix} c_{A_{i+1}^+}(f) \\ c_{A_{i+1}^-}(f) \end{bmatrix} = \frac{\Gamma(s+4i+1)}{\sqrt{2\pi}}$$

$$(6.2) \qquad \times \begin{bmatrix} \exp\left(-\frac{\pi}{2}\sqrt{-1}(s+1)\right), & \exp\left(\frac{\pi}{2}\sqrt{-1}(s+1)\right) \\ \exp\left(\frac{\pi}{2}\sqrt{-1}(s+1)\right), & \exp\left(-\frac{\pi}{2}\sqrt{-1}(s+1)\right) \end{bmatrix} \begin{bmatrix} c_{A_i^+}(f) \\ c_{A_i^-}(f) \end{bmatrix},$$

provided that $s \neq -(4i+1), -(4i+1)-1, \cdots$. *When* $s=-(4i+1), -(4i+1)-1, \cdots$, *the inverse matrix of*

$$\frac{\Gamma(s+4i+1)}{\sqrt{2\pi}} \times \begin{bmatrix} \exp\left(-\frac{\pi}{2}\sqrt{-1}(s+1)\right), & \exp\left(\frac{\pi}{2}\sqrt{-1}(s+1)\right) \\ \exp\left(\frac{\pi}{2}\sqrt{-1}(s+1)\right), & \exp\left(-\frac{\pi}{2}\sqrt{-1}(s+1)\right) \end{bmatrix}$$

is well defined and the relation of the coefficients are given by this inverse matrix.

Proof. As proved in [Ka-Mi], the relations between the coefficients are obtained from the normal relation matrix corrected by the Maslov indices of Lagrangian subvarieties and their intersections. See [Ka-Mi] p. 139, formula 3.5. In our case there is no effect of correction by the Maslov indices. Following the method presented by [Ka-Mi], we have:

i) In the division case.

$$\begin{bmatrix} c_{A_{i+1}^{j-1,k+1}}(f) \\ c_{A_{i+1}^{j-1,k}}(f) \end{bmatrix} = \frac{\Gamma(s+4i-3)}{\sqrt{2\pi}}$$

$$(6.3) \qquad \times \begin{bmatrix} \exp\left(-\frac{\pi}{2}\sqrt{-1}(s+4i+1)\right), & \exp\left(\frac{\pi}{2}\sqrt{-1}(s+4i+1)\right) \\ \exp\left(\frac{\pi}{2}\sqrt{-1}(s+4i+1)\right), & \exp\left(-\frac{\pi}{2}\sqrt{-1}(s+4i+1)\right) \end{bmatrix}$$

$$\times\begin{bmatrix}\exp(2\pi\sqrt{-1}(i-2k)), & 0 \\ 0 & , \exp(-2\pi\sqrt{-1}(i-2k))\end{bmatrix}$$

$$\times\begin{bmatrix}c_{A_i^{j,k}}(f) \\ c_{A_i^{j-1,k}}(f)\end{bmatrix}.$$

ii) In the split case:

$$\begin{bmatrix}c_{A_{i+1}^+}(f) \\ c_{A_{i+1}^-}(f)\end{bmatrix}=\frac{\Gamma(s+4i-3)}{\sqrt{2\pi}}$$

$$\times\begin{bmatrix}\exp\left(-\frac{\pi}{2}\sqrt{-1}(s+4i+1)\right), & \exp\left(\frac{\pi}{2}\sqrt{-1}(s+4i+1)\right) \\ \exp\left(\frac{\pi}{2}\sqrt{-1}(s+4i+1)\right), & \exp\left(-\frac{\pi}{2}\sqrt{-1}(s+4i+1)\right)\end{bmatrix} \tag{6.4}$$

$$\times\begin{bmatrix}\exp\left(\frac{\pi}{4}\sqrt{-1}\,0\right), & 0 \\ 0 & , \exp\left(\frac{\pi}{4}\sqrt{-1}\,0\right)\end{bmatrix}\begin{bmatrix}c_{A_i^+}(f) \\ c_{A_i^-}(f)\end{bmatrix}.$$

These relations (6.3) and (6.4) imply (6.1) and (6.2), respectively. q.e.d.

Remark. Conversely for a given set of the coefficients $\{c_{A_i^*}\}$ satisfying the relations (6.1) and (6.2), there exists a unique hyperfunction solution to $\mathfrak{M}_s$ whose coefficients $c_{A_i^*}(f)$ are the given ones. This fact is proved as a consequence derived from the main theorem in [Mr 3].

§ 7. Real principal symbols on the zero section and on the conormal bundle of the origin

In this section, we shall prove that the real principal symbol of a hyperfunction solution of $\mathfrak{M}_s$ has a specific expression especially on the zero-section and the conormal bundle of the origin. Moreover the real principal symbol on the conormal bundle is expressed in terms of the Fourier transform of $f(x)$. This enables us to reduce the calculation of the Fourier transform of $f(x)$ to the computation of the coefficients of $f(x)$ on the conormal bundle of the origin.

For a hyperfunction solution $f(x)$ of $\mathfrak{M}_s$, the real principal symbols of $f(x)$ on the zero section: $\Lambda_{i\mathrm{R}}^{\circ}:=(\boldsymbol{X}_{\mathrm{R}}-\boldsymbol{S}_{\mathrm{R}})\times\{0\}\subset\boldsymbol{X}_{\mathrm{R}}\times\boldsymbol{X}_{\mathrm{R}}^*$ (resp. the conormal bundle of the origin: $\Lambda_{3\mathrm{R}}^{\circ}:=\{0\}\times(\boldsymbol{X}_{\mathrm{R}}^*-\boldsymbol{S}_{\mathrm{R}})\subset\boldsymbol{X}_{\mathrm{R}}\times\boldsymbol{X}_{\mathrm{R}}^*$) have expression in terms of the coordinate system on $\boldsymbol{X}_{\mathrm{R}}$ (resp. $\boldsymbol{X}_{\mathrm{R}}^*$). In fact, let Λ_0^j ($\boldsymbol{j}\in\{(p,0);p=0,1,2,3\}$ in the division case and $\boldsymbol{j}\in\{+,-\}$ in the split case) be a connected component of Λ_{0R}° and let Λ_3^k ($k \in \{(0, q); q= 0, 1, 2, 3\}$ in the division case and $k \in \{+, -\}$ in the split case) be the connected component decomposition of Λ_{3R}°. Then we have:

(7.1) 1) $\sigma_{\Lambda_0^j}(sp(f(x))=c_0^j(f)\cdot|P(x)|_j^s$

2) $\sigma_{\Lambda_3^k}(sp(f(x))=c_3^k(f)\cdot|K_0|^s\cdot|K_1|^{1/2}\cdot|P(y)|_k^{-s-9}\sqrt{|dy|}/\sqrt{|dx|}$

where $c_0^j(f)$ (resp. $c_3^k(f)$) are the coefficients of $f(x)$ on Λ_0^j (resp. Λ_3^k) with respect to $|P_{\Lambda_{0R}}|^s\sqrt{|\omega_{\Lambda_{0R}}|}/\sqrt{|dx|}$ (resp. $|P_{\Lambda_{3R}}|^s\sqrt{|\omega_{\Lambda_{3R}}|}/\sqrt{|dx|}$) and the constants $|K_0|$ and $|K_1|$ are defined by:

$$|K_0|:=|P(y)\cdot P(\text{grad}(\log P(y)))|=1,$$
$$|K_1|:=|P(y)|^{-18}|\text{Hessian}(\log P(y))|=2^{24}.$$

The equalities (7.1) 1) and 2) are proved by a direct calculation applying the equation (5.6). Namely substituting $|P_{\Lambda_{0R}}(x, y)|:=|P(x)|$ and $|\omega_{\Lambda_{0R}}|:=|dx|$, we have (7.1), 1). For $P_{\Lambda_{3R}}$ and $\omega_{\Lambda_{3R}}$, we see that

$$\begin{aligned}|P_{\Lambda_{3R}}(x, y)|&=|P(s\cdot \text{glad}(\log P(y)))/s^3|\\ &=|P(y)P(\text{grad}(\log P(y)))\cdot P(y)^{-1}|=|K_0|\cdot|P(y)^{-1}|\\ &=|P(y)^{-1}|.\\ |\omega_{\Lambda_{3R}}(x, y)|&=d(s\cdot\text{grad}(\log P(y)))/s^{27}\\ &=|P(y)|^{18}|\text{Hessian}(\log P(y))|\cdot|P(y)|^{-18}\cdot|dy|\\ &=|K_1|\cdot|P(y)|^{-18}|dy|=2^{24}\cdot|P(y)|^{-18}|dy|.\end{aligned}$$

Thus by using (5.6), we get (7.1), 1) and 2).

