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T-Complexes and Ogata's Zeta Zero Values

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Dedicated to Prof. Ichiro Satake and Prof. Friedrich Hirzebruch on their sixtieth birthdays

Introduction

In [T1], Tsuchihashi defined the notion of cusp singularities in arbitrary dimension. They include the Hilbert modular cusp singularities as a special case. In this paper, we will show the rationality of the zeta zero value $Z(C, \Gamma; 0)$ of the zeta function associated to a Tsuchihashi cusp singularity which was defined by Ogata [Og]. He gave a formula for the zero value as a sum of integrals of C^∞-functions described by the characteristic function of the convex cone C. By this formula, he showed that the value is a half-integer in odd-dimensional case [Og, Theorem 2.3]. In two dimensional case, the singularity is a Hilbert modular cusp and 12 times the zeta zero value is an integer by [Z].

By the construction of Tsuchihashi cusp singularities, they have toroidal resolutions and the exceptional sets are toric divisors in the sense of [S2]. In order to describe toric divisors, we introduce the notion of *T-complexes* which is essentially equal to that of the weighted dual graphs which appear in [T1]. A T-complex Σ is a category with a finite number of objects. We define a functor $D^0_{\boldsymbol{Q}}$ from Σ to the category of $\boldsymbol{Q}$-vector spaces. We show that the rational number field $\boldsymbol{Q}$ has a natural injection into the inductive limit $\operatorname{ind\,lim}_\Sigma D^0_{\boldsymbol{Q}}$. We define a special element ω_Σ of $\operatorname{ind\,lim}_\Sigma D^0_{\boldsymbol{Q}}$. When Σ is the T-complex associated to a toroidal resolution of a Tsuchihashi cusp singularity (C, Γ), Ogata's formula means that there exists an explicit retraction $\operatorname{ind\,lim}_\Sigma D^0_{\boldsymbol{Q}} \otimes \boldsymbol{R} \to \boldsymbol{R}$ and the zero value $Z(C, \Gamma; 0)$ is the image of ω_Σ in $\boldsymbol{R}$. By using an equality in Section 1 for a nonsingular complete fan, we show that ω_Σ is in the image of $\boldsymbol{Q}$ in $\operatorname{ind\,lim}_\Sigma D^0_{\boldsymbol{Q}}$ for any T-complex Σ. This implies that the image of ω_Σ in $\boldsymbol{R}$ is independent of the retraction and is a rational number.

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§ 1. An equality for a nonsingular complete fan

Let N be a free $\boldsymbol{Z}$-module of finite rank and let $N_{\boldsymbol{R}}=N\otimes_{\boldsymbol{Z}}\boldsymbol{R}$. A nonempty subset σ of $N_{\boldsymbol{R}}$ is said to be a *strongly convex rational polyhedral cone* (*s.c.r.p. cone* for short) if there exists a finite subset $\{n_1, \cdots, n_s\}\subset N$ such that $\sigma=\boldsymbol{R}_0 n_1+\cdots+\boldsymbol{R}_0 n_s$ with $\sigma\cap(-\sigma)=\{0\}$, where $\boldsymbol{R}_0=\{c\in\boldsymbol{R};\ c\geq 0\}$. An s.c.r.p. cone σ is said to be *nonsingular* if there exists a $\boldsymbol{Z}$-basis $\{n_1, \cdots, n_r\}$ of N such that $\sigma=\boldsymbol{R}_0 n_1+\cdots+\boldsymbol{R}_0 n_s$ for an integer $0\leq s\leq r$. We call $\{n_1, \cdots, n_s\}$ *the canonical set of generators for* σ and denote it by $\operatorname{gen}\sigma$ when σ is nonsingular. This set is uniquely determined by σ.

A nonempty collection Δ of s.c.r.p. cones is said to be a *fan* if (1) $\tau\in\Delta$ and $\sigma\prec\tau$, which means that σ is a face of τ, imply $\sigma\in\Delta$, and (2) if $\sigma, \tau\in\Delta$ then $\sigma\cap\tau$ is a common face of σ and τ. For an s.c.r.p. cone π in $N_{\boldsymbol{R}}$, we denote by $\Gamma(\pi)$ the set of faces of π. Clearly, $\boldsymbol{0}:=\{0\}$ and π are in $\Gamma(\pi)$, and $\Gamma(\pi)$ is a fan in $N_{\boldsymbol{R}}$. A fan Δ in $N_{\boldsymbol{R}}$ is said to be *complete* if it is finite and the union $\bigcup_{\sigma\in\Delta}\sigma$ is equal to $N_{\boldsymbol{R}}$. It is said to be *nonsingular* if it consists of nonsingular cones.

For an s.c.r.p. cone ρ, we denote $N[\rho]:=N/(N\cap(\rho+(-\rho)))$. $N[\rho]$ is a free $\boldsymbol{Z}$-module with its rank equal to $\operatorname{codim}\rho$. For a cone σ with $\rho\prec\sigma$, we denote by $\sigma[\rho]$ the image of σ in the quotient $N[\rho]_{\boldsymbol{R}}=N_{\boldsymbol{R}}/(\rho+(-\rho))$. If ρ is an element of a fan Δ, then $\Delta[\rho]=\{\sigma[\rho];\ \sigma\in\Delta \text{ and } \rho\prec\sigma\}$ is a fan in $N[\rho]_{\boldsymbol{R}}$.

For each nonzero element x in N, we denote $\gamma(x):=\boldsymbol{R}_0 x$. The cone $\gamma(x)$ is nonsingular of dimension one. If x is primitive, then $\operatorname{gen}\gamma(x)=\{x\}$.

Let Δ be a nonsingular fan in $N_{\boldsymbol{R}}$. For a subset Φ of Δ, we set

$$f(\Phi):=\sum_{\sigma\in\Phi}\prod_{x\in\operatorname{gen}\sigma}\frac{1}{\exp(x)-1},$$

where we understand $\prod_{x\in\operatorname{gen}\sigma}(1/(\exp(x)-1))=1$ for $\sigma=\{0\}$. Let $M:=\operatorname{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})$ be the $\boldsymbol{Z}$-module dual to N. Since $x\in N$ is a linear function on the complex space $M_{\boldsymbol{C}}=M\otimes_{\boldsymbol{Z}}\boldsymbol{C}$, $\exp(x)$ is understood to be a holomorphic function on $M_{\boldsymbol{C}}$. Hence $f(\Phi)$ is a meromorphic function on $M_{\boldsymbol{C}}$. Note that the meromorphic function $1/(\exp(z)-1)$ of a complex variable z has poles of order one at each point of $2\pi i\boldsymbol{Z}$ and it has no pole elsewhere. Since no $\operatorname{gen}\sigma$ contains both x and $-x$ for any $x\in N$, $f(\Phi)$ may have poles of order at most one along the hyperplanes $H_{x,d}=(x=2\pi id)\subset M_{\boldsymbol{C}}$ for $x\in G(\Delta)$ and $d\in\boldsymbol{Z}$, where $G(\Delta):=\bigcup_{\sigma\in\Delta}\operatorname{gen}\sigma$. Note that $G(\Delta)$ is in one-to-one correspondence with the set $\Delta(1)$ of cones of dimension one in Δ by $x\mapsto\gamma(x)$.

