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A Note on Zeta Functions Associated with Certain Prehomogeneous Affine Spaces

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Dedicated to Prof. Ichiro Satake on his sixtieth birthday

§ 0. Introduction

The theory of zeta functions associated with prehomogeneous vector spaces (briefly P.V.) was founded by M. Sato and studied by several authors. In [5], Sato and Shintani established the analytic continuation and the functional equations of zeta functions associated with irreducible P.V.'s, and F. Sato [4] extended their results to regular P.V.'s under some mild assumptions. (See also Shintani [6, 7] and Suzuki [8].)

In this paper we shall study some zeta functions associated with *non-regular* P.V. in a special case and prove their analytic continuation and functional equations. Recall that the dual of a regular P.V. is also a regular P.V. and that functional equations hold between zeta functions associated with a regular P.V. and its dual. In our case, however, the dual of our non-regular P.V. is not even a P.V. Thus, instead of the dual, we are led to introduce some prehomogeneous *affine* spaces, the precise definition of which is given below.

Let G be a complex connected linear algebraic group, V a finite dimensional vector space and ρ a rational homomorphism from G into the group of affine transformations of V. We call a triple (G, ρ, V) a *prehomogeneous affine space* (briefly P.A.) if there exists a proper algebraic subset S of V such that $V-S$ is a single G-orbit. The set S is called the *singular set* of (G, ρ, V). In particular, when $\mathrm{Im}\,\rho$ is contained in $GL(V)$, such a triple is called a *prehomogeneous vector space* (briefly P.V.).

In § 1, we define a non-reductive algebraic group $\boldsymbol{G}$ and introduce a pair of non-regular P.V. and P.A. with $\boldsymbol{G}$-action. Zeta functions associated with them are defined and studied in § 2. Though our P.V. and P.A. are not dual to each other, we can prove functional equations between these two types of zeta functions. The next section (§ 3) is devoted to the preparation for the last one. Finally we prove that some contribution to the

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dimension formula for Jacobi cusp forms of degree n is expressed in terms of special values of our zeta functions (associated with P.A.).

We note here that our result is an analogue of Shintani's work on zeta functions associated with the space of quadratic forms ([7]). In fact, most of our results are obtained by a slight modification of Shintani's argument.

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Notation. As usual, $\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}$ and $\boldsymbol{C}$ are the ring of rational integers, the rational number field, the real number field and the complex number field. For any complex number x, we put $\boldsymbol{e}[x]=\exp(2\pi\sqrt{-1}x)$. We denote by $M_{m,n}$ and Sym_n the space of matrices with m rows and n columns and the space of symmetric matrices of degree n respectively. For any finite dimensional vector space V over $\boldsymbol{R}$, $\mathscr{S}(V)$ is the space of rapidly decreasing functions on V. When X is a smooth manifold, we denote by $C_0^\infty(X)$ the space of smooth functions with compact support. For any complex number λ we define a holomorphic function $\det(Z)^\lambda$ on $\{Z \in \mathrm{Sym}_n(\boldsymbol{C}) \mid \mathrm{Re}\, Z>0\}$ so that it coincides with the usual one on $\{X \in \mathrm{Sym}_n(\boldsymbol{R}) \mid X>0\}$.

§ 1. Prehomogeneous affine spaces

Fix once for all positive integers n, m and put $V=\mathrm{Sym}_n$, $W=M_{m,n}$ and $\boldsymbol{V}=V\times W$. Let $\boldsymbol{G}$ be the semi-direct product of W and $G=GL_n$ with composition rule

$$(\xi, g)\cdot(\xi', g')=(\xi+\xi' g^{-1}, gg') \qquad (\xi, \xi' \in W, g, g' \in G).$$

Note that $\boldsymbol{G}$ is isomorphic to the subgroup

$$\left\{\begin{bmatrix} 1_m & \xi \\ & 1_n \end{bmatrix}\begin{bmatrix} 1_m & \\ & g \end{bmatrix} \middle| \xi \in W, g \in G\right\}$$

of GL_{m+n}.

As is well-known, the triple $(G, \cdot, V)$ is a P.V. with G-action $\cdot$ on V given by $x\cdot g={}^t gxg$ $(x \in V, g \in G)$, whose singular set is $\{x \in V \mid \det x=0\}$. We now introduce two types of prehomogeneous affine structure to $(\boldsymbol{G}, \boldsymbol{V})$. We fix once for all a semi-integral positive definite symmetric matrix S of degree m. For simplicity we write $S(u, v)$ and $S[u]$ for ${}^t uSv$ and ${}^t uSu$ respectively $(u, v \in W)$. For $X=[x, u] \in \boldsymbol{V}$ and $\boldsymbol{g}=(\xi, g) \in \boldsymbol{G}$, set

$$X\cdot\boldsymbol{g}=[g^{-1}x{}^t g^{-1}, (u-\xi x){}^t g^{-1}] \qquad \text{(dot action)}$$

and

$$X * \boldsymbol{g} = [{}^t g(x + S[\xi] + S(u, \xi) + S(\xi, u))g, (u+\xi)g] \qquad \text{(star action).}$$

It is easily verified that $(X, \boldsymbol{g}) \mapsto X \cdot \boldsymbol{g}$, $X * \boldsymbol{g}$ give rise to right affine actions of $\boldsymbol{G}$ on $\boldsymbol{V}$. We put $P(X) = \det x$ and $P^*(X) = \det(x - S[u])$ $(X = [x, u] \in \boldsymbol{V})$. Then

$$P(X \cdot \boldsymbol{g}) = (\det g)^{-2} P(X)$$

and

$$P^*(X * \boldsymbol{g}) = (\det g)^2 P^*(X) \qquad (\boldsymbol{g} = (\xi, g) \in \boldsymbol{G}, X \in \boldsymbol{V}).$$

Lemma 1.1. *The triple* $(\boldsymbol{G}, \cdot, \boldsymbol{V})$ *(resp.* $(\boldsymbol{G}, *, \boldsymbol{V})$*) is a prehomogeneous affine space with the singular set* $\boldsymbol{S} = \{X \in \boldsymbol{V} \mid P(X) = 0\}$ *(resp.* $\boldsymbol{S}^* = \{X \in \boldsymbol{V} \mid P^*(X) = 0\}$*).*

Remark 1.2. In fact $(\boldsymbol{G}, \cdot, \boldsymbol{V})$ is a prehomogeneous *vector* space and its relative invariants are $P(X)^n$ $(n \in \boldsymbol{Z})$. Thus $(\boldsymbol{G}, \cdot, \boldsymbol{V})$ is not regular and the general theory of F. Sato is not applicable to our case.