Next we shall see that the real principal symbols of a solution $u(x)$ on the conormal bundle of the origin are written in terms of the Fourier transform of $u(x)$. For a tempered distribution $u(x)$ on X_R we put:

$$\text{(7.2)}\qquad \begin{aligned}u^{\vee}(y)&:=\int u(x)\cdot\exp(-2\pi\sqrt{-1}\langle x, y\rangle)dx\\ u^{\wedge}(x)&:=\int u(y)\cdot\exp(2\pi\sqrt{-1}\langle x, y\rangle)dy.\end{aligned}$$

Then we have:

$$\text{(7.3)}\qquad (u^{\vee})^{\wedge}(x)=(u^{\wedge})^{\vee}(x)=2^{24}\cdot u(x).$$

We can write the real principal symbol on the conormal bundle of the origin by using the Fourier transform. Namely we have:

Proposition 7.1. *Let $f(x)$ be a solution of a holonomic system $\mathfrak{M}_s$.*

Then the Fourier transform $f^{\vee}(y)$ is a real analytic function on $X_{\mathrm{R}}^{*}-S_{\mathrm{R}}$ and the real principal symbol on the conormal bundle of the origin is given by:

(7.4) $$\sigma_{\Lambda_3^k}(sp(f))=(2\pi)^{3s+(27/2)}\cdot 2^{24}\cdot f^{\vee}(y)|_{\Lambda_3^k}\sqrt{|dy|}/\sqrt{|dx|}.$$

Here we regard Λ_3^k as a subset of X_{R}^{*}.

Proof. By utilizing the same method as Proposition 3.5 in [Mr 3], we see that $f^{\vee}(y)$ is a solution to the holonomic system $\mathfrak{M}_{-s-9}$ when we identify X_{R} and X_{R}^{*}. Thus $f^{\vee}(y)$ is real analytic on $X_{\mathrm{R}}-S_{\mathrm{R}}$. The distribution $f(x)$ is χ^s-invariant. As stated in Proposition 4.2, any χ^s-invariant hyperfunction is written as

(7.5) $$\sum_{i\in I_0} a_i(r)\cdot|P(x)|_i^r|_{r=s}$$

where $a_i(r)$'s are holomorphic functions in $r\in\mathbb{C}$ defined near $r=s$. For the χ^s-invariant hyperfunction $|P(x)|_i^s$ $(i\in I_0)$ with $s\notin \mathrm{Cri}\,(P(x)^s)$, we may prove (7.4) in the same way as the proof of Proposition 3.6, Formula (3.35) in [Mr 3]. Thus for any function $f(x)$ written in the form (7.5), we may prove by analytic continuation that the principal symbol of $f(x)$ on Λ_3^k is expressed as (7.4). q.e.d.

Comparing (7.1), 2) and (7.4), we have:

Proposition 7.2. *Let $f(x)$ be a solution of the holonomic system $\mathfrak{M}_s$. Then the Fourier transform is given by*:

(7.6) $$f^{\vee}(y)|_{X_{\mathrm{R}}-S_{\mathrm{R}}}=(2\pi)^{-3s-(27/2)}\cdot 2^{-12}\cdot\sum_{k} c_3^k(f)\cdot|P(x)|^{-s-9}|_{\Lambda_3^k},$$

where $\boldsymbol{k}$ runs through the set $\{(0, q);\ q=0, 1, 2, 3\}$ in the division case or the set $\{+, -\}$ in the split case.

§ 8. The Fourier transforms of local zeta functions

In this section, we shall give the formula of the Fourier transform of the relatively invariant tempered distributions $|P(x)|_i^s$ explicitly. As proved in the preceding section, we have to compute the real principal symbol on the conormal bundle of the origin. More precisely, our problem is to compute the coefficients of $|P(x)|_i^s$ on the conormal bundle of the origin (Proposition 7.2).

Theorem 8.1. (The formula of the Fourier transform of $|P(x)|_i^s$)

1) (*In the division case*): *We have*:

(8.1)
$$\int |P(x)|_i^s \cdot \exp(-2\pi\sqrt{-1}\langle x, y\rangle)dx$$
$$=(2\pi)^{-3s-(27/2)} \cdot 2^{-12} \cdot \sum_{j=0}^{3} c_{ij}(s) \cdot |P(y)|_j^{-s-9}, \qquad (i=0, 1, 2, 3)$$

where $c_{ij}(s) := A_{i+1, j+1}$ *with the* 4×4 *matrix* $A := (A_{p,q})_{1\leq p,q\leq 4}$ *defined by*:

(8.2)
$$A=(2\pi)^{-(3/2)} \cdot \Gamma(s+1) \cdot \Gamma(s+5) \cdot \Gamma(s+9) \times (-\sqrt{-1})$$
$$\times \begin{bmatrix} -\exp\left(-\frac{3\pi}{2}\sqrt{-1}s\right), & 3\exp\left(\frac{\pi}{2}\sqrt{-1}s\right), & -3\exp\left(-\frac{\pi}{2}\sqrt{-1}s\right), & \exp\left(\frac{3\pi}{2}\sqrt{-1}s\right) \\ \exp\left(-\frac{\pi}{2}\sqrt{-1}s\right), & -\exp\left(-\frac{3\pi}{2}\sqrt{-1}s\right)-2\exp\left(\frac{\pi}{2}\sqrt{-1}s\right), & \exp\left(\frac{3\pi}{2}\sqrt{-1}s\right)+2\exp\left(-\frac{\pi}{2}\sqrt{-1}s\right), & -\exp\left(\frac{\pi}{2}\sqrt{-1}s\right) \\ -\exp\left(\frac{\pi}{2}\sqrt{-1}s\right), & \exp\left(\frac{3\pi}{2}\sqrt{-1}s\right)+2\exp\left(-\frac{\pi}{2}\sqrt{-1}s\right), & -\exp\left(-\frac{3\pi}{2}\sqrt{-1}s\right)-2\exp\left(\frac{\pi}{2}\sqrt{-1}s\right), & \exp\left(-\frac{\pi}{2}\sqrt{-1}s\right) \\ \exp\left(\frac{3\pi}{2}\sqrt{-1}s\right), & -3\exp\left(-\frac{\pi}{2}\sqrt{-1}s\right), & 3\exp\left(\frac{\pi}{2}\sqrt{-1}s\right), & -\exp\left(-\frac{3\pi}{2}\sqrt{-1}s\right) \end{bmatrix}.$$

2) (*In the split case*); *We have*:

