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Zeta Functions of Finite Graphs and Representations of p-Adic Groups

Ki-ichiro Hashimoto*)

Dedicated to Prof. Friedrich Hirzebruch and Prof. Ichiro Satake on their sixtieth birthdays

Contents

§ 0. Introduction

0-1. In this paper we shall be concerned with the two different subjects, which have been developed separately. One is a combinatorial problem in algebraic graph theory, and the other is an arithmetic of discrete subgroups of p-adic groups and their representations.

Suppose that X is a finite (multi)graph, which is not a tree. We always assume that X is *non-oriented*. A closed path C in X is called reduced, if C and $C^2 = C.C$ have no backtracking. Then obviously the set $\mathscr{C}_l^{red}(X)$ of reduced closed paths of length l is finite, and $\sharp(\mathscr{C}_l^{red}(X)) \to \infty$ $(l \to \infty)$ if X is not homotopic to a circuit, i.e., S^1. (See § 1 for

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precise definitions). The main problems to be studied in this paper can be stated as follows:

(0.1) Count the number N_l of elements of $\mathscr{C}_l^{red}(X)$; can one give a closed formula for it?

(0.2) To which extent does the data $(N_l)_{l\in \boldsymbol{N}}$ reflect basic properties of X?

We are especially interested in studying the properties of the generating function

$$Z_X(u) := \exp\left\{\sum_{l=1}^{\infty} (N_l/l)u^l\right\}, \qquad N_l := \#(\mathscr{C}_l^{red}(X)). \tag{0.3}$$

More generally, let $f\colon EX \to \boldsymbol{C}$ be a function on the set EX of edges of X. For a reduced closed path $C=(e_1, \cdots, e_l)$ $(e_j \in EX)$, we put $f(C) := \prod_{k=1}^{l} f(e_k)$, and define

$$Z_X(f;u) = \exp\left\{\sum_{l=1}^{\infty} \sum_{C\in\mathscr{C}_l^{red}(X)} (f(C)/l)u^l\right\}. \tag{0.4}$$

These functions are expected to keep many properties of X, as in the case of the congruence zeta functions of algebraic varieties over finite fields. One of our main results (cf. (2.22), (2.27)) implies that $Z_X(f;u)$ is always a rational function of u. In fact, the first of which implies that $Z_X(u)^{-1}$ is a polynomial with integral coefficients:

Theorem (0.5). *Suppose that X is a connected multigraph with m edges and n vertices $\{P_j;\ 1\le j\le n\}$, and denote by q_j+1 the valency of P_j. Then there exist $2m$ complex numbers (in fact algebraic integers) α_i $(1\le i \le 2m)$, such that*

$$N_l = \sum_{i=1}^{2m} \alpha_i^l \qquad (l=1, 2, \cdots), \tag{0.6}$$

$$\prod_{j=1}^{2m} \alpha_j = (-1)^n q_1 q_2 \cdots q_n . \tag{0.7}$$

Moreover,

(i) *the number r of indices such that $\alpha_j = 1$ satisfies*

$$r = \dim_{\boldsymbol{C}} H_1(X, \boldsymbol{C}), \tag{0.8}$$

except for the trivial case that X is homotopic to a circuit Cir_n (i.e., $H_1(X, \boldsymbol{C}) \simeq \boldsymbol{C}$); *in such case one has $r=2$.*

(ii) *the number r' of indices such that $\alpha_j = -1$ satisfies*

$$
\begin{aligned}
(0.9)\qquad r' &= \dim_{\boldsymbol{C}} H_1(X, \boldsymbol{C}) - 1, && \text{if } X \text{ is non-bipartite,}\\
&= \dim_{\boldsymbol{C}} H_1(X, \boldsymbol{C}), && \text{if } X \text{ is bipartite and } \dim_{\boldsymbol{C}} H_1(X, \boldsymbol{C}) > 1,\\
&= \dim_{\boldsymbol{C}} H_1(X, \boldsymbol{C}) + 1 = 2, && \text{if } X \text{ is homotopic to } \mathrm{Cir}_{2n}.
\end{aligned}
$$

We note, among others, that the last result (0.9) gives a new characterization of finite bipartite multigraphs. If X is assumed to have some regularity, then one has much more strong results:

Theorem (0.10). (i) *Suppose that X is regular, i.e., $q_j = q$ (>1) for $1 \leq j \leq n$. Then exactly one of the α_i's is equal to q. Moreover, those α_i's s.t. $\alpha_i \neq \pm 1$, $\pm q$ are divided into the union of pairs $\{\alpha_k, \alpha'_k\}$ which satisfy*

$$(0.11)\qquad \alpha_k \alpha'_k = q, \quad \alpha_k + \alpha'_k \in \boldsymbol{R} \quad (\alpha_k, \alpha'_k \neq 1, q).$$

(ii) *Suppose that X is a semi-regular bipartite multigraph of valency $(q+1, q'+1)$, with $q \geq q'$, $qq' > 1$; and put $n_1 = \sharp\{j;\ q_j = q\}$, $n_2 = \sharp\{j;\ q_j = q'\}$. Then one has $\alpha_{i+m} = -\alpha_i$ $(1 \leq i \leq m)$, and*

$$
\begin{aligned}
(0.12)\qquad &\{\alpha_1^2, \cdots, \alpha_m^2\}\\
&= \{\overbrace{1, \cdots, 1}^{r};\ qq';\ \overbrace{-q', \cdots, -q'}^{n_2 - n_1},\ \beta_k, \beta'_k\ (1 \leq k \leq n_1 - 1)\},
\end{aligned}
$$

where each pair $\{\beta_k, \beta'_k\}$ satisfies

$$(0.13)\qquad \beta_k \beta'_k = qq', \quad \beta_k + \beta'_k \in \boldsymbol{R} \quad (\beta_k, \beta'_k \neq 1, qq').$$

These results imply that our "zeta function $Z_X(u)$" of X has in fact many properties which are strikingly analogous to the congruence zeta functions of algebraic curves over a finite field $\boldsymbol{F}_q$.

0-2. On the other hand, an analogue of the Selberg zeta function $Z_\Gamma(u)$ has been introduced by Ihara [I-1], for the discrete subgroups of SL(2, K) over a $\boldsymbol{p}$-adic field K. It is defined by an infinite formal product

$$(0.14)\qquad Z_\Gamma(u) := \prod_{\{r\}_\Gamma} (1 - u^{\deg\{r\}_\Gamma})^{-1},$$

extended over the set of "primitive hyperbolic" Γ-conjugacy classes. As is remarked in [I-2], this $Z_\Gamma(u)$ coincides with a congruence zeta function of a modular curve over $\boldsymbol{F}_p$ for some arithmetically defined Γ. Moreover, as it was pointed out by Serre [Ser], $Z_\Gamma(u)$ has an interpretation as a zeta function of certain regular graph of valency p+1; the above Theorem (0.10), (i) is then a consequence of a result of [I-1]. However, these relations have been seldom taken up seriously until recently. It is an interesting problem to ask for a generalization of such relations to a wider class of groups and graphs, and possibly, varieties (curves).

0–3. In our previous paper [H-H], Ihara's zeta function has been generalized to a class of groups, containing those defined over a p-adic field K, with K-rank one. We have given it an expression as a rational function. Actually, $Z_\Gamma(u)^{-1}$ was shown to be a polynomial of u, which is essentially a Hecke polynomial for an element $T(p)$ of $\mathscr{H}(G, U)$, the Hecke algebra of G with respect to a maximal open compact subgroup U. This raises some interesting questions. It is known that a simply connected semisimple group over a local field K with K-rank one has two non-conjugate maximal open compact subgroups, say U_1 and U_2. The formula for $Z_\Gamma(u)$ is [H-H] is not symmetric in U_1, U_2, reflecting that the calculation has been done depending on a choice of one of them. How can one explain the difference between the two expressions? $\cdots$ This has been a motivation for the present paper.

Another problem, which is more important and closely related to it, is the relation between $Z_\Gamma(u)$ and the spectral decomposition of $L^2(G/\Gamma)$. Recall that in the case of real Lie groups of $\boldsymbol{R}$-rank one, to determine the Selberg zeta function is equivalent to know the eigenvalues of the Laplacian on $L^2(G/\Gamma)$.

0–4. In studying these problems we were led to introduce the third expression for $Z_\Gamma(u)$, which was based on $B = U_1 \cap U_2$, the Iwahori subgroup of G. The Hecke algebra $\mathscr{H}(G, B)$ is generated by two elements T_1, T_2. Now the main result of the present paper states that our zeta function $Z_\Gamma(u; \rho)$ with additional parameter ρ (= unitary representation of Γ of finite degree), can be expressed:

$$Z_\Gamma(u; \rho)^{-1} = \det\{I - \rho^*(T_1 T_2)u\}, \tag{0.15}$$

where ρ^* is a representation of $\mathscr{H}(G, B)$ associated with ρ. A simple argument on the representations of $\mathscr{H}(G, B)$ now makes the situation clear, and one can reproduce quite simply from (0.15) the main result in [H-H]. Moreover, if one combines this result with the theory of Borel [Bo-2] on the admissible representations of p-adic groups having fixed vectors under B, one easily gets a complete answer to the above questions. We shall describe it in Section 6. Among others, we shall prove the following result, which is also a generalization of [I-1].

Theorem (0.16). *The zeta function $Z_\Gamma(u)$ describes the spectral decomposition in $L^2(G/\Gamma)$, of those components which have B-fixed vectors. Namely, let (π, V_π) be the irreducible unitary representation of G such that*

$V^B \neq \{0\}$, *and let* φ *be the induced representation of* $\mathcal{H}(G, B)$ *on* V^B. *Then the multiplicity of* (π, V_π) *in* $L^2(G/\Gamma)$ *is equal to that of the characteristic polynomial of* $\varphi(T_1T_2)$ *in* $Z_\Gamma(u)^{-1}$.

0–5. Quite surprisingly, it turned out that the expression (0.15) holds for an arbitrary finite multigraph X, if one only assume that it is of bipartite type. The non-bipartite case can easily be reduced to this case. Here we need no regularity condition on X. T_1, T_2 are interpreted as the correspondences on the edges EX. Based on this result, one can study various combinatorial properties of finite graphs in terms of our zetafunctions. Especially, we shall describe the relation between the spectrum of a finite multigraph X and $Z_X(u)$. This has a number of interesting applications. Among others, we shall give a formula which relates the complexity $\kappa(X)$ of X and the value of $(1-u)^r Z_X(u)$ at $u=1$; this is an analogue of the class number formula for a number field, or a function field.

Finally, it is a great pleasure for the author to remark that the present paper, as well as the previous one [H-H], grew out of the effort to understand the important (mysterious) paper of Ihara [I-1], and to ask for the possible generalization of his results. He also would like to make the following:

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Notation. As usual, we denote by $\boldsymbol{Z}$ the ring of integers, and by $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ the field of rational, real, and complex numbers, respectively. If A is a ring with unit 1, then $M(n, A)$ is the ring of $n \times n$ matrices, and $GL(n, A)$ the group of invertible matrices in $M(n, A)$. The unit matrix is denoted by I_n, or simply by I. For a finite set S, $\sharp(S)$ denotes its cardinality. We shall often write the collection of indexed elements set theoretically as $\{a_1, \cdots, a_n\}$, even if a_j's are not necessarily distinct.

Since there are several different terminology even for basic objects in the graph theory, we collect in Section 1 those which we use, to avoid confusion.

§ 1. Graphs and multigraphs

The purpose of this section is to fix the notation and basic definitions in our graph theory, and to state our main points of interests. We remark first that the (multi) graphs X we are interested are all assumed to be *non-oriented*, unless otherwise stated.

A *multigraph* X is a triple (VX, EX, ε), consisting of two sets VX, EX, whose elements are called *vertices*, and *edges* respectively, and a map $\varepsilon = \varepsilon_X$, called the *incident map* of X:

$$(1.1) \qquad \varepsilon: EX \longrightarrow VX \times VX, \qquad \varepsilon(y) = (o(y), t(y)),$$

where the vertex $o(y)$ (resp. $t(y)$) is called the *origin* (resp. *terminus*) of y, respectively. Moreover, we require that there is an involution ι_X; $EX \to EX$, $y \to y^{-1}$ $(y \neq y^{-1})$ such that $o(y^{-1}) = t(y)$, $t(y^{-1}) = o(y)$. We call the pair $e = \{y, y^{-1}\}$ a *non-oriented edge*, or simply an *edge* if there is no fear of confusion, and write $\varepsilon(e) = \{o(y), t(y)\}$. The set of (non-oriented) edges are also denoted by EX. An edge $e \in EX$ is said to *join* its *ends* P, $Q \in VX$, or often called to be *incident* to P, Q if $\varepsilon(e) = \{P, Q\}$; and two vertices are then called to be *joined* by the edge e, or *adjacent*. Also, we call two distinct edges e_1, $e_2 \in EX$ to be *adjacent*, if $\varepsilon(e_1) \cap \varepsilon(e_2) \neq \phi$. We do not assume that ends of an edge to be distinct; if $\varepsilon(e) = \{P, P\}$, e is called a *loop*. Also, more than one non-oriented edges may have the same pair $\{P, Q\}$ as their ends, in which case X is said to have a *multiple edge*. If X has no loop and no multiple edge, it is a *combinatorial graph*, or simply a *graph*. The number of the non-oriented edges e which is incident to a vertex $P \in VX$ is called the *valency* of P. The vertex of valency one is called an *end point*. A multigraph X is called to be *s-partite*, if VX is divided into the disjoint union of s subsets V_i $(1 \leq i \leq s)$ such that no two vertices of the same V_i are adjacent. 2-partite multigraph is simply called *bipartite*.

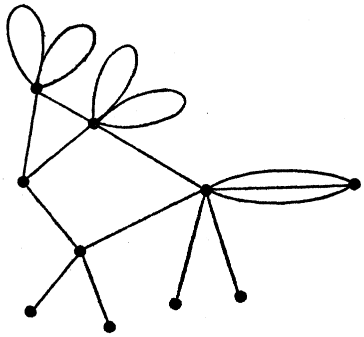


Fig. 1

A *morphism* of a multigraph φ: $X = (VX, EX, \varepsilon_X) \to Y = (VY, EY, \varepsilon_Y)$ is a pair of mappings φ_0: $VX \to VY$, φ_1: $EX \to EY \cup VY$ such that the following diagram, together with the involutions ι_X, ι_Y, is commutative, where φ_0^*: $VX\times VX\to VY\times VY$ is the map induced naturally from φ.

(1.2)
$$\begin{tikzcd} EX \arrow[r, "\varphi_1"] \arrow[d, "\varepsilon_X"'] & EY\cup VY \arrow[d, "\varepsilon_Y\cup\Delta_Y"] \\ VX\times VX \arrow[r, "\varphi_0^*"] & VY\times VY \end{tikzcd}$$

Here Δ_Y: $VY\to VY\times VY$ is the diagonal map. X and Y are called *isomorphic*, if there is a morphism $\varphi=(\varphi_0, \varphi_1)$ such that φ_0 and φ_1: $EX\to EY$ are bijective.

A *path C of length n* (≥ 0) on a multigraph X is a sequence

(1.3) $$C=(P_0, y_1, P_1, y_2, \cdots, P_{n-1}, y_n, P_n)$$

of $n+1$ vertices and n *oriented* edges such that $\varepsilon(y_i)=(P_{i-1}, P_i)$ for $i=1, 2, \cdots, n$. Here we do not require that the vertices or edges are distinct. The length of C is denoted by $|C|$. We write $P_0=o(C)$, $P_n=t(C)$, and call them *the origin*, *the terminus* of C, respectively. The inverse of a path C is defined by

$$C^{-1}=(P_n, y_n^{-1}, P_{n-1}, y_{n-1}^{-1}, \cdots, P_1, y_1^{-1}, P_0).$$

Also the composition $C.C'$ of two paths C, C' satisfying $t(C)=o(C')$ is defined by an obvious way. If C satisfies $y_i\neq y_{i+1}^{-1}$ $(1\leq i\leq n-1)$, it is called a *proper path*, or to have *no backtracking*. We always regard a path of length 1 to be proper. A path C can be written without any ambiguity as $C=(y_1, y_2, \cdots, y_n)$. If X is a graph, then C is determined also by giving a sequence of $n+1$ vertices $(P_0, P_1, \cdots, P_n)$ where P_i and P_{i+1} are adjacent for $0\leq i\leq n-1$. One sees in this case that C is a proper path if and only if $P_{i-1}\neq P_{i+1}$ for $1\leq i\leq n-1$. We shall often use these simplified notation. A multigraph is called *connected*, if for any two distinct vertices P, Q, there exists a path C such that $o(C)=P$, $t(C)=Q$. We assume, unless otherwise stated, that all multigraphs we treat are connected. Then we can define a distance d_X on VX; $d_X(P, Q)$ is the length of the shortest (proper) path C with origin P, and terminus Q, where we note that $d_X(P, Q)=0$ if $P=Q$, and conversely. The maximum of $d_X(P, Q)$ $(P, Q\in VX)$ is called the *diameter* of X, and denoted by $d(X)$. A path C is called *closed*, if $o(C)=t(C)$.

A typical example of the closed proper path of length n is an *n-circuit*; it is a closed path where the vertices $P_0, P_1, \cdots, P_{n-1}$ are all distinct. In other words, n-circuit is an isomorphic image of the following graph Cir_n:

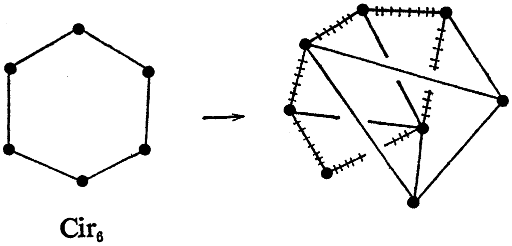


Fig. 2

The minimum length of the circuits of X is called, if it exists, the *girth* of X, and written by $g(X)$. X is called a *tree*, if it has no circuit, and we put $g(X)=\infty$. Note that $g(X)\geq 2$ unless X has a loop, and $g(X)\geq 3$ if X is a graph.

Definition. A closed proper path $C=(y_1, \cdots, y_n)$ on X is called *reduced*, if either $n=1$, or $y_1\neq y_n^{-1}$. In other words, C is reduced if and only if C and $C.C=C^2$ are both proper. We put

(1.4) $\mathscr{C}_n=\mathscr{C}_n(X):=\{C;\ C=\text{a proper closed path},\ |C|=n\}$,

(1.5) $\mathscr{C}_n^{red}=\mathscr{C}_n^{red}(X):=\{C\in\mathscr{C}_n;\ C=\text{reduced}\}$.

As illustrated in the following figure, to each closed proper path $C\in\mathscr{C}_n$ is assigned a unique reduced path $C^*\in\mathscr{C}_{n-2k}^{red}$ by removing the edges $\{y_1, y_n\}$, $\{y_2, y_{n-1}\}, \cdots, \{y_k, y_{n-k+1}\}$, when $y_i^{-1}=y_{n-i+1}$ $(1\leq i\leq k)$ and $y_{k+1}\neq y_{n-k}^{-1}$.

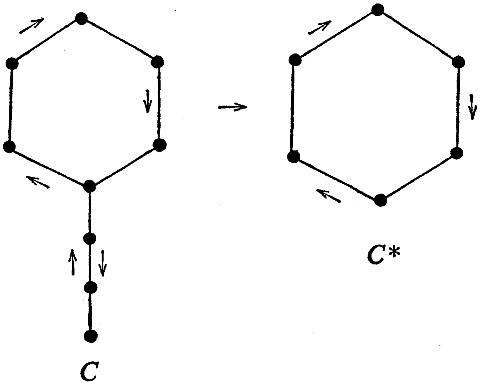


Fig. 3

If $C\in\mathscr{C}_n^{red}(X)$ is reduced, then for each k $(1\leq k\leq n-1)$, the closed path

$$C_k:=(y_{k+1}, y_{k+2}, \cdots, y_n, y_1, \cdots, y_k),$$

which is derived from C by shifting the origin by k steps, is also reduced. We call C_k to be conjugate to C, and write $C_k \sim C$. This defines an equivalence relation in the set $\mathscr{C}_n^{red}$ of the reduced closed paths of length n. We call an equivalence class of C *a cycle* of X, and denote it by [C]. A cycle, or a reduced closed path C of length n which represents it, is called *non-primitive*, if there exists a positive integer k $(1 \leq k < n)$ such that $C = C_k$; and otherwise it is called *primitive*. In other words, C is primitive if and only if it is not of the form $C = D^m$ with $m > 1$. We denote by $\mathscr{C}_n^{red,pr}$ the set of all primitive reduced closed paths of length n. Then one sees that there is an obvious decomposition:

$$\mathscr{C}_n^{red} = \bigcup_{d|n} (\mathscr{C}_d^{red,pr})^{n/d} \qquad \text{(disjoint)}. \tag{1.6}$$

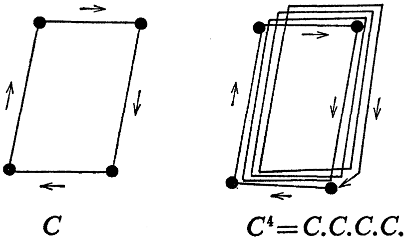


Fig. 4

Finally we recall the basic properties of the fundamental group. Let $P_0 \in VX$ be a vertex which we fix once and for all. Then, as usual, the fundamental group $\pi_1(X, P_0)$ of X with base point P_0 is defined as the group of homotopy classes $\langle C \rangle$ of the closed paths C in X such that $o(C) = t(C) = P_0$, where the product in $\pi_1(X, P_0)$ is induced from the composition of the paths. As illustrated in Fig. 3, each C determines the cycle $[C^*]$ of reduced path C^* which is homotopic to C. Moreover, it is easy to see that the mapping $\langle C \rangle \rightarrow [C^*]$ induces the bijection

$$\{\text{conjugacy classes in } \pi_1(X, P_0)\} \simeq \{\text{cylces in } X\}. \tag{1.7}$$

Note that $\pi_1(X, P_0)$ is a free group of rank $r = \dim_C H_1(X, C)$. Hence the centralizer $Z(\gamma)$ of any element $\gamma \neq 1$ of $\pi_1(X, P_0)$ is an infinite cyclic group. It is called *primitive*, if $Z(\gamma)$ is generated by itself. It is also easy to see that, in the bijection (1.7), the primitive conjugacy classes correspond to the primitive cycles.