Let $G_{\boldsymbol{R}}^+ = \{g \in G_{\boldsymbol{R}} \mid \det g > 0\}$ and $\boldsymbol{G}_{\boldsymbol{R}}^+ = W_{\boldsymbol{R}} G_{\boldsymbol{R}}^+$. We now define two representations r and R of $\boldsymbol{G}_{\boldsymbol{R}}^+$ on $\mathscr{S}(\boldsymbol{V}_{\boldsymbol{R}})$ in the following manner:

$$\begin{aligned} r(\boldsymbol{g})f(X) &= f(X \cdot \boldsymbol{g})\boldsymbol{e}[\operatorname{tr}(xS[\xi] - 2S(u, \xi))], \\ R(\boldsymbol{g})f(X) &= (\det g)^{m+n+1} f(X * \boldsymbol{g}), \\ &\quad (\boldsymbol{g} = (\xi, g) \in \boldsymbol{G}_{\boldsymbol{R}}^+, X = [x, u] \in \boldsymbol{V}_{\boldsymbol{R}}, f \in \mathscr{S}(\boldsymbol{V}_{\boldsymbol{R}})). \end{aligned}$$

The Fourier transform $\mathscr{F}f$ of $f \in \mathscr{S}(\boldsymbol{V}_{\boldsymbol{R}})$ is defined by

$$\mathscr{F}f(Y) = \int_{\boldsymbol{V}_{\boldsymbol{R}}} f(X)\boldsymbol{e}[\langle X, Y\rangle]dX.$$

Here the inner product $\langle \ , \ \rangle$ on $\boldsymbol{V}_{\boldsymbol{R}}$ is given by

$$\langle [x, u], [y, v]\rangle = \operatorname{tr}(xy + 2S(u, v))$$

and dX is the usual Lebesgue measure on $\boldsymbol{V}_{\boldsymbol{R}}$; for $X = [x, u] \in \boldsymbol{V}_{\boldsymbol{R}}$

$$dX = dx\,du, \quad dx = \prod_{1 \le i \le j \le n} dx_{ij}, \quad du = \prod_{\substack{1 \le k \le m \\ 1 \le l \le n}} du_{kl}.$$

The following formula is easily verified.

Lemma 1.3. $r(\boldsymbol{g})\mathscr{F} = \mathscr{F}R(\boldsymbol{g})$ $(\boldsymbol{g} \in \boldsymbol{G}_{\boldsymbol{R}}^+)$.

For $0 \le i \le n$, let $V_i = \{x \in V_{\boldsymbol{R}} \mid \operatorname{sgn} x = (i, n-i)\}$, $\boldsymbol{V}_i = V_i \times W_{\boldsymbol{R}}$ and $\boldsymbol{V}_i^* = \{[x, u] \in \boldsymbol{V}_{\boldsymbol{R}} \mid x - S[u] \in V_i\}$. Then we have

$$V_R - S_R = \coprod_{i=0}^{n} V_i, \qquad V_R - S_R^* = \coprod_{i=0}^{n} V_i^* \quad \text{(disjoint union).}$$

For $f \in \mathscr{S}(V_R)$ and $s \in C$, we put

$$\Phi_i(s, f) = \int_{V_i} |P(X)|^s \, e[\mathrm{tr}\,(x^{-1}S[u])] f(X) dX$$

and

$$\Phi_i^*(s, f) = \int_{V_i^*} |P^*(X)|^s f(X) dX$$

The integrals are absolutely convergent for $\mathrm{Re}\, s > 0$. It is clear that

$$\Phi_i(s, r(g)f) = (\det g)^{2s+m+n+1} \Phi_i(s, f)$$

and

$$\Phi_i^*(s, R(g)f) = (\det g)^{-2s} \Phi_i^*(s, f) \qquad (g \in G_R^+, f \in \mathscr{S}(V_R)).$$

Proposition 1.4 (cf. Lemma 15 in [7]). *Let $f \in \mathscr{S}(V_R)$.*

(i) *$\Phi_i(s, f)$ and $\Phi_i^*(s, f)$ are continued to meromorphic functions on the whole s-plane.*

(ii) $$\Phi_i(s-(m+n+1)/2, \mathscr{F}f) = C_{n,S}^{-1/2}(2\pi)^{n(n-1)/4-ns} e[m(2i-n)/8 + ns/4] \times \Gamma_n(s) \sum_{j=0}^{n} u_{ij}(s) \Phi_j^*(-s, f),$$

where $C_{n,S} = (\det 2S)^n 2^{n(n-1)/2}$, $\Gamma_n(s) = \prod_{k=0}^{n-1} \Gamma(s-k/2)$ and $u_{ij}(s)$ are polynomials in $e[-s/2]$ given by (2.4) *in* [7].

Proof. Let T be the linear operator of $\mathscr{S}(V_R)$ given by $Tf([x, u]) = f([x - S[u], u])$. For $F \in \mathscr{S}(V_R)$, we define its Fourier transform $\hat{F}$ by

$$\hat{F}(x) = \int_{V_R} F(y) e[\mathrm{tr}\,(xy)] dy$$

and put

$$\phi_i(s, F) = \int_{V_i} |\det x|^s F(x) dx.$$

In view of Lemma 15 in [7], our proof is reduced to the following formula.