(8.3)
$$\int |P(x)|_i^s \cdot \exp(-2\pi\sqrt{-1}\langle x, y\rangle)dx$$
$$=(2\pi)^{-3s-(27/2)} \cdot 2^{-12} \cdot \sum_{j\in\{+,-\}} c_{ij}(s) \cdot |P(y)|_j^{-s-9}, \qquad (i=+, -)$$

where

(8.4)
$$\begin{bmatrix} c_{++}(s), & c_{+-}(s) \\ c_{-+}(s), & c_{--}(s) \end{bmatrix} = (2\pi)^{-(3/2)} \cdot \Gamma(s+1) \cdot \Gamma(s+5) \cdot \Gamma(s+9) \times -\sqrt{-1}$$
$$\times \begin{bmatrix} -3\exp\left(\frac{\pi}{2}\sqrt{-1}s\right)-\exp\left(-\frac{3\pi}{2}\sqrt{-1}s\right), \\ 3\exp\left(-\frac{\pi}{2}\sqrt{-1}s\right)+\exp\left(\frac{3\pi}{2}\sqrt{-1}s\right), \end{bmatrix}$$

$$\left.\begin{array}{r} 3\exp\left(-\frac{\pi}{2}\sqrt{-1}s\right)+\exp\left(\frac{3\pi}{2}\sqrt{-1}s\right) \\ -3\exp\left(\frac{\pi}{2}\sqrt{-1}s\right)-\exp\left(-\frac{3\pi}{2}\sqrt{-1}s\right) \end{array}\right].$$

Proof. 1) First we suppose that s is not a negative integer. As we have seen in (7.1), the coefficients of $|P(x)|_i^s$ $(i=0, 1, 2, 3)$ on $\Lambda_{0\mathrm{R}}^{\circ}$ are given by:

$$c_{\Lambda_0^{p,0}}(|P(x)|_i^s)=\begin{cases}1 & \text{if } p=i,\\ 0 & \text{if } p\neq i.\end{cases}$$

Applying the relation (6.1), we can compute the coefficients of $|P(x)|_i^s$ on $\Lambda_{1\mathrm{R}}^{\circ}$, $\Lambda_{2\mathrm{R}}^{\circ}$ and $\Lambda_{3\mathrm{R}}^{\circ}$ explicitly. Especially the coefficient on $\Lambda_3^{0,j}$ is the coefficient $c_3^{(0,j)}(|P(x)|_i^s)$ appearing in (7.6). By Proposition 7.2, we get the formula of the Fourier transform, namely the coefficient $c_3^{(0,j)}(|P(x)|_i^s)$ is $c_{ij}(s)$ in (8.1). Thus, by computing the coefficients on $\Lambda_{0\mathrm{R}}^{\circ}$ explicitly, we have the Fourier transform (8.1) for every $s \in \mathbb{C}$ except when it is a negative integer. When s is a negative integer, we also get (8.1) by analytic continuation. After all we obtain (8.1) for all $s \in \mathbb{C}$.

2) In the case of 2), we may prove (8.3) in the same way. q.e.d.

§ 9. Invariant measures on singular orbits and their Fourier transforms

As proved in Proposition 3.2, the singular set $\boldsymbol{S}_{\mathrm{R}}$ consists of three real loci $\boldsymbol{S}_{1\mathrm{R}}$, $\boldsymbol{S}_{2\mathrm{R}}$ and $\boldsymbol{S}_{3\mathrm{R}}$. Each $\boldsymbol{S}_{i\mathrm{R}}$ decomposes into a finite number of $\boldsymbol{G}_{\mathrm{R}}^{+}$-orbits denoted by $\coprod_{j\in \boldsymbol{I}_i}\boldsymbol{S}_i^j$ $(i=1, 2, 3)$. Here, in the division case the index set is $\boldsymbol{I}_i:=\{0, 1, \cdots, 3-i\}$ and in the split case it is $\boldsymbol{I}_i:=\{0\}$. We have seen that each $\boldsymbol{G}_{\mathrm{R}}^{+}$-orbit $\boldsymbol{S}_i^j$ is a $\boldsymbol{G}_{\mathrm{R}}^{1}$-orbit in Proposition 3.2 3). By investigating the action of Lie algebra of $\boldsymbol{G}_{\mathrm{R}}^{1}$ on $\boldsymbol{S}_i^j$, we see that each $\boldsymbol{S}_i^j$ admit a non-trivial $\boldsymbol{G}_{\mathrm{R}}^{1}$-invariant measure, which is uniquely determined up to a constant multiple. We denote it by $d\nu_i^j$.

Proposition 9.1. *Let $\mathcal{S}(\boldsymbol{X}_{\mathrm{R}})$ be the space of rapidly decreasing functions on $\boldsymbol{X}_{\mathrm{R}}$ and let $d\nu_i^j$ $(i=1, 2, 3,$ and $j \in \boldsymbol{I}_i)$ be a non-trivial $\boldsymbol{G}_{\mathrm{R}}^{1}$-invariant measure on $\boldsymbol{S}_i^j$. For any element $f(x)$ in $\mathcal{S}(\boldsymbol{X}_{\mathrm{R}})$, the integral $\int f(x)d\nu_i^j(x)$ is absolutely convergent and the functional,*

$$f\longmapsto\int f(x)d\nu_i^j(x), \tag{9.1}$$

defines a tempered distribution on $\boldsymbol{X}_{\mathrm{R}}$ supported on $\boldsymbol{S}_i^j$.

Proof. If $i=3$, $d\nu_i^j(x)$ is a measure supported on the origin $\{0\}$. Then (9.1) is apparently defines a tempered distribution on $\boldsymbol{X}_{\mathbb{R}}$. So we confine ourselves to the cases $i=1$ and $i=2$. Note that $d\nu_i^j(x)$ is a $\boldsymbol{G}_{\mathbb{R}}^1$-invariant measure on the orbit $\boldsymbol{S}_i^j$ and $\boldsymbol{S}_i^j$ is stable under the multiplication operation of positive numbers. From the definition of $d\nu_i^j(x)$, we see that it is homogeneous of degree $36-12i$, i.e., $d\nu_i^j(r\cdot x)=r^{36-12i}d\nu_i^j(x)$ for any $r>0$. Thus the integral $\int f(x)d\nu_i^j(x)$ is written in terms of the polar coordinate $(r, \omega)\in\mathbb{R}_+\times\boldsymbol{S}^{(26)}$ where $\mathbb{R}_+ :=\{r\in\mathbb{R}; r\geq 0\}$ and $\boldsymbol{S}^{(26)}$ is a 26-dimensional sphere. The integral $\int f(x)d\nu_i^j(x)$ is written as

$$\int_{\boldsymbol{X}_{\mathbb{R}}} f(x)d\nu_i^j(x)=\int_{\mathbb{R}_+\times\boldsymbol{S}^{(26)}} f(r\cdot\omega)\cdot r^{35-12i}dr\cdot d\tilde{\nu}_i^j(\omega), \tag{9.2}$$

where $d\tilde{\nu}_i^j(\omega)$ is the restriction of $d\nu_i^j(x)$ on the unit sphere in $\boldsymbol{X}_{\mathbb{R}}$ and dr is an Euclidean measure on $\mathbb{R}_+$. When $f(x)\in\mathscr{S}(\boldsymbol{X}_{\mathbb{R}})$, the integral

$$\int_{\boldsymbol{S}^{(26)}} f(r\cdot\omega)d\tilde{\nu}_i^j(\omega)$$

is a continuous bounded function on $\mathbb{R}_+$, hence the left hand side is absolutely integrable. It is a routine work to show that (9.1) is a continuous linear functional on $\mathscr{S}(\boldsymbol{X}_{\mathbb{R}})$. q.e.d.

