Advanced Studies in Pure Mathematics 15, 1989
Automorphic Forms and Geometry of Arithmetic Varieties
pp. 365–372

The Structure of the Icosahedral Modular Group

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Dedicated to Prof. Friedrich Hirzebruch and Prof. Ichiro Satake on their sixtieth birthdays

Introduction

By the group in the title we mean the Hilbert modular group PSL(2, $\mathcal{O}$) where $\mathcal{O}$ is the ring of integers in the number field $\boldsymbol{Q}(\sqrt{5})$. This naming here comes from the following fact: Hirzebruch [H] studied the irreducible action over $H\times H$ ($H:=\{z\in \boldsymbol{C};\ \mathrm{Im}(z)>0\}$) of the principal congruence subgroup $\Gamma=\Gamma(2)$ of $SL(2, \mathcal{O})$ associated with the prime ideal $(2)\subseteq\mathcal{O}$ and showed that thecom pactified quotient $\overline{H\times H/\Gamma}$ is equivariantly birational to the projective plane $P_2(\boldsymbol{C})$ acted nontrivially by the icosahedral group $\mathfrak{A}_5\cong PSL(2, \mathcal{O}/(2))$ ($\mathcal{O}/(2)\cong F_4$; $\mathfrak{A}_5$: the alternating group of degree five). Γ can be regarded as a subgroup of $PSL(2, \mathcal{O})$ since $-1\notin\Gamma$. It acts even freely on $H\times H$ and the description in [H] is so explicit that one can describe the arithmetic group Γ as some quotient group of π_1 of the complement of the icosahedral arrangement. This arrangement consists of the reflecting lines of fifteen involutions in $\mathfrak{A}_5$ and is therefore mapped into itself by the group. Thus, if one calculates π_1 of the quotient by $\mathfrak{A}_5$ of the complement, then one succeeds in determining the structure of $PSL(2, \mathcal{O})$. Since we have only a few Hilbert modular groups for which the group-structure is combinatorially described, this is naturally of interest and is the purpose of this note. The main result is stated as follows:

Theorem. *The icosahedral modular group $PSL(2, \mathcal{O})$ is isomorphic to the group generated by three elements α, β, γ with the fundamental relation;*

$$\begin{cases}\alpha\beta=\beta\alpha \quad \alpha\gamma\alpha=\gamma\alpha\gamma \quad \beta\gamma\beta\gamma\beta=\gamma\beta\gamma\beta\gamma \\ (\beta\gamma)^2=1 \quad (\alpha\gamma)^3=1.\end{cases}$$

Moreover, if we define the homomorphism $h: PSL(2, \mathcal{O})\to\mathfrak{A}_5$ by setting $h(\alpha)=(1,4)(2,5)$, $h(\beta)=(1,2)(4,5)$, $h(\gamma)=(1,4)(2,3)$, then we obtain the exact sequence:

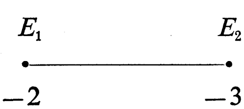
Received April 24, 1987.
Revised March 2, 1988.

$$1 \to \Gamma \to PSL(2, \mathcal{O}) \xrightarrow{h} \mathfrak{A}_5 \to 1.$$

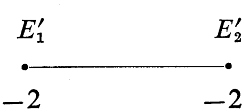
The motivation to this work was the hope that this description will serve the purpose of giving a fundamental region for the action over $H \times H$, which is a much harder problem.

§ 1. Quotient of $P_2(C)$ by $\mathfrak{A}_5$.