Lemma 1.5. *Assume that* $\mathrm{Re}\, s > 0$ *and* $f \in \mathscr{S}(V_R)$. *Then*

$$\Phi_j^*(s, Tf) = \phi_j(s, F_f),$$
$$\Phi_j(s, \mathscr{F}Tf) = (\det 2S)^{-n/2} \boldsymbol{e}[m(2j-n)/8]\phi_j(s+m/2, \hat{F}_f),$$

where

$$F_f(x) = \int_{W_R} f([(x, u)]du \in \mathscr{S}(V_R).$$

Proof. The first part is obvious from the definitions. The Lebesgue convergence theorem implies

$$\Phi_j(s, \mathscr{F}Tf)$$
$$= \int_{V_j} |\det x|^s \, dx \left\{ \lim_{\varepsilon \downarrow 0} \int_{W_R} \boldsymbol{e}[\operatorname{tr}(x^{-1} + \sqrt{-1}\varepsilon) S[u]] \mathscr{F}Tf([x, u]) du \right\}.$$

Changing the order of integration, we have

$$\int_{W_R} \boldsymbol{e}[\operatorname{tr}(x^{-1} + \sqrt{-1}\varepsilon) S[u]] \mathscr{F}Tf([x, u]) du$$
$$= \int_{W_R} du \int_{V_R} dy dv f([y, v]) \boldsymbol{e}[\operatorname{tr}\{(x^{-1} + \sqrt{-1}\varepsilon) S[u + v(x^{-1} + \sqrt{-1}\varepsilon)^{-1}]$$
$$+ \sqrt{-1}\varepsilon x (x^{-1} + \sqrt{-1}\varepsilon)^{-1} S[v] + xy\}]$$
$$= \det(\varepsilon - \sqrt{-1}x^{-1})^{-m/2} (\det 2S)^{-n/2}$$
$$\int_{V_R} \boldsymbol{e}[\operatorname{tr}\{xy + \sqrt{-1}\varepsilon x (x^{-1} + \sqrt{-1}\varepsilon)^{-1} S[v]\}] f([y, v]) dy dv.$$

Taking the limit, we obtain

$$\lim_{\varepsilon \downarrow 0} \int_{W_R} \boldsymbol{e}[\operatorname{tr}(x^{-1} + \sqrt{-1}\varepsilon) S[u]] \mathscr{F}Tf([x, u]) dx du$$
$$= \boldsymbol{e}[m(2j-n)/8] |\det x|^{m/2} (\det 2S)^{-n/2} \hat{F}_f(x),$$

which completes the proof of the lemma. Q.E.D.

§ 2. Zeta functions associated with prehomogeneous affine spaces

In this section we study zeta functions associated with P.A. introduced in the previous section.

For $x \in V_R$ ($\det x \neq 0$) put $G_x = \{g \in G_R^+ \mid x \cdot g = x\} = SO(x)$. Similarly for $X = [x, u] \in \boldsymbol{V}_R - \boldsymbol{S}_R$ (resp. $\boldsymbol{V}_R - \boldsymbol{S}_R^*$), we put $\boldsymbol{G}_X = \{\boldsymbol{g} \in \boldsymbol{G}_R^+ \mid X \cdot \boldsymbol{g} = X\}$ (resp. $\boldsymbol{G}_R^* = \{\boldsymbol{g} \in \boldsymbol{G}_R^+ \mid X * \boldsymbol{g} = X\}$). Then G_x (resp. $G_{x-S[u]}$) is isomorphic to $\boldsymbol{G}_X$ (resp. $\boldsymbol{G}_X^*$) through the mapping ι (resp. ι^*), where $\iota(h) = (u(1-h^{-1})x^{-1}, {}^t h^{-1})$ (resp. $\iota^*(h) = (u(h^{-1}-1), h)$). We normalize a Haar measure dg (resp. $d\xi$) on G_R^+ (resp. W_R) by

$$dg=(\det g)^{-n}\prod dg_{ij} \quad (\text{resp. } d\xi=\prod d\xi_{ij}).$$

Then $d_r\boldsymbol{g}=d\xi dg$ ($\boldsymbol{g}=(\xi, g)\in \boldsymbol{G}_{\boldsymbol{R}}^+$) gives a right invariant measure on $\boldsymbol{G}_{\boldsymbol{R}}^+$. We let $x\in V_i$ (resp. $X\in \boldsymbol{V}_i$, $X\in \boldsymbol{V}_i^*$) and normalize a Haar measure $d\nu_x$ (resp. $d\nu_X$, $d\nu_X^*$) on G_x (resp. $\boldsymbol{G}_X$, $\boldsymbol{G}_X^*$) by the following formula:

$$\int_{G_{\boldsymbol{R}}^+}\phi(g)dg=\int_{G_x\backslash G_{\boldsymbol{R}}^+}|\det x\cdot g|^{-(n+1)/2}d(x\cdot g)\int_{G_x}\phi(hg)d\nu_x(h)$$

$$(\text{resp. } \int_{\boldsymbol{G}_{\boldsymbol{R}}^+}\phi(\boldsymbol{g})d_r\boldsymbol{g}=\int_{\boldsymbol{G}_X\backslash \boldsymbol{G}_{\boldsymbol{R}}^+}|P(X\cdot \boldsymbol{g})|^{-(m+n+1)/2}d(X\cdot\boldsymbol{g})\int_{\boldsymbol{G}_X}\phi(\boldsymbol{hg})d\nu_X(\boldsymbol{h}),$$

$$\int_{\boldsymbol{G}_X^*\backslash \boldsymbol{G}_{\boldsymbol{R}}^+}|P^*(X*\boldsymbol{g})|^{-(m+n+1)/2}d(X*\boldsymbol{g})\int_{\boldsymbol{G}_X^*}\phi(\boldsymbol{hg})d\nu_X^*(\boldsymbol{h})),$$

where $d(x\cdot g)$ (resp. $d(X\cdot\boldsymbol{g})$, $d(X*\boldsymbol{g})$) is the usual Lebesgue measure on $V_i=x\cdot G_{\boldsymbol{R}}^+$ (resp. $\boldsymbol{V}_i=X\cdot\boldsymbol{G}_{\boldsymbol{R}}^+$, $\boldsymbol{V}_i^*=X*\boldsymbol{G}_{\boldsymbol{R}}^+$). It is easy to see that

$$d\nu_X(\iota(h))=|\det x|^{-m/2}\,d\nu_x(h) \qquad (X=[x, u],\ h\in G_x),$$

and

$$d\nu_X^*(\iota^*(h))=|\det y|^{m/2}\,d\nu_y(h) \qquad (X=[x, u],\ y=x-S[u],\ h\in G_y).$$