As proved in Proposition 9.1, (9.1) defines a homogeneous distribution of degree $36-12i$ supported on $\boldsymbol{S}_i^j$. We may denote it by $T_i^j(x)$, i.e.,

$$\int f(x)d\nu_i^j(x)=\int T_i^j(x)f(x)dx \qquad \text{for any } f(x)\in\mathscr{S}(\boldsymbol{X}_{\mathbb{R}}).$$

Since $d\nu_i^j(x)$ is $\boldsymbol{G}_{\mathbb{R}}^1$-invariant and homogeneous of degree $36-12i$, we have

$$\int f(\rho(g)\cdot x)\cdot T_i^j(x)dx=\chi(g)^{4i-12}\int f(x)T_i^j(x)dx,$$

which means that

$$T_i^j(\rho(g)\cdot x)=\chi(g)^{-(4i-3)}\cdot T_i^j(x), \tag{9.3}$$

for all $g\in\boldsymbol{G}_{\mathbb{R}}^+$. Thus the distribution defined by (9.1) is a $\chi(g)^{-(4i-3)}$-invariant tempered distribution $T_i^j(x)$ supported on $\boldsymbol{S}_i^j$.

The problem we want to solve in this section is to calculate explicitly the image of the Fourier transform of $T_i^j(x)$ on $\boldsymbol{X}_{\mathbb{R}}^*-\boldsymbol{S}_{\mathbb{R}}$. We normalize $d\nu_i^j$ suitably and compute the Fourier transform on $\boldsymbol{X}_{\mathbb{R}}^*-\boldsymbol{S}_{\mathbb{R}}$ in the same manner as in § 8.

(In the division case)

Let $d\nu_i^j$ be a $G_{\mathbb{R}}^1$-invariant measure on S_i^j. We shall normalize $d\nu_i^j$ in the following way. We put

$$x_i^j := \begin{bmatrix} I_{3-i}^{(j)} & \\ & 0_i \end{bmatrix},$$

a point in S_i^j. We identify the tangent space $(TX_{\mathbb{R}})_{x_i^j}$ of $X_{\mathbb{R}}$ at x_i^j with $X_{\mathbb{R}}$. We introduce a coordinate on $X_{\mathbb{R}}$ by

$$X = \begin{bmatrix} \xi_1, & x_3, & \bar{x}_2 \\ \bar{x}_3, & \xi_2, & x_1 \\ x_2, & \bar{x}_1, & \xi_3 \end{bmatrix}, \tag{9.4}$$

with $\xi_p \in \mathbb{R}$ and $x_q := (x_p^1 + x_p^2 u + x_p^3 v + x_p^4 w) + (x_p^5 + x_p^6 u + x_p^7 v + x_p^8 w)e \in \mathfrak{C}_{\mathbb{R}}^d$. Then the tangent space of S_i^j at x_i^j is a subspace of $(TX_{\mathbb{R}})_{x_i^j}$, which is given by:

(9.5) 1) $(TS_1^j)_{x_1^j} = \{X \in X_{\mathbb{R}};\ \xi_3 = 0\}$ $(j=0, 1, 2)$,

2) $(TS_2^j)_{x_2^j} = \{X \in X_{\mathbb{R}};\ \xi_2 = \xi_3 = 0,\ x_1 = 0\}$ $(j=0, 1)$,

3) $(TS_3^j)_{x_3^j} = \{X \in X_{\mathbb{R}};\ \xi_1 = \xi_2 = \xi_3 = 0,\ x_1 = x_2 = x_3 = 0\}$ $(j=0)$.

Now we normalize $d\nu_i^j$ so that

$$d\nu_i^j|_{x_i^j} = \begin{cases} |d\xi_1 \wedge d\xi_2 \wedge dx_1 \wedge dx_2 \wedge dx_3| & i=1 \\ |d\xi_1 \wedge dx_2 \wedge dx_3| & i=2 \\ 1 & i=3 \end{cases} \tag{9.6}$$

where $dx_p := \bigwedge_{q=1}^{8} dx_p^q$.

Definition (In the division case). Let $T_i^j(x)$ be the tempered distribution defined by the linear functional (9.1) where $d\nu_i^j$ is normalized by (9.6).

Theorem 9.2 (In the division case). *The Fourier transform of the tempered distribution $T_i^j(x)$ is given by*:

$$\int T_i^j(x) \exp(-2\pi\sqrt{-1}\langle x, y\rangle)dx|_{X_{\mathbb{R}}^* - S_{\mathbb{R}}} = \sum_{k=0}^{3} b_{ik}^j \cdot |P(x)|^{-4i-12}|_{O_k}. \tag{9.8}$$

where b_{ik}^j's are:

$$b^2_{1k} := (2\pi)^{-12}\cdot 2^{-12}\cdot \Gamma(4)\Gamma(8),$$
$$b^1_{1k} := (2\pi)^{-12}\cdot 2^{-11}\cdot \Gamma(4)\Gamma(8),$$
$$b^0_{1k} := (2\pi)^{-12}\cdot 2^{-12}\cdot \Gamma(4)\Gamma(8),$$
$$b^1_{2k} := (2\pi)^{-4}\cdot 2^{-12}\cdot \Gamma(4),$$
$$b^0_{2k} := (2\pi)^{-4}\cdot 2^{-12}\cdot \Gamma(4),$$
$$b^0_{3k} := 2^{-12}.$$

Proof. The coefficients of the solution $T^j_1(x)$ on $\Lambda^\circ_{0\mathrm{R}}$ and on $\Lambda^\circ_{1\mathrm{R}}$ are from the definition

$$c_{\Lambda^p_0,0}(T^j_1(x)) = 0 \qquad \text{for all } p = 0, 1, 2, 3.$$
$$c_{\Lambda^j_1,q}(T^j_1(x)) = (2\pi)^{-(1/2)} \qquad \text{for all } q = 0, 1.$$
$$c_{\Lambda^k_1,q}(T^j_1(x)) = 0 \qquad \text{for } k \neq j \text{ and for all } q = 0, 1.$$

By using the relation of the coefficients (6.1) for $s = -1$, we can easily compute the coefficients of $T^j_1(x)$ on $\Lambda^\circ_{3\mathrm{R}}$ from the data of coefficients on $\Lambda^\circ_{1\mathrm{R}}$. Namely we have

$$c_{\Lambda^0_3,k}(T^j_1(x)) = (2\pi)^{-1}\Gamma(4)\Gamma(8)(2\pi)^{-(1/2)} \qquad \text{for all } k.$$

Similarly, for $T^j_2(x)$ $(j = 0, 1)$, the coefficients of $T^j_2(x)$ on $\Lambda^\circ_{0\mathrm{R}}$ and on $\Lambda^\circ_{1\mathrm{R}}$ are zero, $c_{\Lambda^j_2,q}(T^j_2(x)) = (2\pi)^{-5}$ $(q = 0, 1, 2)$ and $c_{\Lambda^k_2,q}(T^j_2(x)) = 0$ for $k \neq j$ $(q = 0, 1, 2)$. Calculating the coefficients by using the relation matrix of the coefficients (6.1) for $s = -5$, we have

$$c_{\Lambda^0_3,k}(T^j_2(x)) = (2\pi)^{-(1/2)}\Gamma(4)(2\pi)^{-5} \qquad \text{for all } k.$$

Thus, by Proposition 7.2, we have the above formula (9.8) of the Fourier transform.

For $T^0_3(x)$, the coefficients of $T^0_3(x)$ on $\Lambda^\circ_{0\mathrm{R}}$, on $\Lambda^\circ_{1\mathrm{R}}$ and on $\Lambda^\circ_{2\mathrm{R}}$ are zero. On $\Lambda^\circ_{3\mathrm{R}}$, we see that

$$c_{\Lambda^0_3,k}(T^0_3(x)) = (2\pi)^{-(27/2)} \qquad \text{for all } k.$$

Thus, by Proposition 7.2, we have the above formula (9.8) of the Fourier transform of $T^0_3(x)$. q.e.d.