In the rest of this section, we devote ourselves to proving the following theorem.

Theorem 1.1. *Let n be the rank of N. If $n \geq 1$ and Δ is a nonsingular complete fan, then $f(\Delta)$ is equal to zero.*

We prove the theorem by induction on n.

If $n=1$, then there is only one complete fan and the theorem is true since

$$1/(\exp(x)-1)+1/(\exp(-x)-1)+1=0.$$

Let $r \geq 2$ be an integer. We assume that the theorem is true for $1 \leq n \leq r-1$. We now suppose $n=r$.

Lemma 1.2. *In the above situation, $f(\Delta)$ is an entire function on $M_{\boldsymbol{C}}$.*

Proof. Since $f(\Delta)$ is periodic with respect to the subgroup $2\pi i M \subset M_{\boldsymbol{C}}$, it is sufficient to show that $f(\Delta)$ has no pole along the hyperplane $H_{x,0}$ for each $x \in G(\Delta)$. Let $\gamma := \gamma(x)$ and $\Delta(\gamma \prec) := \{\sigma \in \Delta; \gamma \prec \sigma\}$. By definition, we have $f(\Delta) = f(\Delta(\gamma \prec)) + f(\Delta \backslash \Delta(\gamma \prec))$. It is easy to see that the restriction $xf(\Delta(\gamma \prec))|_{H_{x,0}}$ is equal to $f(\Delta[\gamma])$, where $\Delta[\gamma]$ is the fan in $N[\gamma]_{\boldsymbol{R}} = N_{\boldsymbol{R}}/(\gamma+(-\gamma))$ induced by Δ. Since $\Delta[\gamma]$ is also a nonsingular complete fan and $N[\gamma] = N/(N_{\boldsymbol{R}} \cap (\gamma+(-\gamma)))$ is of rank $r-1$, $f(\Delta[\gamma])$ is equal to zero by the induction assumption. Hence $f(\Delta(\gamma \prec))$ has no pole along $H_{x,0}$. If $-\gamma$ is not an element of Δ, then $f(\Delta \backslash \Delta(\gamma \prec))$ has no pole at $H_{x,0}$ by definition. Hence so is $f(\Delta)$. Suppose $-\gamma \in \Delta$. Then $f(\Delta \backslash \Delta(\gamma \prec)) = f(\Delta(-\gamma \prec)) + f(\Delta \backslash (\Delta(\gamma \prec) \cup \Delta(-\gamma \prec)))$ has no pole at $H_{x,0}$, since $(-x)f(\Delta(-\gamma \prec))$ is zero on $H_{x,0}$ similarly as above. q.e.d.

Let m be an element of M. We define the $\boldsymbol{C}$-linear mapping $\varphi_m : \boldsymbol{C} \to M_{\boldsymbol{C}}$ by $\varphi_m(t) = tm$ for $t \in \boldsymbol{C}$. We denote by $g_m(t)$ the pull-back $\varphi_m^*(f(\Delta))$. By the above lemma, $g_m(t)$ is an entire function on $\boldsymbol{C}$.

Lemma 1.3. *Suppose m is not in $x^{\perp} := \{v \in M_{\boldsymbol{R}}; \langle v, x\rangle = 0\}$ for any $x \in G(\Delta)$. Then $g_m(t) = 0$.*

Proof. By the definition of $f(\Delta)$, we have

$$g_m(t) = \sum_{\sigma \in \Delta} \prod_{x \in \mathrm{gen}\, \sigma} \{(1/(\exp(a_x t)-1)\},$$

where $a_x = \langle m, x\rangle$ for each $x \in G(\Delta)$. The integers a_x's are not zero by assumption. By Lemma 1.2, $g_m(t)$ is an entire function on $\boldsymbol{C}$. Since a_x's are integers, $g_m(t)$ has the periodicity $g_m(t+2\pi i) = g_m(t)$. Hence, in order to prove $g_m(t)=0$, it is sufficient to show that

$$|g_m(t)| \longrightarrow 0, \qquad \text{as } |\mathrm{Re}\, t| \longrightarrow \infty.$$

Let $H(m)$ be the hyperplane $\{u \in N_{\boldsymbol{R}};\ \langle m, u\rangle = 0\}$ in $N_{\boldsymbol{R}}$, and let $H^+(m) := \{u \in N_{\boldsymbol{R}};\ \langle m, u\rangle \geq 0\}$ and $H^-(m) := \{u \in N_{\boldsymbol{R}};\ \langle m, u\rangle \leq 0\}$. It is clear that

$$1/(\exp(a_x t)-1) \longrightarrow 0, \quad \text{as } \operatorname{Re} a_x t \longrightarrow \infty, \quad \text{and}$$
$$1/(\exp(a_x t)-1) \longrightarrow -1, \quad \text{as } \operatorname{Re} a_x t \longrightarrow -\infty.$$

Since $\sigma \in \Delta$ is contained in $H^+(m)$(resp. $H^-(m)$) if and only if $a_x > 0$ (resp. $a_x < 0$) for every $x \in \operatorname{gen} \sigma$, we have

$$g_m(t) \longrightarrow \sum_{\sigma \in \Delta^-}(-1)^{\dim \sigma}, \quad \text{as } \operatorname{Re} t \longrightarrow \infty, \quad \text{and}$$
$$g_m(t) \longrightarrow \sum_{\sigma \in \Delta^+}(-1)^{\dim \sigma}, \quad \text{as } \operatorname{Re} t \longrightarrow -\infty,$$

where $\Delta^- := \{\sigma \in \Delta;\ \sigma \subset H^-(m)\}$ and $\Delta^+ := \{\sigma \in \Delta;\ \sigma \subset H^+(m)\}$. We set $\Phi := \{\sigma \cap H^+(m);\ \sigma \in \Delta \backslash \Delta^-\}$ and $\Phi_0 := \{\sigma \cap H(m);\ \sigma \in \Delta\}$. Then $\Phi \cap \Phi_0 = \emptyset$ and $\Phi \cup \Phi_0$ is a finite polyhedral decomposition of $H^+(m)$. By [1, Lemma 1.6], for example, we have $\sum_{\tau \in \Phi}(-1)^{\dim \tau} = (-1)^r$. The same lemma also implies that $\sum_{\sigma \in \Delta}(-1)^{\dim \sigma} = (-1)^r$. Since the map $\Delta \backslash \Delta^- \to \Phi$ which sends σ to $\sigma \cap H^+(m)$ is bijective and preserves the dimension of cones, the limit $\sum_{\sigma \in \Delta^-}(-1)^{\dim \sigma}$ in the first case is equal to zero. The second limit is also zero since the equality $\sum_{\sigma \in \Delta^+}(-1)^{\dim \sigma} = 0$ is proved similarly. q.e.d.

By Lemma 1.3, the entire function $\varphi(\Delta)$ is zero on rational lines $\boldsymbol{R}m$ in $M_{\boldsymbol{R}}$ which are not contained in $\bigcup_{x \in G(\Delta)} x^{\perp}$. Clearly, the union of such lines is dense in $M_{\boldsymbol{R}}$. Thus $\varphi(\Delta)$ is zero on $M_{\boldsymbol{R}}$. Since $\varphi(\Delta)$ is an analytic function, it is also zero on $M_{\boldsymbol{C}}$. q.e.d.