Let $L=V\cap M_n(\boldsymbol{Z})$, $L^*=\{x=(x_{ij})\in V_{\boldsymbol{R}}\,|\,x_{ij}\in 2^{-1}\boldsymbol{Z},\ x_{ii}\in \boldsymbol{Z}\}$, $M=M_{m,n}(\boldsymbol{Z})$, $M^*=(2S)^{-1}M$, $\boldsymbol{L}=L\times M$, $\boldsymbol{L}^*=L^*\times M^*$, $\Gamma=SL_n(\boldsymbol{Z})$ and $\boldsymbol{\Gamma}=\{(\xi, g)\in \boldsymbol{G}_{\boldsymbol{R}}\,|\,\xi\in M,\ g\in\Gamma\}$. Then $\boldsymbol{L}$ and $\boldsymbol{L}^*$ are dual to each other with respect to the inner product $\langle\ ,\ \rangle$ and $\boldsymbol{L}$ (resp. $\boldsymbol{L}^*$) is $\boldsymbol{\Gamma}$-invariant under the dot action (resp. the star action). For $x\in V_{\boldsymbol{Q}}$ ($\det x\neq 0$) (resp. $X\in \boldsymbol{V}_{\boldsymbol{Q}}-\boldsymbol{S}_{\boldsymbol{Q}}$, $X\in\boldsymbol{V}_{\boldsymbol{Q}}-\boldsymbol{S}_{\boldsymbol{Q}}^*$) we put $\Gamma_x=\Gamma\cap G_x$ (resp. $\boldsymbol{\Gamma}_X=\boldsymbol{\Gamma}\cap\boldsymbol{G}_X$, $\boldsymbol{\Gamma}_X^*=\boldsymbol{\Gamma}\cap\boldsymbol{G}_X^*$). It is easy to see that $\Gamma_x\backslash G_x$ (resp. $\boldsymbol{\Gamma}_X\backslash\boldsymbol{G}_X$, $\boldsymbol{\Gamma}_X^*\backslash\boldsymbol{G}_X^*$) has a finite volume and we define

$$\mu(x)=\int_{\Gamma_x\backslash G_x}d\nu_x$$

$$(\text{resp. } \mu(X)=\int_{\boldsymbol{\Gamma}_X\backslash\boldsymbol{G}_X}d\nu_X,\ \mu^*(X)=\int_{\boldsymbol{\Gamma}_X^*\backslash\boldsymbol{G}_X^*}d\nu_X^*)$$

unless $n=2$ and $-(\det x)$ (resp. $-P(X)$, $-P^*(X)$) has a square root in $\boldsymbol{Q}$. In this exceptional case we put $\mu(x)=0$ (resp. $\mu(X)=0$, $\mu^*(X)=0$).

For $0\leq i\leq n$, we put $\boldsymbol{L}_i=\boldsymbol{L}\cap\boldsymbol{V}_i$ and $\boldsymbol{L}_i^*=\boldsymbol{L}^*\cap\boldsymbol{V}_i^*$. Let $\boldsymbol{L}_i/\cdot$ (resp. $\boldsymbol{L}_i^*/*$) be a complete set of representatives of $\boldsymbol{L}_i$ (resp. $\boldsymbol{L}_i^*$) under the dot action (resp. the star action) of $\boldsymbol{\Gamma}$. We now define Dirichlet series $\xi_i(s, \boldsymbol{L})$ and $\xi_i^*(s, \boldsymbol{L}^*)$ by

$$\xi_i(s, \boldsymbol{L})=\sum_{X=[x,u]\in\boldsymbol{L}_i/\cdot}\mu(X)\boldsymbol{e}[-\mathrm{tr}\,(x^{-1}S[u])]\,|P(X)|^{-s}$$

and

$$\xi_i^*(s, \boldsymbol{L}^*)=\sum_{X\in \boldsymbol{L}_i^*/*}\mu^*(X)|P^*(X)|^{-s}.$$

Lemma 2.1. *The Dirichlet series* $\xi_i(s, \boldsymbol{L})$ *and* $\xi_i^*(s, \boldsymbol{L}^*)$ *converge absolutely if* $\operatorname{Re} s>(m+n+1)/2$.

Proof. On accounting the relation between $d\nu_X^*$ and $d\nu_y$ we have

$$\mu^*([x, u])=|\det y|^{m/2}[\Gamma_y: \iota^{*-1}(\boldsymbol{\Gamma}_X^*)]\mu(y),$$

where $y=x-S[u]$. For each fixed $y\in V_Q$, we say that two elements u and u' in M^* are equivalent if there exists a $\gamma\in\boldsymbol{\Gamma}_y$ such that $u'\equiv u\gamma \pmod M$. Take a complete set T_y^* of representatives of such equivalence classes in M^*. Let $\mathscr{R}_i$ be a complete set of representatives of V_i under the action of Γ. Then we can take

$$\{[y+S[u], u]\,|\, y\in\mathscr{R}_i,\ y+S[u]\in L^*,\ u\in T_y^*\}$$

as $\boldsymbol{L}_i^*/*$. Since

$$\sum_{\substack{u\in T_y^*\\ S[u]+y\in L^*}}[\Gamma_y: \iota^{*-1}(\boldsymbol{\Gamma}_{[y+S[u],u]})]=\sum_{\substack{u\in M^*/M\\ S[u]+y\in L^*}}1,$$

we have

$$\sum_{X\in \boldsymbol{L}_i^*/*}\mu(X)|P^*(X)|^{-s}=\sum_{y\in\mathscr{R}_i}\sum_{\substack{u\in M^*/M\\ S[u]+y\in L^*}}\mu(y)|\det y|^{-(s-m/2)}.$$

Noting that $\{y\in V_i\,|\,S[u]+y\in L^* \text{ for some } u\in M^*\}$ is contained in a lattice of $V_{\boldsymbol{R}}$, we see that $\xi_i^*(s, \boldsymbol{L}^*)$ converges absolutely if $\operatorname{Re}(s-m/2)>(n+1)/2$ (cf. (2.6) in [7]). The convergence of $\xi_i(s, \boldsymbol{L})$ is similarly proved. Q.E.D.