It is also known that there exists a tree $\tilde{X}$ and an action of $\pi_1(X, P_0)$ on $\tilde{X}$ such that the quotient $\tilde{X}/\Gamma$ is isomorphic to X. In other words, $\tilde{X}$ together with the natural morphism $p\colon \tilde{X} \rightarrow X$ is a universal covering of X.

Let $\tilde{P}_0$ be a vertex of $\tilde{X}$ such that $p(\tilde{P}_0)=P_0$, and let, for each $\gamma \in \pi_1(X, P_0)$, $\tilde{C}_\gamma$ be the unique proper path in $\tilde{X}$ with origin $\tilde{P}_0$ and terminus $\tilde{P}_0 \cdot \gamma$. Then the image $p(\tilde{C}_\gamma)$ of $\tilde{C}_\gamma$ is a closed proper path in X with origin P_0, and we recover γ as the class of $p(\tilde{C}_\gamma)$.

For the more detailed description of $\pi_1(X, P_0)$ and alternative definitions, we refer to [Ser].

§ 2. Zeta functions of finite multigraphs

We begin with a generalization of the definition of zeta function. Let X be a finite connected multigraph, and let $\Gamma := \pi_1(X, P_0)$ be the fundamental group of X. As remarked in Section 1, Γ is a free group, and its rank r is given by

$$r = \#(EX) - \#(VX) + 1 = \dim_{\boldsymbol{C}} H_1(X, \boldsymbol{C}). \tag{2.1}$$

Let $EX = \{e_1, \cdots, e_m\}$ be the set of non-oriented edges of X. By a *labelling* on EX, we mean an assignment $e_j \to u_j$ $(1 \le j \le m)$, where $u_1, \cdots, u_m$ are (independent) variables. We denote them simply by $\boldsymbol{u} = (u_1, u_2, \cdots, u_m)$.

Definition. For each Γ-conjugacy class $\boldsymbol{P} = \{\gamma\}_\Gamma$, we put

$$\boldsymbol{u}^{\boldsymbol{P}} = \boldsymbol{u}^{C_\gamma} := \prod_{k=1}^{d} u_{i_k} \tag{2.2}$$

where $C_\gamma = (y_{i_1}, y_{i_2}, \cdots, y_{i_d})$ is a reduced closed path corresponding to $\boldsymbol{P}$, such that $y_{i_k} \in e_{i_k}$ $(1 \le k \le d)$. Also we define the degree of $\boldsymbol{P}$ by

$$\begin{aligned} \deg \boldsymbol{P} &:= d = \deg \boldsymbol{u}^{\boldsymbol{P}} \\ &= |C_\gamma| \quad (= \text{the length of the reduced path } C_\gamma). \end{aligned} \tag{2.3}$$

We denote by $\langle C \rangle$ the Γ-conjugacy class, or an element in the class, corresponding to the cycle $[C]$, with $C \in \mathscr{C}_d^{red}(X)$. Let $\rho\colon \Gamma \to U(n)$ be an n-dimensional unitary representation of Γ. Then the zeta function of X attached to $(\rho, \boldsymbol{u})$ is defined by either one of the following equivalent (formal) infinite products:

$$\begin{aligned} Z_X(\boldsymbol{u}; \rho) &:= \prod_{\boldsymbol{P}=\{\gamma\}_\Gamma:\text{primitive}} \det\{I_n - \rho(\gamma)\boldsymbol{u}^{\boldsymbol{P}}\}^{-1} \\ &= \prod_{[C]:\text{primitive}} \det\{I_n - \rho(\langle C \rangle)\boldsymbol{u}^{C}\}^{-1} \end{aligned} \tag{2.4}$$

where $\boldsymbol{P} = \{\gamma\}_\Gamma$ (resp. $[C]$) runs over the set of primitive Γ-conjugacy classes (resp. cycles). Taking the logarithm of (2.4), we have

$$\begin{aligned}\log Z_X(\boldsymbol{u};\rho)&=\sum_{[C]}-\log\det\{I_n-\rho(\langle C\rangle)\boldsymbol{u}^C\}\\&=\sum_{[C]}\sum_{i=1}^{n}-\log(1-\alpha_i\boldsymbol{u}^C)\\&=\sum_{[C]}\sum_{i=1}^{n}\sum_{k=1}^{\infty}(\alpha_i^k/k)\boldsymbol{u}^{Ck},\\&=\sum_{k=1}^{\infty}\sum_{d=1}^{\infty}\sum_{\substack{[C]\\|C|=d}}\left\{\sum_{i=1}^{n}(\alpha_i^k/k)\boldsymbol{u}^{Ck}\right\},\end{aligned}$$

where $\{\alpha_i;\ (1\leq i\leq n)\}$ are the eigenvalues of $\rho(\langle C\rangle)$. Notice that the last sum in the bracket is homogeneous of degree kd. Applying the Euler operator, we obtain

$$\sum_{j=1}^{m}u_j(\partial/\partial u_j)\log Z_X(\boldsymbol{u};\rho)=\sum_{k=1}^{\infty}\sum_{d=1}^{\infty}d\sum_{\substack{[C]\\|C|=d}}\left\{\sum_{i=1}^{n}\alpha_i^k\right\}\boldsymbol{u}^{Ck}. \tag{2.5}$$

We note, from the definition of the cycles, that the natural map $C\to[C]$, from $\mathscr{C}_d^{red,pr}$ to the set of primitive cycles, is d-to-one. Putting $kd=l$, we see that the right hand side of (2.5) is reformed to

$$\sum_{l=1}^{\infty}\sum_{d|l}\sum_{C\in\mathscr{C}_d^{red,pr}}\operatorname{tr}\rho(\langle C\rangle^{l/d})\boldsymbol{u}^{C^{(l/d)}}=\sum_{l=1}^{\infty}\sum_{C\in\mathscr{C}_l^{red}}\operatorname{tr}\rho(\langle C\rangle)\boldsymbol{u}^{C}.$$

The last equality follows from the obvious decomposition (1.6). Put

$$N_{l,\rho}(\boldsymbol{u}):=\sum_{C\in\mathscr{C}_l^{red}}\operatorname{tr}\rho(\langle C\rangle)\boldsymbol{u}^C, \tag{2.6}$$

$$N_{l,\rho}:=\sum_{C\in\mathscr{C}_l^{red}}\operatorname{tr}\rho(\langle C\rangle)=N_{l,\rho}(1,\cdots,1). \tag{2.7}$$

Then we can express our zetafunction as follows:

$$\sum_{j=1}^{m}u_j(\partial/\partial u_j)\log Z_X(\boldsymbol{u};\rho)=\sum_{l=1}^{\infty}N_{l,\rho}(\boldsymbol{u}). \tag{2.8}$$

Note that $Z_X(u,\cdots,u;\rho)$ agrees with $Z_X(u;\rho)$ defined by (0.3). Also, for $\rho=\mathbf{1}$ (= trivial representation) and for a function f on EX, we get $Z_X(f(e_1),\cdot\ \cdot,f(e_m);\mathbf{1})=Z_X(f;u)$ (cf. (0.4)). Finally we have

$$N_l=N_{l,\mathbf{1}}=\sharp(\mathscr{C}_l^{red}), \tag{2.9}$$

hence $Z_X(u,\cdots,u;\mathbf{1})=Z_X(u)$ (cf. (0.3)). Here we note that in (2.9), we count, if any, a loop C twice, distinguishing C and C^{-1}.

To proceed further, we first assume that X is a *bipartite* multigraph. Then it is easy to see that any closed path has an even length. Let $\tilde{X}$ be the universal covering tree of X as above. Since X is assumed to be of bipartite, so is $\tilde{X}$, and one has a partition $V\tilde{X}=\tilde{V}_1 \cup \tilde{V}_2$ (disjoint), which is preserved by the action of $\Gamma=\pi_1(X, P_0)$. For each edge $e=\{P_1, P_2\} \in E\tilde{X}$, such that $P_i \in \tilde{V}_i$, let $\tilde{E}_i(e)$ be the set of edges which are incident to P_i $(i=1, 2)$.

Definition (2.10). Let $\boldsymbol{Z}[E\tilde{X}]$ be the free $\boldsymbol{Z}$-module over the set $E\tilde{X}$. We define the correspondences T_1, T_2 on $E\tilde{X}$ to be the elements in $\mathrm{End}(\boldsymbol{Z}[E\tilde{X}])$ given by

$$T_i(e):=\sum_{\substack{e'\in\tilde{E}_i(e)\\ e'\neq e}} e' \qquad (i=1, 2). \tag{2.11}$$

The following lemma is immediately seen (see Fig. 5).

Lemma (2.12). *One has for each* $e \in E\tilde{X}$, *and* $l \geq 0$

$$(T_2T_1)^l(e)=\sum_{(*)} e^{2l+1}.$$

Here the sum is extended over the edges $e^{2l+1} \in E\tilde{X}$ *such that there exists* $C=(y_{(1)}, y_{(2)}, \cdots, y_{(2l+1)}) \in \tilde{\mathscr{C}}^{(+)}_{2l+1}(e)$ *with* $y_{(1)} \in e$, $o(y_{(1)}) \in V_2$, $t(y_{(1)}) \in V_1$ *and* $y_{(2l+1)} \in e^{2l+1}$; $\tilde{\mathscr{C}}^{(+)}_{2l+1}(e)$ *is the set of all such proper paths* $C=(y_{(1)}, y_{(2)}, \cdots, y_{(2l+1)})$ *of length* $2l+1$. *Similar formula holds for* $(T_1T_2)^l(e)$, *with* $C \in \tilde{\mathscr{C}}^{(-)}_{2l+1}(e)$ *running to the other direction, starting with* $y_{(1)}^{-1}$.

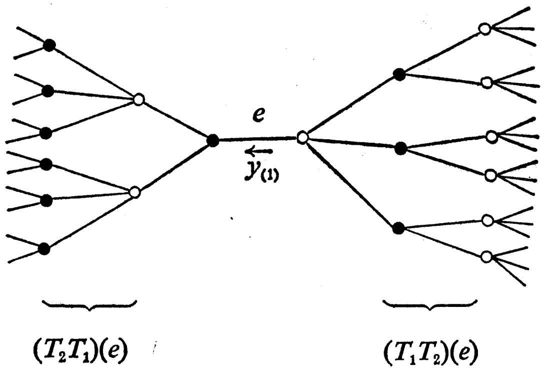


Fig. 5

Now we consider the $\boldsymbol{C}$-vector space M_ρ consisting of the $\boldsymbol{C}^n$-valued functions on $E\tilde{X}$ satisfying

$$f(e\cdot\gamma)=f(e)\cdot\rho(\gamma) \qquad \text{for any } \gamma \in \Gamma=\pi_1(X, P_0). \tag{2.13}$$

It is easy to see that $\dim_{\boldsymbol{C}} M_\rho = n. \sharp(EX)$. In fact, let $\mathscr{E} = \{e^{(i)}; (1 \leq i \leq m = \sharp(EX))\}$ be a complete set of representatives of $E\tilde{X}/\Gamma$, and let $\{\boldsymbol{v}_j\ (1 \leq j \leq n)\}$ an orthonormal basis of $\boldsymbol{C}^n$, with respect to the standard inner product. Then from the fact that Γ acts on $\tilde{X}$ without fixed point, it follows that the functions $f_{i,j}$ determined by

$$(2.14) \qquad \begin{aligned} f_{i,j}(e) &= \boldsymbol{v}_j \cdot \rho(\gamma) && (\text{if } e = e^{(i)} \cdot \gamma,\ \exists \gamma \in \Gamma), \\ &= 0 && (\text{otherwise}), \end{aligned}$$

form a basis of M_ρ. Moreover, one can introduce an inner product on M_ρ in such a way that these basis form an orthonormal system. In other words, we put

$$(2.15) \qquad (f, f') := \sum_{e^{(i)} \in \mathscr{E}} (f(e^{(i)}), f'(e^{(i)})),$$

where those in the right hand side denote the standard inner product on $\boldsymbol{C}^n$.

Since the action of Γ on $\tilde{X}$ preserves the incidence relation and the partition of $V\tilde{X}$, the correspondences T_i induce naturally the endomorphisms on M_ρ,

$$(2.16) \qquad (\rho^*(T_i)f)(e) := \sum_{e' \in \tilde{E}_i(e)} f(e') - f(e) \qquad (i = 1, 2).$$

In fact it is easy to see that ρ^* is an anti-representation of $\mathrm{End}(\boldsymbol{Z}[E\tilde{X}])$. One also sees immediately that $\rho^*(T_i)$ are hermitian operators with respect to the inner product (2.15).

Let $A := \boldsymbol{C}[\boldsymbol{u}] = \boldsymbol{C}[u_1, \cdots, u_m]$ be the polynomial ring of $\boldsymbol{u}$ over $\boldsymbol{C}$, and define the elements $\rho_{\boldsymbol{u}}^*(T_i)$ of $\mathrm{End}(M_\rho \otimes_{\boldsymbol{C}} A)$:

$$(2.17) \qquad (\rho_{\boldsymbol{u}}^*(T_i)F)(e) := \sum_{\substack{e' \in \tilde{E}_i(e) \\ e' \neq e}} F(e')\boldsymbol{u}^{e'} \qquad (F \in M_\rho \otimes_{\boldsymbol{C}} A),$$

where $\boldsymbol{u}^{e'} = u_i$ if $p(e') = e^{(i)} (1 \leq i \leq m)$, and we regard F naturally as a $\boldsymbol{C}^n \otimes_{\boldsymbol{C}} A = A^n$-valued function on $E\tilde{X}$. Note that these are well-defined endomorphisms over A, since one has $\boldsymbol{u}^{e \cdot \gamma} = \boldsymbol{u}^e$ for any $e \in E\tilde{X}$, $\gamma \in \Gamma$. Moreover, one can extend the inner product (2.15) to a nondegenerate hermitian pairing $(M_\rho \otimes_{\boldsymbol{C}} A) \times (M_\rho \otimes_{\boldsymbol{C}} A) \to A$, and see that $\rho_{\boldsymbol{u}}^*(T_i)$ are self-adjoint operators on a finite A-module $M_\rho \otimes_{\boldsymbol{C}} A$. From Lemma (2.12), we obtain

$$(2.18) \qquad (\rho_{\boldsymbol{u}}^*(T_1)\rho_{\boldsymbol{u}}^*(T_2))^l F(e) = \sum_{C \in \tilde{\mathscr{C}}_{2l+1}^{(+)}(e)} F(e')\boldsymbol{u}^C.$$

Now the following lemma is a key to our study:

Lemma (2.19). *As endomorphisms over A, one has*

$$\operatorname{tr}(\rho_{\boldsymbol{u}}^*(T_2)\rho_{\boldsymbol{u}}^*(T_1))^l=\operatorname{tr}(\rho_{\boldsymbol{u}}^*(T_1)\rho_{\boldsymbol{u}}^*(T_2))^l=(1/2)\cdot N_{2l,\rho}(\boldsymbol{u}).$$

Proof. First note that the elements $F_{i,j}:=f_{i,j}\otimes 1$ $(1\leq i\leq m,\ 1\leq j\leq n)$ form a basis of $M_\rho\otimes_C A$ over A. Therefore we have

$$\begin{aligned}(2.20)\qquad \operatorname{tr}(\rho_{\boldsymbol{u}}^*(T_2)\rho_{\boldsymbol{u}}^*(T_1))^l&=\sum_{i=1}^{m}\sum_{j=1}^{n}((\rho_{\boldsymbol{u}}^*(T_2)\rho_{\boldsymbol{u}}^*(T_1))^l(F_{i,j}),\ F_{i,j})\\ &=\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{m}((\rho_{\boldsymbol{u}}^*(T_2)\rho_{\boldsymbol{u}}^*(T_1))^l(F_{i,j})(e^{(k)}\otimes 1),\ F_{i,j}(e^{(k)}\otimes 1))\\ &=\sum_{i=1}^{m}\sum_{j=1}^{n}((\rho_{\boldsymbol{u}}^*(T_2)\rho_{\boldsymbol{u}}^*(T_1))^l(F_{i,j})(e^{(i)}\otimes 1),\ \boldsymbol{v}_j\otimes 1).\end{aligned}$$

Using (2.18), we see that

$$(\rho_{\boldsymbol{u}}^*(T_2)\rho_{\boldsymbol{u}}^*(T_1))^l(F_{i,j})(e^{(i)}\otimes 1)=\sum_{C\in\tilde{\mathscr{C}}_{2l+1}^{(+)}(e^{(i)})} f_{i,j}(e')\boldsymbol{u}^C,$$

where the sum $\sum(*)$ is extended over the proper path $C=(y^{(i)},\cdots,y')\in\mathscr{C}_{2l+1}^{(+)}(e^{(i)})$ such that $y'=y^{(i)}\cdot\gamma$ $(y^{(i)}\in e^{(i)})$ for some $\gamma\in\Gamma$. Denote the set of such paths by $\tilde{\mathscr{C}}_{2l+1}^{(+,0)}(e^{(i)})$, and the similar subset of $\tilde{\mathscr{C}}_{2l+1}^{(-)}(e^{(i)})$ by $\tilde{\mathscr{C}}_{2l+1}^{(-,0)}(e^{(i)})$. Since γ is determined uniquely by C, it follows from (2.14) that the last expression of (2.20) can be reformed as

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{C\in\tilde{\mathscr{C}}_{2l+1}^{(+,0)}(e^{(i)})}(\boldsymbol{v}_j\cdot\rho(\gamma)\otimes 1,\ \boldsymbol{v}_j\otimes 1)\boldsymbol{u}^C=\sum_{i=1}^{m}\sum_{C\in\tilde{\mathscr{C}}_{2l+1}^{(+,0)}(e^{(i)})}\operatorname{tr}\rho(\gamma)\boldsymbol{u}^C$$

Now the morphism $p:\tilde{X}\to X$ induces the following bijection:

$$(2.21)\qquad \bigcup_{i=1}^{m}[\tilde{\mathscr{C}}_{2l+1}^{(+,0)}(e^{(i)})\cup\tilde{\mathscr{C}}_{2l+1}^{(-,0)}(e^{(i)})]\underset{p}{\simeq}\mathscr{C}_{2l}^{red},$$

where the $\cup$'s in the left are all disjoint. Thus we have

$$\operatorname{tr}(\rho_{\boldsymbol{u}}^*(T_2)\rho_{\boldsymbol{u}}^*(T_1))^l+\operatorname{tr}(\rho_{\boldsymbol{u}}^*(T_1)\rho_{\boldsymbol{u}}^*(T_2))^l=\sum_{C\in\mathscr{C}_{2l}^{red}}\operatorname{tr}\rho(\langle C\rangle)\boldsymbol{u}^C.$$

This completes the proof. Q.E.D.

Now we have the following

Main theorem (I). *Let X be a finite connected multigraph of bipartite type, and let ρ, $\boldsymbol{u}$ be a unitdry representation of $\Gamma=\pi_1(X,P_0)$ of degree n, and a labelling of EX, respectively. Then the zetafunction $Z_X(\boldsymbol{u};\rho)$ attached to $(\boldsymbol{u},\rho)$ is a rational function of $\boldsymbol{u}$, and it is given by*

$$(2.22)\qquad Z_X(\boldsymbol{u};\rho)=\det(I_{nm}-\rho_{\boldsymbol{u}}^*(T_2)\rho_{\boldsymbol{u}}^*(T_1))^{-1}\qquad (m=\sharp(EX)).$$

In particular, one has (putting $\boldsymbol{u}=(u,\cdots,u)$)

$$(2.23)\qquad Z_X(u;\rho)=\det(I_{nm}-\rho^*(T_2T_1)u^2)^{-1}.$$

Proof. Taking log of the right hand side of (2.22), one get

$$\log\det(I_{nm}-\rho_{\boldsymbol{u}}^*(T_2)\rho_{\boldsymbol{u}}^*(T_1))^{-1}=\sum_{l=1}^{\infty}(1/l)\operatorname{tr}(\rho_{\boldsymbol{u}}^*(T_2)\rho_{\boldsymbol{u}}^*(T_1))^l.$$

Notice that $\operatorname{tr}(\rho_{\boldsymbol{u}}^*(T_2)\rho_{\boldsymbol{u}}^*(T_1))^l$ is a homogeneous polynomial in $\boldsymbol{u}$ of degree $2l$, and apply the Euler's operator. One obtains

$$\sum_{j=1}^{m}u_j(\partial/\partial u_j)\log\det(I_{nm}-\rho_{\boldsymbol{u}}^*(T_2)\rho_{\boldsymbol{u}}^*(T_1))^{-1}=2\sum_{l=1}^{\infty}\operatorname{tr}(\rho_{\boldsymbol{u}}^*(T_2)\rho_{\boldsymbol{u}}^*(T_1))^l.$$

The assertion follows by comparing this and (2.8), (2.19). Q.E.D.

Remark (2.24). (i) We note that, in both (2.19) and (2.22), the operator $\rho^*(T_1T_2)$ plays the same role as the Frobenius endomorphism in the case of congruence zeta functions of algebraic curves over finite fields.

(ii) For a bipartite multigraph X, the zeta function $Z_X(\boldsymbol{u};\rho)^{-1}$ is an even polynomial of $\boldsymbol{u}$ i.e., $Z_X(-\boldsymbol{u};\rho)=Z_X(\boldsymbol{u};\rho)$. For the zeta function of single variable, it is often convenient to modify it and put

$$(2.25)\qquad Z_{X,b}(u;\rho):=Z_X(u^{1/2};\rho)=\det(I_{nm}-\rho^*(T_2T_1)u)^{-1}.$$

Next we consider a general multigraph X. Let $X^{(2)}$ be the barycentric subdivision of X. This means that we add a new vertex to the middle point of each edge of X, hence the edge is divided into two edges. It can be seen immediately that $X^{(2)}$ is a graph of bipartite type. Namely $VX^{(2)}=V_1\cup V_2$, where $V_1=VX$ and $V_2\simeq EX$ is the set of new vertices.

Definition (2.26). Let $EX^{(2)}=\{e_1^{(2)},\cdots,e_{2m}^{(2)}\}$ be the edges of $X^{(2)}$ such that $e_{2j-1}^{(2)}$, $e_{2j}^{(2)}$ correspond to the edge e_j of X. For a labelling $\boldsymbol{u}$ of EX, we put $\boldsymbol{u}^{(2)}:=(u_1,u_1,u_2,u_2,\cdots,u_m,u_m)$ and regard it a (reduced) labelling of $EX^{(2)}$. We also put $\boldsymbol{u}^2:=(u_1^2,\cdots,u_m^2)$.