Remark 1.4. Let $\sum_{k=0}^{\infty} B_k t^{k-1}/k!$ be the power series expansion of $1/(\exp(t)-1)$. The coefficients B_k's are known as the Bernoulli numbers. The above theorem for $n=1$ means the wellknown fact that $B_1 = -1/2$ and $B_{2k+1} = 0$ for $k > 0$.

§ 2. *T*-complexes

We denote by $\mathscr{C}$ the category of pairs (N, σ) of a free $\boldsymbol{Z}$-module N of finite rank and an s.c.r.p. cone σ in $N_{\boldsymbol{R}}$. For an object $\alpha \in \mathscr{C}$, we denote $\alpha = (N(\alpha), \sigma(\alpha))$. For two objects $\alpha, \beta \in \mathscr{C}$, a morphism $u: \alpha \to \beta$ consists of an isomorphism $u_{\boldsymbol{Z}}: N(\alpha) \to N(\beta)$ such that $u_{\boldsymbol{R}}(\sigma(\alpha))$ is a face of $\sigma(\beta)$, where $u_{\boldsymbol{R}} = u_{\boldsymbol{Z}} \otimes 1_{\boldsymbol{R}}: N(\alpha)_{\boldsymbol{R}} \xrightarrow{\sim} N(\beta)_{\boldsymbol{R}}$. Since any morphism u is determined by the isomorphism $u_{\boldsymbol{Z}}$, the following lemma is obvious.

Lemma 2.1. *Every morphism $u: \alpha \to \beta$ in $\mathscr{C}$ is epimorphic and monomorphic.*

For an object $\alpha \in \mathscr{C}$, we denote $r(\alpha):=\operatorname{rank} N(\alpha)$ and $d(\alpha):=\dim \sigma(\alpha)$. Clearly, $0 \leq d(\alpha) \leq r(\alpha)$ for any α. For each nonnegative integer r, we denote by $\mathscr{C}_r$ the subcategory of $\mathscr{C}$ consisting of $\alpha \in \mathscr{C}$ with $r(\alpha)=r$. It is obvious that the category $\mathscr{C}$ is the disjoint union of $\mathscr{C}_r$'s.

Definition 2.2. A subcategory Σ of $\mathscr{C}$ is said to be a *graph of cones* if the class $\operatorname{mor} \Sigma$ of morphisms in Σ is a finite set.

If Σ is a graph of cones, then Σ consists of finite objects, since $1_\alpha \in \operatorname{mor} \Sigma$ for each $\alpha \in \Sigma$.

Example 2.3. Let Δ be a finite fan of $N_{\boldsymbol{R}}$. Then we regard Δ as a graph of cones by identifying Δ with $\{(N, \sigma); \sigma \in \Delta\}$ and by defining morphism $u:(N, \sigma) \rightarrow (N, \tau)$ to be in $\operatorname{mor} \Delta$ if and only if $u_{\boldsymbol{Z}}$ is the identity. Furthermore, any subset Φ of Δ is similarly considered to be a graph of cones.

Let Σ be a graph of cones and let ρ be an element of Σ. We denote by $\Sigma(\rho\prec)$ the comma category consisting of the pairs $\beta'=(\beta, v)$ of an element $\beta \in \Sigma$ and a morphism $v: \rho \rightarrow \beta$ in $\operatorname{mor} \Sigma$. A morphism $u': \beta'=(\beta, v) \rightarrow \gamma'=(\gamma, w)$ in the category $\Sigma(\rho\prec)$ consists of a morphism $u: \beta \rightarrow \gamma$ such that $u \circ v=w$. By defining $N(\beta')=N(\beta)$, $\sigma(\beta')=\sigma(\beta)$ and $u'_{\boldsymbol{Z}}=u_{\boldsymbol{Z}}$, we see that $\Sigma(\rho\prec)$ is also a graph of cones and is contained in $\mathscr{C}_{r(\rho)}$. Similarly, the comma category $\Sigma(\prec\rho)$ consists of pairs $\beta'=(\beta, v)$ of $\beta \in \Sigma$ and $(v: \beta \rightarrow \rho) \in \operatorname{mor} \Sigma$. $\Sigma(\prec\rho)$ is also a graph of cones contained in $\mathscr{C}_{r(\rho)}$ if we define $N(\beta')$ and $\sigma(\beta')$ in the same way as above.

Let $\beta'=(\beta, v)$ be an element of $\Sigma(\rho\prec)$. Then $v_{\boldsymbol{R}}(\sigma(\rho)) \subset N(\beta)_{\boldsymbol{R}}$ is an s.c.r.p. cone of dimension $d(\rho)$. We denote by $\beta'[\rho]$ the object of $\mathscr{C}_{r(\rho)-d(\rho)}$ with $N(\beta'[\rho]):=N(\beta)[v_{\boldsymbol{R}}(\sigma(\rho))]$ and $\sigma(\beta'[\rho]):=\sigma(\beta)[v_{\boldsymbol{R}}(\sigma(\rho))]$. Let $u': \beta' \rightarrow \gamma'=(\gamma, w)$ be a morphism in $\Sigma(\rho\prec)$. Then the isomorphism $u'_{\boldsymbol{Z}}: N(\beta) \rightarrow N(\gamma)$ induces a morphism $\beta'[\rho] \rightarrow \gamma'[\rho]$ in $\mathscr{C}_{r(\rho)-d(\rho)}$ which we denote by $u'[\rho]$. Hence if we set $\Sigma[\rho]:=\{\beta'[\rho]: \beta' \in \Sigma(\rho\prec)\}$ and $\operatorname{mor} \Sigma[\rho]:=\{u'[\rho]; u' \in \operatorname{mor} \Sigma(\rho\prec)\}$, $\Sigma[\rho]$ is a graph of cones contained in $\mathscr{C}_{r(\rho)-d(\rho)}$ which is naturally isomorphic to $\Sigma(\rho\prec)$ as categories. We call $\Sigma[\rho]$ *the link of* Σ *at* ρ.

A subcategory Φ of Σ is said to be *star closed* if $\Phi(\rho\prec)=\Sigma(\rho\prec)$ for every $\rho \in \Phi$, and *star open* if $\Phi(\prec\rho)=\Sigma(\prec\rho)$ for every $\rho \in \Phi$. Since star closed or star open subcategories are full subcategories, we also call them star closed or star open *subsets* of Σ, respectively.

Definition 2.4. A *homomorphism* $\varphi: \Sigma \rightarrow \Sigma'$ of graphs of cones consists of a functor $\bar{\varphi}: \Sigma \rightarrow \Sigma'$ and a collection $\{\varphi_\alpha; \alpha \in \Sigma\}$ of morphisms $\varphi_\alpha: \alpha \rightarrow \bar{\varphi}(\alpha)$ such that the diagram

$$\begin{array}{ccc} \alpha & \xrightarrow{u} & \beta \\ {\scriptstyle\varphi_\alpha}\downarrow & & \downarrow{\scriptstyle\varphi_\beta} \\ \bar{\varphi}(\alpha) & \xrightarrow{\bar{\varphi}(u)} & \bar{\varphi}(\beta) \end{array}$$

is commutative for every $u:\alpha\to\beta$ in mor Σ. φ is said to be an *isomorphism* if $\bar{\varphi}$ is isomorphic, i.e., $\bar{\varphi}$ induces a bijection mor $\Sigma\to$mor Σ', and all φ_α are isomorphisms.