Remark 2.2. The above proof implies that

$$\xi_n^*(s, \boldsymbol{L}^*)=2^{-(n+1)}C_n\xi_n^*(s-m/2; S),$$

where

$$\xi_n^*(s; S)=\sum_{u\in M^*/M}\sum_{\substack{y\in\mathscr{R}_n\\ y+S[u]\in L^*}}|\Gamma_y|^{-1}(\det y)^{-s}$$

and

$$C_n=\prod_{k=1}^n\frac{2\pi^{k/2}}{\Gamma(k/2)}.$$

To show the analytic continuation and the functional equation of $\xi_i(s, \boldsymbol{L})$ and $\xi_i^*(s, \boldsymbol{L})$, consider the integrals

$$Z(f, \boldsymbol{L}, s)=\int_{\Gamma\backslash G_{\boldsymbol{R}}^{+}} \det g^{-2s} \sum_{X \in \boldsymbol{L}'} r(\boldsymbol{g})f(X)d_r\boldsymbol{g},$$

$$Z^*(f, \boldsymbol{L}^*, s)=\int_{\Gamma\backslash G_{\boldsymbol{R}}^{+}} \det g^{2s-(m+n+1)} \sum_{X \in \boldsymbol{L}^{*\prime}} R(\boldsymbol{g})f(X)d_r\boldsymbol{g}.$$

Here f is a function on $\boldsymbol{V_R}$, $\boldsymbol{L}'=\boldsymbol{L}\cap(\boldsymbol{V_R}-\boldsymbol{S_R})$ and $\boldsymbol{L}^{*\prime}=\boldsymbol{L}^*\cap(\boldsymbol{V_R}-\boldsymbol{S}_{\boldsymbol{R}}^*)$.

Lemma 2.3 (cf. Lemma 16 in [7]). *When $n\neq 2$, the integral $Z(f, \boldsymbol{L}, s)$ and $Z^*(f, \boldsymbol{L}^*, s)$ ($f \in \mathcal{S}(\boldsymbol{V_R})$) are absolutely convergent if* $\operatorname{Re} s>(m+n+1)/2$. *Furthermore we have*

$$Z(f, \boldsymbol{L}, s)=\sum_{0\leq i\leq n} \xi_i(s, \boldsymbol{L})\Phi_i(s-(m+n+1)/2, f),$$

$$Z^*(f, \boldsymbol{L}^*, s)=\sum_{0\leq i\leq n} \xi_i^*(s, \boldsymbol{L}^*)\Phi_i^*(s-(m+n+1)/2, f).$$

If $f \in C_0^\infty(V_n)$ (resp. $f \in C_0^\infty(V_n^*)$) this is true even for $n=2$.

Proof. It is easy to see that for real s

$$\int_{\Gamma\backslash G_{\boldsymbol{R}}^{+}} |\det g|^{-s} \sum_{X \in \boldsymbol{L}'} |r(\boldsymbol{g})f(X)|d_r\boldsymbol{g}$$

$$\leq \int_{\Gamma\backslash G_{\boldsymbol{R}}^{+}} \det g^{2s-m} \sum_{x \in L'} F_{|f|}({}^t gxg)dg,$$

where $L'=\{x \in L \mid \det x\neq 0\}$. Thus the convergence of $Z(f, \boldsymbol{L}, s)$ is an immediate consequence of Lemma 16 in [7]. For $X \in \boldsymbol{V_R}-\boldsymbol{S_R}$, put

$$P_s(X)=|\det x|^s \boldsymbol{e}[\operatorname{tr}(x^{-1}S[u])].$$

Then

$$P_s(X\cdot\boldsymbol{g})/P_s(X)=(\det g)^{-2s} \boldsymbol{e}[\operatorname{tr}(xS[\xi]-2S(u, \xi))]$$

and we have

$$Z(f, \boldsymbol{L}, s)=\int_{\Gamma\backslash G_{\boldsymbol{R}}^{+}} \sum_{X \in \boldsymbol{L}'} P_s(X\cdot\boldsymbol{g})/P_s(X)f(X\cdot\boldsymbol{g})d_r\boldsymbol{g}$$

$$=\sum_{0\leq i\leq n} \sum_{X \in \boldsymbol{L}_i/\cdot} \mu(X)P_s(X)^{-1}\int_{G_X\backslash G_{\boldsymbol{R}}^{+}} P_{s-(m+n+1)/2}(X\cdot\boldsymbol{g})d(X\cdot\boldsymbol{g}),$$

which proves the assertion on $Z(f, \boldsymbol{L}, s)$. The remaining part is checked in quite a similar manner. Q.E.D.

Set

$$Z_+(f, \boldsymbol{L}, s)=\int_{\substack{\Gamma\backslash G_{\boldsymbol{R}}^{+} \\ \det g\leq 1}} (\det g)^{-2s} \sum_{X \in \boldsymbol{L}'} r(\boldsymbol{g})f(X)d_r\boldsymbol{g}$$

and

$$Z_+^*(f, L^*, s)=\int_{\substack{\Gamma\backslash G_R^+ \\ \det g\geq 1}} (\det g)^{2s-(m+n+1)} \sum_{X\in L^{*\prime}} R(g)f(X)d_r g.$$

These integrals are absolutely convergent for any $s \in C$ and hence define entire functions if $f \in \mathcal{S}(V_R)$ and $n\neq 2$.

Theorem 2.4 (cf. Theorem 5, in [7]). *Assume $n\neq 2$.*
(i) *Dirichlet series $\xi_i(s, L)$ and $\xi_i^*(s, L^*)$ $(0\leq i\leq n)$ are continued to meromorphic functions in the whole s-plane.*
(ii) *They are holomorphic except for possible simple poles at $s=(m+k+1)/2$ $(1\leq k\leq n)$.*
(iii) *They satisfy the following functional equations*:

$$\xi_i^*((m+n+1)/2-s, L^*)=C_{n,S}^{1/2}\, e\left[\frac{ns}{4}\right](2\pi)^{n(n-1)/4-ns}\,\Gamma_n(s)$$
$$\times\sum_{j=0}^{n} u_{ji}(s)e\left[\frac{m(2j-n)}{8}\right]\xi_j(s, L).$$