Main theorem (II). *Suppose that X is an arbitrary finite connected multigraph, and let $\boldsymbol{u}$, ρ be as above. Then we have*

$$(2.27)\qquad Z_X(\boldsymbol{u}^2;\rho)=Z_{X^{(2)}}(\boldsymbol{u}^{(2)};\rho),$$

where the right hand side is the zeta function of a bipartite multigraph, *evaluated in* (2.22).

Proof. First it should be noted that the geometric realizations of X, $X^{(2)}$ as CW-complexes are the same, and the fundamental groups of them are identified in the obvious way. Note also that there is a natural bijection $\mathscr{C}_l^{red}(X) \simeq \mathscr{C}_{2l}^{red}(X^{(2)})$, so that, if C and $C^{(2)}$ are the corresponding reduced closed paths, then one has

$$(2.28) \qquad \boldsymbol{u}^{2C} = (\boldsymbol{u}^{(2)})^{C^{(2)}}.$$

The assertino follows from this and (2.4). Q.E.D.

We conclude this section with the following observation, which follows either from (2.8) or from (2.22). Let X^* be the connected multigraph obtained from X by removing its endpoints. Note that the fundamental groups of X and X^* can be identified. Then we have

Proposition (2.29).

$$Z_{X^*}(\boldsymbol{u};\, \rho) = Z_X(\boldsymbol{u};\, \rho).$$

In particular, $Z_X(\boldsymbol{u};\, \rho)$ *does not involve the variable* u_i, *if the corresponding edge* e_i *is incident to an endpoint of* X.

From (2.22), one sees that $Z_X(\boldsymbol{u};\, \rho)^{-1}$ is a polynomial of $\boldsymbol{u}$ of degree at most $2n.\,\sharp(EX)$, where $n = \deg(\rho)$. For a multigraph X $(= X^*)$ having no endpoint, we have the following fact, which follows from (5.23):

Proposition (2.30). *Suppose that* $X = X^*$. *Then* $Z_X(\boldsymbol{u};\, \rho)^{-1}$ *is a polynomial of* $\boldsymbol{u}$ *of degree* $2n.\,\sharp(EX)$. *If* X *is bipartite,* $Z_{X,b}(u)^{-1}$ *is a polynomial of degree* $n.\,\sharp(EX)$.

§ 3. Spectrum of a finite multigraph

In this section we study the relation between our zeta functions and the spectra of the finite multigraphs X, under the assumption that X has certain regularity. We shall be mostly concerned with the simplest zeta function $Z_X(u) = Z_X(u;\, \mathbf{1})$.

Let $VX = \{P_1, \cdots, P_n\}$, and $EX = \{e_1, \cdots, e_m\}$ be the sets of vertices and edges of a multigraph X.

Definition (3.1). The $n \times n$ matrix $A = A(X) = (a_{ij})$, defined by

$$a_{ij} := \sharp\{y \in EX;\, o(y) = P_i,\, t(y) = P_j\},$$

is called *the adjacency matrix.* Applying the involution $\iota_X \colon EX \to EX$, we see that $A(X)$ is a symmetric matrix. The polynomial

$$(3.2) \qquad \phi_X(z) := \det(z.I_n - A)$$

is called the *characteristic polynomial* of X, and the set of its roots, counted with multiplicities, is called *the spectrum* of X, and denoted by $\mathrm{Spec}(X)$.

Note that A being a real symmetric matrix implies that the spectrum consists of real numbers; $a_{ii}=0$ $(1 \leq i \leq n)$ if X has no loop, which implies that $\mathrm{tr}(A)=0$. Moreover, if X is a graph then all nonzero entries of A are equal to 1. Note also that the (i, i)-entry of A^m is equal to the number of closed paths having P_i as their origin, hence we have

$$(3.3) \qquad \mathrm{tr}(A^m) \geq N_m.$$

Let λ be the eigenvalue of A with maximal absolute value. Then we see immediately that the power series in (0.3) converges absolutely for $|u|<|\lambda|^{-1}$. This already shows that our zeta function is connected with the spectrum of X. Thus it is interesting to ask how two polynomials $\phi_X(u)$ and $Z_X(u)^{-1}$ are related. We shall give an answer in the case where X satisfies some regularity conditions.

Definition (3.4). Let $P_i \in VX$ be a vertex of a multigraph X. Then one has

$$\sum_{j=1}^{n} a_{ij} = \text{the number of edges incident to } P_i,$$

and this number is called the *valency* of P_i; if the valency, say k, is the same for all vertices P_i, then X is called *regular* of valency k, or *k-valent*. The following fact has been known for a long time:

Proposition (3.5). *Suppose that X is a connected regular k-valent multigraph. Then*

(i) *k belongs to* $\mathrm{Spec}(X)$ *with multiplicity one.*

(ii) *Moreover, k is the eigenvalue of $A(X)$ with maximal absolute value.*

(iii) $-k \in \mathrm{Spec}(X)$ *if and only if X is of bipartite type.*

Proof. See [Bi]. We shall give a proof, based on the same idea as in [Bi], in Section 4 (see (4.26)). Q.E.D.

Also the zeta function $Z_X(u)$ of regular graphs has been known. However, their simple relation, described below, has been seldom noticed until recently.

Theorem (3.6) (Ihara [I-1]). *Suppose that X is a connected regular multigraph with valency $q+1$, and the adjacency matrix A. Then one has*

$$Z_X(u)^{-1}=(1-u^2)^{r-1}\det[I_n-Au+qu^2], \tag{3.7}$$

where $r=(q-1)n/2$ is the rank of $\Gamma=\pi_1(X, P_0)$, and $n=\sharp(VX)$. In other words,

$$Z_X(u)^{-1}=(1-u^2)^{r-1}u^n\phi_X\left(\frac{1+qu^2}{u}\right). \tag{3.8}$$

Proof. This can be viewed as a special case of the following Main theorem (III), in two ways. First, if X is of bipartite type, it is directly included. To see it in other way, we take the barycentric subdivison $X^{(2)}$ of X, and apply (2.27). Q.E.D.

Remark (3.9). In [I-1], (3.7) was proved for a slightly different context. It was Serre [Ser] who pointed out that Ihara's result could be interpreted in terms of graphs. The detail has been given by Sunada [Su-1, 2].

The above theorem has a number of interesting consequences, since the spectra of regular graphs have been studied by many authors, and their results can be translated to the properties of zeta function $Z_X(u)$. We shall describe some of them in Sections 8, 9.

Next we proceed to the case of bipartite multigraphs.

Definition (3.10). Let X be an *s-partite* multigraph, and let $VX=\bigcup_{i=1}^{s}V_i$ be the corresponding decomposition of VX. It is also called *bipartite*, if $s=2$. An s-partite graph is called *semi-regular* of valency $(k_1, \cdots, k_s)$, if each vertex $P\in V_i$ is incident to exactly k_i edges for each $i=1, 2, \cdots, s$.

Note that a multi-partite multigraph has no loop. As we have seen in the previous section, the bipartite mutigraphs play an essential role in our study. For such graphs, one has the following

Lemma (3.11). *Suppose that X is a bipartite mutigraph, and let $VX=V_1\cup V_2$ be the corresponding partition, with $\sharp(V_i)=n_i$ $(i=1, 2)$, $n_2\geq n_1$. Then the spectrum of X has the form*

$$\mathrm{Spec}(X)=\{\pm\lambda_1, \pm\lambda_2, \cdots, \pm\lambda_{n_1}, 0, \cdots, 0\ (n_2-n_1\ \textit{times})\},$$

with $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{n_1}\geq 0$.

Proof. From the definition, the adjacency matrix A of X has the form

$$\text{(3.12)} \qquad A = \left[\begin{array}{c|c} \begin{matrix} 0 \cdots 0 \\ \cdots \\ 0 \cdots 0 \end{matrix} & B \\ \hline {}^tB & \begin{matrix} 0 \cdots 0 \\ \cdots \\ 0 \cdots 0 \end{matrix} \end{array}\right] \qquad \text{with } B \in M(n_1, n_2; \boldsymbol{Z}).$$

One can find an orthogonal matrix $W \in O(n_2)$ in such a way that

$$BW = \begin{array}{c} \begin{array}{cc} \longleftarrow n_1 \longrightarrow & \leftarrow n_2 - n_1 \rightarrow \end{array} \\ \left[\begin{array}{cccc|ccc} \mu_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & \ddots & & \vdots & \vdots & & \vdots \\ * & \mu_2 & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ * & & * & \mu_{n_1} & 0 & \cdots & 0 \end{array}\right] \end{array} = (C \,|\, 0), \text{ say.}$$

Then one sees that A is conjugate to the matrix

$$\begin{array}{ccc} \longleftarrow n_1 \longrightarrow & \longleftarrow n_1 \longrightarrow & \leftarrow n_2 - n_1 \rightarrow \\ \end{array}$$
$$\left[\begin{array}{c|c|c} 0 & C & 0 \\ \hline {}^tC & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right].$$

Now the last matrix has the characteristic polynomial

$$\begin{aligned} x^{n_2-n_1} \det\begin{pmatrix} x.I & -C \\ -{}^tC & x.I \end{pmatrix} &= x^{n_2-n_1} \det\begin{pmatrix} x.I & -C \\ -{}^tC & x.I \end{pmatrix}\begin{pmatrix} I & x^{-1}C \\ O & I \end{pmatrix} \\ &= x^{n_2-n_1} \det\begin{pmatrix} x.I & O \\ -{}^tC & x.I - x^{-1}\,{}^tC.\,C \end{pmatrix} \\ &= x^{n_2-n_2} \det(x^2 I - {}^tC.C). \end{aligned}$$

Since ${}^tC.C$ is a positive semi-definite symmetric matrix, its eigenvalues are of the form $\{\lambda_i^2;\ \lambda_i \geq 0\ (1 \leq i \leq n_1)\}$. Q.E.D.

Now we state a theorem which generalizes Ihara's result (3.7) to the connected regular bipartite multigraphs. First we need some notation. Let X be such a graph, and let $VX = V_1 \cup V_2$ as above. Then one can construct a new multigraphs $X^{[i]}$ $(i=1, 2)$ as follows:

$$\text{(3.13)} \quad VX^{[i]} := V_i,\ EX^{[i]} := \{C = \text{proper path};\ |C| = 2,\ o(C), t(C) \in V_i\},$$
$$\varepsilon\colon C \to \{o(C),\ t(C)\}.$$

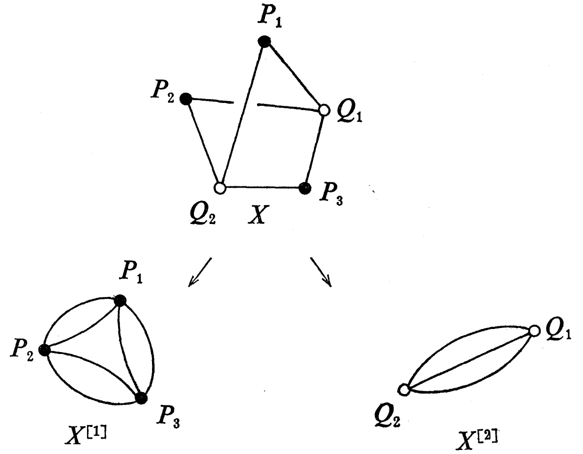


Fig. 6

Main theorem (III). *Suppose that X is a connected semiregular bipartite multigraph of valency* (q_1+1, q_2+1), $\sharp(V_i)=n_i$ $(i=1, 2)$, $q_1 \geq q_2$, *and let* $A^{[i]}$ *be the adjacency matrix of the associated multi graph* $X^{[i]}$ $(i=1, 2)$. *Then one has*

$$(3.14)\quad \begin{aligned} &Z_{X,b}(u)^{-1}\\ &=(1-u)^{(r-1)}(1+q_2u)^{(n_2-n_1)}\times\det[I_{n_1}-(A^{[1]}-q_2+1)u+q_1q_2u^2],\\ &=(1-u)^{(r-1)}(1+q_1u)^{(n_1-n_2)}\times\det[I_{n_2}-(A^{[2]}-q_1+1)u+q_1q_2u^2],\end{aligned}$$

where $Z_{X,b}(u)$ *is defined by* (2.25), *and* $r=n_1q_1-n_2+1=n_2q_2-n_1+1$ *is the rank of* $\Gamma=\pi_1(X, P_0)$.

In particular, if $\mathrm{Spec}\,(X)=\{\pm\lambda_1, \pm\lambda_2, \cdots, \pm\lambda_{n_1}, 0, \cdots, 0\}$ *are as in Lemma* (3.11), *one has*

$$(3.15)\quad \det[I_{n_1}-(A^{[1]}-q_2+1)u+q_1q_2u^2]=\prod_{j=1}^{n_1}\{1-(\lambda_j^2-q_1-q_2)u+q_1q_2u^2\}.$$

Proof. A proof of (3.14) has been given in [H-H]; see Remark (3.19) below. We shall give in Section 5 a different proof, which describes that it can be derived from our Main Theorem (II). To show (3.15), one notes first that the last equality in (3.14) is equivalent to the following relation between $A^{[1]}$ and $A^{[2]}$:

$$(3.16)\qquad A^{[2]}\simeq\left(\begin{array}{c|c} A^{[1]}+(q_1-q_2)I_{n_1} & 0\\ \hline 0 & -(q_2+1)I_{n_2-n_1}\end{array}\right).$$

On the other hand, one can see easily that

$$A^2=\left(\begin{array}{c|c} A^{[1]}+(q_1+1)I_{n_1} & 0 \\ \hline 0 & A^{[2]}+(q_2+1)I_{n_2} \end{array}\right).$$

From these follows

$$(3.17)\qquad A^{[1]}\simeq\begin{pmatrix} \lambda_1^2-(1+q_1) & & 0 \\ & \ddots & \\ 0 & & \lambda_{n_1}^2-(1+q_1) \end{pmatrix}.$$

This proves (3.15). Q.E.D.

From the above proof, one has the following

Corollary (3.18). *Suppose X is a regular connected multigraph of valency $q+1$, with $n=\sharp(VX)$, $m=\sharp(EX)$, and let* $\mathrm{Spec}(X^{(2)})=\{\pm\lambda_1, \pm\lambda_2, \cdots, \pm\lambda_n, 0, \cdots, 0$ *($m-n$ times)\} be the spectrum of the barycentric subdivision $X^{(2)}$ of X (cf.* (3.11)). *Then the spectrum of X is given by*

$$\mathrm{Spec}(X)=\{\lambda_1^2-q-1, \cdots, \lambda_n^2-q-1\}.$$

Remark (3.19). The expression (3.14) has been given in [H-H] for a zeta function $Z_\Gamma(u;\rho)$ attached to a subgroup Γ of G, which satisfies axioms (G, l, I), (G, l, II) (cf. (6.1)). That these two zeta functions are the same thing can be seen as follows. Consider the universal covering tree $\tilde{X}$ and the action of the fundamental group $\Gamma=\pi_1(X, P_0)$, which is faithful so that it can be regarded as a subgroup of $G:=\mathrm{Aut}(\tilde{X})$. Now it is known and easy to see that G acts transitively on $E\tilde{X}$, hence also on $V_1\tilde{X}$, $V_2\tilde{X}$. Thus we can apply the results of our previous paper [H-H] to get (3.14). In fact it suffices to note that the matrix $A_{1,\rho}$ in [H-H] is nothing but our $A^{[1]}$, for $\rho=\mathbf{1}$; and similarly for $A^{[2]}$.

Now one can prove the following generalization of Proposition (3.5):

Proposition (3.20). *Suppose that X is a connected semiregular bipartite multigraph of valency (k_1, k_2). Then $\sqrt{k_1k_2}$ belong to* $\mathrm{Spec}(X)$, *with multiplicity one; moreover, $\sqrt{k_1k_2}$ is the maximal absolute value of the eigenvalues of A.*

Proof. It is easy to see that $X^{[1]}$ (resp. $X^{[2]}$) is a regular multigraph of valency $q_2(q_1+1)$ (resp. $q_1(q_2+1)$), where we put $k_i=q_i+1$ $(i=1, 2)$. Therefore one sees from Proposition (3.5) that $A^{[1]}$ has $q_2(q_1+1)$ as its eigenvalue of maximal absolute value, with multiplicity one. This implies by (3.14), that $\det[I-(A^{[1]}-q_2+1)u+q_1q_2u^2]$ has a simple factor

$$1-\{q_2(q_1+1)-q_2+1\}u+q_1q_2u^2=(1-u)(1-q_1q_2u)$$

which is, by (3.15), equal to $1-\{\lambda_1^2-(q_1+q_2)\}u+q_1q_2u^2$. Hence we get

$$\lambda_1^2=1+q_1+q_2+q_1q_2=(q_1+1)(q_2+1)=k_1k_2,$$

as asserted. Similarly, we have $\lambda_j^2-(q_1+1)<q_2(q_1+1)$ ($j\geq 2$), hence $\lambda_j^2<k_1k_2$. Q.E.D.

See Proposition (4.26) for an interpretation of the eigenvalue $\sqrt{k_1k_2}$. Finally we make the following

Definition (3.21). Let X be a finite connected semiregular bipartite multigraph of valency (q_1+1, q_2+1), such that $q_1\geq q_2>1$. We call X a *weak Ramanujan graph*, if

$$\operatorname{ord}_{(1+q_2u)}Z_{X,b}(u)^{-1}=n_2-n_1, \tag{3.22}$$

where $n_i=\sharp(VX_i)$. Note that we have $n_2\geq n_1$. X is called a *Ramanujan graph*, if

$$|\lambda^2-q_1-q_2|\leq 2\sqrt{q_1q_2}, \tag{3.23}$$

for any $\lambda\in\operatorname{Spec}(X)$ such that $\lambda^2\neq(1+q_1)(1+q_2)$.

Observe that (3.23) is equivalent to the condition that the nonlinear factors $1-(\lambda^2-q_1-q_2)u+q_1q_2u^2$ of $Z_{X,b}(u)^{-1}$ have the imaginary roots α, α' such that $|\alpha|=|\alpha'|=\sqrt{q_1q_2}$. Also note that this definition of the Ramanujan graph agrees with the one given in [L-P-S], in the case when X is regular bipartite, as one sees easily. Also it is easy to prove the following assertions:

Lemma (3.24). *Suppose that X is as in* (3.21), *and that it is a Ramanujan graph. Then one has*

(i) *X is a weak Ramanujan graph.*

(ii) *If $q_1=q_2$, then $X^{(2)}$ (= the barycentric subdivision of X) is also a Ramanujan graph.*

Conversely, if $q_1=q_2$ and $X^{(2)}$ is a Ramanujan graph, then X is a Ramanujan graph.

Proof. Omitted.

We shall give in Section 9 a number of examples of multigraphs which are *not* Ramanujan graphs.

§ 4. Harmonic functions and the Hodge decomposition

In the previous two sections we gave two essentially different expressions of $Z_X(u)$. While the first one (2.23) (and (2.27)) is completely general, it does not give a direct connection with the spectrum of X. On the other hand the second expression (3.14), which applies only to the semiregular bipartite graphs, shows that $Z_{X,b}(u)$ is determined by the spectrum of X, together with its topological invariants r, n_2-n_1. Moreover, the latter expression has two distinguished factors, $(1-u)$ and $(1+q_2u)$. In this section, we study the interpretation of these factors.

Throughout this section, we always assume that our multigraph X is of bipartite type, and $VX=V_1\cup V_2$ is the corresponding partition. (This assumption does not cause any loss of generality; take the barycentric subdivision of X if necessary).

We introduce, only in this section, an orientation on X. Thus we call an oriented edge y to be *positive*, if $o(y)\in V_1$ and $t(y)\in V_2$. If $e=\{y, y^{-1}\}$ and y is positive, we write $y=e^+$ and $y^{-1}=e^-$. We define the signature $\operatorname{sgn}(y)$ of an oriented edge y by

$$\operatorname{sgn}(y)=\begin{cases} 1 & \text{if } y \text{ is positive} \\ -1 & \text{if } y \text{ is negative.}\end{cases} \tag{4.1}$$

Recall that if $\rho=\mathbf{1}$ is the trivial representation of $\pi_1(X, P_0)$, the C-space $M^1(X):=M_\rho(X)$ is regarded as consisting of C-valued functions on EX, and it is equipped with an inner product (2.15).

We first describe an important decomposition of $M^1(X)$, called the *Hodge decomposition*, which distinguishes the subspace corresponding to the factor $(1-u)$. This is the one dimensional version of Garland and Borel [Gar], [Bo-1]. Let $M^0(X)$ denote the space of functions on VX. Then we can write $M^0(X)=M^0(V_1)\oplus M^0(V_2)$, where $M^0(V_i)$ is the space of functions on V_i $(i=1, 2)$. We define an inner product on $M^0(X)$ by putting

$$(f, g):=\sum_{P\in V} k(P)f(P)\overline{g(P)} \qquad (k(P):=\sharp(E(P))=\text{valency of } P). \tag{4.2}$$

Define the linear maps d, δ by

$$\begin{aligned} &d\colon M^0(X)\to M^1(X),\ (df)(e):=f(t(e^+))-f(o(e^+)).\\ &\delta\colon M^1(X)\to M^0(X),\ (\delta f)(P):=(-1)^i\sum_{e\in E(P)} f(e)/k(P) \quad (P\in V_i).\end{aligned} \tag{4.3}$$

Then it is easily checked that d and δ are the adjoint of each other:

$$(\delta f, g)=(f, dg) \qquad (f\in M^1(X),\ g\in M^0(X)). \tag{4.4}$$

Finally define the Laplace operator by $\Delta = d\delta\colon M^1(X) \to M^1(X)$, and put $\boldsymbol{H}^1(X) := \mathrm{Ker}(\Delta)$, and call it the space of *harmonic* functions.

Lemma (4.5).
(i) $\mathrm{Ker}(d)$ $(:= C^0(X)) = \{$*constant functions on* $VX\}$.
(ii) $\boldsymbol{H}^1(X) = \mathrm{Ker}(\delta)$.