For a morphism $u:\alpha\to\beta$ in $\mathscr{C}$, we denote $i(u):=\alpha$ and $f(u):=\beta$. The connectedness of a graph of cones is defined in the same way as that of usual graphs. Namely, Σ is connected if and only if the equivalence relation generated by $i(u)\sim f(u)$ for $u\in$ mor Σ has at most one equivalence class.

Now we are ready to define the notion of T-complexes.

Definition 2.5. A graph of cones Σ is said to be a *T-complex* if

(1) Σ is nonempty and connected,

(2) for any $\rho\in\Sigma$, the comma category $\Sigma(\prec\rho)$ is isomorphic to $\Gamma(\sigma(\rho))\backslash\{\mathbf{0}\}$ as graphs of cones, and

(3) for any $\rho\in\Sigma$ the link $\Sigma[\rho]$ is isomorphic to a complete fan.

Furthermore, Σ is said to be *nonsingular* if $\sigma(\alpha)\subset N(\alpha)_{\boldsymbol{R}}$ is a nonsingular cone for every $\alpha\in\Sigma$.

All the known examples of T-complexes are essentially written as follows:

There exist a fan $\tilde{\Sigma}$ of $N_{\boldsymbol{R}}$ and a subgroup $\Gamma\subset\mathrm{Aut}_{\boldsymbol{Z}}(N)$ such that

(1) $U=(\bigcup_{\sigma\in\tilde{\Sigma}}\sigma)\backslash\{0\}$ is a nonempty connected open cone of $N_{\boldsymbol{R}}$, and

(2) Γ induces a free action on $\tilde{\Sigma}\backslash\{\mathbf{0}\}$ and $\sharp(\tilde{\Sigma}\backslash\{\mathbf{0}\})$ is finite modulo Γ.

Let Σ be a set of representatives of $\tilde{\Sigma}\backslash\{\mathbf{0}\}$ modulo Γ. For each element $\alpha\in\Sigma$, we set $N(\alpha):=N$ and $\sigma(\alpha):=\alpha$. For elements $\alpha,\beta\in\Sigma$, a morphism $u:\alpha\to\beta$ consists of an element $u_{\boldsymbol{Z}}\in\Gamma$ such that $u_{\boldsymbol{R}}(\alpha)\subset N_{\boldsymbol{R}}$ is a face of the cone β. Then we see that Σ is a T-complex. Σ is nonsingular if and only if so is $\tilde{\Sigma}$.

Examples 2.6. We will give examples of T-complexes of the above form.

(1) *Toric variety type.* Let $\tilde{\Sigma}$ be a complete fan of $N_{\boldsymbol{R}}$ and $\Gamma=\{1_N\}$. Then $\Sigma=\tilde{\Sigma}\backslash\{\mathbf{0}\}$ is a T-complex.

(2) *Degenerate Abelian variety type.* Let $N=\boldsymbol{Z}^{n+1}$, $C=\{(x_1,\cdots,x_{n+1})\in\boldsymbol{R}^{n+1};\ x_{n+1}>0\}$ and Γ be a subgroup of finite index of the group of the matrices of the form

$$\begin{bmatrix} 1 & & 0 & b_1 \\ & \ddots & & \vdots \\ 0 & & 1 & b_n \\ 0 & \cdots & 0 & 1 \end{bmatrix} \quad (b_i \in \boldsymbol{Z}).$$

Then, for a Γ-invariant polyhedral decomposition $\tilde{\Sigma}$ of $C\cup\{0\}$, we get a T-complex Σ.

(3) *Tsuchihashi cusp singularity type.* When C is an open convex cone which contains no lines in $N_{\boldsymbol{R}}$, such a pair (C, Γ) induces an isolated singularity which is independent of the choice of $\tilde{\Sigma}$. A Hilbert modular cusp singularity is a special case of this type of singularities. Other cases and some explicit examples were studied by Tsuchihashi [T1].

(4) *Inoue-Kato manifold type.* Let A be an $n\times n$-matrix of positive integers with the determinant ± 1, and let $N=\boldsymbol{Z}^n$ and $\pi=\{(x_1, \cdots, x_n)\in N_{\boldsymbol{R}}; x_1, \cdots, x_n\geq 0\}$. Then $\bigcup_{m\in \boldsymbol{Z}}A^m(\pi)\backslash\{0\}$ is an open half-space and $\bigcap_{m\in\boldsymbol{Z}}A^m(\pi)$ is a closed half-line. Let $C:=(\bigcup_{m\in\boldsymbol{Z}}A^m(\pi))\backslash(\bigcap_{m\in\boldsymbol{Z}}A^m(\pi))$ and $\Gamma=\{A^m; m\in \boldsymbol{Z}\}$. Then there exists a nonsingular Γ-invariant polyhedral decomposition $\tilde{\Sigma}$ of $C\cup\{0\}$. By these data, we can construct a compact non-Kähler manifold of dimension n with the fundamental group $\boldsymbol{Z}$ [T2]. When $n=2$, this is known as a hyperbolic Inoue surface (see [MO, Sec. 15]). C is connected if $n\geq 3$. The associated T-complex corresponds to an anti-canonical divisor of the manifold if $\det A=1$.

§ 3. Functors on a graph of cones

We denote by $\mathscr{C}^{n.s.}$ the full subcategory of $\mathscr{C}$ consisting of $\alpha\in\mathscr{C}$ such that the cone $\sigma(\alpha)$ is nonsingular. We denote the canonical set of generators $\operatorname{gen}\sigma(\alpha)\subset N(\alpha)$ simply by $\operatorname{gen}\alpha$. For a morphism $u:\alpha\to\beta$ in $\mathscr{C}^{n.s.}$, we have $u_{\boldsymbol{Z}}(\operatorname{gen}\alpha)\subset\operatorname{gen}\beta$. For each $\alpha\in\mathscr{C}^{n.s.}$, we denote $x(\alpha)=\prod_{x\in\operatorname{gen}\alpha}x$ which is an element of the symmetric power $S^{d(\alpha)}N(\alpha)$ over $\boldsymbol{Z}$. For a morphism $u:\alpha\to\beta$ in $\mathscr{C}^{n.s.}$, we set $x(u):=\prod_{x\in\operatorname{gen}\beta\backslash u_{\boldsymbol{Z}}(\operatorname{gen}\alpha)}x\in S^{d(\beta)-d(\alpha)}N(\beta)$.