As the icosahedral group acts on $P_1(C) \cong S^2$, it acts also over $P_1(C) \times P_1(C)$, commuting the transposition τ of product components. It thus acts on $P_2(C) \cong P_1(C) \times P_1(C)/\tau$. This action was thoroughly investigated by F. Klein [K]. It is induced by the three-dimensional irreducible representation of $\mathfrak{A}_5$ which is unique up to automorphisms, and the most reasonable generators for the ring of invariants are given by him. They are denoted by A, B, C, D and are of degree 2, 6, 10, 15. For the present section we are mainly interested in A and D. In this note we denote the curves $A=0$, $D=0$ etc. by $[A]$, $[D]$, etc. $[A]$ is the image of the diagonal of $P_1(C) \times P_1(C)$ and $[D]$ is the union of the fifteen reflecting lines mentioned in the introduction. We sometimes call this set of lines the icosahedral arrangement. Outside $[D]$ there are exactly two orbits of points, at which $\mathfrak{A}_5$ has a non-trivial isotropy subgroup: they are isolated points. They all lie on the invariant conic $[A]$. At one of the orbits the isotropy subgroup is of order 5 and it is of order 3 at the other orbit. Thus the quotient $S := P_2(C)/\mathfrak{A}_5$ has exactly two quotient singular points which are both cyclic. One is resolved by the exceptional set of the graph:



and the graph for the other is:



E_i, E_i' $(i=1, 2)$ are all rational curves on the minimal desingularization of S, which we denote by $\hat{S}$. We denote the images on S of $[A]$, $[D]$ etc. by $[\![A]\!]$, $[\![D]\!]$ etc. and let $[\![\hat{A}]\!]$, $[\![\hat{D}]\!]$ stand for their strict transforms on $\hat{S}$. $[\![\hat{A}]\!]$ is nonsingular; $[\![\hat{D}]\!]$ has three singular points which are rational double of Types A_1, A_2, A_4; they correspond to the intersection points of the icosahedral arrangement with multiplicity 2, 3, 5. $[\![\hat{D}]\!]$ is disjoint from E_i, E_i'; but $[\![\hat{A}]\!]$ cuts once the exceptional sets at E_2, E_2' transversally (E_i'

are so indexed). One sees moreover that $[\![\hat{A}]\!]$ is an exceptional curve of the first kind. This implies that one can successively blow down the curves $[\![\hat{A}]\!]$, E_2', E_2 and E_1, and that, by this, E_1' is mapped to a rational curve of self-intersection number 1. We denote this blowing down of $\hat{S}$ by $\check{S}$ and the image of E_1' by L. By computing directly we see further that the Euler number of $\check{S}$ is equal to 3. Thus $\check{S}$ is also isomorphic to the plane $P_2(C)$ and L is a line on it. Now we denote the image to $\check{S}$ of the curve $[\![\hat{D}]\!]$ by $[\![\check{D}]\!]$; it has four singular points of Types A_1, A_2, A_3, A_4 in which the last one arises from the blowing down. As the image of a line in $[D]$, $[\![\check{D}]\!]$ is rational; so it is a quintic curve on $\check{S}\cong P_2(C)$. $[\![\check{D}]\!]$ and L intersect only at the last A_4-singular point with multiplicity 5.

§ 2. Computation of the fundamental group

In this section we preserve the notation in the previous section. Our object here is to compute the fundamental group of $P_2(C)-[D]$. Note that $[A]$ intersects $[D]$ transversally, Thus, by using [O-S] and by putting a line l in $[D]$ at infinity, we have the isomorphism:

$$\text{(2.1)} \qquad \pi_1(P_2(C)-([A]\cup[D]),\ *)\cong\pi_1(P_2(C)-[D],\ *)\times Z$$

where $*$ denotes the reference point chosen. Here the product is direct and we have used $\pi_1(P_2(C)-([A]\cup l),\ *)\cong Z$. Thus it suffices that we compute the left hand side of (2.1). Now recall that $\mathfrak{A}_5$ acts freely on the complement $P_2(C)-([A]\cup[D])$. On the other hand we have seen in Section 1 that

$$\begin{aligned}\{P_2(C)-([A]\cup[D])\}/\mathfrak{A}_5 &\cong P_2(C)-(L\cup[\![\check{D}]\!])\\ &\cong C^2-([\![\check{D}]\!]\cap C^2)\end{aligned}$$

where we have introduced the identifications $\check{S}=P_2(C)$, $P_2(C)-L=C^2$, Denoting for brevity the affine quintic curve $[\![\check{D}]\!]\cap C^2$ by $\tilde{D}$, we thus obtain the exact sequence:

$$\text{(2.2)} \quad 1\to\pi_1(P_2(C)-([A]\cup[D]),\ *)\to\pi_1(C^2-\tilde{D},\ *)\xrightarrow{\ h\ }\mathfrak{A}_5\to 1.$$