Proof. (i) is trivial. We have for $f \in M^1(X)$,

$$f \in \boldsymbol{H}^1(X) \Longleftrightarrow (\Delta f, g) = (\delta f, \delta g) = 0 \qquad (\forall g \in M^1(X))$$
$$\Longleftrightarrow \delta f = 0 \qquad (\text{Take } g := f).$$

Q.E.D.

From this and (4.4) one one can easily prove the following result, which is called the *Hodge decomposition*:

Corollary (4.6). *We have the orthogonal decompositions*:

$$M^1(X) = \boldsymbol{H}^1(X) \oplus dM^0(X), \qquad M^0(X) = \mathrm{Ker}(d) \oplus \delta M^1(X).$$

Let $C^0(V_i)$ denote the space of constant functions on V_i, and let $M_0^0(V_i) := (M^0(V_i) \cap (C^0(V_i))^{\perp}$ be the orthogonal complement in $M^0(V_i)$ of $C^0(V_i)$. Then we see immediately that

$$\mathrm{Ker}(d) = C^0(X) \subset C^0(V_1) \oplus C^0(V_2), \qquad d(C^0(V_1) \oplus C^0(V_2)) = C^1(X),$$

and that d is injective on $M_0^0(V_1) \oplus M_0^0(V_2)$.

Definition (4.7). The subspace $d(M_0^0(V_1) \oplus M_0^0(V_2))$ of $M^1(X)$ is called the space of *cusp forms* on X, and denoted by $M^1_{\mathrm{cusp}}(X)$.

Lemma (4.8). *One has the orthogonal decomposition*

$$d(M^0(X)) = M^1_{\mathrm{cusp}}(X) \oplus C^1(X).$$

Proof. This follows from the following formula: for $f, g \in M^0(V)$, one has

$$(4.9) \quad (df, dg) = (f, g) - \sum_{e \in EX} \{f(o(e^+))\overline{g(t(e^+))} + f(o(e^-))\overline{g(t(e^-))}\}.$$

Q.E.D.

Now recall that $M^1(X)$ is acted upon by the hermitian operators $\rho^*(T_i)$ $(i = 1, 2)$. For the real numbers $\alpha, \beta \in \boldsymbol{R}$, we denote by $M(\alpha, \beta)$ the subspace of $M^1(X)$ consisting of the functions satisfying

$$(4.10) \qquad \rho^*(T_1)f = \alpha f, \quad \rho^*(T_2)f = \beta f. \qquad (f \in M^1(X)).$$

For each vertex $P \in VX$, we denote by $E(P)$ the set of edges of X which are incident to P, i.e., $E(P) := \{e \in EX;\ P \in \varepsilon(e)\}$.

Lemma (4.11). *The following conditions are equivalent*:

(i) $f \in M(-1, -1)$.

(ii) $f \in \boldsymbol{H}^1(X)$; *i.e., f is a harmonic function.*

(iii) $\sum_{e \in E(P)} f(e) = 0$ *for any* $P \in VX$.

Proof. If one puts $\varepsilon(e) = \{P_1, P_2\}$ $(P_i \in V_i)$, one has $E(P_i) = E_i(e)$. From the definitions (2.16), (4.3), it follows

$$(\rho^*(T_i)f)(e) = \sum_{e' \in E(P_i)} f(e') - f(e) \tag{4.12}$$
$$= (-1)^i k(P_i)(\delta f)(P_i) - f(e).$$

The assertion follows immediately from this. Q.E.D.

Proposition (4.13). *There is a canonical isomorphism*

$$\eta\colon H_1(X, \boldsymbol{C}) \simeq \boldsymbol{H}^1(X) = M(-1, -1).$$

In particular, one has

$$\dim_{\boldsymbol{C}} \boldsymbol{H}^1(X) = \dim_{\boldsymbol{C}} H_1(X, \boldsymbol{C}) = r = \textit{rank of } \pi_1(X, P_0). \tag{4.14}$$

Proof. Let $C = (y_1, \cdots, y_{2l})$ be a closed path in X, so that $t(y_{j-1}) = o(y_j)$ $(1 \leq j \leq 2l)$, with the convention $y_0 = y_{2l}$. Define a function $f = f_C \in M^1(X)$ by

$$f_C(e) := \sum_{y_j \in e} \operatorname{sgn}(y_j). \tag{4.15}$$

It is easily seen that f_C satisfies (4.11), (iii), hence $f_C \in \boldsymbol{H}^1(X) = M(-1, -1)$. It is also clear that f_C depends only on the reduced path to which C corresponds, or the homology class of C. Thus the map $\eta\colon C \to f_C$, is a well defined linear map on $H_1(X, \boldsymbol{Z})$. That it is injective on $H_1(X, \boldsymbol{Z})$ is obvious, since any element of this group is represented by a single closed path. It follows easily that η can be extended to an injective linear map on $H_1(X, \boldsymbol{C}) = H_1(X, \boldsymbol{Z}) \otimes_{\boldsymbol{Z}} \boldsymbol{C}$. We shall show that $\dim_{\boldsymbol{C}} H_1(X, \boldsymbol{C}) = \dim_{\boldsymbol{C}} M(-1, -1)$. First note that by Lemma (4.11), $M(-1, -1)$ is the orthogonal complement of the subspace generated by the characteristic functions f_P of $E(P)$, $P \in VX$. Call this subspace N. We claim that $\dim_{\boldsymbol{C}} N = \#(VX) - 1$. To prove this, suppose that $\Sigma c_P f_P = 0$ is a linear relation among f_P's. Let $e \in EX$ be an edge such that $o(e^+) = P$, $t(e^+) = Q$. Evaluating at e, we get $c_P + c_Q = 0$. Since X is connected, this implies that there exists a constant c such that $c_P = c$,

$c_Q=-c$ for any $P\in V_1$, $Q\in V_2$. Thus, the functions f_P's have exactly one linear relation, which proves our claim. Now we have $\dim_C M(-1,-1)=\dim_C M^1(X)-(\sharp(VX)-1)=\dim_C H_1(X,C)$ (cf. 2.1). This completes the proof. Q.E.D.

Thus the space of harmonic functions is acted upon by $\rho^*(T_i)$ as the scalar -1; and this part of $M^1(X)$ contributes in $Z_X(u)^{-1}$ to the factor $(1-u)$. Namely one has:

Corollary (4.16). *Suppose that X is an arbitrary connected multigraph, which is not necessarily of bipartite type. Then the multiplicity of the factor* $(1-u)$ *in* $Z_X(u)^{-1}$ *is at least* $r=\operatorname{rank}\pi_1(X,P_0)$; *moreover, it is exactly r if X is regular, or semiregular bipartite, with valency* (q_1+1, q_2+1), $q_1q_2>1$.

Proof. The first assertion is an immediate consequence of (2.23). The second statement is already shown in the proof of Proposition (3.20); indeed, by (3.15), we see that the multiplicity of $(1-u)$ in $\det[I-(A^{[1]}-q_2+1)u+q_1q_2u^2]$ is equal to that of $\sqrt{(1+q_1)(1+q_2)}$ in $\mathrm{Spec}(X)$ (see also Proposition (4.26)). Q.E.D.

Remark (4.17). In Section 5, we shall prove that the multiplicity of $(1-u)$ in $Z_X(u)^{-1}$ is equal to r also for any non-regular multigraph X which is not homotopic to a circle (cf. (0.8)).

Next we consider the factor $(1+q_2u)$.

Proposition (4.18). *Suppose that X is a semiregular bipartite multigraph of valency* (q_1+1, q_2+1). *Then*

(i) $M(q_1,-1)=dM^0(V_1)\cap(dM^0(V_2))^\perp\subset M^1_{\mathrm{cusp}}(X)$,
$M(-1,q_2)=dM^0(V_2)\cap(dM^0(V_1))^\perp\subset M^1_{\mathrm{cusp}}(X)$.

(ii) $M(-1,q_2)$ *(resp,* $M(q_1,-1)$*) consists of the functions satisfying the local conditions*:

$$\sum_{e\in E(P)} f(e)=0 \qquad \textit{for any } P\in V_1 \ (\textit{resp. } P\in V_2), \tag{4.19}$$

$$f \textit{ is constant on each } E(Q),\ Q\in V_2\ (\textit{resp. } Q\in V_1). \tag{4.20}$$

(iii) *One has the following equalities*:

$$\dim_C M(-1,q_2)=\textit{the multiplicity of } (1+q_2u) \textit{ in } Z_X(u), \tag{4.21}$$
$$\dim_C M(q_1,-1)=\textit{the multiplicity of } (1+q_1u) \textit{ in } Z_X(u).$$

Proof. We first prove (i), (ii). We first show the equivalence

$$\rho^*(T_1)f=-f \Longleftrightarrow (4.19) \Longleftrightarrow f\in dM^0(V_1)^\perp.$$

The first one is easy from (4.12). On the other hand, one has for any $g \in M^0(V_1)$

$$(f, dg) = \sum_{e \in EX} f(e)\overline{(dg)(e)} = -\sum_{P \in V_1} \overline{g(P)}\Big(\sum_{e \in E(P)} f(e)\Big),$$

from which follows the second equivalence. Similarly one has

$$\rho^*(T_2)f = q_2 f \Longleftrightarrow (4.20) \Longleftrightarrow f \in dM^0(V_2).$$

In fact, one has

$$\rho^*(T_2)f = q_2 f \Longleftrightarrow \sum_{e' \in E(Q)} f(e') = (q_2+1)f(e) \qquad (Q \in V_2,\ t(e^+) = Q).$$

Since the sum in the left side depends only on Q, we see this is equivalent to (4.20). The second one is easily shown. The least assertion (iii) will be proved in Section 5. Q.E.D.

Remark (4.22). Counting the number of equations in (4.19), (4.20), we know, without using (3.14), the following estimate

$$\dim_C M(-1, q_2) \geq \#(EX) - [\#(V_1) + q_2\#(V_2)] = n_2 - n_1.$$

However, with the knowledge of Main theorem (III), one sees that this is a consequence of the following:

Proposition (4.23). *Suppose that $q_1 \geq q_2$. Then one has*

$$\dim_C M(-1, q_2) - \dim_C M(q_1, -1) = n_2 - n_1.$$

Proof. From (3.14), (3.15), we see that the extra factors $(1+q_2u)$ of $Z_{X,b}(u)^{-1}$, other than those $n_2 - n_1$ one's, come from

$$\{1 - (\lambda_j^2 - q_1 - q_2)u + q_1q_2u^2\} \qquad \text{for } \lambda_j \in \mathrm{Spec}(X), \quad (1 \leq j \leq n_1).$$

This implies that the quadratic polynomial has a root $-q_2^{-1}$, hence the other root is $-q_1^{-1}$. Q.E.D.

Corollary (4.24). *Let the assumptions be as above. Then the following assertions are equivalent.*

(i) *X is a weak Ramanujan graph.*

(ii) *$Z_{X,b}(u)^{-1}$ is not divisible by $(1+q_1u)$.*

If $q_2 = 1$, one can, in analogy of Proposition (4.8), construct a map

$$\eta^{\#}: H_1^{(0)}(X, Z) \longrightarrow M(-1, 1) \text{ modulo } \{\pm 1\},$$

where $H_1^{(0)}(X, Z)$ is a Z-submodule of $H_1(X, Z)$ generated by the closed paths of length divisible by 4. In fact, if $C=(y_1, \cdots, y_{4l})$ is such a path, one defines $\eta^{\#}(C)=f \in M^1(X)$ by

$$f(e):=\sum_{y_j \in e}(-1)^{[(j-1)/2]+j-1}\operatorname{sgn}(y_j). \tag{4.25}$$

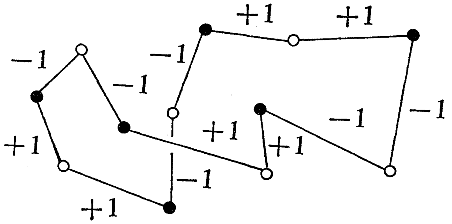


Fig. 7

It is immediately seen that $f=\eta^{\#}(C)$ belongs to $M(-1, 1)$. One can prove, as in the proof of Theorem (5.32), that the image of $\eta^{\#}$ spans $M(-1, 1)$.

We close this section with the following observation:

Proposition (4.26). *Suppose X is a semiregular bipartite multigraph of valency (q_1+1, q_2+1). Then*

$$M(q_1, q_2)=C^1(X) \ (:=\textit{constant functions}),$$

and if $q_1q_2>1$, one has

$$\dim_C M(q_1, q_2)=1=\textit{multiplicity of } (1-q_1q_2u) \textit{ in } Z_{X,b}(u)^{-1}.$$

Proof. The first assertion is proved similarly as in Proposition (4.18). In view of (2.25), in order to prove the last assertion, it suffices to show that q_1q_2-eigenspace of $\rho^*(T_1T_2)$ is $M(q_1, q_2)$.

In fact, let $f \in M^1(X)$ be such that $\rho^*(T_1T_2)f=q_1q_2f, f\neq 0$, Let $e \in EX$ be an edge such that $f(e)$ has a maximal absolute value. By Lemma (2.12), we have

$$(\rho^*(T_1T_2)f)(e)=\sum_{(*)} f(e'),$$

where the sum in the right consists of q_1q_2 terms. It follows that

$$q_1q_2|f(e)|\leq \sum_{(*)}|f(e')|\leq q_1q_2|f(e)|,$$

hence we see that f is a constant function on EX. This proves our assertion. Q.E.D.

§ 5. Representation of $C[T_1, T_2]$; a proof of (3.14)

In this section X is always assumed to be a bipartite multigraph such that $VX = V_1 \cup V_2$. We first treat the case that X is semiregular of valency (q_1+1, q_2+1).

Let $\tilde{X}$ be, as in Section 2, the universal covering tree, and let $T_1, T_2 \in \mathrm{End}(Z[E\tilde{X}])$ be the correspondences on $E\tilde{X}$ as in (2.10). We denote by $C[T_1, T_2]$ the subalgebra over C generated by T_i $(i=1, 2)$.

Lemma (5.1). *$C[T_1, T_2]$ is the non-commutative ring of polynomials of T_1, T_2 with the fundamental relations*

$$(5.2) \qquad T_i^2 = (q_i - 1)T_i + q_i \qquad (i = 1, 2).$$

In particular, we have

$$(5.3) \qquad C[T_1, T_2] = \sum_{(i_1, i_2, \cdots, i_l)} CT_{i_1}T_{i_2}\cdots T_{i_l} \qquad (i_k = 1, 2),$$

where the sum is extended over $(i_1, \cdots, i_l)$, $l \geq 0$, such that $i_k \neq i_{k+1}$.

Proof. The relations (5.2) is an immediate consequence of the definition (2.10). On the other hand, from Lemma (2.12) and (Fig. 5), we see that the monomials $T_{i_1}T_{i_2}\cdots T_{i_l}$; $i_k \neq i_{k+1}$, $l = 0, 1, 2, \cdots$, are linearly independent. Q.E.D.

Recall that (2.16) defines an anti-representation ρ^* of $C[T_1, T_2]$ on the space $M^1(X)$ of C-valued functions on EX. Since $\rho^*(T_1)$, $\rho^*(T_2)$ are hermitian operators, we see that ρ^* is semi-simple. Throughout the following, we shall omit the prefix "anti-", and refer ρ^* as a representation of $C[T_1, T_2]$.

Lemma (5.4). *Suppose that φ: $C[T_1, T_2] \to \mathrm{End}_C(W)$ be an irreducible representation of finite dimension $n = \dim_C(W)$. Then we have $n \leq 2$.*

Proof. From (5.2), we see that either $\varphi(T_1)$ is a scalar, or it has $(X+1)(X-q_1)$ as the minimal polynomial. In the first case, we get $\dim_C(W) = n = 1$. Suppose that $n \geq 2$, in which case we have the following two decompositions of W into eigenspaces:

$$W = W_1 \oplus W_1', \quad \varphi(T_1) \mid W_1 = q_1, \quad \varphi(T_1) \mid W_1' = -1,$$

$$W = W_2 \oplus W_2', \quad \varphi(T_2) \mid W_2 = q_2, \quad \varphi(T_2) \mid W_2' = -1,$$

with $\dim_C(W_i)$, $\dim_C(W_i') \geq 1$. From our assumption that φ is irreducible of dimension $n \geq 2$, it follows that the intersection of any two of the subspaces W_1, W_2, W'_1, W'_2 is the zero space. Then we see that they all have the same dimension, and that

$$W = W_1 \oplus W_2 = W_1 \oplus W'_2. \tag{5.5}$$

Corresponding to the first decomposition, we see that φ is expressed as

$$\varphi(T_1) = \left(\begin{array}{c|c} q_1 I & 0 \\ \hline C_1 & -I \end{array}\right), \quad \varphi(T_2) = \left(\begin{array}{c|c} -I & C_2 \\ \hline 0 & q_2 I \end{array}\right). \tag{5.6}$$

Moreover, taking the conjugation with a matrix $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$, $X, Y \in GL(n/2, \boldsymbol{C})$, we can transform C_1, C_2 to YC_1X^{-1}, XC_2Y^{-1}, respectively. Now it is easy to see that φ = irreducible implies that $n/2 = 1$ and $C_1, C_2 \neq 0$.

Q.E.D.

Proposition (5.7). *Suppose that* $q_1q_2 > 1$. *The irreducible representations* φ *of* $\boldsymbol{C}[T_1, T_2]$ *are classified as follows*:
(*we put* $p_\varphi(u) := \det[I - \varphi(T_1T_2)u]$).

(i) *Degree one*;

$$\varphi(T_1) = q_1, \quad \varphi(T_2) = q_2; \quad \varphi(T_1T_2) = q_1q_2, \quad p_\varphi(u) = 1 - q_1q_2u,$$
$$\varphi(T_1) = q_1, \quad \varphi(T_2) = -1; \quad \varphi(T_2T_2) = -q_1, \quad p_\varphi(u) = 1 + q_1u,$$
$$\varphi(T_1) = -1, \quad \varphi(T_2) = q_2; \quad \varphi(T_1T_2) = -q_2, \quad p_\varphi(u) = 1 + q_2u,$$
$$\varphi(T_1) = -1, \quad \varphi(T_2) = -1; \quad \varphi(T_1T_2) = +1, \quad p_\varphi(u) = 1 - u.$$

(ii) *Degree two*; φ *is parametrized by* $c \in \boldsymbol{C}$, $c \neq 0$, $(q_1+1)(q_2+1)$, *with*

$$\varphi(T_1) = \left(\begin{array}{c|c} q_1 & 0 \\ \hline c & -1 \end{array}\right), \quad \varphi(T_2) = \left(\begin{array}{c|c} -1 & 1 \\ \hline 0 & q_2 \end{array}\right); \tag{5.8}$$
$$\varphi(T_1T_2) = \left(\begin{array}{c|c} -q_1 & q_1 \\ \hline -c & c - q_2 \end{array}\right), \quad p_\varphi(u) = 1 - (c - q_1 - q_2)u + q_1q_2u^2.$$

Proof. (i) follows immediately from (5.2). (ii): From the proof of Lemma (5.4), one sees that φ is equivalent to the representation given by (5.8). A direct computation shows that (5.8) is reducible if and only if $c = 0$, or $(1+q_1)(1+q_2)$, or equivalently

$$p_\varphi(u) = (1 + q_1u)(1 + q_2u), \qquad \text{or } (1-u)(1-q_1q_2u). \tag{5.9}$$

Q.E.D.

Corollary (5.10). *Suppose that* (φ, W) *is a* 2-*dimensional irreducible* *subspace of* $M^1(X)$ *s.t.* $p_\varphi(u)=1-(c-q_1-q_2)u+q_1q_2u^2$. *Then* W *has a basis* f_1, f_2 *which satisfies the following relations*:

$$(5.11)\qquad f_1(e)=\sum_{e'\in E(o(e+))} f(e'),\qquad cf_2(e)=\sum_{e'\in E(t(e+))} f(e').$$

Proof. This follows easily from (5.8) and (4.12). Q.E.D.

Corollary (5.12). *The irreducible representation* φ *of* $C[T_1, T_2]$ *is determined by the characteristic polynomial* $p_\varphi(u)$ *of* $\varphi(T_1T_2)$.

We note that the above Proposition (5.7), together with (5.9), gives the proof of (4.21).

A Proof of (3.14). Now we shall give a new proof of (3.14), which is a simple consequence of the general formula (2.22), and which is independent of our previous proof in [H-H]. We know already that $Z_{X,b}(u)^{-1}$ is a polynomial of u, which is divisible by

$$(1-u)^r(1-q_1q_2u)(1+q_2u)^{n_2-n_1}.$$

Moreover, we know from the above results, that each of these factors corresponds to one-dimensional subspace of $M^1(X)$, invariant by $C[T_1, T_2]$. Denote by M_* the orthogonal complement in $M^1(X)$ of the direct sum of them.

We have $\dim_C(M_*)=\sharp(EX)-[r+1+n_2-n_1]=2(n_1-1)$.

Lemma (5.13). $\det\rho^*(T_i)=(-1)^{n_iq_i}(q_i)^{n_i}$ $(i=1, 2)$.

Proof. We prove this for T_1. The assumption on X implies that EX is the disjoint union of $E(P)$'s $(P\in V_1)$, hence we have the decomposition $M^1(X)=\oplus_P M^1(E(P))$, where $M^1(E(P))$ consists of the functions on $E(P)$. Since T_1 preserves $Z[E(P)]$, it follows that $\rho^*(T_1)$ preserves the above decomposition of $M^1(X)$. Now from (4.18), (5.2), we see easily that on each $M^1(E(P))$, $\rho^*(T)$ has a simple eigenvalue q_1, and the eigenvalue -1 with multiplicity q_1. The assertion follows from this.

Q.E.D.