Proof. Let $\mathcal{D}=\det((1/(2-\delta_{ij}))(\partial/\partial x_{ij}))$ be a differential operator on V_R. For $f_i^{(0)} \in C_0^\infty(V_i^*)$ put $f_i=\mathcal{D}f_i^{(0)}$. Note that

$$\Phi_i^*(s,f_i)=(-1)^i\prod_{k=0}^{n-1}(s+k/2)\Phi_i^*(s-1,f_i^{(0)})$$

and that for any $s \in C$ there exists an $f_i^{(0)}$ such that $\Phi_i^*(s,f_i^{(0)})$ is not zero. It is easy to see that

$$\mathcal{F}f_i(X)=(-2\pi\sqrt{-1})^n P(X)\mathcal{F}f_i^{(0)}(X).$$

Thus f_i (resp. $\mathcal{F}f_i$) vanishes on S_R^* (resp. S_R). The Poisson summation formula and Lemma 1.2 now imply

$$Z(\mathcal{F}f_i, L, s)=Z_+(\mathcal{F}f_i, L, s)+C_{n,S}^{-1}Z_+^*(f_i, L^*, (m+n+1)/2-s),$$
$$Z^*(f_i, L^*, s)=Z_+^*(f_i, L^*, s)+C_{n,S}Z_+(\mathcal{F}f_i, L, (m+n+1)/2-s).$$

Hence $Z(\mathcal{F}f_i, L, s)$ and $Z^*(f_i, L^*, s)$ are entire functions of s and satisfy the functional equation

$$Z^*(f_i, L^*, (m+n+1)/2-s)=C_{n,S}Z(\mathcal{F}f_i, L, s).$$

Applying Lemma 2.3 and Proposition 1.4 to both sides, we have the analytic continuation of ξ_i^* and the functional equation (iii) in $\mathrm{Re}\, s>0$. Taking account of the fact

$$(2\pi)^{-n}e[n(n+1)/8]\Gamma_n(s)\Gamma_n((n+1)/2-s)$$
$$\sum_{0\leq l\leq n} u_{i,l}(s)u_{n-l,j}((n+1)/2-s) \qquad =\delta_{i,j},$$

we have the analytic continuation of ξ_i. Therefore (iii) holds for any $s \in \boldsymbol{C}$. Q.E.D.

Remark 2.5. As we can see from Lemma 3.1 and Theorem 3.2 below, even when $n=2$, $\xi_2^*(s, \boldsymbol{L}^*)$ has an analytic continuation to a meromorphic function on $\boldsymbol{C}$ and is holomorphic except for possible simple poles at $s=(m+2)/2$ and $(m+3)/2$.

§ 3. Integrals $Z(f_n(X; \lambda), \boldsymbol{L}, s)$ and $Z^*(f_n^*(X; \lambda), \boldsymbol{L}^*, s)$

For $\lambda \in \boldsymbol{C}$, we define functions $f_n(X; \lambda)$ and $f_n^*(X; \lambda)$ on $V_{\boldsymbol{R}}$ by

$$f_n([x, u]; \lambda)=\det\left(1+\frac{1}{2\sqrt{-1}}x\right)^{-\lambda} \boldsymbol{e}[-\operatorname{tr}(2\sqrt{-1}+x)^{-1}S[u]]$$

and

$$f_n^*([x, u]; \lambda)=\begin{cases}\det(x-S[u])^{\lambda-(m+n+1)/2}\exp(-4\pi\operatorname{tr}(x)) & \text{if } x>S[u]\\ 0 & \text{otherwise.}\end{cases}$$

Put

$$d_{n,S}(\lambda)=C_{n,S}^{1/2}(2\pi)^{n(n-1)/4-n(\lambda-m/2)}2^{-n\lambda}\Gamma_n(\lambda-m/2).$$

A direct calculation shows

Lemma 3.1. *The integral $Z^*(f_n^*(X; \lambda), \boldsymbol{L}^*, s)$ is absolutely convergent and is equal to*

$$C_{n,S}^{-1}d_{n,S}(s+\lambda-(m+n+1)/2)\xi_n^*(s, \boldsymbol{L}^*),$$

if $\operatorname{Re} s>(m+n+1)/2$ *and* $\operatorname{Re}(\lambda+s)>m+n$.

Thus $Z^*(f_n^*(X; \lambda), \boldsymbol{L}^*, s)$ is continued to a meromorphic function of (λ, s) on the whole $\boldsymbol{C}^2$. It is easily verified that the Fourier transform $\mathscr{F}f_n^*(X; \lambda)$ exists if $\operatorname{Re}\lambda>(m+n+1)/2$ and is equal to $C_{n,S}^{-1}d_{n,S}(\lambda)f_n(X; \lambda)$. Furthermore the identity

$$\sum_{X\in \boldsymbol{L}^*} f_n^*(X; \lambda)=d_{n,S}(\lambda)\sum_{X\in \boldsymbol{L}} f_n(X; \lambda)$$

holds if $\operatorname{Re} s>m+n+1$. By a similar argument to that in the proof of Lemma 21 in [7], we obtain the following theorem.

Theorem 3.2 (cf. Lemma 21 in [7]).
(i) *Assume that* $\operatorname{Re} s>(n-1)/2$ *and that the pair* (λ, s) *satisfies the following inequalities*:

$$\begin{aligned} &\operatorname{Re}\lambda>m+1,\ \operatorname{Re}s<\operatorname{Re}\lambda-m/2 && \text{if } n=1,\\ &\operatorname{Re}\lambda>\operatorname{Max}(13/2,\ 2\operatorname{Re}s+7/2)+m && \text{if } n=2,\\ &\operatorname{Re}\lambda>m+n+7/2,\ \operatorname{Re}s<\operatorname{Re}\lambda-(m+n-1)/2 && \text{if } n\geq 3. \end{aligned}$$

Then the integral $Z(f_n(X;\lambda), \boldsymbol{L}, s)$ *is absolutely convergent.*
(ii) *Both* $Z(f_n(X;\lambda), \boldsymbol{L}, s)$ *and* $Z^*(f_n(X;\lambda), \boldsymbol{L}^*, s)$ *are continued to meromorphic functions of* (λ, s) *on* $\boldsymbol{C}^2$ *and satisfy*

$$Z^*(f_n^*(X;\lambda), \boldsymbol{L}^*, s)=d_{n,S}(\lambda)Z(f_n(X;\lambda), \boldsymbol{L}, (m+n+1)/2-s).$$

§ 4. The contribution of purely parabolic conjugacy classes to the dimension formula for the space of Jacobi cusp forms

In this section, we show that special values of $\xi_n^*(s, \boldsymbol{L}^*)$ appear in the calculation of the dimension formula for the space of Jacobi cusp forms.