By this we see that it suffices to compute $\pi_1(C^2-\tilde{D},\ *)$. Now it is necessary to write down the equation of $\tilde{D}$ explicitly i.e. we have to recall the construction of Section 1 with explicit expressions. It is summarized in the following:

$$\begin{array}{ccc} P_2(C) & \overset{\delta}{\dashrightarrow} & \check{S}\cong P_2(C) \\ {\scriptstyle q}\downarrow & & \uparrow{\scriptstyle \beta}\ \wr \\ S=P_2(C)/\mathfrak{A}_5 & \underset{\pi}{\overset{\sim}{\longleftarrow}} & \hat{S} \end{array}$$

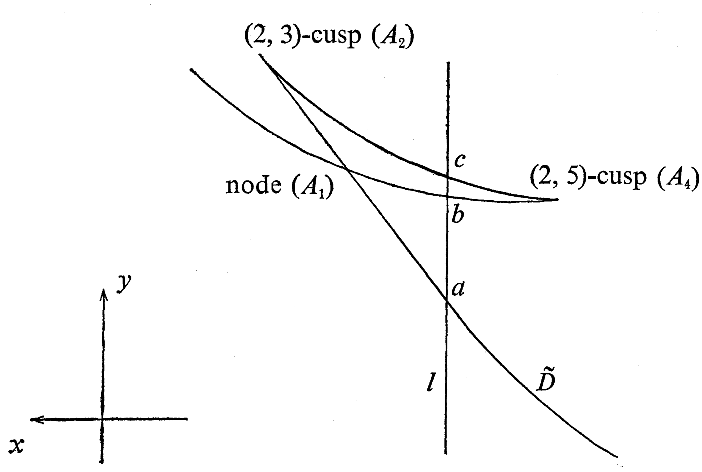
where $\xrightarrow{\sim}$ means a birational morphism, q is the natural map, π is the minimal desingularization of S, β is the blowing-down mentioned in Section 1 and δ is the rational mapping defined by the diagarm and to be written down explicitly. Now, using the properties of Klein's invariants A, B, C, D, we can see that the mapping δ is given by assigning the proportionality $(A^5 : A^2B : C) \in P_2(\boldsymbol{C})$ to the generic point of the original $P_2(\boldsymbol{C})$ on which the curves $[A]$, $[D]$ sit. The line L is obviously given by putting the first coordinate to be zero. If we introduce the affine coordinates x, y for $\check{S}-L$ by identifying (x, y) with $(1 : x : y)$ in the target plane of δ, then δ is given by $x=B/A^3$, $y=C/A^5$. Now, by [K], we have the following identity:

$$144D^2 = -1728B^2 + 720ACB^3 - 80A^2C^2B + 64A^3(5B^2 - AC)^2 + C^3.$$

This implies that the equation of the affine curve $\tilde{D}$ is:

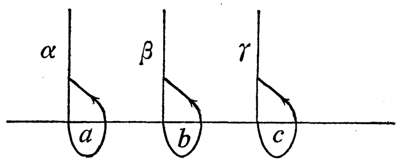
$$-1728x^5 + 720x^3y - 80xy^2 + 64(5x^2 - y)^2 + y^3 = 0. \tag{2.3}$$