It follows that $\det(\rho^*(T_i)\,|\,M_*)=(-q_i)^{n_1-1}$. Now the comparison of (5.7) and (5.13), together with Proposition (4.13), implies that the restriction of $\rho^*\,|\,M_*$ is decomposed into a direct sum of two-dimensional invariant subspaces on which neither one of $\rho^*(T_i)$ acts as a scalar. Let W be such a subspace. Then as in the proof of Lemma (5.4), we see that W has a one-dimensional subspace W_1 such that $\rho^*(T_1)\,|\,W_1=q_1$. From (5.2), we see that $\rho^*(1+T_1)$ gives the projection of W onto W_1, and that

$$\rho^*(1+T_1)^2=(q_1+1)\cdot\rho^*(1+T_1),$$

Now define an element A of $\boldsymbol{C}[T_1, T_2]$ by

$$A:=(1+q_1)^{-1}(1+T_1)T_2(1+T_1). \tag{5.14}$$

Recall that $X^{[1]}$ is the multigraph derived from X such that $VX^{[1]}=V_1$ and $EX^{[1]}$ consists of the proper paths of the form (P, e_1, Q, e_2, P') of length 2 ($P, P' \in V_1$). We identify $M^0(X^{[1]})$ with $M^0(V_1)$, and regard the adjacency matrix $A^{[1]}$ of $X^{[1]}$ as an endomorphism of $M^0(V_1)$ as follows. Let $VX^{[1]}=V_1=\{P_1, \cdots, P_{n_1}\}$, and let $f \in M^0(V_1)$. Then we have

$$(A^{[1]}f)(P_i):=\sum_j a_{ij}f(P_j), \qquad (A^{[1]}=(a_{ij})).$$

From the above remark, we see that $M_*(\subset dM^0(X))$ is decomposed as $M_*=M_*^{[1]}\oplus M_*^{[2]}$, where $M_*^{[i]}$ is the subspace such that $\rho^*(T_i)\,|\,M_*^{[i]}=q_i$. Now from the proof of Proposition (4.18), we can regard $M_*^{[i]}$ as a space of functions on V_i, i.e., $M_*^{[i]}\subset M^0(V_i)=M^0(X^{[i]})$. Moreover, one can see easily that $M_*^{[1]}$ is stable under $A^{[1]}$, and that $A\,|\,M_*^{[1]}=A^{[1]}\,|\,M_*^{[1]}$. Similar assertion holds for $A^{[2]}$, $M_*^{[2]}$.

Lemma (5.15). *Notation and assumptions being as above, we have*
(i) $W_1\subset dM^0(V_1)$.
(ii) W_1, *regarded as a subspace of* $M^0(X^{[1]})$, *is stable by* $A^{[1]}$, *and*

$$A^{[1]}\,|\,W_1=A\,|\,W_1=\operatorname{tr}(\rho^*(T_1T_2)\,|\,W)+q_2-1. \tag{5.16}$$

Proof. Since W_1 is the eigenspace of $\rho^*(T_1)$ with eigenvalue q_1, (i) follows from the proof of Proposition (4.18). From the invariance of W by $\boldsymbol{C}[T_1, T_2]$, and the above remark, we see that W_1 is invariant by $A^{[1]}$, and that $A^{[1]}\,|\,W_1=A\,|\,W_1$. Now we have

$$\begin{aligned}A^{[1]}\,|\,W_1=\rho^*(A)\,|\,W_1&=(q_1+1)^{-1}\rho^*[(1+T_1)T_2(1+T_1)]\,|\,W_1\\&=(q_1+1)^{-1}\rho^*[(1+T_1)^2T_2]\,|\,W_1\\&=\operatorname{tr}(\rho^*(T_1T_2)\,|\,W)+q_2-1.\end{aligned}$$

Q.E.D.

It follows that

$$\begin{aligned}\det(I-(\rho^*(T_1T_2)\,|\,M_*)u)&=\prod_W \det(I_2-(\rho^*(T_1T_2)\,|\,W)u)\\&=\prod_W (1-\operatorname{tr}(\rho^*(T_1T_2)\,|\,W)u+q_1q_2u^2)\\&=\prod_W \{1-((A^{[1]}\,|\,W^{[1]})-q_2+1)u+q_1q_2u^2\}.\end{aligned}$$

Now the equality (3.14) follows from this and (2.25).

Next we consider the general case, where X is a bipartite multigraph, which is not necessarily semiregular, such that

$$VX = V_1 \cup V_2, \quad V_1 = \{P_j;\ 1 \leq j \leq n_1\}, \quad V_2 = \{Q_j;\ 1 \leq j \leq n_2\},$$

and the valency of P_j (resp. Q_j) is $q_1^{(j)}+1$ (resp. $q_2^{(j)}+1$). In such general case, the algebra $C[T_1, T_2]$ has a complicated structure, and it seems difficult to classify the equivalence classes of irreducible representations of it. Nevertheless, it would be interesting to ask the following questions:

(5.17) Are the irreducible representations of $C[T_1, T_2]$ of degree ≤ 2? (see example (5.20)).

(5.18) Can one classify them, or, if the answer to (5.17) is negative, those of degree one and two, as in Proposition (5.7)?

(5.19) Does the characteristic polynomial $p_\varphi(u)$ of $\varphi(T_1 T_2)$ determine the (irreducible) representation φ?

Example (5.20). The following example shows that for $X = Y^{(2)}$, the barycentric subdivision of $Y = X_1(4, 5)$, $(\rho^*, M^1(X))$ has an irreducible component of degree 3.

$Y := X_1(4.5)$

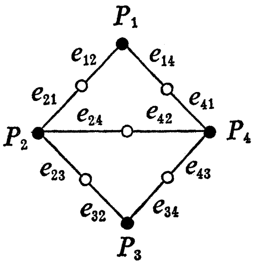


Fig. 8

Arrange the edges of EX is the following order:

$$EX = \{e_{12}, e_{14};\ e_{21}, e_{23}, e_{24};\ e_{32}, e_{34};\ e_{41}, e_{42}, e_{43}\}.$$

We identify $M^1(X)$ with C-space spanned by EX. Then taking EX as the basis, we have the following decomposition of $M^1(X)$:

$$M^1(X) = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5; \quad \rho^* = \varphi_1 \oplus \varphi_2 \oplus \varphi_3 \oplus \varphi_4 \oplus \varphi_5.$$

$$W_1 = \{(-a-b, a+b, a+b, -b, -a, b, -b, -a-b, a, b);\ a, b \in C\}$$

$$\varphi_1(T_1) = -1, \quad \varphi_1(T_2) = -1, \quad p_\varphi(u) = (1-u)^2.$$

$$W_2 = \{(a, -a, a, -a, 0, -a, a, -a, 0, a);\ a \in C\}$$

$$\varphi_2(T_1)=-1,\quad \varphi_2(T_2)=1,\quad p_\varphi(u)=(1+u).$$

$$W_3=\{(-a,\ -a,\ b,\ -b,\ 0,\ a,\ a,\ b,\ 0,\ -b);\ a,\ b\in \boldsymbol{C}\}$$

$$\varphi_3(T_1)\simeq\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \varphi_3(T_2)\simeq\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix},\quad p_\varphi(u)=1+u^2.$$

$$W_4=\{(a-b,\ -a+b,\ -b,\ -b,\ -a,\ a-b,\ -a+b,\ b,\ a,\ b);\ a,\ b\in \boldsymbol{C}\}$$

$$\varphi_4(T_1)\simeq\begin{pmatrix}0 & 1\\ 2 & 1\end{pmatrix},\quad \varphi_4(T_2)\simeq\begin{pmatrix}-1 & -1\\ 0 & 1\end{pmatrix},\quad p_\varphi(u)=1+u+2u^2.$$

$$W_5=\{(a,\ a,\ a+b-c,\ a+b-c,\ b+c,\ a,\ a,\ a+b-c,\ b+c,\ a+b-c);\ a,\ b,\ c\in \boldsymbol{C}\}$$

$$\varphi_5(T_1)\simeq\begin{pmatrix}1 & 1 & 1\\ 0 & 2 & 0\\ 0 & -1 & -1\end{pmatrix},\quad \varphi_5(T_2)\simeq\begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ -1 & 1 & 0\end{pmatrix},\quad p_\varphi(u)=1-u^2-2u^3.$$

$$\begin{aligned}Z_Y(u)^{-1}=Z_{X,b}(u)^{-1}&=1-4u^3-2u^4+4u^6+4u^7+u^8-4u^{10}\\ &=(1-u)^2(1+u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3).\end{aligned}$$

We begin with the following observation. In $\mathrm{End}_{\boldsymbol{Z}}(\boldsymbol{Z}[E\tilde{X}])$, the element T_1 (resp. T_2) preserves the decomposition

$$(5.21)\qquad E\tilde{X}=\bigcup_{p(\tilde{P}_j)=P_j} E(\tilde{P}_j)\qquad (\text{resp. } E\tilde{X}=\bigcup_{p(\tilde{Q}_k)=Q_k} E(\tilde{Q}_k)).$$

where $E(\tilde{P}_j)$ denotes the set of edges incident to $\tilde{P}_j$. And as the element of $\mathrm{End}_{\boldsymbol{Z}}(\boldsymbol{Z}[E(\tilde{P}_j)])$ (resp. $\mathrm{End}_{\boldsymbol{Z}}(\boldsymbol{Z}[E(\tilde{Q}_k)])$), it satisfies

$$(5.22)\qquad T_1^2=(q_1^{(j)}-1)T_1+q_1^{(j)}\qquad (\text{resp. } T_2^2=(q_2^{(k)}-1)T_2+q_2^{(k)}).$$

It follows from this that the only possible linear representations φ of $\boldsymbol{C}[T_1,\ T_2]$ are:

$$\begin{aligned}
&\varphi(T_1)=q_1^{(j)},\quad \varphi(T_2)=q_2^{(k)};\quad \varphi(T_1T_2)=q_1^{(j)}q_2^{(k)},\quad p_\varphi(u)=1-q_1^{(j)}q_2^{(k)}u,\\
&\varphi(T_1)=q_1^{(j)},\quad \varphi(T_2)=-1;\quad \varphi(T_1T_2)=-q_1^{(j)},\quad p_\varphi(u)=1+q_1^{(j)}u,\\
&\varphi(T_1)=-1,\quad \varphi(T_2)=q_2^{(k)};\quad \varphi(T_1T_2)=-q_2^{(k)},\quad p_\varphi(u)=1+q_2^{(k)}u,\\
&\varphi(T_1)=-1,\quad \varphi(T_2)=-1;\quad \varphi(T_1T_2)=+1,\quad p_\varphi(u)=1-u.
\end{aligned}$$

Now we consider the representation ρ^* of $\boldsymbol{C}[T_1,\ T_2]$ on $M^1(X)$. From (5.21), (5.22), we easily get the following equalities:

$$(5.23)\qquad \begin{aligned}\det\rho^*(T_1)&=\prod_{j=1}^{n_1}(-1)^{q_1^{(j)}}q_1^{(j)}=(-1)^{m-n_1}q_1^{(1)}\cdots q_1^{(n_1)},\\ \det\rho^*(T_2)&=\prod_{k=1}^{n_2}(-1)^{q_2^{(k)}}q_2^{(k)}=(-1)^{m-n_2}q_2^{(1)}\cdots q_2^{(n_2)}.\end{aligned}$$

Proposition (5.24). *Suppose that X is not homotopic to a single circle, and that (φ, W) is an irreducible component of $\rho^*: \boldsymbol{C}[T_1, T_2] \to \mathrm{End}_{\boldsymbol{C}}(M^1(X))$, such that $p_\varphi(1)=0$. Then φ is linear* (*i.e.* $\dim W=1$) *and $W \subset M(-1, -1)$: $\varphi(T_1)=\varphi(T_2)=-1$.*

Proof. We first note that, for any $f \in M^1(X)$, one has

$$(5.25)\quad (\rho^*(T_1T_2)f)(e) = \sum_{Q' \in V(P)} \{\sum_{e' \in E(Q')} f(e')\} - \sum_{e' \in E(Q)} f(e') - \sum_{e' \in E(P)} f(e') + f(e),$$

where we put $o(e^+)=P$, $t(e^+)=Q$, and $V(P)$ denotes the set of vertices Q' adjacent to P. Now suppose that $f \in W \subset M^1(X)$ is an eigenfunction of $\varphi(T_1T_2)$ such that $\varphi(T_1T_2)f=f$. By (5.25), this implies that the following equality holds for any $P \in V_1$ and $Q \in V(P)$:

$$\sum_{Q' \in V(P)} \{\sum_{e' \in E(Q')} f(e')\} - \sum_{e' \in E(P)} f(e') = \sum_{e' \in E(Q)} f(e').$$

Since the left hand side depends only on P, it follows from this that, for any $P=P^{(j)} \in V_1$,

$$f^{**}(Q) := \sum_{e' \in E(Q)} f(e') \qquad \text{is a constant function on } V(P^{(j)}),$$

and that

$$f^*(P^{(j)}) := \sum_{e' \in E(P^{(j)})} f(e') = q_1^{(j)} f^{**}(Q) \qquad (Q \in V(P^{(j)})).$$

Since X is connected, one sees that $f^{**}(Q)$ is constant, say c, on V_2. Now from the disjoint union $EX = \cup E(P_j) = \cup E(Q_k)$, one obtain

$$cn_2 = \sum_{e \in EX} f(e) = \sum_{P_j \in V_1} q_1^{(j)} f^{**}(Q) = c \sum_{j=1}^{n_1} q_1^{(j)},$$

hence from (2.1),

$$c\left(\sum_{j=1}^{n_1} (q_1^{(j)}+1) - n_1 - n_2\right) = c(\#(EX) - \#(VX)) = c(r-1) = 0.$$

It follows that, if $r>1$, then $f^*(P)=f^{**}(Q)=0$ for any $P \in V_1$, $Q \in V_2$. From Lemma (4.11), this implies that $f \in M(-1, -1)$. Q.E.D.

As an immediate consequence, we obtain the following result which is a generalization of (4.16).

Theorem (5.26). *Suppose that Y is a connected finite multigraph, and* *let* $r := \dim_{C} H_1(Y, C)$ *be the number of independent cycles of* Y. *Then one has*

$$\begin{aligned} \operatorname{ord}_{(1-u)} Z_Y(u)^{-1} &= r, \qquad \text{if } r>1, \\ &= r+1=2, \qquad \text{if } r=1. \end{aligned}$$

Proof. Let $X := Y^{(2)}$ be the barycentric subdivision of Y. Then by (2.25), (2.27), we have $Z_Y(u)^{-1} = Z_{X,b}(u)^{-1} = \det(I - \rho^*(T_1T_2)u)$. Therefore, the assertion follows from (5.24), if $r>1$. Suppose that $r=1$, and let $C=(y_1, y_2, \cdots, y_{2l})$ be the circuit contained in X, which is unique up to the orientation and shifting the origin. Then it is easy to see that the two linearly independent functions

$$f_1 : f_1(e_i) = (-1)^i, \quad \text{and} \quad f_2 : f_2(e_i) = 1 \qquad (1 \le i \le 2l),$$

which vanish outside C, form a basis of 1-eigenspace of $\rho^*(T_1T_2)$. Q.E.D.

In what follows, we keep our assumption that φ is an irreducible component of $\rho^* : C[T_1, T_2] \to \operatorname{End}_C(M^1(X))$.

Proposition (5.27). *Suppose* X *is as above, and* (φ, W) *is a linear representation such that* $\varphi(T_1) = q_1^{(j)}$, $\varphi(T_2) = q_2^{(k)}$ *for some indices* $1 \le j \le n_1$, $1 \le k \le n_2$. *Then* φ *occurs in* $(\rho^*, M^1(X))$ *if and only if* X *is semiregular; moreover, it occurs exactly once.*

Proof. Let $f \in M(q_1^{(j)}, q_2^{(k)})$ be a function which spans W. Then as in the proof of Proposition (4.18), one sees that f is constant on each subsets $E(P)$, $E(Q)$ $(P \in V_1, Q \in V_2)$. Since X is connected, this implies that f is a constant function, say c, on EX. Now for any $P_i \in V_1$, and $e \in E(P_i)$, one has $(\rho^*(T_1)f)(e) = q_1^{(i)} f$, hence $q_1^{(i)} = q_1^{(j)}$; and similarly for $Q_i \in V_2$. Q.E.D.

Next we describe the condition under which the linear representation $\varphi(T_1) = -1$, $\varphi(T_2) = q_2^{(k)}$ occurs in $(\rho^*, M^1(X))$. Let $\{Q_{k_1}, \cdots, Q_{k_t}\}$ be the vertices of V_2 such that $q_2^{(k_i)} = q_2^{(k)}$, and let $X(q_2^{(k)})$ be the subgraph of X whose edges are the union of $E(Q_{k_i})$. Now we remove from $X(q_2^{(k)})$ the edges of $E(Q_{k_i})$, if one of it has an end point. Repeating this procedure, we finally get a sub-multigraph $X_0(q_2^{(k)})$ of X, which is not necessarily connected. The following assertion can be proved by a similar argument as above, and we omit the detail.

Proposition (5.28). *Notation being as above, the linear representation* φ *occurs in* $M^1(X)$ *if and only if it occurs in* $M^1(X_0(q_2^{(k)}))$. *Moreover, the* *multiplicity of* φ *in* $M^1(X)$ *is the sum of multiplicities of it in the connected components of* $X_0(q_2^{(k)})$.

Example (5.29). $X = X_{10}(6, 8) \to X(3) = X_0(3) \simeq K(3, 2)$.

$$Z_{X,b}(u)^{-1} = (1-u)^3(1+2u)^2(1-u-4u^2-4u^3)$$
$$Z_{X_0(3),b}(u) = (1-u)^2(1+u)^2(1-2u)(1+2u).$$

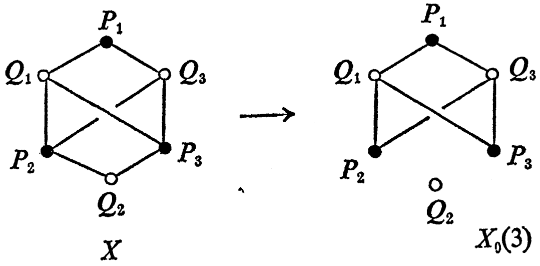


Fig. 9

Next we shall prove the following result, which characterize the factor $(1+u)$ in $Z_X(u)$.

Proposition (5.30). *Suppose that* Y *is a non-bipartite multigraph,* $X := Y^{(2)}$ *is the barycentric subdivision of* Y, *and* (φ, W) *is an irreducible component of* $(\rho^*, M^1(X))$ *such that* $p_\varphi(-1) = 0$. *Then* φ *is linear, and* $W \subset M(-1, 1)$.

Proof. By the assumption, there exists a nonzero function $f \in W \subset M^1(X)$ such that $\rho^*(T_1T_2)f = -f$. By a similar argument using (5.25), we see that the function $f^*(P) := \sum_{e \in E(P)} f(e)$ of $P \in V_1$ satisfies

$$(5.31) \qquad f^*(P) = f(e') - f(e) \qquad \text{if } e \in E(P),\ \{e, e'\} = E(Q),\ Q \in V_2.$$

Now the assumption that Y is nonbipartite implies the existence of a closed path $C = (P_0, \cdots, P_l)$ in Y of *odd* length such that $P_0 = P_l = P$. From (5.31), one obtains $f^*(P) = 0$. By (4.12), this implies that $\rho^*(T_1)f = -f$, hence also $\rho^*(T_2)f = f$. Since (φ, W) is assumed to be irreducible, we have $W = Cf$, which completes the proof. Q.E.D.

Using the above result, and (2.27), we can now prove the following theorem, which gives a new characterization of the bipartite multigraphs.

Theorem (5.32). *Suppose that* Y *is a connected finite multigraph, and let* $r := \dim_C H_1(Y, C)$ *be the number of independent cycles of* Y. *Suppose moreover that* $r > 1$. *Then one has*

$$\begin{aligned} \operatorname{ord}_{(1+u)} Z_Y(u)^{-1} &= r-1, && \textit{if } Y \textit{ is non-bipartite,} \\ &= r, && \textit{if } Y \textit{ is bipartite.} \end{aligned}$$

Proof. If Y is bipartite, the assertion is a consequence of (5.26) and the fact $Z_Y(u)^{-1}$ is a polynomial of u^2. So suppose that Y is non-bipartite, and put $X=Y^{(2)}$. The above proposition (5.30) shows that $\operatorname{ord}_{(1+u)} Z_Y(u)^{-1}$ is equal to $\dim_C M(-1, 1)$. By (4.18), (ii), we can identify $M(-1, 1)$ with the space M_Y^0 of functions on EY, which satisfy $\sum_{e\in E(P)} f(e)=0$ for all $P\in VY$. Now with respect to the inner product (4.2), M_Y^0 is the orthogonal complement of the subspace spanned by $\{f_P;\ P\in VY\}$, where f_P is the characteristic function of $E(P)$. We claim that the f_P's are linearly independent. In fact, let $\sum_P c_P f_P=0$ be a linear relation among them. Let $C=(C_0, \cdots, P_l)$ be as in the proof of Proposition (5.30). Evaluating at the edge $[P_i, P_{i+1}]$, we obtain $c_{P_i}+c_{P_{i+1}}=0$ for $0\le i\le l-1$, from which follows $c_P=0$ for all $P\in VY$. Therefore we obtain

$$\operatorname{ord}_{(1+u)} Z_Y(u)^{-1}=\dim_C M(-1, 1)=\#(EY)-\#(VY)=r-1. \qquad \text{Q.E.D.}$$

Remark (5.33). From the above results, we see that, if $X=Y^{(2)}$ for a non-bipartite multigraph Y, then we always have $M(1, -1)=\{0\}$.

§ 6. Representations of p-adic groups

Let G be an abstract group to which is given a mapping $l\colon G\to \mathbf{N}\cup\{0\}$, called the length function, satisfying the following conditions (G, l, I), (G, l, II).

(G, l, I) For each $l\ge 0$, $G_l:=\{x\in G;\ l(x)=l\}$ is non-empty, and $U:=G_0$ is a subgroup; moreover, $UG_lU=G_l=G_l^{-1}$, and $\#(U\backslash G_l)<\infty$.

Under these conditions, one can define the Hecke algebra $\mathscr{H}(G, U)$ of the pair (G, U), and one can regard G_l as an element of $\mathscr{H}(G, U)$.

(G, l, II) There exist two positive integers q_1, q_2 such that one has the following relations in $\mathscr{H}(G, U)$.

$$\begin{aligned} (G_1)^2 &= G_2+(q_2-1)G_1+q_2(q_1+1)U, \\ G_1G_l &= G_{l+1}+(q_2-1)G_l+q_1q_2G_{l-1} \qquad (l\ge 2). \end{aligned} \tag{6.1}$$

We describe two typical classes of groups satisfying these conditions.

(6.2) **Example 1.** Let X be a semiregular bipartite multigraph of valency (q_1+1, q_2+1), and let $\tilde{X}$ be the universal covering tree of X.