(In the split case)

Let $d\nu^0_i$ be a $\boldsymbol{G}^1_{\mathrm{R}}$-invariant measure on $\boldsymbol{S}^0_i$. We shall normalize $d\nu^0_i$ in the following way. We put

$$x^0_i := \begin{bmatrix} I_{3-i} & \\ & 0_i \end{bmatrix},$$

a point in $\boldsymbol{S}_i^0$. We identify the tangent space $(T\boldsymbol{X}_{\mathrm{R}})_{x_i^0}$ of $\boldsymbol{X}_{\mathrm{R}}$ at x_i^0 with $\boldsymbol{X}_{\mathrm{R}}$. We introduce a coordinate on $\boldsymbol{X}_{\mathrm{R}}$ on by (9.4) where each x_p is an element in the split Cayley algebra $\mathfrak{C}_{\mathrm{R}}^s$. Then the tangent space of $\boldsymbol{S}_i^0$ at x_i^0 is a subspace of $(T\boldsymbol{X}_{\mathrm{R}})_{x_i^0}$, which is given by

(9.9) 1) $(T\boldsymbol{S}_1^0)_{x_1^0}=\{X\in \boldsymbol{X}_{\mathrm{R}};\ \xi_3=0\}$.

2) $(T\boldsymbol{S}_2^0)_{x_2^0}=\{X\in \boldsymbol{X}_{\mathrm{R}};\ \xi_2=\xi_3=0,\ x_1=0\}$.

3) $(T\boldsymbol{S}_3^0)_{x_3^0}=\{X\in \boldsymbol{X}_{\mathrm{R}};\ \xi_1=\xi_2=\xi_3=0,\ x_1=x_2=x_3=0\}$.

We normalize $d\nu_i^0$ so that

$$(9.10)\qquad d\nu_i^0|_{x_i^0}=\begin{cases}|d\xi_1\wedge d\xi_2\wedge dx_1\wedge dx_2\wedge dx_3| & i=1\\ |d\xi_1\wedge dx_2\wedge dx_3| & i=2\\ \quad 1 & i=3\end{cases}$$

where $dx_p:=\bigwedge_{q=1}^{8}dx_p^q$.

Definition (In the split case). Let $T_i^0(x)$ be the tempered distribution defined by the linear functional (9.1) where $d\nu_i^0$ is normalized by (9.10).

Theorem 9.3 (In the split case). *The Fourier transform of the tempered distribution $T_i^0(x)$ is given by*:

$$(9.8)\qquad \begin{aligned}&\int T_i^0(x)\exp(-2\pi\sqrt{-1}\langle x,y\rangle)dx|_{X_{\mathrm{R}}^*-S_{\mathrm{R}}}\\ &=\sum_{\varepsilon=\pm} b_{ik}^0\cdot|P(x)|^{-4i-12}|_{O_k}.\end{aligned}$$

where b_{ik}^0's are:

$$b_{1k}^0:=(2\pi)^{-12}\cdot 2^{-10}\cdot\Gamma(4)\Gamma(8),$$
$$b_{2k}^0:=(2\pi)^{-4}\cdot 2^{-11}\cdot\Gamma(4),$$
$$b_{3k}^0:=2^{-12}.$$

The proof of this theorem is the same as that of Theorem 9.2.

Chapter II. Zeta functions associated with the exceptional group of type E_6

In this chapter we shall give a definition of zeta functions $\xi_i(\Gamma, L, s)$ and $\xi_j^*(\Gamma, L^*, s)$ $(i, j\in \boldsymbol{I}_0)$ which are defined for a given discrete subgroup in $\boldsymbol{G}_{\mathrm{R}}^+$ and a $\mathbb{Z}$-lattice in $\boldsymbol{X}_{\mathrm{R}}$. They are combined with one another through a functional equation. They have a finite number of simple poles. Our purpose is to give them explicitly. After defining the zeta functions as Dirichlet series, we will prove the possibility of their analytic continuation and compute the functional equation explicitly in § 10. In § 11, we will show that computation of their residues is reduced to calculation of the Fourier transforms of invariant measures on singular orbits and we will give their values.

§ 10. Zeta functions and functional equations

Let $(G_{\mathbb{R}}^{+}, \rho, X_{\mathbb{R}})$ be one of the real forms of the complex prehomogeneous vector space $(G_{\mathbb{C}}, \rho, X_{\mathbb{C}})=(GL_1(\mathbb{C})\times E_{6\mathbb{C}}, \rho, \mathbf{Her}_3(\mathfrak{C}_{\mathbb{C}}))$. One is 1) the division case and the other is 2) the split case in (3.3). Let $\coprod_{i\in I_0} O_i$ be the connected component decomposition of $X_{\mathbb{R}}-S_{\mathbb{R}}$, where the index set $I_0:=\{0, 1, 2, 3\}$ in the division case and $I_0:=\{+, -\}$ in the split case (Proposition 3.2). Let $X_{\mathbb{Q}}$ be a $\mathbb{Q}$-vector space in $X_{\mathbb{R}}$ such that $\mathbb{R}\otimes_{\mathbb{Q}} X_{\mathbb{Q}}=X_{\mathbb{R}}$. We put $G_{\mathbb{Q}}^1:=\{g\in G_{\mathbb{R}}^1;\ \rho(g)\in GL(X_{\mathbb{Q}})\}$. It is easily checked that $\rho^*(G_{\mathbb{Q}}^1)=\rho(G_{\mathbb{Q}}^1)$ where ρ^* is the contragredient representation of ρ with respect to the inner product $\langle x, y\rangle=\mathrm{tr}(x\cdot y)$.

Let L be a $\mathbb{Z}$-lattice in $X_{\mathbb{Q}}$ and let L^* be its dual lattice. Take a subgroup Γ in $G_{\mathbb{Q}}^1$ such that L is *Γ-stable*, i.e. $\rho(\Gamma)\cdot L=L$. Then L^* is also a Γ-stable lattice, i.e. $\rho^*(\Gamma)\cdot L^*=L^*$. For a function $f\in\mathcal{S}(X_{\mathbb{R}})$ and for a Γ-stable subset M in L (resp. L^*), we put:

$$(10.1)\qquad Z(f, M, s):=\int_{G_{\mathbb{R}}^{+}/\Gamma}|\chi(g)|^{s}\cdot\sum_{x\in M}f(\rho(g)\cdot x)dg,$$

$$\left(\text{resp. } Z^*(f, M, s):=\int_{G_{\mathbb{R}}^{+}/\Gamma}|\chi(g)|^{-s}\cdot\sum_{y\in M}f(\rho^*(g)\cdot y)dg,\right)$$

where dg is a non-trivial Haar measure on $G_{\mathbb{R}}^{+}$. In particular, when we put $L_i:=L\cap O_i$ and $L_i^*:=L^*\cap O_i$, they are Γ-stable subsets. We suppose that:

(10.2) *If the real part* $\mathrm{Re}(s)$ *is sufficiently large, then* $Z(f, L_i, s)$ *and* $Z^*(f, L_i^*, s)$ *are absolutely convergent for any* $f\in\mathcal{S}(X_{\mathbb{R}})$ *and for any* $i\in I_0$.