The line-family $x=$const. are exaclty the lines passing through the A_4-singular point of $[\![\check{D}]\!]$ at infinity. They intersect $[\![\check{D}]\!]$ at this double point with multiplicity 2, since L, the unique line with higher contact, is not included in the family. This explains why the left hand side is of degree 3 with respect to y. Note also that all the singular points of $\tilde{D}$ are in the real part $\boldsymbol{R}^2 \subseteq \boldsymbol{C}$. This suggests that the real configuration is essential for the computation of $\pi_1(\boldsymbol{C}^2 - \tilde{D}, *)$: It looks like the following figure:



The vertical line, denoted by l, is the standard line which we fix for the computation of $\pi_1(\boldsymbol{C}^2 - \tilde{D}, *)$, and a, b, c are the intersection points of l

and $\tilde{D}$, ordered from below. Now we take three paths α, β, γ (issuing from the base point $*$ in the upper half-plane which one can for example take to be i) in the complex feature of the line l as follows:



Then we apply the well known method, which is attributed to Zariski and Van Kampen, to the computation of the required fundamental group. The first fact to be mentioned is that α, β, γ generate $\pi_1(C^2-\tilde{D}, *)$; so we have only to get adequate relations among them. For this, the method above suggests us to move the standard line l within the family $x=\text{const.}$ along the real axis in the x-plane so far as it does not meet any singular point of $\tilde{D}$, to surround or avoid the singular point by escaping in a sufficient small size into the complex part, and to turn back to the original position of l; the paths α, β, γ are by this kind of processes deformed to other paths say $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, which can of course be written as some products of α, β, γ. Since $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ are obviously identical with α, β, γ in $\pi_1(C^2-\tilde{D}, *)$, we get the corresponding relations among α, β, γ. Skipping detailed discussions, we simply assert that the relations:

$$(2.4)\qquad \begin{cases} \alpha\beta=\beta\alpha \qquad \alpha\gamma\alpha=\gamma\alpha\gamma \\ \beta\gamma\beta\gamma\beta=\gamma\beta\gamma\beta\gamma, \end{cases}$$

are obtained by the above procedure, corresponding to the node, the (2, 3)-cusp and the (2, 5)-cusp of the curve $\tilde{D}$ respectively. There is no essential relation other than (2.4) since we do not have any line $x=\text{const.}$ which is tangent to $\tilde{D}$. Now we note that the composite path $\alpha\beta\gamma$ surrounds once the (2, 5)-cusp at infinity. According to the description of the blowing down β in Section 1, this implies that $\alpha\beta\gamma$ surrounds only the exceptional curve E_1 in $\hat{S}$. Since $E_1\cup E_2$ is the exceptional set of the cyclic quotient singularity on $[\![A]\!]$ of order 5, the power $(\alpha\beta\gamma)^5$ is the image of a path which surrounds the conic $[A]$ once. By (2.1), (2.2) this should lie in the center of $\pi_1(C^2-\tilde{D})$, which one can in fact check directly by using the relations (2.4). We have also seen by this argument that $\alpha\beta\gamma$ is mapped to an element of order five by the homomorphism $h:\pi_1(C^2-\tilde{D})\to\mathfrak{A}^5$ of (2.2). We may of course assume that $h(\alpha\beta\gamma)=(1, 2, 3, 4, 5)$. Then $h(\alpha)$, $h(\beta)$, $h(\gamma)$ are almost uniquely determined:

$$\begin{cases} h(\alpha)=(1, 4)(2, 5) \\ h(\beta)=(1, 2)(4, 5) \\ h(\gamma)=(1, 4)(2, 3). \end{cases} \tag{2.5}$$