Then we know that $\tilde{G}=\mathrm{Aut}(\tilde{X})$ satisfies the above conditions, with $l(x)=(1/2)d_{\tilde{X}}(P_0, P_0x)$ where $d_{\tilde{X}}$ is th distance on $\tilde{X}$ and $P_0 \in \tilde{V}_1$ is a fixed vertex.

Moreover, any subgroup G of $\tilde{G}$ which acts transitively on $\tilde{V}_1$ satisfies (G, l, I), (G, l, II) (c.f. [H-H]).

(6.3) **Example 2.** The algebraic groups G over a local field K (i.e., locally compact field with respect to a discrete valuation of K), which is simply connected and has K-rank one.

In both examples, the group G is a topological group which is locally compact, totally disconnected and unimodular. Moreover, it has a Tits system (G, B, N, S) of affine type, where the Weyl group is the infinite dihedral group, and the associated building is isomorphic to $\tilde{X}$. Here, if one choose two adjacent vertices $P_0 \in \tilde{V}_1$, $Q_0 \in \tilde{V}_2$, their stabilizers $U=U_1:=\mathrm{Stab}_G(P)$ and $U_2:=\mathrm{Stab}_G(Q_0)$ are representatives of the two maximal parahoric subgroups of G, up to G-conjugation. Recall that a subgroup is called parahoric, if it contains a conjugate of B. We may assume that $B=U_1 \cap U_2$. Then the set of edges $E\tilde{X}$ can be naturally identified with $B\backslash G$, on which G acts by right translation.

The following facts are well known:

Lemma (6.4). $\mathscr{H}(G, B) \simeq C[T_1, T_2]$, $U_i \to 1+T_i$ $(i=1, 2)$, *where* $C[T_1, T_2]$ *is as in* (5.2).

Lemma (6.5). $\mathscr{H}(G, U)=C[G_1]$, *and there is a monomorphism* $\mathscr{H}(G, U) \to \mathscr{H}(G, B)$ *which maps* $I_U \to (1+q_1)I_B$.

Now we recall the basic facts on the representation theory. Let G be a locally compact, totally disconnected unimodular group. A representation (π, V) of G is a homomorphism $\pi\colon G \to GL(V)$. It is called *smooth*, if

(6.6) for each $v \in V$, the stabilizer $\{x \in G;\, \pi(x)v=v\}$ is an open subgroup of G.

We note that this is equivalent to the assertion $V=\bigcup_B V^B$, where B runs over the open compact subgroups of G, and $V^B:=\{v \in V;\, \pi(g)v=v\ (g \in B)\}$ is the space of B-fixed vectors. (π, V) is called *admissible* if, in addition,

$$\text{(6.7)} \qquad \dim_C V^B < \infty \qquad \text{for any open compact subgroup } B.$$

If (π, V) is smooth, then one gets a representation of the algebra $\mathscr{H}(G)$ of locally constant and compactly supported functions f on G, the product being defined by the convolution

$$\text{(6.8)} \qquad (f_1 * f_2)(g):=\int_G f_1(x)f_2(x^{-1}g)dx, \qquad (f_1, f_2 \in \mathscr{H}(G)).$$

with respect to a Haar measure of G. Namely it is defined by

$$(6.9) \qquad \pi(f)\cdot v := \int_G f(x)\pi(x)\cdot v\,dx \qquad (f\in\mathscr{H}(G),\ v\in V).$$

One of the basic fact here is that the correspondence $(\pi, V)\to(\pi, \mathscr{H}(G))$ induces an equivalence of the category of smooth representations of G and the non-degenerate $\mathscr{H}(G)$-modules. In particular a subspace V_1 of V is G-invariant if and only if it is $\mathscr{H}(G)$-invariant. If, moreover, (π, V) is admissible, then $\pi(f)$ is of finite rank for any $f\in\mathscr{H}(G)$, and conversely.

Let B be an open compact subgroup of G. Then we can define the Hecke algebra $\mathscr{H}(G, B)$ to be the subalgebra of $\mathscr{H}(G)$, which consists of the bi-B-invariant compactly supported functions on G. If (π, V) is a smooth representation, then V^B is stable under the action of $\mathscr{H}(G, B)$. In this way, one obtains a natural map

$$(6.10) \qquad \rho\colon \mathrm{Hom}_G(V, W) \longrightarrow \mathrm{Hom}_{\mathscr{H}(G,B)}(V^B, W^B),$$

where (π, V), (π', W) are smooth representations of G. Let (φ, E) be a representation of $\mathscr{H}(G, B)$. Put

$$(6.11) \qquad I(E) = I(\varphi) = C_c(G/B)\otimes_{\mathscr{H}(G,B)}E,$$

where $C_c(G/B)$ denotes the space of compactly supported right B-invariant functions on G, which is acted upon by $\mathscr{H}(G, B)$ from the right, and regard it as a G-module by left translation. It is easily seen that $I(E)$ is a smooth G-module.

Proposition (6.12) ([Bo-2], (2.5)). *Suppose that* $\dim_C E<\infty$. *Then the natural map* $\rho_I\colon \mathrm{Hom}_G(I(E), V)\to\mathrm{Hom}_{\mathscr{H}(G,B)}(E, V^B)$ *is bijective.*

Now suppose, in addition, that G has a Tits system (G, B, N, S) of affine type, where B is an open compact subgroup of G, called the Iwahori subgroup. The following facts have been proved in [Bo-2].

Theorem (6.13) ([Bo-2], (4.4) (4.10)). *Assumption being as above, let* (φ, E) *be a finite dimensional* $\mathscr{H}(G, B)$*-module. Then* $I(E)$ *is an admissible* G*-module, and it is irreducible if and only if* E *is an irreducible* $\mathscr{H}(G, B)$*-module. The assignment* $E\to I(E)$ *is an exact functor from finite dimensional* $\mathscr{H}(G, B)$*-modules to admissible* G*-modules.*

From this theorem and (6.12), it follows that the irreducible admissible G-modules V such that $V^B\neq\{0\}$ are parametrized by the irreducible $\mathscr{H}(G, B)$-modules. To apply this to the spectral decomposition of

$L^2(G/\Gamma)$, we need the following fact which is also well known. Let (π, V) be a unitary rerpesentation of G, where V is a Hilbert space. A vector $v \in V$ is called smooth, if the isotropy group of v is an open subgroup of G. The set V_∞ of smooth vectors forms a vector subspace of V, which is stable under G. In this way, one gets a G-module (π_∞, V_∞).

Theorem (6.14) (cf. [Car-2]). *Suppose that* (π, V) *is an irreducible unitary representation of* G. *Then* V_∞ *is dense in* V, *and the representation* (π_∞, V_∞) *is admissible.*

Now let G be a locally compact, totally disconnected unimodular group, and assume that G has a Tits system (G, B, N, S) of affine type such that the Weyl group $W := \langle S \rangle$ $(S = \{s_1, s_2\})$ is the infinite dihedral group. Let Γ be a discrete subgroup such that

$$\Gamma \text{ is torsion free, and } G/\Gamma \text{ is compact.} \tag{6.15}$$

Let, as usual, $L^2(G/\Gamma)$ be the Hilbert space of square integrable functions on G, which are right Γ-invariant. Then one can define the left regular representation of G on $L^2(G/\Gamma)$ by

$$(\pi(g)f)(x) := f(g^{-1}x) \qquad (g \in G,\ x \in G/\Gamma, f \in L^2(G/\Gamma)). \tag{6.16}$$

This gives a unitary representation of G. It is well known that the assumption of the compactness of G/Γ implies that $L^2(G/\Gamma)$ decomposes into an orthogonal direct sum

$$L^2(G/\Gamma) = \bigoplus_{[\pi]} V(\pi), \tag{6.17}$$

where $[\pi]$ is extended over the set of equivalent classes of irreducible unitary representations of G, and $V(\pi)$ is the closed G-invariant subspace of $L^2(G/\Gamma)$ which is π-isotypic. Moreover, each $V(\pi)$ is isomorphic to a direct sum of a finite copies of an irreducible G-module V_π which belongs to $[\pi]$. Denote by $m_\Gamma(\pi)$ the multiplicity of π in $V(\pi)$:

$$V(\pi) \simeq m_\Gamma(\pi) \cdot V_\pi. \tag{6.18}$$

Now we can state the second half of our main results in this paper. It is concerned with the multiplicities $m_\Gamma(\pi)$ of π in $L^2(G/\Gamma)$. Let G and Γ be as above, and let $\tilde{X}$ be the building associated with the Tits system (G, B, N, S). By our assumption, $\tilde{X}$ is a tree of semiregular bipartite type with valency (q_1+1, q_2+1), and the quotient, say X, is a finite graph with the same property. Moreover, Γ can be identified with the fundamental group of X as a CW-complex. Let $Z_\Gamma(u) = Z_X(u)$ be the zetafunction of Γ or X which has been evaluated in Sections 2, 3.

Main Theorem (IV). *Notation and assumption being as above, let* (π, V) *be an irreducible unitary representation of* G, *and let* (φ, V_∞^B) *be the corresponding irreducible representation of* $\mathscr{H}(G, B)$ *as in Theorem* (6.13). *Assume that* $V_\infty^B \neq \{0\}$. *Then one has*

$$(6.19) \qquad m_\Gamma(\pi) = \text{the multiplicity of } p_\varphi(u) \text{ in } Z_\Gamma(u)^{-1},$$

where $p_\varphi(u) = \det(I - \varphi(T_1 T_2)u)$ *is a linear or quadratic polynomial as in Proposition* (5.7). *In other words, the zeta function describes the spectral decomposition in* $L^2(G/\Gamma)$, *those components of which have* (*nonzero*) *B-fixed vectors.*

Proof. This is a consequence of the above quoted facts and the obvious equality

$$(6.20) \qquad L^2(G/\Gamma)^B = L^2(B\backslash G/\Gamma) \simeq M^1(X) \qquad (X = \tilde{X}/\Gamma). \qquad \text{Q.E.D.}$$

The above result has a number of applications. First of all, recall that $\mathscr{H}(G, B) = \boldsymbol{C}[T_1, T_2]$ has exactly 4 linear representations (cf. Proposition (5.9)). Each of such representation corresponds to an admissible representation of G. In [Bo-2], Borel determined, in a more general context, the linear representations which correspond to those representations of G which are square integrable. Among such representations, there is a distinguished one, called the Steinberg representation, which corresponds to the linear representation $T_i \to -1$ $(i = 1, 2)$, hence to $p_\varphi(u) = 1 - u$. That this is square integrable was first noted by Matsumoto [Ma]. The following result was first proved by Ihara [I-1] for $G = SL_2(K)$. We note also that the first equality has been proved by Garland [Gar] for a general $\boldsymbol{p}$-adic group.

Corollary (6.21) (cf. [Gar], [Bo-2]). *For the Steinberg representation* π_{St}, *one has*

$$(6.22) \qquad \begin{aligned} m_\Gamma(\pi_{St}) &= r = \dim_{\boldsymbol{C}} H^1(\Gamma, \boldsymbol{C}) \\ &= -\operatorname{ord}_{(1-u)} Z_\Gamma(u). \end{aligned}$$

Proof. This follows immediately from (6.19) and (4.16). Q.E.D.

Now suppose that $q_1 > q_2$. Then the linear representation such that $p_\varphi(u) = 1 + q_2 u$ is also known to be square integrable, whereas the one which corresponds to $p_\varphi(u) = 1 + q_1 u$, is not (cf. [Bo-2]. We note that there is a misprint in the statement of [Bo-2], pp 254, (ii)). This fact would be compared with the following result, which is an immediate consequence of Proposition (4.23).

Corollary (6.23). *Suppose that* $q_1 > q_2$, *and* (π, V) *corresponds to a linear representation of* $\mathscr{H}(G, B)$ *such that* $p_\varphi(u) = 1 + q_2 u$. *Then one has*

$$m_\Gamma(\pi) = -\mathrm{ord}_{(1+q_2u)} Z_\Gamma(u) \geq n_2 - n_1 > 0, \tag{6.24}$$

where for $i = 1, 2$

$$n_i = \sharp(U_i \backslash G / \Gamma) = \text{the number of vertices of } X \text{ of valency } q_i + 1. \tag{6.25}$$

Moreover, the equality $m_\Gamma(\pi) = n_2 - n_1$ *holds if and only if* $X = \tilde{X}/\Gamma$ *is a weak Ramanujan graph.*

§ 7. Special values of zeta functions

As in the cases of zeta functions which appear in number theory, we can expect that the special values of our $Z_X(u)$ are related with properties of the graph X. In fact the results (4.16), (4.21), and (4.26) are regarded as giving such relations. Moreover, recall that in the special case where X can be derived from an arithmetic subgroup of $SL(2, \boldsymbol{Q}_p)$, the essential part of $Z_X(u)$ is nothing but the congruence zeta function of the reduction modulo p of a modular curve $X_0(N)$ (see Ihara [I-2]). In this case, the class number of the function field of $X_0(N) \otimes \boldsymbol{F}_p$ is expressed as the residue of the congruence zeta function.

In this section, we shall extend this fact to $Z_{X,b}(u)$, and give an interpretation of the residue of it at $u = 1$.

Definition (7.1). Let X be an arbitrary connected multigraph. A *spanning tree* T is a subgraph of X, which is a tree and such that $VT = VX$. If X is finite, the number of spanning trees is called the *complexity* of X, and denoted by $\kappa(X)$.

We shall show that the complexity is an analogue of the class numbers of the global fields. Let X be a finite connected multigraph such that $VX = \{P_j;\ 1 \leq j \leq n\}$, and let $A \in M(n, \boldsymbol{Z})$ be its adjacency matrix (cf. (3.1)). Let D be the diagonal matrix such that

$$D := \mathrm{diag}(k_1, \cdots, k_n), \tag{7.2}$$

with $\quad k_j := \text{valency of } P_j = \sharp\{e \in EX;\ P_j \in \varepsilon(e)\} \qquad (1 \leq j \leq n).$

Also put

$$J = J_n := \text{the } n \times n \text{ matrix whose entries are all } 1, \tag{7.3}$$

$$Q := D - A. \tag{7.4}$$

Now the following result is fundamental:

Theorem (cf. [Bi]). *With the above notation, we have*

(i) *The matrix of cofactors* (*adjugate*) *of Q is a multiple of J, by* $\kappa(X)$:

$$\operatorname{adj}(Q)=\kappa(X)\cdot J. \tag{7.5}$$

(ii) *The complexity of X is given by the formula*

$$\kappa(X)=n^{-2}\det(J+Q). \tag{7.6}$$

Theorem (7.7). *Suppose that X is a regular connected multigraph of valency $q+1$, such that $q>1$, $\sharp(VX)=n$, and let $r=\dim_C H_1(X, C)$. Then one has the following formula*:

$$\kappa(X)=\frac{-1}{n(q-1)2^{r-1}}\cdot\frac{1}{(1-u)^r Z_X(u)}\bigg|_{u=1}.$$

Proof. The assumption on X implies that $D=(q+1)I_n$, so that $Q=(q+1)I-A$. Moreover from the regularity, we see immediately that $JA=AJ$. Now the constant function $f\in M^0(X)$ is a common eigenfunction of A and J: $Af=(q+1)f$, $Jf=nf$, hence $Qf=0$. Note also that J is 0 on the orthogonal complement of $\boldsymbol{C}f$. Thus one see that

$$\det(J+Q)=\det[J+(q+1)I-A]=n\prod_{j=1}^{n-1}(q+1-\eta_j),$$

where $\eta_1, \cdots, \eta_{n-1}$, and $\eta_n=q+1$ are the eigenvalues of A. Applying (3.7) to the above result, we thus obtain

$$\begin{aligned}\kappa(X)&=n^{-1}\prod_{j=1}^{n-1}(q+1-\eta_j)\\&=n^{-1}\{\det[I_n-Au+qu^2]/(1-u)(1-qu)\}|_{u=1}\\&=\frac{-1}{n(q-1)2^{r-1}}\cdot\frac{1}{(1-u)^r Z_X(u)}\bigg|_{u=1}.\end{aligned}\tag{7.8}$$

Q.E.D.

The above result has an interesting application to the class number of the function field of the modular curve $X_0(l)$ over $\boldsymbol{F}_p$. Let l be a prime such that $(p, l)=1$, and let $\boldsymbol{B}$ be the definite quaternion algebra over $\boldsymbol{Q}$, which ramifies exactly at the places ∞, l. For a subset S of $\boldsymbol{B}$, we put $S^{(1)}:=\{s\in S;\ \mathrm{Nr}(s)=1\}$, where $\mathrm{Nr}(s)$ is the reduced norm of s. Let O be a maixmal order of $\boldsymbol{B}$, and put

$$\Gamma:=\boldsymbol{B}^{(1)}\cap[\boldsymbol{B}_p^{(1)}\times\prod_{q\neq p}O_q^{(1)}]\qquad(O_\infty=\boldsymbol{B}_\infty), \tag{7.9}$$

and regard Γ as a subgroup of $G := \boldsymbol{B}_p^{(1)} \simeq SL(2, \boldsymbol{Q}_p)$ through the projection to the first component. Since $O_q^{(1)}$ $(q \neq p)$ are compact groups, Γ is a discrete subgroup of G. Moreover, by the strong approximation theorem, one has

$$\begin{aligned} O_A^{(1)} \backslash \boldsymbol{B}_A^{(1)} / \boldsymbol{B}^{(1)} &= O_A^{(1)} \backslash [\boldsymbol{B}_p^{(1)} \times \prod_{q \neq p} O_q^{(1)}] \cdot \boldsymbol{B}^{(1)} / \boldsymbol{B}^{(1)} \\ &= O_A^{(1)} \backslash [\boldsymbol{B}_p^{(1)} \times \prod_{q \neq p} O_q^{(1)}] / \Gamma \\ &= O_p^{(1)} \backslash \boldsymbol{B}_p^{(1)} / \Gamma \simeq SL(2, \boldsymbol{Z}_p) \backslash G / \Gamma . \end{aligned}$$

Since $\sharp[O_A^{(1)} \backslash \boldsymbol{B}_A^{(1)} / \boldsymbol{B}^{(1)}] = \sharp[O_A^{\times} \backslash \boldsymbol{B}_A^{\times} / \boldsymbol{B}^{\times}] = h =$ the class number of $\boldsymbol{B}$ is known to be finite (see (7.10) below), we see that G/Γ is compact. Now a famous result of Eichler [Ei] states that there is an isomorphism

$$L^2(O_A^{(1)} \backslash \boldsymbol{B}_A^{(1)} / \boldsymbol{B}^{(1)}) = M^0(V_1) \simeq S_2(\Gamma_0(l)) \oplus \boldsymbol{C} \tag{7.10}$$

as modules over the Hecke algebra $\otimes_{q \neq l} \mathscr{H}(\boldsymbol{B}_q^{(1)}, O_q^{(1)})$, where $S_2(\Gamma_0(l))$ is the space of holomorphic cusp forms of weight 2 for the group $\Gamma_0(l)$. In particular we have $n = h = \dim_{\boldsymbol{C}} S_2(\Gamma_0(l)) + 1$. It follows that the main part of our zeta function $Z_X(u)$ attached to the graph $X = \tilde{X}/\Gamma$ is

$$\det[I_h - Au + pu^2] = (1-u)(1-pu) \cdot \det[I - uT(p) \,|\, S_2(\Gamma_0(l)) + pu^2]. \tag{7.11}$$

On the other hand, from Eichler-Shimura's congruence relation ([Sh]), it is well known that the congruence zetafunction $Z(X_0(l)/\boldsymbol{F}_p; u)$ of the modular curve $X_0(l)$ over $\boldsymbol{F}_p$ is equal to

$$Z(X_0(l)/\boldsymbol{F}_p; u) = \det[I - uT(p) \,|\, S_2(\Gamma_0(l)) + pu^2]/(1-u)(1-pu). \tag{7.12}$$

It is also well known, and easy to show that

Lemma (7.13). *The class number $H_{l,p}$ of the function field of the modular curve $X_0(l)$ over $\boldsymbol{F}_p$ $(p \neq l)$, is equal to the number of $\boldsymbol{F}_p$-rational points of the jacobian variety of $X_0(l)$, and it is given by*

$$H_{l,p} = (1-p)[(1-u)Z(X_0(l)/\boldsymbol{F}_p; u)]|_{u=1}.$$

Proof. See [Sch].

Comparing this with (7.7), we get the following observation for the class number $H_{l,p}$:

(7.14) *Suppose that $l \equiv 1$ (mod 12). Then one has $H_{l,p} = (g+1)\kappa(X)$,*

where $g = \dim_{\boldsymbol{C}}(S_2(\Gamma_0(l))$ is the genus of $X_0(l)$, and is independent of p. In particular $H_{l,p}$ is always a multiple of $g+1 = (l-1)/12$.

Proof. We prove that, under the assumption $l\equiv 1 \pmod{12}$, the group $\bar{\Gamma}=\Gamma/\{\pm 1\}$ is torsion free, hence it is a free group. Let γ be an element of Γ of order n. Then, as an element of $\boldsymbol{B}_{\infty}^{(1)}=SU(2)$, γ is conjugate to an element of its maximal torus $SO(2)$. It follows that γ has the characteristic polynomial $X^2-2\cos(2\pi/n)X+1$. By the definition of Γ, we see that $2\cos(2\pi/n)\in \boldsymbol{Z}_q$ $(\subset \boldsymbol{Q}_q)$ for any $q\neq l$. In particular, it follows that each q has only the prime divisors of degree 1 in $K=\boldsymbol{Q}(\zeta_n+\zeta_n^{-1})$, where ζ_n denotes a primitive n-th root of unity. This is possible only for $K=\boldsymbol{Q}$, hence for $n=1, 2, 3, 4, 6$. On the other hand, the assumption on l implies that $\boldsymbol{B}_l$ has no element of order 3, 4, 6. This proves that $\{\pm 1\}$ is the only torsion subgroup of Γ. Since this $\{\pm 1\}$ acts trivially on the tree $\tilde{X}$ (= the Tits building of $G=SL_2(\boldsymbol{Q}_p)$), we have $\tilde{X}/\Gamma=\tilde{X}/\bar{\Gamma}=X$. Now applying (7.7) and (7.13), we obtain the result.

Q.E.D.