Let k be a commutative ring with unity. For each nonnegative integer m, we define the functor $D_k^m:\mathscr{C}^{n.s.}\to(k\text{-modules})$ as follows. For each $\alpha\in\mathscr{C}^{n.s.}$, we set $D_k^m(\alpha):=S^{d(\alpha)+m}N(\alpha)_k$ where $N(\alpha)_k:=N(\alpha)\otimes_{\boldsymbol{Z}}k$ and the symmetric power is taken over the ring k. For a morphism $u:\alpha\to\beta$, we define the homomorphism $D_k^m(u)(z):=x(u)\cdot S^{d(\alpha)+m}u_k(z)$, where $S^d u_k: S^dN(\alpha)_k\to S^dN(\beta)_k$ is the symmetric power of $u_k=u_{\boldsymbol{Z}}\otimes 1_k$. It is easy to see that D_k^m satisfies the axiom of functors. We denote by $k^\sim$ the constant functor defined by $k^\sim(\alpha):=k$ and $k^\sim(u):=1_k$ for all $\alpha\in\mathscr{C}^{n.s.}$ and $u\in\operatorname{mor}\mathscr{C}^{n.s.}$. We define the morphism of functors $\varepsilon: k^\sim\to D_k^0$ by $\varepsilon(\alpha)(a):=ax(\alpha)\in D_k^0(\alpha)$ for $\alpha\in\mathscr{C}^{n.s.}$ and $a\in k$. Since $x(u)\cdot S^{d(\alpha)}u_k(x(\alpha))=x(\beta)$, for $u:\alpha\to\beta$, this is indeed a morphism of functors.

Let Φ be a graph of cones. Then, for a functor $V: \mathscr{C}^{n.s.} \to (k\text{-modules})$, the inductive limit ind $\lim_\Phi V$ is described as the cokernel

$$\oplus_{u \in \mathrm{mor}\,\Phi} V(i(u)) \underset{q}{\overset{p}{\rightrightarrows}} \oplus_{\alpha \in \Phi} V(\alpha) \longrightarrow \mathrm{ind}\lim_\Phi V$$

where p consists of the identities $1_{V(i(u))}: V(i(u)) \to V(i(u)) \subset \oplus_{\alpha \in \Phi} V(\alpha)$ and q consists of the homomorphisms $V(u): V(i(u)) \to V(f(u)) \subset \oplus_{\alpha \in \Phi} V(\alpha)$. For a graph of cones Φ, we get a homomorphism ind $\lim_\Phi \varepsilon$: ind $\lim_\Phi k^\sim \to$ ind $\lim_\Phi D_k^0$. Note that ind $\lim_\Phi k^\sim = k$ if Φ is nonempty and connected.

Lemma 3.1. *Let Σ be a nonsingular T-complex. Then, there exists a morphism of functors $\nu: D_k^0|_\Sigma \to k^\sim|_\Sigma$ such that $\nu \circ \varepsilon$ is the identity on Σ. In particular,* ind $\lim_\Sigma \varepsilon$ *defines an injection $k \hookrightarrow$* ind $\lim_\Sigma D_k^0$ *and the image is a direct summand.*

Proof. Let $\Sigma_1 = \{\gamma \in \Sigma; d(\gamma) = 1\}$. By the condition (2) in Definition 2.5, Σ does not contain zero-dimensional cone. Hence Σ_1 is a star open subset of Σ. Let γ be an element of Σ_1 and let gen $\gamma = \{x\}$. Since x is a primitive element of $N(\gamma)$, kx is a direct summand of $N(\gamma)_k$. Hence, there exists a k-homomorphism $\nu(\gamma): N(\gamma)_k \to k$ such that $\nu(\gamma)(x) = 1$. By the condition (2) in Definition 2.5, there is no morphism $u: \gamma \to \gamma'$ if $\gamma, \gamma' \in \Sigma_1$ and $\gamma \neq \gamma'$. Hence we get a morphism of functors $\nu: D_k^0|_{\Sigma_1} \to k^\sim|_{\Sigma_1}$ which satisfies $\nu \circ \varepsilon = \mathrm{id}$ on Σ_1. Let Φ be a maximal star open subset of Σ such that $\Sigma_1 \subset \Phi$ and that there exists a morphism of functors $\nu: D_k^0|_\Phi \to k^\sim|_\Phi$ with $\nu \circ \varepsilon = \mathrm{id}$ on Φ. Assume $\Phi \neq \Sigma$ and let ρ be an element of $\Sigma \backslash \Phi$ with the smallest $d(\rho) =: d$. By Definition 2.5, (2), we have an isomorphism $\Sigma(\prec \rho) \simeq \Gamma(\rho) \backslash \{\mathbf{0}\}$. By the minimality of $d(\rho)$, ν induces a morphism of functors $\nu': D_k^0|_{\Gamma(\rho)\backslash\{0,\rho\}} \to k^\sim|_{\Gamma(\rho)\backslash\{0,\rho\}}$. Let $N = N(\rho)$ and let $\{x_1, \cdots, x_r\}$ be a basis of N such that gen $\rho = \{x_1, \cdots, x_d\}$. The free k-module $S^d N_k$ has $\{x_1^{a_1} \cdots x_r^{a_r} : a_1, \cdots, a_r \geq 0, a_1 + \cdots + a_r = d\}$ as a basis. For each face α of ρ, gen α is a subset of $\{x_1, \cdots, x_d\}$ and the image of $S^{d(\alpha)} N_k$ in $S^d N_k$ is generated by monomials of degree d which is divisible by $x(\rho/\alpha) = \prod_{x \in \mathrm{gen}\,\rho \backslash \mathrm{gen}\,\alpha} x$. For each monomial z of degree d, we define $\nu'(\rho)(z) := \nu'(\alpha)(y)$ if $z = x(\rho/\alpha)y$ for some $\alpha \in \Gamma(\rho) \backslash \{\mathbf{0}, \rho\}$ and for a monomial y of degree $d(\alpha)$, and we define $\nu'(\rho)(z) := 0$ otherwise. Since $d = d(\rho) \geq 2$, $\Gamma(\rho) \backslash \{\mathbf{0}, \rho\}$ is nonempty. Since $x(\rho) = x(\rho/\alpha)x(\alpha)$, we have $\nu'(\rho)(x(\rho)) = 1$. We see easily that the definition does not depend on the choice of α and hence $\nu'(\rho) \circ D_k^0(u) = \nu'(\alpha)$ for every $u: \alpha \to \rho$. Thus the morphism of functors ν' is extended to $\Gamma(\rho) \backslash \{\mathbf{0}\}$. Since there is no morphism $\rho \to \alpha$ in Σ with $\alpha \in \Phi$, we can combine this extended ν' with ν, and we get an extension of ν to $\Phi \cup \{\rho\}$. This contradicts the maximality of Φ and we have $\Phi = \Sigma$. q.e.d.

We call ν in the above lemma a *retraction* of $D_k^0|_\Sigma$ to $k^{\sim}|_\Sigma$. In the proof of the above lemma, the extension of ν' to $\Gamma(\rho)\backslash\{\mathbf{0}\}$ depends on the choice of the basis $\{x_1, \cdots, x_r\}$ of N. Hence the retraction ν is neither unique nor canonical. We will see in Section 5 that there exists an explicit retraction for $k=\boldsymbol{R}$ in the case of Thuchihashi cusp singularities.