(From the definition of the arrangement D, the elements α, β, γ should be mapped onto some involutions of $\mathfrak{A}_5$.) We have thus completely described $\pi_1(C^2-\tilde{D})$, and hence also $\pi_1(P_2(C)-([A]\cup[D]), *)$, etc. In particular we obtain the exact sequence:

$$1 \longrightarrow \pi_1(P_2(C)-[D], *) \longrightarrow G_1 \xrightarrow{h} \mathfrak{A}_5 \longrightarrow 1$$

where G_1 is the group generated by α, β, γ with the fundamental relations (2.4) and

$$(\alpha\beta\gamma)^5=1. \tag{2.6}$$

§ 3. The icosahedral modular group

As in the introduction we let $\mathcal{O}$ be the ring of integers of the number field $Q(\sqrt{5})$, $\mathfrak{p}=(2)$ the prime ideal generated by 2 and Γ the principal congruence subgroup of $SL(2, \mathcal{O})$ for the level $\mathfrak{p}$. We begin by recalling Hirzebruch's description [H] of the compactified quotient $\check{Y}=\overline{H\times H/\Gamma}$; it has exactly five cusps, each of which is resolved minimally by a triangle of rational curves with self-intersection number -3. He shows that the closure of the image of diagonal $\{(z, z); z \in H\}$ is lifted to an exceptional curves of the first kind on the minimal desingularization Y of $\check{Y}$ and that there are exactly ten transforms of it by $\mathfrak{A}_5\cong PSL(2, \mathcal{O}/\mathfrak{p})$ which are disjoint from each other. The separate blowing down of these ten curves is the cubic diagonal surface of Clebsch and Klein (X: $\sum_i y_i=\sum_i y_i^3=0$ in $P_4(C)$: $(y_0, y_1, \cdots, y_4)$) with their image being the Eckardt points of the surface. (See [S] for this terminaology.) The five exceptional sets give rise to fifteen lines on the cubic surface and the remaining twelve lines on the surface separate themselves into two orbits under the action of $\mathfrak{A}_5$ which form a "double six" [S]. We take one of the two and blow down the lines belonging to it in order to get the plane $P_2(C)$ and the action over it. This last action is identical with the one discussed in Section 1 and the fifteen lines on X are mapped onto the icosahedral arrangement. The inverse procedure of getting quotient $H\times H/\Gamma$ from the arrangement is also clear for us: We blow up the six fivefold intersection points of the arrangement to get the diagonal surface and then we blow up the ten threefold intersection points of the arrangement to get Y. To obtain $H\times H/\Gamma$ one needs only delete the strict transforms on Y of the lines in the arrange ment. Since we also know by [H] that Γ acts freely on $H\times H$, the group

Γ is described to be π_1 of this complement in Y. By the first blowing up we should add to (2.4), (2.6) the following relation:

$$(\beta\gamma)^5=1 \tag{3.1}$$

and by the second blowing up the relation:

$$(\alpha\gamma)^3=1. \tag{3.2}$$

The paths $(\beta\gamma)^5$ and $(\alpha\gamma)^3$ correspond to paths surrounding the exceptional curves for the two kinds of blowings-up, which therefore become upstairs homotope to zero. (Check that $(\beta\gamma)^5$, $(\alpha\gamma)^3$ both lie in the kernel of the homomorphism h in (2.2).) Thus we have arrived at the exact sequence

$$1 \longrightarrow \Gamma \longrightarrow G_2 \xrightarrow{h} \mathfrak{A}_5 \longrightarrow 1 \tag{3.3}$$

where G_2 is the group generated by α, β, γ with the fundamental relations (2.4), (2.6), (3.1), (3.2) and h is again defined by (2.5). It is obvious that we have also prove the isomorphism

$$G_2 \cong PSL(2, \mathcal{O}). \tag{3.4}$$

This and the sequence (3.3) should be regarded as the main result of the note. It is now an easy work to find the matrix elements of the group in the right hand side corresponding to α, β, γ:

$$\alpha=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

$$\beta=\begin{pmatrix}1 & u^2\\ 0 & 1\end{pmatrix} \qquad u=\frac{\sqrt{5}-1}{2}$$

$$\gamma=\begin{pmatrix}0 & 1\\ -1 & 2\end{pmatrix}$$

We finally remark that these are all in $SL(2, \mathcal{O})$, that the relations (2.4), (2.6) are satisfied in $SL(2, \mathcal{O})$-level while we have $(\beta\gamma)^5=(\alpha\gamma)^3=-1$.

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