The above result is a special case of (slightly stronger form of) a theorem of Doi and Brumer ([D-M]):

Theorem (7.15). *Let l be an odd prime. Then for any prime $p\neq l$, one has*

$$H_{l,p}=\det[I-uT(p)\,|\,S_2(\Gamma_0(l))+pu^2]|_{u=1}\equiv 0 \qquad (\mathrm{mod}\ e_0\cdot(l-1)/12),$$

where $e_0=1, 3, 2, 6$ (resp. 6) *according as* $l\equiv 1, 5, 7, 11 \bmod 12$ (resp. $l=3$).

Proof. This is proved as a consequence of the basic properties (7.16) of the Brandt matrix $B(p)$, and was proved in [Ha]. For the convenience of the reader, we quote it as Lemma (7.17) below. Note that we have $\sum_i (1/e_i)=(l-1)12$, by Eichler's mass formula. Q.E.D.

Suppose that $A=(a_{ij})\in M(n, \boldsymbol{Z})$ is an integral matrix and $e_1, \cdots, e_n$ are positive integers satisfying the conditions:

(7.16) (i) $\sum_{j=1}^{n} a_{ij}=k$ for all $i=1, \cdots, n$.

(ii) $e_i a_{ij}=e_j a_{ji}$ for any i, j.

Then regarding A as an endomorphism of $W=\boldsymbol{R}^n$, one gets $A\boldsymbol{u}=k\boldsymbol{u}$ for $\boldsymbol{u}={}^t(1, \cdots, 1)$. Let W_0 be the orthogonal complement of $\boldsymbol{u}$ in W, and denote by A_0 the restriction of A to W_0. Put

$$M:=\sum_{i=1}^{n} 1/e_i, \qquad e_0:=\mathrm{l.c.m}(e_1, \cdots, e_n).$$

Lemma (7.17). *Assumption being as above, one has*

$$\det(kI_{n-1}-A_0)\equiv 0 \qquad (\operatorname{mod} e_0 M).$$

Proof. See [Ha-1], Theorem 2. Q.E.D.

It seems to be possible to obtain the above result (7.15) using graphs also in the case when $\Gamma/\{\pm 1\}$ has torsion elements. Now we shall generalize Theorem (7.7) to the case where X is an arbitrary connected semiregular multigraph with valency $(q_1+1,\ q_2+1)$.

Theorem (7.18). *Suppose X is a connected semiregular bipartite multigraph of valency (q_1+1, q_2+1), such that $\sharp(V_i)=n_i$, and $q_1q_2>1$. Then we have*

$$\kappa(X)=\frac{1+q_2}{n_1}\cdot\frac{1}{1-q_1q_2}\cdot\frac{1}{(1-u)^r Z_{X,b}(u)}\bigg|_{u=1}.$$

Proof. We have to evaluate $\det(J+Q)$, where $J=J_n$, $n=n_1+n_2$, and $Q=D-A$ is as in (7.4). From (7.5), it follows again that $JQ=QJ$, and the constant function $f\in M^0(X)$ satisfies $Jf=nf$, $Qf=0$. Thus, it suffices to evaluate the product of the eigenvalues of Q, excluding the last one, which is 0. For this we arrange the vertices of X as $\{P_1, \cdots, P_{n_1}, P_{n_1+1}=Q_1, \cdots, P_n=Q_{n_2};\ P_i\in V_1, Q_j\in V_2\}$, so that Q is expressed as

$$Q=\left(\begin{array}{c|c} (q_1+1)I_{n_1} & -B \\ \hline -{}^tB & (q_2+1)I_{n_2}\end{array}\right).$$

One can find an orthogonal matrix $W\in O(n_2)$ in such a way that

$$BW=\begin{array}{c} \begin{array}{cc} \longleftarrow\ n_1\ \longrightarrow & \longleftarrow\ n_2-n_1\ \longrightarrow \end{array}\\ \left[\begin{array}{cccc|ccc} \mu_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ * & \mu_2 & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ * & & 0 & \mu_{n_1} & 0 & \cdots & 0 \end{array}\right]\end{array}=(C\,|\,0),\ \text{say}.$$

Then one sees that Q is conjugate to the matrix

$$\begin{array}{ccc} \longleftarrow n_1 \longrightarrow & \longleftarrow n_1 \longrightarrow & \longleftarrow n_2-n_1 \longrightarrow \end{array}$$
$$\left[\begin{array}{c|c|c} (q_1+1)I & -C & 0 \\ \hline -{}^tC & (q_2+1)I & 0 \\ \hline 0 & 0 & (q_2+1)I \end{array}\right].$$

Now the last matrix has the characteristic polynomial

$$\{x-(q_2+1)\}^{n_2-n_1}\det\left(\begin{array}{c|c} (x-(q_1+1))I_{n_1} & C \\ \hline {}^tC & (x-(q_2+1))I_{n_1} \end{array}\right)$$

$$=\{x-(q_2+1)^{n_2-n_1}\det[\{x-(q_1+1)\}\{x-(q_2+1)\}I_{n_1}-C\,{}^tC]$$

$$=\{x-(q_2+1)\}^{n_2-n_1}\prod_{j=1}^{n_1}[\{x-(q_1+1)\}\{x-(q_2+1)\}-\lambda_j^2].$$

From this follows that the eigenvalues of Q are

$$q_2+1 \quad (n_2-n_1 \text{ times}), \qquad \alpha_j,\ \alpha_j' \quad (1\leq j\leq n_1)$$

with $\alpha_j\alpha_j'=(q_1+1)(q_2+1)-\lambda_j^2$. Here $\{\pm\lambda_j\ (1\leq j\leq n_1), 0, \cdots, 0\}$ are the eigenvalues of A, and we know from Proposition (3.19) that

$$\lambda_1^2=(q_1+1)(q_2+1)>\lambda_2^2\geq\cdots\geq\lambda_{n_1}^2.$$

It follows from (7.6), that

$$\kappa(X)=n_1^{-1}(1+q_2)^{n_2-n_1+1}$$
$$\times\{\det[I_{n_1}-(A^{[1]}-q_2+1)u+q_1q_2u^2]/(1-u)(1-q_1q_2u)\}|_{u=1}$$

The assertion now follows from (3.15). Q.E.D.

Remark (7.19). The above formula gives a generalization of (7.7). In fact, if X is a regular multigraph of valency $q+1$, we take the barycentric subdivision $X^{(2)}$, which can be regarded as a semiregular bipartite multigraph of valency $(q+1, 2)$. Then (7.7), (7.18), and (2.27) gives

$$\kappa(X^{(2)})=2^r\cdot\kappa(X). \tag{7.20}$$

This relation can be explained as follows: A spanning tree of X is obtained by removing an edge from each of the r independent set of cycles in X. On the other hand, to each edge $e\in EX$ which are to be removed, we have two choices of edges in $EX^{(2)}$, to get a spanning tree of $X^{(2)}$. Thus, (7.20) holds for any multigraphs X and $X^{(2)}$.

Problem (7.21). Can one show the relation for $\kappa(X)$ and the values of $(1-u)^rZ_X(u)$ at $u=1$, analogous to (7.18), in the general case where no regularity of X is assumed?

Finally we prove the following result which generalize (7.20) partially.

Proposition (7.22). *Suppose that X is a semiregular bipartite multigraph with valency (q_1+1, q_2+1), such that $q_1\geq q_2$, $q_1q_2>1$, and $n_i=\sharp(V_i)$*

($i=1, 2$). *Them* $\kappa(X)$ *is a multiple of* $(1+q_2)^{n_2-n_1+1}$.

Proof. This follows from the formula for $\kappa(X)$ given in the proof of Theorem (7.18), applying Lemma (7.17) to $A^{[1]}$. Q.E.D.

Note that, if in Lemma (7.17) one assumes that $a_{ij}\geq 0$ for any i, j, and that $e_i=1$ for all i, then A is regarded as the adjacency matrix of a regular multigraph X of valency k. The assertion then is also a consequence of (7.8), since $\kappa(X)$ is an integer.

§ 8. Miscellaneous results

Here we describe some of the general results on the relation between the spectra of two or more finite (multi)-graphs, which are immediately interpreted in terms of our zeta functions. They are the complementary graphs, line graphs, and those derived from the general compositions. The last ones are useful to compute $Z_X(u)$ for various families of graphs, or to construct them with prescribed properties.

(8.1) Complementary graphs.

Let X be a finite connected graph with n vertices. Then there exists an injective morphism $\iota\colon X\to K(n)$, where $K(n)$ is the complete graph with n vertices (i.e., any two vertices in $K(n)$ are joined by an edge). Since ι is unique up to the permutation of VX, one can identify X with its image $\iota(X)$, and regard it a subgraph of $K(n)$. If one removes from $K(n)$ all edges of X, one gets a new graph with n vertices VX, which is called the *complementary graph* of X, and denoted by X^c. It is in general not connected, and it may have isolated points (i.e., no edge incident to it), or endpoints. If we put $r=\dim_C H_1(X, C)$, $r^c=\dim_C H_1(X^c, C)$, then we have the following relation which follows from (2.1):

(8.2) $$r+r^c=(n-1)(n-4)/2, \qquad \text{if } X^c \text{ is connected.}$$

Proposition (8.3). *Suppose that X is a connected regular graph with valency k, and let* $\mathrm{Spec}(X)=\{\lambda_1=k, \lambda_2, \cdots, \lambda_n\}$ *be the spectrum of X. Then X^c is again regular of valency $n-1-k$, and one has*

(8.4) $$\mathrm{Spec}(X^c)=\{n-1-k, -\lambda_2-1, -\lambda_3-1, \cdots, -\lambda_n-1\}.$$

If, moreover, X^c is assumed to be connected, one has

(8.5) $$Z_{X^c}(u)^{-1}=(1-u^2)^{n(n-3-k)/2}(1-u)[1-(n-1-k)u] \times\prod_{j=2}^{n}[1+(\lambda_j+1)u+(n-2-k)u^2].$$

Proof. This can be proved similarly as Proposition (8.7), and we omit the detail. Q.E.D.

Suppose next that X is a connected bipartite graph such that $\#(V_1) = n_1$, $\#(V_2) = n_2$. Then an analogue of X^c, denoted by X^{bc}, is defined as follows. X^{bc} has the same set $VX = V_1 \cup V_2$ of vertices as X, and an edge joins the vertices $P \in V_1$ and $Q \in V_2$ if and only if they are not adjacent in X. Put $r' := \dim_C H_1(X^{bc}, C)$, and observe

$$(8.6) \qquad r + r' = n_1 n_2 - 2n_1 - 2n_2 + 2, \qquad \text{if } X^{bc} \text{ is connected.}$$

Proposition (8.7). *Notation being as above, suppose that X is semiregular of valency* (k_1, k_2), $k_1 \geqq k_2 > 1$, *and that* X^{bc} *is connected. Let*

$$\mathrm{Spec}(X) = \{\pm\sqrt{k_1 k_2}, \pm\lambda_2, \cdots, \pm\lambda_{n_1}, 0, \cdots, 0 \ (n_2 - n_1 \text{ times})\}$$

be the spectrum of X as in (3.11). *Then* X^{bc} *is again semiregular bipartite of valency* $(k_1', k_2') = (n_2 - k_1, n_1 - k_2)$, *and one has*

$$(8.8) \quad \mathrm{Spec}(X^{bc}) = \{\pm\sqrt{k_1' k_2'} = \pm\lambda_1, \pm\lambda_2, \cdots, \pm\lambda_{n_1}, 0, \cdots, 0 \ (n_2 - n_1 \ \textit{times})\},$$

$$(8.9) \quad Z_{X^{bc},b}(u)^{-1} = (1-u)^{r'-1}(1+q_2' u)^{n_2-n_1} \prod_{j=1}^{n_1} [1 - (\lambda_j^2 + q_1' - q_2')u - q_1' q_2' u^2],$$

where we put $q_i' := k_i' - 1$ $(i = 1, 2)$.

Proof. We shall prove (8.8), since (8.9) follows from it and (3.15). Let $A^{[1]}$, $A_{bc}^{[1]}$ be the adjacency matrices of $X^{[1]}$, $(X^{bc})^{[1]}$ respectively (cf. (3.13)). In view of the equation (3.17), it suffices to evaluate the eigenvalues of $A_{bc}^{[1]}$. Let A, A_{bc} be the adjacency matrices of X, X^{bc}, and express A as in (3.12). Then it follows from the definition that

$$A_{bc} = \left(\begin{array}{c|c} 0 & J \\ \hline {}^tJ & 0 \end{array}\right) - A = \left(\begin{array}{c|c} 0 & J - B \\ \hline {}^t(J-B) & 0 \end{array}\right).$$

where $J = J_{n_1, n_2}$ is the $n_1 \times n_2$ matrix whose entries are all 1. We have

$$A_{bc}^2 = \left(\begin{array}{c|c} (J-B)\,{}^t(J-B) & 0 \\ \hline 0 & {}^t(J-B)(J-B) \end{array}\right),$$

and, by the assumption that X is semiregular of valency (k_1, k_2), we have $B\,{}^tJ = J\,{}^tB = k_1 J_{n_1}$, hence

$$(J-B)\,{}^t(J-B) = J\,{}^tJ - 2k_1 J_{n_1} + B\,{}^tB = (n_2 - 2k_1)J_{n_1} + B\,{}^tB.$$

Similarly we have

$$(B\,{}^tB)J_{n_1}=k_1k_2J_{n_1}=J_{n_1}(B\,{}^tB).$$

Now observing that $(B\,{}^tB)$ and J_{n_1} has the common eigen vector ${}^t(1, \cdots, 1)$, one sees that, from the assumption

$$B\,{}^tB\simeq\operatorname{diag}(\lambda_1^2, \cdots, \lambda_{n_1}^2)$$

it follows that

$$(J-B)\,{}^t(J-B)\simeq\operatorname{diag}(\lambda_1^2+(n_2-2k_1)n_1, \lambda_2^2, \cdots, \lambda_{n_1}^2).$$

The assertion (8.8) now follows, if one observes the relations (3.17) and $\lambda_1^2+(n_2-2k_1)n_1=k_1'k_2'$, which follows from

$$k_1+k_1'=n_2, \quad k_2+k_2'=n_1, \quad \text{and} \quad k_1k_2'=k_1'k_2. \qquad \text{Q.E.D.}$$

(8.10) Line graphs.

The line graph $L(X)$ of a (connected) graph X is defined as follows. As the set of vertices, put $VL(X)=EX$, and define two vertices in $L(X)$ to be adjacent, if the corresponding edges in X have a common vertex. The following result is well known:

Proposition (8.11) (Sachs 1967). *Let X be a regular graph of valency k, such that $n=\#(VX)$, $m=nk/2=\#(EX)$. Then $L(X)$ is again regular of valency $2k-2$, and*

(8.12) $\operatorname{Spec}(L(X))$

$$=\{\lambda_j+k-2\ (1\leq j\leq n),\ -2,\ -2,\ \cdots,\ -2\ (m-n \textit{ times})\}.$$

where $\operatorname{Spec}(X)=\{\lambda_j\ (1\leq j\leq n)\}$. *Therefore if X is connected, one has*

(8.13) $Z_{L(X)}(u)^{-1}$

$$=(1-u^2)^{nk(k-2)/2}(1+2u+q'u^2)^{m-n}\prod_{j=1}^{n}[1-(k-2+\lambda_j)u+q'u^2],$$

where $q'=2k-3$.

Proof. For (8.12), see [Bi], Theorem 3.8. (8.13) is a direct consequence of this and (3.7). Q.E.D.

(8.14) Cartesian product $X_1\times X_2$.

The cartesian product of two graphs $X_1\times X_2$ is defined to be a graph X which has $VX_1\times VX_2$ as the set of vertices and two vertices (P_1, P_2),

(Q_1, Q_2) are defined to be adjacent if either (i) $P_1=Q_1$ and P_2, Q_2 are adjacent, or (ii) $P_2=Q_2$ and P_1, Q_1 are adjacent. From this definition we see that the valency of $(P, Q)\in V(X_1\times X_2)$ is the sum of the valencies of P and Q. It is also easy to see that, when X_1, X_2 are connected, then so is $X_1\times X_2$, and

$$\begin{aligned}(8.15)\qquad R:&=\dim_C H_1(X_1\times X_2, C)\\ &=\sharp(E(X_1\times X_2))-\sharp(V(X_1\times X_2))+1\\ &=\sharp(VX_1)\sharp(EX_2)+\sharp(VX_2)\sharp(EX_1)-\sharp(VX_1)\sharp(VX_2)+1.\end{aligned}$$

Proposition (8.16). *Suppose that* X_i *is a connected graph and let* $\mathrm{Spec}(X_1)=\{\lambda_j;\ 1\leq j\leq n_1\}$, $\mathrm{Spec}(X_2)=\{\mu_j;\ 1\leq j\leq n_2\}$ *be the spectrum of* X_1, X_2 *respectively. Then one has*

$$(8.17)\qquad \mathrm{Spec}(X_1\times X_2)=\{\lambda_i+\mu_j;\ 1\leq i\leq n_1,\ 1\leq j\leq n_2\}.$$

If, moreover, X_i *is assumed to be regular of valency* k_i $(i=1, 2)$, *then one has*

$$(8.18)\qquad Z_{X_1\times X_2}(u)^{-1}=(1-u^2)^{(q-1)n_1n_2/2}\prod_{i=1}^{n_1}\prod_{j=1}^{n_2}[1-(\lambda_i+\mu_j)u+qu^2],$$

where $q=k_1+k_2-1$.

Proof. Although (8.17) is well known (see [Scw]), we give here a simple proof. Arrange the vertices of $X_1\times X_2$ in n_2 blocks:

$$(P_1, Q_1), \cdots, (P_{n_1}, Q_1);\ (P_1, Q_2), \cdots,\\ (P_{n_1}, Q_2);\ \cdots;\ (P_1, Q_{n_2}), \cdots, (P_{n_1}, Q_{n_2}).$$

Then the adjacency matrix C of $X_1\times X_2$ is written as a matrix with $n_2\times n_2$ blocks

$$C=\begin{array}{r|ccccccc|}
 & & \underset{\smile}{i} & & & & \underset{\smile}{j} & \\
\hline
 & A & & & & & & \\
i) & & & & & & I & \\
 & & & A & & & & \\
 & & & & A & & & \\
 & & & & & A & & \\
j) & & I & & & & & \\
 & & & & & & & A \\
\hline
\end{array}\in M(n_1n_2, Z),$$

where A is the adjacency matrix of X_1, and the unit matrix $I=I_{n_1}$ appears at the (i, j) block if and only if Q_i and Q_j are adjacent in X_2. Let $W=(w_{ij})\in GL_{n_2}(C)$ be a nonsingular matrix which transforms the adjacency matrix B of X_2 into a diagonal matrix: $X^{-1}BX=\mathrm{diag}(\mu_1, \cdots, \mu_{n_2})$, and let $\tilde{W}$ be the matrix having the same type as C, whose (i,j) block is $w_{ij}I_{n_1}$. Then one can easily see that

$$\tilde{W}^{-1}C\tilde{W}=\begin{bmatrix} \mu_1 I+A & & & 0 \\ & \mu_2 I+A & & \\ 0 & & \ddots & \\ & & & \mu_{n_2}I+A \end{bmatrix},$$

and the assertion (8.17) follows. Q.E.D.

(8.19) Conjunction $X_1\wedge X_2$.

The conjunction $X_1\wedge X_2$ of two graphs X_1, X_2 is defined as follows. Again we have $V(X_1\wedge X_2)=V(X_1)\times V(X_2)$, and (P_1, Q_1), (P_2, Q_2) are defined to be adjacent if and only if both pairs $\{P_1, P_2\}$ and $\{Q_1, Q_2\}$ are adjacent. We see easily that if X_1, X_2 are connected, then so is $X_1\wedge X_2$, and

$$\begin{aligned}(8.20)\qquad R:&=\dim_{\boldsymbol{C}} H_1(X_1\wedge X_2, \boldsymbol{C})\\ &=\#(E(X_1\wedge X_2))-\#(V(X_1\wedge X_2))+1\\ &=2\#(EX_1)\#(EX_2)-\#(VX_1)\#(VX_2)+1.\end{aligned}$$

Proposition (8.21). *Suppose that* X_i *and* Spec(X_i) *are as above* $(i=1, 2)$. *Then one has*

$$(8.22)\qquad \mathrm{Spec}(X_1\wedge X_2)=\{\lambda_i\mu_j;\ 1\le i\le n_1,\ 1\le j\le n_2\}.$$

If, moreover, X_i *is assumed to be regular of valency* k_i $(i=1, 2)$, *then* $X_1\wedge X_2$ *is also regular with valency* k_1k_2, *and one has*

$$(8.23)\qquad Z_{X_1\wedge X_2}(u)^{-1}=(1-u^2)^{(q-1)n_1n_2/2}\prod_{i=1}^{n_1}\prod_{j=1}^{n_2}[1-(\lambda_i\mu_j)u+qu^2],$$

where $q=k_1k_2-1$.

Proof. With the same notation as in the proof of Proposition (8.16), we see that the adjacency matrix C of $X_1\wedge X_2$ now is

$$(8.24)\qquad C=\begin{array}{cc} & \begin{matrix} & \underset{\smile}{i} & & & \underset{\smile}{j} & \end{matrix} \\ \begin{matrix} \\ i) \\ \\ \\ \\ j) \\ \\ \end{matrix} & \begin{bmatrix} 0 & & & & & \\ & & & & A & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & A & & & & \\ & & & & & 0 \end{bmatrix} \end{array}\in M(n_1, n_2, \boldsymbol{Z}),$$

where we put A in (i, j) block if Q_i and Q_j are adjacent. In other words, $C = A \otimes B$. Now the assertion follows from this by a similar argument as above. Q.E.D.

(8.25) Strong product $X_1 * X_2$.