Let Σ be a graph of nonsingular cones, i.e., a graph of cones contained in $\mathscr{C}^{n.s.}$, and let ρ be an element of Σ. We are going to define the restriction

$$\text{ind lim}_\Sigma D_k^0 \longrightarrow \text{ind lim}_{\Sigma[\rho]} D_k^{d(\rho)}$$

of the inductive limit of D_k^0 to the link of ρ.

We define the homomorphism

$$h'_\rho : \oplus_{\alpha\in\Sigma} D_k^0(\alpha) \longrightarrow \oplus_{\alpha'[\rho]\in\Sigma[\rho]} D_k^{d(\rho)}(\alpha'[\rho])$$

by $h'_\rho((y_\alpha)):=(\bar{y}_{\alpha'})$ where $y_\alpha \in S^{d(\alpha)}N(\alpha)_k$ and $\bar{y}_{\alpha'}$ is the image of y_α by the natural homomorphism $S^{d(\alpha)}N(\alpha)_k \to S^{d(\alpha)}N(\alpha'[\rho])_k$ if $\alpha'=(\alpha, u)\in\Sigma(\rho\prec)$ for a morphism $u:\rho\to\alpha$. Note that $d(\alpha)=d(\alpha'[\rho])+d(\rho)$ in this case. Similarly, we define the homomorphism

$$h''_\rho : \oplus_{u\in\text{mor}\,\Sigma} D_k^0(i(u)) \longrightarrow \oplus_{u'[\rho]\in\text{mor}\,\Sigma[\rho]} D_k^{d(\rho)}(i(u'[\rho]))$$

by $h''_\rho((z_u)):=(\bar{z}_{u'})$ for $z_u \in S^{d(i(u))}N(i(u))_k$ and $\bar{z}_{u'}$ is the image of z_u in $S^{d(i(u'))}N(i(u'[\rho]))_k$ if $u'\in\text{mor}\,\Sigma(\rho\prec)$ is defined by $u\in\text{mor}\,\Sigma$.

Proposition 3.2. *Let Σ be a nonsingular T-complex and let ρ be an element of Σ. Then the diagram*

$$\begin{array}{ccc} \oplus_{u\in\text{mor}\,\Sigma} D_k^0(i(u)) & \overset{p}{\underset{q}{\rightrightarrows}} & \oplus_{\alpha\in\Sigma} D_k^0(\alpha) \\ \downarrow h''_\rho & & \downarrow h'_\rho \\ \oplus_{u'[\rho]\in\text{mor}\,\Sigma[\rho]} D_k^{d(\rho)}(i(u'[\rho])) & \overset{p}{\underset{q}{\rightrightarrows}} & \oplus_{\alpha'[\rho]\in\Sigma[\rho]} D_k^{d(\rho)}(\alpha'[\rho]) \end{array}$$

commutes for p and q, respectively.

Proof. Let v be in mor Σ and let z be an element of the direct summand $D_k^0(i(v))$ of $\oplus_{u\in\text{mor}\,\Sigma} D_k^0(i(u))$. We have $h'_\rho(p(z))=p(h''_\rho(z))$, since their components for $\alpha'[\rho]\in\Sigma[\rho]$ are both equal to the image z_u of z in $S^{d(i(v))}N(\alpha'[\rho])_k$ if $\alpha'=(i(v), u)$ for some $u:\rho\to i(v)$ and are both zero otherwise. Hence the diagram is commutative for p.

Now we prove the commutativity for q. Let $\beta=f(v)$. The component for $\beta'[\rho]$ of $h'_\rho(q(z))$ is equal to the image $v(z)_w$ of $v(z):=D_k^0(v)(z)$ in $S^{d(\beta)}N(\beta'[\rho])_k$ if $\beta'=(\beta, w)$ for some $w:\rho\to\beta$ and zero otherwise. On the other hand, the same component of $q(h''_\rho(z))$ is equal to $v_t[\rho](z_t)$ if $\beta'=(\beta, v\circ t)$ for some $t:\rho\to i(v)$ and is zero otherwise, where $v_t:(i(u), t)\to(\beta, v\circ t)$ is defined by v and z_t is the image of z in $S^{d(i(v_t))}N(i(v_t)[\rho])_k$. Here note that such t is unique by Lemma 2.1. Clearly, these components $v(z)_w$ and $v_t[\rho](z_t)$ for $\beta'[\rho]$ are equal if there exists such a t. By the condition (2) in Definition 2.5, such a t exists if and only if $w_{\boldsymbol{Z}}(\text{gen}\,\rho)\subset v_{\boldsymbol{Z}}(\text{gen}\,i(v))$. If t does not exist, there exists $x\in\text{gen}\,\rho$ such that $w_{\boldsymbol{Z}}(x)\notin v_{\boldsymbol{Z}}(\text{gen}\,i(v))$. Hence, $x(v)$ is divisible by $w_{\boldsymbol{Z}}(x)$ and $v(z)_w\in S^{d(\beta)}N(\beta'[\rho])_k$ is also zero since $N(\beta'[\rho])=N(\beta)/\boldsymbol{Z}w_{\boldsymbol{Z}}(\text{gen}\,\rho)$. q.e.d.

We denote by $h_{\rho,\Sigma}$, or simply h_ρ, the homomorphism ind $\lim_\Sigma D_k^0\to$ ind $\lim_{\Sigma[\rho]}D_k^{d(\rho)}$ induced by the diagram in the above proposition.

Let N be a free $\boldsymbol{Z}$-module of rank $r\geq 0$. We denote by $B(N_k)$ the total ring of homogeneous quotients of the symmetric algebra S^*N_k. $B(N_k)$ is written as the direct sum $\bigoplus_{m\in\boldsymbol{Z}}B(N_k)_m$ of the k-vector spaces consisting of the homogeneous elements of degree m.

Let Δ be a nonsingular fan in $N_{\boldsymbol{R}}$, and let m be a nonnegative integer. For each $\alpha\in\Delta$, we define the homomorphism $\lambda_\Delta^\alpha: D_k^m(\alpha)\to B(N_k)_m$ by $\lambda_\Delta^\alpha(z):=z/x(\alpha)$ for $z\in S^{d(\alpha)+m}N_k$. It is easy to see that these homomorphisms commute with $D_k^m(u)$ for every morphism $u:\alpha\to\beta$ in Δ. Hence we get the limit homomorphism λ_Δ : ind $\lim_\Delta D_k^m\to B(N_k)_m$.

For a nonsingular T-complex Σ and for an element $\rho\in\Sigma$, we denote by $\bar{h}_\rho$ the composite $\lambda_{\Sigma[\rho]}\circ h_\rho$: ind $\lim_\Sigma D_k^0\to B(N[\rho]_k)_{d(\rho)}$, where $N[\rho]=N(\rho)[\rho]$.

Lemma 3.3. *Let Σ be a nonsingular T-complex. Then an element z in* ind $\lim_\Sigma D_k^0$ *is in the image of* ind $\lim_\Sigma\varepsilon$ *if and only if $\bar{h}_\rho(z)=0$ for every $\rho\in\Sigma$.*

Proof. The image of the morphism $\varepsilon(\alpha): k^\sim(\alpha)\to D_k^0(\alpha)$ is equal to $kx(\alpha)\subset S^{d(\alpha)}N(\alpha)_k$. Since the image of $x(\alpha)$ in $S^{d(\alpha)}N[\alpha]_k$ is zero for every α, the necessity of the condition is obvious.