We recall the definition of Jacobi cusp forms from Shintani's unpublished work (for more precise treatment refer to [2]; see also Yamazaki [9]). Let $\tilde{G}=\tilde{G}_{n,m}$ be the subgroup of Sp_{m+n} consisting of

$$(\xi,\eta,\kappa)g=\left(\begin{array}{c|cc} 1_{m+n} & \begin{matrix}\kappa & \eta\\ {}^t\eta & 0\end{matrix} \\ \hline & 1_{m+n}\end{array}\right)\left(\begin{array}{cc|cc} 1_m & \xi & & \\ & 1_n & & \\ \hline & & 1_m & \\ & & -{}^t\xi & 1_n\end{array}\right)\left(\begin{array}{cc|cc} 1_m & & & \\ & a & & b \\ \hline & & 1_m & \\ & c & & d\end{array}\right),$$

where $\xi, \eta\in M_{m,n}$, $\kappa\in \mathrm{Sym}_m$ and $g=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in Sp_n$. Then $\tilde{G}$ is the semi-direct product of $H=\{(\xi,\eta,\kappa)\}$ and $G=Sp_n$. The center Z of $\tilde{G}$ is $\{(0,0,\kappa)\}$ and hence $\tilde{G}$ is not reductive if $m\geq 1$. We let $\tilde{G}_{\boldsymbol{R}}$ act on $\mathscr{D}=\mathscr{D}_{n,m}=\mathfrak{H}_n\times M_{m,n}(\boldsymbol{C})$ ($\mathfrak{H}_n$ = the upper half plane of degree n) as follows:

$$\tilde{g}\langle Z\rangle=(g\langle z\rangle,\ wj(g,z)^{-1}+\xi\cdot g\langle z\rangle+\eta)$$
$$(\tilde{g}=(\xi,\eta,\kappa)g\in\tilde{G}_{\boldsymbol{R}},\ Z=(z,w)\in\mathscr{D}),$$

where $g\langle z\rangle=(az+b)(cz+d)^{-1}$ and $j(g,z)=cz+d$. We define an automorphy factor $J_{S,l}(\tilde{g},Z)$ by

$$\begin{aligned} J_{S,l}(\tilde{g},Z)=\det j(g,z)^l\boldsymbol{e}[\operatorname{tr}\{&-S\kappa+S[w]j(g,z)^{-1}c\\ &-2S(\xi,w)j(g,z)^{-1}-S[\xi]g\langle z\rangle\}], \end{aligned}$$

where S is a positive definite semi-integral symmetric matrix of degree m and l is a positive integer. Then the space $\mathfrak{S}(S,l)$ of Jacobi cusp forms of index S and weight l with respect to $\tilde{\Gamma}=\tilde{G}(\boldsymbol{Z})$ is defined to be the $\boldsymbol{C}$-vector space of holomorphic functions f on $\mathscr{D}$ which satisfy

(i) $f(\tilde{\gamma}\langle Z\rangle)=J_{S,l}(\tilde{\gamma}, Z)f(Z)$ for $\forall\tilde{\gamma}\in\tilde{\Gamma}$, $\forall Z\in\mathscr{D}$,
(ii) as a function on $\tilde{G}_{\boldsymbol{R}}$, $J_{S,l}(\tilde{g}, Z_0)^{-1}f(\tilde{g}\langle Z_0\rangle)$ is bounded $(Z_0=(\sqrt{-1}1_n, 0)\in\mathscr{D})$.

For $Z=(z, w)$ and $Z'=(z', w')\in\mathscr{D}$, we put

$$K_{S,l}(Z, Z')=A_{S,l}\det\left(\frac{z-\bar{z}'}{2\sqrt{-1}}\right)^{-l}\boldsymbol{e}[-\operatorname{tr}(z-\bar{z}')^{-1}S[w-\bar{w}']],$$

where

$$A_{S,l}=(\det 2S)^n 2^{-n(n+3)/2}\pi^{-n(n+1)/2}\prod_{i=0}^{n-1}\prod_{j=1}^{n-i}\left(l-\frac{m+i}{2}-j\right).$$

Let $d\mu(Z)$ be the $\tilde{G}_{\boldsymbol{R}}$-invariant measure on $\mathscr{D}$ given by

$$d\mu(Z)=(\det y)^{-m-n-1}\prod_{1\leq i\leq j\leq n}dx_{ij}dy_{ij}\prod_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}du_{ij}dv_{ij}$$
$$(Z=(z, w),\ z=x+\sqrt{-1}y,\ w=u+\sqrt{-1}v).$$

In view of Satake [3] and applying the Selberg trace formula, we obtain the following dimension formula for $\mathfrak{S}(S, l)$.

Lemma 4.1. *Assume $l>m+2n$. Then*

$$\dim_{\boldsymbol{C}}\mathfrak{S}(S, l)=\int_{\tilde{\Gamma}\backslash\mathscr{D}}\sum_{\tilde{\gamma}\in\tilde{\Gamma}/Z(\tilde{\Gamma})}K_{S,l}(\tilde{\gamma}\langle Z\rangle, Z)J_{S,l}(\tilde{\gamma}, Z)^{-1}$$
$$|J_{S,l}(\tilde{g}_Z, Z_0)|^{-2}\,d\mu(Z).$$

Here $\tilde{g}_Z$ is an element of $\tilde{G}_{\boldsymbol{R}}$ such that $\tilde{g}_Z\langle Z_0\rangle=Z$ and $Z(\tilde{\Gamma})=\{(0, 0, \kappa)\,|\,\kappa\in$ $\mathrm{Sym}_m(\boldsymbol{Z})\}$ is the center of $\tilde{\Gamma}$.

We set $\tilde{\Gamma}'=\{\tilde{\gamma}\in\tilde{\Gamma}\,|\,\boldsymbol{e}[\operatorname{tr}(S\psi_{\tilde{\gamma}}(h))]=1$ for $\forall h\in H(\tilde{\gamma})_{\boldsymbol{R}}\}$, where $H(\tilde{g})=\{h\in H\,|\,h^{-1}\tilde{g}h\tilde{g}^{-1}=(0, 0, \psi_{\tilde{g}}((h))\in Z\}$ for $\tilde{g}\in\tilde{G}$. The following is easily verified.