Then we may express $Z(f, L_i, s)$ and $Z^*(f, L_i^*, s)$ by:

$$(10.3)\qquad Z(f, L_i, s):=\xi_i(\Gamma, L, s)\cdot\int f(x)\cdot|P(x)|_i^{s-9}\,dx,$$

$$\left(\text{resp. } Z^*(f, L_i^*, s):=\xi_i^*(\Gamma, L^*, s)\cdot\int f(x)\cdot|P(x)|_i^{s-9}\,dx\right)$$

with

(10.4) $$\xi_i(\Gamma, L, s) := \sum_{[x]\in\Gamma\backslash L_i} \left(\int_{G^1_{\mathbf{R}x}/\Gamma_x} d\nu_x\right)\cdot|P(x)|^{-s}.$$

$$\left(\text{resp. } \xi^*_i(\Gamma, L^*, s) := \sum_{[x]\in\Gamma\backslash L^*_i} \left(\int_{G^1_{\mathbf{R}x}/\Gamma_x} d\nu^*_x\right)\cdot|P(x)|^{-s}\right).$$

Here $G^+_{\mathbf{R}x}$ is the isotropy subgroup of $G^+_{\mathbf{R}}$ at x; Γ_x is the isotropy subgroup of Γ at x; $d\nu_x$ (resp. $d\nu^*_x$) is the left invariant measure on $G^+_{\mathbf{R}x}$ normalized so that (10.3) is valid.

The absolute convergence of $Z(f, L_i, s)$ (resp, $Z^*(f, L^*_i, s)$) guarantees that $\xi_i(\Gamma, L, s)$ (resp. $\xi_i(\Gamma, L^*, s)$) is absolutely convergent when $\mathrm{Re}(s)$ is sufficiently large. We call $\xi_i(\Gamma, L, s)$ and $\xi^*_i(\Gamma, L^*, s)$ zeta functions associated with the exceptional Lie group of type E_6.

Under the convergence condition (10.2), we may prove that $\xi_i(\Gamma, L, s)$ and $\xi^*_i(\Gamma, L^*, s)$ are continued to the whole complex plane as a meromorphic function with finitely many poles and they are combined with one another by functional equations. We shall explain it briefly following Sato-Shintani [Sm-Sh].

Recall Poisson's summation formula:

(10.5) $$\sum_{x\in L} f(x) = v(L)^{-1}\cdot \sum_{y\in L^*} f^\vee(y),$$

for any $f(x) \in \mathscr{S}(X_{\mathbf{R}})$ with

$$v(L) := \int_{L\backslash X_{\mathbf{R}}} dx.$$

Here

$$f^\vee(y) = \int_{X_{\mathbf{R}}} f(x)\exp(-2\pi\sqrt{-1}\langle x, y\rangle)dx.$$

We may put $f(x)$ belonging to $C^\infty_0(\boldsymbol{O}_i)$:= the space of compactly supported functions on $\boldsymbol{O}_i$. Then for $g \in G^+_{\mathbf{R}}$, $f(\rho(g)\cdot x)$ is also in $C^\infty_0(\boldsymbol{O}_i)$. Substituting $f(\rho(g)\cdot x)$ in (10.5), we have:

(10.6) $$\sum_{x\in L} f(\rho(g)\cdot x) = v(L)^{-1}\cdot \sum_{y\in L^*} f^\vee(\rho^*(g)\cdot y)\cdot \chi(g)^{-9}.$$

Integrating the left hand side of (10.6) on $G^+_{\mathbf{R}}/\Gamma$ with respect to $\chi(g)^s dg$, we see that the integral is absolutely convergent and coincides with $Z(f, L, s)$ if $\mathrm{Re}(s)$ is sufficiently large. In particular, since $f(x) \in C^\infty_0(\boldsymbol{O}_i)$, $Z(f, L, s)$ is actually

$$\xi_i(\Gamma, L, s)\int f(x)\cdot|P(x)|^{s-9}_i\, dx.$$

We divide the integral $Z(f, L, s)$ into the sum of the two integrals $Z^+(f, L, s)$ and $Z^-(f, L, s)$, i.e., $Z(f, L, s)=Z^+(f, L, s)+Z^-(f, L, s)$, with

$$Z^+(f, L, s) := \int_{G_{\mathbf{R}}/\Gamma, \chi(g)\geq 1} \sum_{x\in L} f(\rho(g)\cdot x)\cdot \chi(g)^s dg$$

and

$$Z^-(f, L, s) := \int_{G_{\mathbf{R}}/\Gamma, \chi(g)<1} \sum_{x\in L} f(\rho(g)\cdot x)\cdot \chi(g)^s dg.$$

Then $Z^+(f, L, s)$ is absolutely convergent for any $s \in \mathbb{C}$ and defines an entire function. The next part $Z^-(f, L_i, s)$ is written by using (10.6) as

$$\begin{aligned} Z^-(f, L, s) &= v(L)^{-1}\cdot\int_{G^+_{\mathbf{R}}/\Gamma, \chi(g)<1} \sum_{y\in L^*} f^\vee(\rho^*(g)\cdot y)\cdot \chi(g)^{s-9} dg \\ &= v(L)^{-1}\cdot\int_{G^+_{\mathbf{R}}/\Gamma, \chi(g)<1} \sum_{y\in L^*-S_{\mathbf{R}}} f^\vee(\rho^*(g)\cdot y)\cdot \chi(g)^{s-9} dg \quad (1) \\ &\quad + v(L)^{-1}\cdot\int_{G^+_{\mathbf{R}}/\Gamma, \chi(g)<1} \sum_{y\in L^*\cap S_{\mathbf{R}}} f^\vee(\rho^*(g)\cdot y)\cdot \chi(g)^{s-9} dg \quad (2) \end{aligned} \tag{10.7}$$

It is easily checked that the first term (1) is absolutely convergent for any $s \in \mathbb{C}$ and defines an entire function.

We substitute $P(\partial/\partial x)f(x)$ for $f(x)$. Then the Fourier transform $(P(\partial/\partial x)f(x))^\vee = P(y)f^\vee(y)$ vanishes on $S_{\mathbf{R}}$. Thus the second term (2) in the calculation of $Z^-(P(\partial/\partial x)\cdot f(x), L, s)$ of (10.7) vanishes and hence $Z^-(P(\partial/\partial x)\cdot f(x), L, s)$ is an entire function. This yields that $Z(P(\partial/\partial x)\cdot f(x), L, s)$ is continued to an entire function. Then

$$\begin{aligned} Z\left(P\left(\frac{\partial}{\partial x}\right)f(x), L, s\right) &= \xi_i(\Gamma, L, s)\cdot\int P\left(\frac{\partial}{\partial x}\right)f(x)\cdot|P(x)|_i^{s-9}\,dx \\ &= \xi_i(\Gamma, L, s)\cdot\int (-1)^3\cdot f(x)\cdot P\left(\frac{\partial}{\partial x}\right)|P(x)|_i^{s-9}\,dx \\ &= \text{const.}\cdot\xi_i(\Gamma, L, s)\cdot b(s-10)\cdot\int f(x)\cdot|P(x)|_i^{s-10}\,dx, \end{aligned}$$

with $b(s)=(s+1)(s+5)(s+9)$. Since $\int f(x)\cdot|P(x)|_i^{s-10}\,dx$ is entire for any $f(x) \in C_0^\infty(\boldsymbol{O}_i)$, $\xi_i(\Gamma, L, s)$ is continued to the whole complex plane as a meromorphic function. Furthermore, for any $s \in \mathbb{C}$, we may choose $f(x)$ so that $\int f(x)\cdot|P(x)|_i^{s-10}\,dx$ does not vanish. Thus $\xi_i(\Gamma, L, s)\cdot b(s-10)$ is an entire function. For the integral $Z^*(f, L^*, s)$, the same argument is possible and we can prove that $\xi_i^*(\Gamma, L^*, s)\cdot b(s-10)$ is entire.