Now we suppose $z\in$ ind $\lim_\Sigma D_k^0$ satisfies $\bar{h}_\rho(z)=0$ for every $\rho\in\Sigma$. We may assume $z\neq 0$ because otherwise the assertion is obvious. Let $(z_\alpha)\in\bigoplus_{\alpha\in\Sigma}D_k^0(\alpha)$ be a representative of z such that $d=\max\{d(\alpha);\,z_\alpha\neq 0\}$ is minimal. We will show $d=1$. Assume $d>1$ and take (z_α) so that the cardinality of $\{\alpha\in\Sigma;\,d(\alpha)=d$ and $z_\alpha\neq 0\}$ is the smallest. Let ρ be an element of Σ such that $d(\rho)=d$ and $z_\rho\neq 0$. By the definition of d, we have $z_\alpha=0$ for any $\alpha\in\Sigma(\rho\prec)$ with $\alpha\neq\rho$. Hence the condition $\bar{h}_\rho(z)=0$ implies that the image of $z_\rho\in S^dN(\rho)_k$ in $S^dN[\rho]_k$ is zero. Let gen $\rho=\{x_1,\cdots,x_d\}$. Since $N[\rho]=N(\rho)/(\boldsymbol{Z}x_1+\cdots+\boldsymbol{Z}x_d)$, the kernel of the homomorphism $S^dN(\rho)_k\to S^dN[\rho]_k$ is equal to $\sum_{i=1}^d x_iS^{d-1}N(\rho)_k$. Hence z_ρ is of the form $\sum_{i=1}^{d} x_i y_i$ for $y_i \in S^{d-1}N(\rho)_k$. By the condition (2) in Definition 2.5, there exists $u^i : \mu_i \to \rho$ in Σ such that u^i (gen μ_i)=gen $\rho \backslash \{x_i\}$, i.e., $x(u^i)=x_i$, for each $i=1, \cdots, d$. Let $y'_i \in S^{d-1}N(\mu_i)_k$ be the element which satisfies $S^{d-1}(u^i)(y'_i)=y_i$, for each an i. Let $y'=(y'_u)$ be the element of $\oplus_{u \in \mathrm{mor}\, \Sigma} D_k^0(i(u))$ defined by $y'_u=y'_i$ if $u=u^i$ for some i and $y'_u=0$ otherwise. Then we have $q(y')=\sum x_i y_i=z_\rho$, while the components of $p(y')$ are zero for every α with $d(\alpha)\geq d$. Let $(z'_\alpha)=(z_\alpha)+p(y')-q(y') \in \oplus_{\alpha \in \Sigma} D_k^0(\alpha)$. Then clearly, $\max\{d(\alpha): z'_\alpha \neq 0\} \leq d$ and $z'_\alpha = z_\alpha$ for $\alpha \in \Sigma$ with $d(\alpha)=d$ except when $\alpha=\rho$. Since $z'_\rho=0$ and (z'_α) is also a representative of z, this contradicts the minimality of $\{\alpha \in \Sigma;\, d(\alpha)=d$ and $z_\alpha \neq 0\}$. Hence we have $d=1$.

Let $\rho \in \Sigma$ be an element with $z_\rho \neq 0$. Then since $d(\rho)=1$, we have gen $\rho=\{x(\rho)\}$. By the condition $\bar{h}_\rho(z)=0$, we see that z_ρ is in $kx(\rho)=\ker(N(\rho)_k \to N[\rho]_k)$ which is equal to the image of $\varepsilon(\rho): k^\sim(\rho) \to D_k^0(\rho)$. Hence every z_α is in the image of $k^\sim(\alpha)$. q.e.d.

§ 4. ω-invariant of a T-complex

Let Δ be a nonsingular fan in $N_{\boldsymbol{R}}$ and let m be a nonnegative integer. For each $\alpha \in \Delta$, we set

$$\omega_\alpha^m := \left[\prod_{x \in \mathrm{gen}\, \alpha} \frac{x}{\exp(x)-1} \right]_{d(\alpha)+m},$$

where $[f]_d$ denotes the homogeneous part of degree d of a power series f. Note that $x/(\exp(x)-1)$ is an element of the completion of the symmetric algebra $S^*N_{\boldsymbol{Q}}$ with respect to the natural grading. Hence $(\omega_\alpha^m)_{\alpha \in \Delta}$ is an element of $\oplus_{\alpha \in \Delta} D_{\boldsymbol{Q}}^m(\alpha)$.

Lemma 4.1. *Let ω_Δ^m be the image of (ω_α^m) in* ind $\lim_\Delta D_{\boldsymbol{Q}}^m$. *If Δ is a nonsingular complete fan, then $\lambda_\Delta(\omega_\Delta^m) \in B(N_{\boldsymbol{Q}})_m$ is equal to zero.*

Proof. Since $x(\alpha)=\prod_{x \in \mathrm{gen}\, \alpha} x$, we see that $\lambda_\Delta^\alpha(\omega_\alpha^m)$ is equal to the homogeneous part of degree m of $\prod_{x \in \mathrm{gen}\, \alpha} 1/(\exp(x)-1)$. Hence $\lambda_\Delta(\omega_\Delta^m)$ is equal to that of $\sum_{\alpha \in \Delta} \prod_{x \in \mathrm{gen}\, \alpha} 1/(\exp(x)-1)$, which is zero by Theorem 1.1. q.e.d.

Let Σ be a nonsingular T-complex. For each $\alpha \in \Sigma$, we set $\omega_\alpha := [\prod_{x \in \mathrm{gen}\, \alpha} x/(\exp(x)-1)]_{d(\alpha)} \in D_{\boldsymbol{Q}}^0(\alpha)$.

Proposition 4.2. *In the above situation, let ω_Σ be the class of $(\omega_\alpha)_{\alpha \in \Sigma}$ in* ind $\lim_\Sigma D_{\boldsymbol{Q}}^0$. *Then ω_Σ is in the image of* ind $\lim_\Sigma \varepsilon$.