Lemma 4.2. *Assume $l>m+2n$. Then*

$$\dim_{\boldsymbol{C}}\mathfrak{S}(S, l)=\int_{\tilde{\Gamma}\backslash\mathscr{D}}\sum_{\tilde{\gamma}\in\tilde{\Gamma}'/Z(\tilde{\Gamma})}K_{S,l}(\tilde{\gamma}\langle Z\rangle, Z)J_{S,l}(\tilde{\gamma}, Z)^{-1}$$
$$|J_{S,l}(\tilde{g}_Z, Z_0)|^{-2}\,d\mu(Z).$$

For an integer r $(0\leq r\leq n)$, let $\tilde{\Pi}_r$ be the set consisting of $\tilde{\gamma}\in\tilde{\Gamma}$ which is $\tilde{\Gamma}$-conjugate to $h\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}$ $(h=(\xi, \eta, \kappa)\in\tilde{\Gamma}$, $x\in\mathrm{Sym}_n(\boldsymbol{Z})$, $\operatorname{rank}x=r)$. We set

$$I_{S,l}(\tilde{\Pi}_r)=\int_{\Gamma\backslash\mathscr{D}}\sum_{\tilde{\gamma}\in(\tilde{\Pi}_r\cap\tilde{\Gamma}')/Z(\tilde{\Gamma})} K_{S,l}(\tilde{\gamma}\langle Z\rangle, Z)J_{S,l}(\tilde{\gamma}, Z)^{-1} |J_{S,l}(\tilde{g}_Z, Z_0)|^{-2}\, d\mu(Z).$$

The sum $\sum_{r=0}^{n} I_{S,l}(\tilde{\Pi}_r)$ is called "the contribution of purely parabolic conjugacy classes to the dimension formula for $\mathfrak{S}(S, l)$". Observe that $\tilde{\Pi}_r\cap\tilde{\Gamma}'$ is

$$\left\{\tilde{\gamma}\in\tilde{\Pi}_r \,|\, \tilde{\gamma} \text{ is } \tilde{\Gamma}\text{-conjugate to } (0, (\eta', 0), \kappa)\left(\begin{array}{c|c} 1_n & \begin{matrix} x' & 0 \\ 0 & 0\end{matrix} \\ \hline & 1_n \end{array}\right)\right.$$
$$\left. \eta'\in M_{m,r}(\boldsymbol{Z}),\ x'\in \mathrm{Sym}_r(\boldsymbol{Z})\right\}.$$

Shintani's argument in the proof of Proposition 8 in [7] works also for our case (see also Theorem 3.2 (i)) and we have the following theorem.

Theorem 4.3. *The integral $I_{S,l}(\tilde{\Pi}_r)$ is absolutely convergent if $l\geq 2n+m+3$ and equals*

$$\frac{2^n\omega_{n-r}A_{S,l}}{U_{n-r}C_r}Z\left(f_r(X;l), \boldsymbol{L}^{(r)}, n-\frac{r-1}{2}\right).$$

Here

$$U_r=\prod_{k=1}^{r}\frac{2\pi^k}{\Gamma(k)},\quad C_r=\prod_{k=1}^{r}\frac{2\pi^{k/2}}{\Gamma(k/2)},\quad \omega_l=\zeta(2)\zeta(4)\cdots\zeta(2l)$$

and $\boldsymbol{L}^{(r)}=\mathrm{Sym}_r(\boldsymbol{Z})\times M_{m,r}(\boldsymbol{Z})$.

Combining Theorem 3.2 (ii), Lemma 3.1 and Remark 2.2 we have

Corollary 4.4.

$$I_{S,l}(\tilde{\Pi}_r)=(\det 2S)^{n-r}2^{r(n-r)-1}(2\pi)^{-(n-r)(n-r+1)/2}$$
$$\omega_{n-r}U_{n-r}^{-1}\xi_r^*(r-n;S)\prod_{i=1}^{n-r}\prod_{j=1}^{i}\left(l-\frac{m+n-i}{2}-j\right),$$

where we put $\xi_0^*(s;S)\equiv 2$.

Remark 4.5. Special values of $\xi_n^*(s;S)$ at non-positive integers are known only for $n\leq 2$. If $n=1$, as is well-known, we have

$$\xi_1^*(1-k;S)=-\sum_{u\in M^*/M}B_k(\langle -S[u]\rangle)/k,$$

where $B_k(x)$ is the k-th Bernoulli polynomial and for $x\in\boldsymbol{R}$ we define $\langle x\rangle$

by the condition: $x \equiv \langle x \rangle \pmod{\boldsymbol{Z}}$ and $0 < \langle x \rangle \leq 1$. On the other hand, when $n=2$, $\xi_2^*(s; S)$ is a linear combination of partial zeta functions studied in Arakawa [1]. Applying his results to this case we have $\xi_2^*(1-k; S) \in \boldsymbol{Q}$. (We can also evaluate $\xi_2^*(1-k; S)$ explicitly using formulas in [1]. For example $\xi_2^*(0; 1)=2^{-6}$, if $m=1$ and $S=1$.)

References

[1] Arakawa, T., Special values of L-functions associated with the space of quadratic forms and the representation of $Sp(2n, F_p)$ in the space of Siegel cusp forms, in this volume.

[2] Murase, A., L-functions attached to Jacobi forms of degree, n, Part I. The basic identity, to appear in J. Reine Angew. Math.

[3] Satake, I., A group extension and its unitary representation (in Japanese), Sugaku, **21** (1969), 241–253.

[4] Sato, F., Zeta functions in several variables associated with prehomogeneous vector spaces I: Functional equation, Tôhoku Math. J., **34** (1982), 437–483.

[5] Sato, M. and Shintani, T., On zeta functions associated with prehomogeneous vector spaces, Ann. of Math., **100** (1974), 131–170.

[6] Shintani, T., On Dirichlet series whose coefficients are class-numbers of integral binary cubic forms, J. Math. Soc. Japan, **24** (1972), 132–188.

[7] ——, On zeta functions associated with the vector space of quadratic forms, J. Fac. Sci. Univ. Tokyo, **22** (1975), 25–65.

[8] Suzuki, T., On zeta functions associated with quadratic forms of variable coefficients, Nagoya Math. J., **73** (1979), 117–147.

[9] Yamazaki, T., Jacobi forms and a Maass relation for Eisenstein series, J. Fac. Sci. Univ. Tokyo, **33** (1986), 295–310.

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