Next we shall compute the functional equation combining $\xi_i(\Gamma, L, s)$ with $\xi_i^*(\Gamma, L^*, s)$. For $f(x) \in C_0^\infty(\boldsymbol{O}_i)$,

$$\begin{aligned}
&\xi_i(\Gamma, L, s)\cdot\int P\Big(\frac{\partial}{\partial x}\Big) f(x)\cdot|P(x)|_i^{s-9}\,dx \\
(10.8)\qquad &=Z\Big(P\Big(\frac{\partial}{\partial x}\Big)f, L, s\Big)=Z^+\Big(P\Big(\frac{\partial}{\partial x}\Big)f, L, s\Big) \\
&\quad+Z^-\Big(P\Big(\frac{\partial}{\partial x}\Big)f, L, s\Big).
\end{aligned}$$

We have seen that both $Z^+(P(\partial/\partial x)f, L, s)$ and $Z^-(P(\partial/\partial x)f, L, s)$ are entire functions. By using Poisson's summation formula (10.6), we have:

$$\begin{aligned}
(10.9)\quad 1)\quad &Z^+\Big(P\Big(\frac{\partial}{\partial x}\Big)f, L, s\Big) \\
&=v(L)^{-1}\cdot\int_{G_{\mathbf{R}}^+/\Gamma,\, \chi(g)\geq 1}\ \sum_{y\in L^*}(P\cdot f^\vee)(\rho^*(g)\cdot y)\cdot\chi(g)^{s-9}dg \\
2)\quad &Z^-\Big(P\Big(\frac{\partial}{\partial x}\Big)f, L, s\Big) \\
&=v(L)^{-1}\cdot\int_{G_{\mathbf{R}}^+/\Gamma,\, \chi(g)<1}\ \sum_{y\in L^*}(P\cdot f^\vee)(\rho^*(g)\cdot y)\cdot\chi(g)^{s-9}dg.
\end{aligned}$$

The integrals in the right hand sides of (10.9), 1) and 2) are not absolutely convergent for $\mathrm{Re}(s)\gg 0$. However if $\mathrm{Re}(s)$ is sufficiently small, both of them are absolutely convergent and the sum of the two is

$$\begin{aligned}
&v(L)^{-1}\cdot\int_{G_{\mathbf{R}}^+/\Gamma}\ \sum_{y\in L^*}(P\cdot f^\vee)(\rho^*(g)\cdot y)\cdot\chi(g)^{s-9}dg \\
&=v(L)^{-1}\cdot Z^*(P\cdot f^\vee, L^*, 9-s) \\
(10.10)\qquad &=v(L)^{-1}\cdot\sum_{j\in \boldsymbol{I}_0}\xi_j^*(\Gamma, L^*, 9-s)\cdot\int(P\cdot f^\vee)(x)\cdot|P(x)|_j^{-s}\,dx \\
&=v(L)^{-1}\cdot\sum_{j\in \boldsymbol{I}_0}\xi_j^*(\Gamma, L^*, 9-s)\cdot\int P\Big(\frac{\partial}{\partial y}\Big)f(y)\cdot(|P(x)|_j^{-s})^\vee dy.
\end{aligned}$$

By using the formula of the Fourier transform of $|P(x)|_j^{-s}$:

$$\begin{aligned}
(|P(x)|_j^{-s})^\vee &:=\int\exp(-2\pi\sqrt{-1}\langle x, y\rangle)\cdot|P(y)|_j^{-s}\,dy \\
&=\sum_{i\in \boldsymbol{I}_0}d_{ji}(-s)\cdot|P(y)|_i^{s-9},
\end{aligned}$$

we have:

$$(10.10)=v(L)^{-1}\cdot\sum_{j\in I_0}\xi_j^*(\Gamma, L^*, 9-s)\cdot d_{ji}(-s)\int P\left(\frac{\partial}{\partial y}\right)f(y)\cdot|P(y)|_i^{s-9}\,dy.$$

Comparing this formula with (10.8), we have

$$(10.11)\qquad \xi_i(\Gamma, L, s)=v(L)^{-1}\cdot\sum_{j\in I_0}\xi_j^*(\Gamma, L^*, 9-s)\cdot d_{ji}(-s),$$

which is the functional equation we want to compute. The explicit form of $d_{ji}(-s)$ is given in § 8, Theorem 8.1:

$$d_{ij}(s)=(2\pi)^{-3s-(27/2)}\cdot 2^{-12}\cdot c_{ij}(s).$$

After all we have obtained the following theorem.

Theorem 10.1. *For a lattice L in $X_{\mathbf{Q}}$ and for a subgroup Γ in $G_{\mathbf{Q}}$ satisfying $\rho(\Gamma)\cdot L\subset L$, we suppose the convergence condition* (10.2). *Then the Dirichlet series $\xi_i(\Gamma, L, s)$ and $\xi_i^*(\Gamma, L^*, s)$ $(i\in I_0)$ are absolutely convergent if the real part* Re (s) *is sufficiently large. They are continued to the whole complex plane as meromorphic functions with a finite number of poles. They have the functional equation* (10.11).

Remark. The convergence condition (10.2) is always satisfied in the division case. In fact, Weil [We] gave a sufficient condition for absolute convergence of integrals of the form $Z(f, L, s)$ and Igusa [Ig] actually proved the convergence of $Z(f, L_i, s)$ in the division case by checking the Weil's condition.

§ 11. Computations of residues

In this section we shall explain how we calculate the residues of $\xi_k(\Gamma, L, s)$. The residues of $\xi_k^*(\Gamma, L^*, s)$ are computed in the same way. As proved in the preceding section,

$$\begin{aligned}
&\xi_k(\Gamma, L, s)\cdot\int f(x)\cdot|P(x)|_k^{s-9}\,dx\\
(11.1)\qquad &=Z^+(f, L, s) &(1)\\
&+v(L)^{-1}\cdot\int_{G_{\mathbf{R}}^+/\Gamma,\chi(g)<1}\sum_{y\in L^*-S_{\mathbf{R}}} f^\vee(\rho^*(g)\cdot y)\cdot\chi(g)^{s-9}dg &(2)\\
&+v(L)^{-1}\cdot\int_{G_{\mathbf{R}}^+/\Gamma,\chi(g)<1}\sum_{y\in L^*\cap S_{\mathbf{R}}} f^\vee(\rho^*(g)\cdot y)\cdot\chi(g)^{s-9}dg &(3)
\end{aligned}$$

for $f(x) \in C_0^\infty(\boldsymbol{O}_k)$ and $s \in \mathbb{C}$ with sufficiently large real part. We have seen that the terms (1) and (2) in (11.1) are entire functions for any $f(x) \in C_0^\infty(\boldsymbol{O}_k)$. Then the poles of $\xi_k(\Gamma, L, s)$ appear in the third term (3) if they exist.

We shall compute the term (3) more precisely. Since Γ is a subgroup of $\boldsymbol{G}_{\mathbb{R}}^1$, we may divide the integral on $\boldsymbol{G}_{\mathbb{R}}^+/\Gamma$ into the product space $\boldsymbol{GL}_1(\mathbb{R})^+ \times (\boldsymbol{G}_{\mathbb{R}}^1/\Gamma)$. Thus the term (3) in (11.1) is written as:

$$(11.2)\qquad v(L)^{-1}\cdot\int_1^\infty \chi(g_1)^{s-9}\frac{d\chi(g_1)}{\chi(g_1)}\int_{G_{\mathbb{R}}^1/\Gamma}\sum_{y\in L^*\cap S_{\mathbb{R}}} f^\vee(\rho^*(g_1\cdot g_2)\cdot y)dg_2,$$

where $g=g_1\cdot g_2$ with $(g_1, g_2)\in \boldsymbol{GL}_1(\mathbb{R})^+\times \boldsymbol{G}_{\mathbb{R}}^1/\Gamma$ and dg_2 is a Haar measure on $\boldsymbol{G}_{\mathbb{R}}^1$ satisfying $dg=d\chi(g_1)/\chi(g_1)\times dg_2$. In order to simplify the argument, we suppose that:

$$(11.3)\qquad \int_{G_{\mathbb{R}}^1/\Gamma}\sum_{y\in L^*\cap S_{\mathbb{R}}} f^\vee(\rho^*(g_2)\cdot y)dg_2 \textit{ is absolutely convergent.}$$