Proof. By Lemma 3.3, it is sufficient to show $\bar{h}_\rho(\omega_\Sigma)=0$ for every

$\rho \in \Sigma$. Let $N=N(\rho)$. Since ρ is the initial object of $\Sigma(\rho\prec)$, we may regard $N(\alpha)=N$ for every $\alpha \in \Sigma(\rho\prec)$ and $u_Z=1_N$ for every $u \in \text{mor}\,\Sigma(\rho\prec)$. For each $\alpha \in \Sigma(\rho\prec)$, $\alpha[\rho]$ is the nonsingular cone of $N[\rho]_R$ with $\text{gen}\,\alpha[\rho]=\{\bar{x};\ x \in \text{gen}\,\alpha\backslash\text{gen}\,\rho)$ where $\bar{x}$ denotes the image of $x \in N$ in $N[\rho]$. Since $x/(\exp(x)-1)=1$ on $M[\rho]_C$ for $x \in \text{gen}\,\rho$, the restriction of $\prod_{x\in\text{gen}\,\alpha} x/(\exp(x)-1)$ to $M[\rho]_C$ is equal to $\prod_{\bar{x}\in\text{gen}\,\alpha[\rho]} \bar{x}/(\exp(\bar{x})-1)$. Since $d(\alpha[\rho])=d(\alpha)-d(\rho)$, we have $\omega_\alpha|_{M[\rho]_C}=\omega^{d(\rho)}_{\alpha[\rho]}$. Hence $h_\rho(\omega_\Sigma)=\omega^{d(\rho)}_{\Sigma[\rho]} \in \text{ind}\lim_{\Sigma[\rho]} D_k^{d(\rho)}$. Since $\Sigma[\rho]$ is a nonsingular complete fan by Definition 2.5, (3), $\bar{h}_\rho(\omega_\Sigma)=\lambda_\Delta \circ h_\rho(\omega_\Sigma)$ is zero by Lemma 4.1. q.e.d.

Lemma 3.1 and Proposition 4.2 imply that there exists a unique rational number a for each nonsingular T-complex Σ such that $(\text{ind}\lim_\Sigma \varepsilon)(a)=\omega_\Sigma$. We also denote $\omega_\Sigma:=a$ and call it *the ω-invariant* of the T-complex Σ.

Proposition 4.2 implies obviously the following:

Corollary 4.3. *Let $\nu: D^0_R|_\Sigma \to R^\sim|_\Sigma$ be an arbitrary retraction. Then* $\text{ind}\lim_\Sigma \nu((\omega_\alpha)) \in R$ *is equal to the rational number ω_Σ.*

Remark 4.4. Let Σ be a nonsingular T-complex, and let d be a positive integer such that

(1) $\qquad d\omega_\alpha \in D^0_Z(\alpha) \qquad$ for every $\alpha \in \Sigma$.

Then $(d\omega_\alpha) \in \oplus_{\alpha\in\Sigma} D^0_Z(\alpha)$ satisfies the condition of Lemma 3.3 for $k=Z$. Hence $d\omega_\Sigma$ is an integer. The minimal number satisfying (1) depends only the dimension r of Σ. For example, $d=12$ for $r=2$ and $d=720$ for $r=4$.

When r is odd, we can show that ω_Σ is a half-integer in the same way as Ogata's [Og, Theorem 2.3].

§ 5. Ogata's zeta zero value

Let C, Γ, $\tilde{\Sigma}$ and Σ be as in Example 2.6, (3). The characteristic function ϕ_C on the open convex cone C is given by

$$\phi_C(x):=\int_{C^*} \exp(-\langle x, x^*\rangle)\,dx^*,$$

where $C^* \subset M_R$ is the dual cone of C and dx^* is a Euclidean metric. Ogata [Og] defined the zeta function of the pair (C, Γ) by

$$Z(C, \Gamma; s):=\textstyle\sum_{x\in(N\cap C)/\Gamma} \phi_C(x)^s$$

which converges for complex numbers s with $\text{Re}\,s>1$ and can be extended meromorphically to the whole complex plane. He proved in [Og, Proposition 3.10] that this zeta function is regular at $s=0$ and the value is equal to

$$\sum_{\alpha\in\Sigma}\int_{\alpha}\Big[\prod_{x\in\operatorname{gen}\alpha}(\partial_x/(1-\exp(-\partial_x)))\Big]_{d(\alpha)}G_2(t)dt_\alpha$$

where ∂_k is the first order derivation defined by

$$\partial_x f(t)=\lim_{h\to 0}\{(f(t+hx)-f(t))/h\},$$

dt_α is the Lebesgue measure on α normalized with respect to the basis $\operatorname{gen}\alpha$ and $G_2(t)=\exp(-\phi_C(t)^{-2})$.

Let α be an element of Σ. By extending the correspondence $x\mapsto -\partial_x$ to their products, we get an isomorphism $z\mapsto D_z$ from $D^0_{\boldsymbol{R}}(\alpha)=S^{d(\alpha)}N_{\boldsymbol{R}}$ to the $\boldsymbol{R}$-module of derivations of order $d(\alpha)$ with constant coefficients.

Proposition 5.1. *For each $\alpha\in\Sigma$, we define the homomorphism $F(\alpha)$: $S^{d(\alpha)}N_{\boldsymbol{R}}\to\boldsymbol{R}$ by*

$$F(\alpha)(z):=\int_\alpha D_zG_2(t)dt_\alpha.$$

Then F is a retraction of the morphism of functors $\varepsilon|_\Sigma:\boldsymbol{R}^\sim|_\Sigma\to D^0_{\boldsymbol{R}}|_\Sigma$, i.e., F is a morphism of functors and $F\circ\varepsilon|_\Sigma$ is identity.

Proof. Let α be an element of Σ and let $\operatorname{gen}\alpha=\{x_1,\cdots,x_d\}$. We take a coordinate $(t_1,\cdots,t_r)$ of $N_{\boldsymbol{R}}$ such that $t_i(x_j)=\delta_{i,j}$, where $\delta_{i,j}$ is Kronecker's delta. Since $D_{\varepsilon(\alpha)(1)}=\prod_{i=1}^d(-\partial/\partial t_i)$, we have

$$(F\circ\varepsilon)(\alpha)(1)=\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^d(-\partial/\partial t_i)G_2(t)dt_1\cdots dt_d.$$

By [Og, Lemma 3.5], the partial derivatives of $G_2(t)$ goes to zero at infinity. Hence this integral is equal to $G_2(0)=1$. Hence it is sufficient to show that F is a morphism of functors. Let $u:\beta\to\alpha$ be a homomorphism in Σ. We may regard $\operatorname{gen}\beta=\{x_1,\cdots,x_{d'}\}\subset\operatorname{gen}\alpha$ for an integer $0<d'\leq d$. Furthermore, it is sufficient to show the commutativity in the case $d'=d-1$. Then, for an element $z\in D^0_{\boldsymbol{R}}(\beta)$, we have $D_{z'}=(-\partial/\partial t_d)D_z$ for $z'=D^0_{\boldsymbol{R}}(u)(z)$. Hence

$$\begin{aligned}F(\alpha)(D^0_{\boldsymbol{R}}(u)(z))&=\int_0^\infty\cdots\int_0^\infty\Big(\int_0^\infty(-\partial/\partial t_d)D_zG_2(t)dt_d\Big)dt_1\cdots dt_{d-1}\\&=\int_0^\infty\cdots\int_0^\infty D_zG_2(t)dt_1\cdots dt_{d-}\\&=F(\beta)(z).\end{aligned}$$

q.e.d.

Let α be in Σ. Then, for ω_α in Section 4, we have

$$D_{\omega_\alpha}=[\prod_{x\in \mathrm{gen}\,\alpha}(\partial_x/(1-\exp(-\partial_x)))]_{d(\alpha)}.$$

Hence by Ogata's formula, Corollary 4.3 and Proposition 5.1, we have the following:

Theorem 5.2. *The zeta zero value $Z(C, \Gamma; 0)$ is equal to the ω-invariant ω_Σ of the T-complex Σ. In particular, it is a rational number.*

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