Let $\coprod_j \boldsymbol{S}_j=\boldsymbol{S}_{\mathbb{R}}$ be the $\boldsymbol{G}_{\mathbb{R}}^1$-orbital decomposition of $\boldsymbol{S}_{\mathbb{R}}$, which coincides with the $\boldsymbol{G}_{\mathbb{R}}^1$-orbital decomposition. The integral in (11.3) is calculated as:

$$(11.4)\qquad \begin{aligned}
&\int_{G_{\mathbb{R}}^1/\Gamma}\sum_{y\in L^*\cap S_{\mathbb{R}}} f^\vee(\rho^*(g_2)\cdot y)dg_2\\
&=\int_{G_{\mathbb{R}}^1/\Gamma}\sum_{i=1}^3\sum_{j\in I_i}\sum_{y\in L^*\cap S_i^j} f^\vee(\rho^*(g_2)\cdot y)dg_2\\
&=\sum_{i=1}^3\sum_{j\in I_i}\sum_{[y]\in\Gamma\backslash L^*\cap S_i^j}\sum_{\gamma\in\Gamma/\Gamma_y}\int_{G_{\mathbb{R}}^1/\Gamma} f^\vee(\rho^*(g_2\cdot\gamma)\cdot y)dg_2\\
&=\sum_{i=1}^3\sum_{j\in I_i}\sum_{[y]\in\Gamma\backslash L^*\cap S_i^j}\int_{G_{\mathbb{R}}^1/\Gamma_y} f^\vee(\rho^*(g_2)\cdot y)dg_2\\
&=\sum_{i=1}^3\sum_{j\in I_i}\sum_{[y]\in\Gamma\backslash L^*\cap S_i^j}\int_{S_i^j} f^\vee(x)d\nu_i^j(x)\cdot\int_{G_{\mathbb{R}_y}^1/\Gamma_y} d\nu_y\\
&=\sum_{i=1}^3\sum_{j\in I_i}\left(\sum_{[y]\in\Gamma\backslash L^*\cap S_i^j}\left(\int_{G_{\mathbb{R}_y}^1/\Gamma_y} d\nu_y\right)\right)\cdot\int_{S_i^j} f^\vee(x)d\nu_i^j(x),
\end{aligned}$$

where $d\nu_i^j(x)$ is a $\boldsymbol{G}_{\mathbb{R}}^1$-invariant measure on $\boldsymbol{S}_i^j$ and $d\nu_y$ is a left invariant measure on the isotropy group $\boldsymbol{G}_{\mathbb{R}_y}^1$ of $\boldsymbol{G}_{\mathbb{R}}^1$ at y normalized according to the normalization of $d\nu_i^j$.

As we have seen in § 8, the integral

$$(11.5)\qquad h(x)\longmapsto\int_{S_i^j} h(x)d\nu_i^j(x),\qquad (h(x)\in\mathscr{S}(\boldsymbol{X}_{\mathbb{R}})).$$

is absolutely convergent and defines a $\chi^{-(4i-3)}$-invariant tempered distribution $T_i^j(x)$ on $\boldsymbol{X}_{\mathbb{R}}$. Since $f\in C_0^\infty(\boldsymbol{O}_k)$, we have

$$\begin{aligned}\int_{S_i^j} f^\vee(\rho^*(g)\cdot y)d\nu_i^j(y) &= \int f^\vee(\rho^*(g)\cdot y)\cdot T_i^j(y)dy \\ &= \chi(g)^{9-(4i-3)}\int f(y)\cdot T_i^{j\vee}(y)dy \\ &= \chi(g)^{9-(4i-3)}\int f(y)\cdot b_{ik}^j\cdot|P(x)|_k^{4i-12}\,dy,\end{aligned}$$

where b_{ik}^j are the constants calculated in § 9. See Theorem 9.2 and Theorem 9.3. Thus the value of the integral (11.2) is,

$$\begin{aligned}&v(L)^{-1}\cdot\int_1^\infty \chi(g_1)^{s-(4i-3)}\frac{d\chi(g_1)}{\chi(g_1)} \\ &\quad\times\sum_{i=1}^{3}\sum_{j\in I_i}\left(\sum_{[y]\in\Gamma\backslash L^*\cap S_i^j}\left(\int_{G^1_{\mathbf{R}_y}/\Gamma_y} d\nu_y\right)\right) \\ &\quad\cdot\int_{S_i^j} f(x)\cdot b_{ik}^j\cdot|P(x)|_k^{4i-12}\,dx \\ &= v(L)^{-1}\cdot(-(s-4i+3)^{-1}) \\ &\quad\times\sum_{i=1}^{3}\sum_{j\in I_i}\left(\sum_{[y]\in\Gamma\backslash L^*\cap S_i^j}\left(\int_{G^1_{\mathbf{R}_y}/\Gamma_y} d\nu_y\right)\right) \\ &\quad\cdot\int_{S_i^j} f(x)\cdot b_{ik}^j\cdot|P(x)|_k^{4i-12}\,dx.\end{aligned}\tag{11.6}$$

Thus we have:

$$\begin{aligned}&\xi_k(\Gamma, L, s)\cdot\int f(x)\cdot|P(x)|_k^{s-9}\,dx \\ &\quad=((1)\text{ in }(11.1))+((2)\text{ in }(11.1)) \\ &\quad\quad-\sum_{i=1}^{3}(s-4i+3)^{-1}\cdot v(L)^{-1} \\ &\quad\quad\times\sum_{j\in I_i}\left(\sum_{[y]\in\Gamma\backslash L^*\cap S_i^j}\left(\int_{G^1_{\mathbf{R}_y}/\Gamma_y} d\nu_y\right)\right)\cdot\int_{S_i^j} f(x)\cdot b_{ik}^j\cdot|P(x)|_k^{4i-12}\,dx.\end{aligned}$$

Hence the residues of the pole of $\xi_k(\Gamma, L, s)$ at $s=4i-3$ ($i=1, 2, 3$) is

$$-v(L)^{-1}\cdot\sum_{j\in I_i}\left(\sum_{[y]\in\Gamma\backslash L^*\cap S_i^j}\left(\int_{G^1_{\mathbf{R}_y}/\Gamma_y} d\nu_y\right)\right)b_{ik}^j.\tag{11.7}$$

Thus the explicit computation is reduced to the computation of the sum of the absolutely convergent series:

$$(11.8)\qquad \Big(\sum_{[y]\in \Gamma\backslash L^*\cap S_j^i}\Big(\int_{G^1_{\mathbb{R}y}/\Gamma_y} d\nu_y\Big)\Big),$$

and the computation of the Fourier transform:

$$(11.9)\qquad \begin{aligned}&\int T_i^j(x)\exp(-2\pi\sqrt{-1}\langle x, y\rangle)dx|_{X^*_{\mathbb{R}}-S_{\mathbb{R}}}\\ &= \sum_{k\in I_0} b^j_{ik}\cdot|P(y)|^{4i-12}|_{O_k}.\end{aligned}$$

Theorem 11.1 (Calculation of the residues). *Under the convergence condition* (11.3), *the residue of the simple pole of* $\xi_k(\Gamma, L, s)$ *at* $s=4i-3$ $(i=1, 2, 3)$ *is given by* (11.7). *The values of* b^j_{ik}*'s have been computed in Theorems* 9.2 *and* 9.3.

Remark. The evaluation of the series (11.8) is another problem. It highly depends on the choice of the lattice L. It may be expressed as a special value of the Riemann's zeta function or a zeta function of quadratic forms (so-called Siegel's zeta function), but we can say nothing about it.

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Department of Mathematics
Kochi University
Kochi 780, Japan

Current address:
Faculty of General Education
Gifu University
Gifu 501–11, Japan