The strong product $X_1 * X_2$ of two graphs X_1, X_2 is defined as follows. Again we have $V(X_1 * X_2) = V(X_1) \times V(X_2)$, and (P_1, Q_1), (P_2, Q_2) are defined to be adjacent if and only if $d_{X_1}(P_1, P_2) \leq 1$, $d_{X_2}(Q_1, Q_2) \leq 1$. We easily see that if X_1, X_2 are connected, then so is $X_1 * X_2$, and

$$\begin{aligned}(8.26)\qquad R &:= \dim_C H_1(X_1 * X_2, C)\\ &= \#(E(X_1 * X_2)) - \#(V(X_1 * X_2)) + 1\\ &= 2\#(EX_1)\#(EX_2) + \#(VX_1)\#(EX_2) + \#(VX_2)\#(EX_1)\\ &\qquad\qquad - \#(VX_1)\#(VX_2) + 1.\end{aligned}$$

Proposition (8.27). *Suppose that X_i and* Spec(X_i) *are as above* $(i = 1, 2)$. *Then one has*

$$(8.28)\qquad \mathrm{Spec}(X_1 * X_2) = \{\lambda_i \mu_j + \lambda_i + \mu_j;\ 1 \leq i \leq n_1,\ 1 \leq j \leq n_2\}.$$

If, moreover, X_i is assumed to be regular of valency k_i $(i = 1, 2)$, *then $X_1 * X_2$ is also regular with valency $k_1 k_2 + k_1 + k_2$, and one has*

$$(8.29)\qquad Z_{X_1 * X_2}(u)^{-1} = (1 - u^2)^{(q-1)n_1 n_2 / 2} \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} [1 - (\lambda_i \mu_j + \lambda_i + \mu_j)u + qu^2],$$

where $q = k_1 k_2 + k_1 + k_2 - 1$.

Proof. This is a combination of (8.15) and (8.21). Q.E.D.

(8.30) Join $X_1 + X_2$.

The join of two graphs X_1 and X_2 is a graph $X_1 + X_2$ such that $V(X_1 + X_2) = V(X_1) \cup V(X_2)$ (disjoint), and to $EX_1 \cup EX_2$ we add the edges joining any $P \in V(X_1)$ and any $Q \in V(X_2)$. To get a result for this graph similar to the above ones, we have to assume that both X_1, X_2 are regular, say, of valency k_1, k_2 respectively.

Proposition (8.31) ([Scw]). *Assumption being as above, one has*

$$(8.32)\qquad \mathrm{Spec}(X_1 + X_2) = \{\xi, \xi', \lambda_i\ (2 \leq i \leq n_1),\ \mu_j\ (2 \leq j \leq n_2)\},$$

where we assume that $\lambda_1 = k_1$ $\mu_1 = k_2$, and ξ, ξ' are the roots of $X^2 - (k_1 + k_2)X + (k_1 k_2 - n_1 n_2)$. If, moreover, $X_1 + X_2$ is assumed to be regular, then one has

$$(8.33)\qquad Z_{X_1+X_2}(u)^{-1}=(1-u^2)^{(q-1)(n_1+n_2)/2}(1-u)(1-qu)(1-cu+qu^2)$$
$$\times\prod_{i=2}^{n_1}[1-\lambda_i u+qu^2]\prod_{j=2}^{n_2}[1-\mu_j u+qu^2],$$

where we put $q=k_1+n_2-1$ $(=k_2+n_1-1)$, $c=k_1-n_1=k_2-n_2$.

Proof. (8.32) is a special case of (8.35) below. Note that the regularity assumption on X_1+X_2 is equivalent to the equality $k_1+n_2=k_2+n_1$. Q.E.D.

(8.34) Composition $G[X_1, \cdots, X_p]$.

Let G be a graph such that $VG=\{P_1, \cdots, P_p\}$, and let X_i be a graph for $i=1, 2, \cdots, p$. Then one can form a new graph $X:=G[X_1, \cdots, X_p]$, whose set of vertices is the disjoint union of VX_i $(1\leq i\leq p)$, and as the set of edges, we add to $\cup EX_i$, those joining $Q_i\in VX_i$ and $Q_j\in VX_j$ whenever P_i and P_j are adjacent in G. This supplies a quite general way of constructing new graphs from known ones. Now we assume that each X_i is regular, say, of valency k_i. Also put $n_i:=\sharp(VX_i)$. The following result, due to Schwenk [Scw] is very useful.

Proposition (8.34) ([Scw]). *Assumption being as above, let*

$$\mathrm{Spec}(X_i)=\{\lambda_1^{(i)}=k_i, \lambda_2^{(i)}, \cdots, \lambda_{n_i}^{(i)}\}$$

be the spectrum of X_i. *Then one has*

$$(8.35)\qquad \mathrm{Spec}(G[X_1, \cdots, X_p])=\mathrm{Spec}(C^*)\cup\bigcup_{i=1}^{p}(\mathrm{Spec}(X_i)\backslash\{\lambda_1^{(i)}\})$$

where in the union we count with multiplicities, and Spec(C^*) *is the spectrum of the following matrix*

$$(8.36)\qquad C^*:=\begin{array}{cc} & \begin{array}{ccccc} & & \underbrace{i}_{} & \underbrace{j}_{} & \end{array}\\ \begin{array}{c} \\ \\ i) \\ j) \\ \\ \end{array} & \left[\begin{array}{ccccc} k_1 & & & & \\ & k_2 & & & \\ & & \ddots & n_j & \\ & & n_i & \ddots & \\ & & & & k_p \end{array}\right]\end{array}\in M(p, Z).$$

where we put n_j *in* (i,j) *entry such that* $i\neq j$, *whenever* P_i *and* P_j *are adjacent in* G.

Proof. We shall give a proof along the same idea as [Scw]. We first note that the adjacency matrix C of $X=G[X_1, \cdots, X_p]$ is divided into $p\times p$ blocks according to the partition of $N:=\sharp(VX)=n_1+n_2+\cdots+n_p$, and written as follows:

$$C=\begin{array}{r|ccccccc|}
 & & & \underset{\smile}{i} & & \underset{\smile}{j} & & \\
 & A_1 & & & & & & \\
 & & A_2 & & & & & \\
i) & & & \ddots & & J & & \\
 & & & & \ddots & & & \\
j) & & {}^tJ & & & \ddots & & \\
 & & & & & & A_{p-1} & \\
 & & & & & & & A_p
\end{array} \in M(N, Z), \tag{8.37}$$

From the regularity of X_i, we get

$$A_iJ=k_iJ, \quad {}^tJA_i=k_i{}^tJ \quad (J=J_{n_i,n_j}\in M(n_i, n_j; Z)), \tag{8.38}$$

where J is the matrix of prescribed size whose entries are all 1. Let $W_i=\boldsymbol{C}^{n_i}$ be the $\boldsymbol{C}$-space to which A_i and ${}^tJ$ act. Observe that $\boldsymbol{u}_i:={}^t(1,\cdots,1)$ is a common eigenvector of them: $A_i\boldsymbol{u}_i=k_i\boldsymbol{u}_i$, ${}^tJ\boldsymbol{u}_i=n_i\boldsymbol{u}_j$, and let W_i^0 be the orthogonal complement of $\boldsymbol{u}_i$ in W_i, which consists of the vectors whose sum of coordinates is 0. Then from (8.37), (8.38), C induces an endomorphism C^0 on $\bigoplus_{i=1}^p W_i^0$, and C^* on $\bigoplus_{i=1}^p(W_i/W_i^0)\simeq\boldsymbol{C}^p$. From (8.38), it is easy to see that $C^0\simeq\bigoplus_{i=1}^p(A_i\,|\,W_i^0)$, and that C^* is represented by the matrix (8.36). Since $\mathrm{Spec}(C)=\mathrm{Spec}(C^0)\cup\mathrm{Spec}(C^*)$ (with multiplicity), this completes the proof. Q.E.D.

Corollary (3.39). *Suppose that G* (resp. X_i $(1\leq i\leq p)$) *is a connected regular graph of valency k_0* (resp. k) *and $\sharp(VG)=p$* (resp. $\sharp(VX_i)=n$). *Then $G[X_1,\cdots,X_p]$ is again a regular connected graph of valency $K=k+nk_0$, and if* $\mathrm{Spec}(G)=\{\mu_j\ (1\leq j\leq p)\}$ (resp. $\mathrm{Spec}(X_i)=\{\lambda_j^{(i)};\ 1\leq j\leq n\}$) *is the spectrum of G with $\mu_1=k_0$* (resp. $\lambda_1^{(i)}=k$), *one has*

$$\begin{aligned} &Z_{G[X_1,\cdots,X_p]}(u)^{-1}\\ &\quad=(1-u^2)^{(q-1)pn/2}(1-u)(1-qu)\prod_{j=2}^{p}[1-(n\mu_j+k)u+qu^2]\\ &\quad\times\prod_{i=1}^{p}\prod_{j=2}^{n}[1-\lambda_j^{(i)}u+qu^2], \end{aligned} \tag{8.40}$$

where $q=K-1=k+nk_0-1$.

Proof. This follows immediately from (8.35) and (3.7). Q.E.D.

§ 9. Zeta functions of well known families of graphs

In this section, we shall compute the zeta functions $Z_X(u)$ for some well known families of graphs. These computations give us many examples of graphs which are not Ramanujan graphs, and also the examples which answer the questions raised in Section 5. Most of them are derived from (3.7), (3.15), and the knowledge of Spec(X), which has been studied by various authors.

(9.1) Cayley graphs of finite groups.

Let G be a group and let S be a symmetric set of generators, i.e., $S=S^{-1}$ and $G=\langle S\rangle$. We assume moreover that $1\notin S$. Now a connected graph $X=X(G,S)$, called the *Cayley graph* of (G,S), is defined as follows. Put $VX=G$, and define two vertices g, h to be adjacent, if $g=hs$ for an element $s\in S$. This gives a connected regular graph of valency $\sharp(S)$. $X(G,S)$ is a bipartite graph if and only if no product of odd number of element of S is equal to 1. As a matter of fact, many of the graphs which have interesting properties are Cayley graphs. A simple example is given by taking $S=G\backslash\{1\}$, in which case we obtain the complete graph $K(n)$ $(n=\sharp(G))$, which will be treated below.

However, it is a difficult problem to determine the zeta function, or the spectrum of it, without specifying G and S. This is reduced to the following problem. Let f be a function on G, and let $A(G,f)\in M(n,\boldsymbol{C})$ $(n=\sharp(G))$ be the matrix, indexed by G, such that

$$(9.2)\quad A(G,f)=(f(g^{-1}h))_{g,h\in G}.$$

Can one determine the characteristic polynomial of $A(G,f)$? In fact the adjacency matrix of the Cayley graph $X(G,S)$ is nothing but $A(G,f)$ with f being the characteristic function of the set S.

Here, we restrict ourselves to the case where G is an abelian group. In this case it is easy to determine the eigenvalues of $A(G,f)$, using a well known result on group determinant:

Lemma (9.3). *The eigenvalues of* $A(G,f)$ *are*

$$\{\sum_{g\in G}\chi(g)f(g)\ (\chi\in\hat{G}:=\text{the group of characters of } G)\}.$$

Thus using (3.7), we obtain

Proposition (9.4). *For the Cayley graph* $X=X(G,S)$ *of a finite abelian group* G, *we have*

$$\mathrm{Spec}(X)=\{\sum_{s\in S}\chi(s)\,;\ \chi\in\hat{G}\},$$

$$Z_X(u)^{-1}=(1-u^2)^{(q-1)n/2}\prod_{\chi\in\hat{G}}[1-(\sum_{s\in S}\chi(s))u+qu^2],$$

where we put $\sharp(G)=n$, $\sharp(S)=q+1$.

Note that $\sum \chi(s) \in \boldsymbol{R}$, because $S=S^{-1}$. This result supplies many examples of graphs with the second eigenvalue λ_2 as close to $\lambda_1=q+1$ as one wants.

For example, take $G_{l,t}:=\boldsymbol{Z}/l\boldsymbol{Z}\oplus\cdots\oplus\boldsymbol{Z}/l\boldsymbol{Z}$, the elementary abelian l-group of rank t ($n=l^t$), and let

$$S=\{(0,\cdots,0,\overset{\overset{i}{\smile}}{\pm 1},0,\cdots,0);\ 1\le i\le t\},\qquad q=2t-1.$$

Then we have for $X=X(G_{l,t},S)$

$$\mathrm{Spec}(X)=\left\{2\sum_{i=1}^{t}\cos\left(\frac{2\pi a_i}{l}\right);\ \boldsymbol{a}\in G_{l,t}\right\},\qquad (\boldsymbol{a}:=(a_1,\cdots,a_t)),$$

$$(9.5)\qquad Z_X(u)^{-1}=(1-u^2)^{l^t(t-1)}\prod_{\boldsymbol{a}\in G_{l,t}}\left[1-2u\sum_{i=1}^{t}\cos\left(\frac{2\pi a_i}{l}\right)+qu^2\right].$$

Note that X is bipartite if and only if l is even. Choosing l suitably, one can give examples of regular bipartite graphs X for which $Z_{X,b}(u)^{-1}$ has an irreducible factor over $\boldsymbol{Q}$ with arbitrary prescribed degree. We remark finally that, putting $t=1$, we get a circuit C_l of length l:

$$(9.6)\qquad Z_{C_l}(u)^{-1}=\prod_{a=0}^{l-1}\left[1-2u\cos\left(\frac{2\pi a}{l}\right)+u^2\right]=(1-u^l)^2.$$

(9.7) Flower of l petals.

X is a semiregular bipartite multigraph, which consists of $l+1$ vertices: $V_1=\{P\}$, $V_2=\{Q_1,\cdots,Q_l\}$, and each Q_j is joined to P by $q+1$ edges.

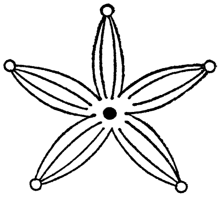


Fig. 10

Thus we see that the adjacent matrix is

$$A=\begin{bmatrix} 0 & q+1 & q+1 & \cdots & q+1 \\ q+1 & 0 & 0 & \cdots & 0 \\ q+1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ q+1 & 0 & 0 & & 0 \end{bmatrix} \in M(n, \boldsymbol{Z}), \qquad n=l+1.$$

It is easy to calculate the eigenvalues of A, and we obtain

$$\text{(9.8)} \qquad \operatorname{Spec}(X)=\{\pm(q+1)\sqrt{l},\ 0,\ \cdots,\ 0\ (l-1 \text{ times})\}.$$

Arranging the order of the edges as

$$EX=\{e_0^{(1)}, e_1^{(1)}, \cdots, e_q^{(1)}, e_0^{(2)}, \cdots, e_q^{(2)}, \cdots, e_0^{(l)}, \cdots, e_q^{(l)}\},$$

we find the following matrix representation of $\rho^*(T_i)$:

$$\rho^*(T_1)=\begin{bmatrix} J-I & I & I & \cdots & I \\ I & J-I & I & \cdots & I \\ I & I & J-I & \cdots & I \\ \vdots & \vdots & \vdots & & \vdots \\ I & I & I & \cdots & J-I \end{bmatrix} \in M(m, \boldsymbol{Z}), \qquad \begin{matrix} m=l(q+1), \\ I=I_{q+1}, \quad J=J_{q+1}, \end{matrix}$$

$$\rho^*(T_2)=\begin{bmatrix} J-I & 0 & 0 & \cdots & 0 \\ 0 & J-I & 0 & \cdots & 0 \\ 0 & 0 & J-I & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & J-I \end{bmatrix} \in M(m, \boldsymbol{Z}), \qquad m=l(q+1),$$

$$\rho^*(T_1T_2)=\begin{bmatrix} (q-1)J+I & qJ & qJ & \cdots & qJ \\ qJ & (q-1)J+I & qJ & \cdots & qJ \\ qJ & qJ & (q-1)J+I & \cdots & qJ \\ \vdots & \vdots & \vdots & & \vdots \\ qJ & qJ & qJ & \cdots & (q-1)J+I \end{bmatrix}.$$

Using the fact that $\det(xI-J)=x^q(x-(q+1))$, or from (3.15), we get

$$\text{(9.9)} \qquad Z_{X,b}^{-1}(u)=(1-u)^{ql}(1+qu)^{l-1}(1-q(lq+l-1)u).$$

Note that X is semiregular bipartite of valency $(q_1+1, q_2+1)=(lq+l, q)$, hence $r=\dim_{\boldsymbol{C}} H_1(X, \boldsymbol{C})=ql$. Thus we have

$$\begin{array}{lccccccc} & M^1(X) = & M(-1,-1) & \oplus & M(-1,q) & \oplus & M(lq+l-1,q), \\ \text{dim:} & (q+1)l = & ql & + & l-1 & + & 1 \end{array}$$

and see that the space of cuspidal functions vanishes.

(9.11) Complete bipartite graph $K(q_1+1, q_2+1)$.

This is a typical example of semiregular bipartite graph of valency (q_1+1, q_2+1). The vertices are divided into two classes V_1, V_2 such that $\sharp(V_1)=q_2+1$, $\sharp(V_2)=q_1+1$, and each $P\in V_1$ (resp. $Q\in V_2$) is joined to all vertices of V_2 (resp. V_1). To determine the spectrum of this graph, we consider the adjacency matrix $A^{[1]}$ of the multigraph $X^{[1]}$ derived from $X=K(q_1+1, q_2+1)$ as in (3.13).

Since there are exactly (q_1+1) proper paths C of length two such that $o(C)=P_i$, $t(C)=P_j$ for any distinct vertices P_i, $P_j\in V_1$, we get $A^{[1]}=(q_1+1)(J-I)$, where $J, I\in M(q_2+1, \boldsymbol{Z})$ are as above. Hence

$$I-(A^{[1]}-q_2+1)u+q_1q_2u^2=(1+(q_1+q_2)u+q_1q_2u^2)I-(q_1+1)Ju,$$

and we easily get

$$(9.12)\qquad Z_{X,b}^{-1}(u)=(1-u)^{q_1q_2}(1-q_1q_2u)(1+q_1u)^{q_2}(1+q_2u)^{q_1}.$$

(9.13) Complete graph $K(q+2)$ $(q\geq 1)$.

This is the most elementary graph, having $n=q+2$ vertices, any two of which are joined by an edge. One sees immediately that $K(q+2)$ is a regular graph with valency $q+1$. The zetafunction $Z_X(u)$ can be simply derived as above. Namely, the adjacent matrix of $K(n)$ is $A=J_n-I_n$, hence we have $I-Au+qu^2=(1+u+qu^2)I-uJ$. From (3.7), one obtain

$$(9.14)\qquad Z_{K(q+2)}(u)^{-1}=(1-u)^r(1+u)^{r-1}(1-qu)(1+u+qu^2)^{q+1},\quad r=q(q+1)/2.$$

This result is also a consequence of the well known fact that

$$(9.15)\qquad \mathrm{Spec}(K(q+2))=\{q+1, -1, \cdots, -1\ (q+1 \text{ times})\}\quad (\text{cf. [Bi]}).$$

Note that (9.14) implies that the representation of $\boldsymbol{C}[T_1, T_2]$ on the space $M^1_{\mathrm{cusp}}(X^{(2)})$ of cusp forms on the edges of $X^{(2)}$, the barycentric subdivision of $X=K(q+2)$, is a direct sum of $M(-1,1)$, which is the $(q+1)(q-2)/2$ copies of a linear representation, and $q+1$ copies of a 2-dimensional irreducible representation, say φ (cf. (4.7)). This fact has an interesting interpretation in terms of the representations of the symmetric group $\mathfrak{S}_{q+2}$ of degree $q+2$, as follows.

The group $\mathfrak{S}_{q+2}$ acts on $K(q+2)$ through the permutation of its vertices, hence it acts also on $X^{(2)}$, and one gets a representation of $\mathfrak{S}_{q+2}$ on our space $M^1(X^{(2)})$. To consider how it decomposes into the irreducible ones, let us identify $M^1(X^{(2)})$ with the space M of bilinear forms $F\colon \boldsymbol{C}^{q+2}\times\boldsymbol{C}^{q+2}\to\boldsymbol{C}$ which satisfy $F(\boldsymbol{v}_i, \boldsymbol{v}_i)=0$ $(1\leq i\leq q+2)$, where $\{\boldsymbol{v}_i\,(1\leq i\leq q+2)\}$ is the standard orthonormal basis of $\boldsymbol{C}^{q+2}$. This can be done by identifying the function $f\in M^1(X^{(2)})$ with the bilinear form $F(x, y):=\sum_{i,j} f([P_iQ_j])x_iy_j$, where $[P_i, Q_j]$ denotes the edge of $X^{(2)}$ joining $P_i\in VX$ and Q_j, the middle point of $[P_i, P_j]\in EX$. Now if we add M the space of diagonal forms, we get $W^*\otimes W^*$, where $W=\boldsymbol{C}^{q+2}$, and W^* is its dual space:

$$(9.16)\qquad M^1(X^{(2)})\oplus\boldsymbol{C}^{q+2}\simeq M\oplus\boldsymbol{C}^{q+2}\simeq W^*\otimes W^*,\qquad \text{as } \mathfrak{S}_{q+2}\text{-modules.}$$

On the other hand, as is well known, the canonical representation of $\mathfrak{S}_{q+2}$ in W^* decomposes as

$$(9.17)\qquad (\rho, W^*)\simeq\rho_{q+2}\oplus\rho_{q+1,1},$$

where, in general, $\rho_{n_1,n_2,\dots,n_t}$ denotes the irreducible representation of $\mathfrak{S}_{q+2}$ corresponding to the partition $q+2=n_1+n_2+\cdots+n_t$, hence, in particular, ρ_{q+2} denotes the trivial one. Now from (9.16), (9.17), to decompose $M^1(X^{(2)})$ into irreducible $\mathfrak{S}_{q+2}$-submodules is equivalent to decompose $\rho_{q+1,1}\otimes\rho_{q+1,1}$. To this, the following answer has been known:

Lemma (9.18) (Murnaghan [Mu]).

$$\rho_{q+1,1}\otimes\rho_{q+1,1}\simeq\rho_{q+2}\oplus\rho_{q+1,1}\oplus\rho_{q,2}\oplus\rho_{q,1,1}.$$

Using this result, one can reproduce (9.14) without using (3.7), as follows. We first note that the above irreducible components have degrees

$$\deg(\rho_{q+1,1})=q+1,\quad \deg(\rho_{q,2})=(q-1)(q+2)/2,\quad \deg(\rho_{q,1,1})=q(q+1)/2.$$

Next we notice that the actions of $\boldsymbol{C}[T_1, T_2]$ and $\mathfrak{S}_{q+2}$ are commutative. This implies that T_1, T_2 acts as scalars on each irreducible components of $M^1(X^{(2)})$. Also note that $M(q, 1)\simeq\rho_{q+2}$ as $\mathfrak{S}_{q+2}$-modules.

Lemma (9.19). *One has*

$$M(-1, -1)\simeq\rho_{q,1,1},\quad M(-1, 1)\simeq\rho_{q,2}\qquad \textit{as } \mathfrak{S}_{q+2}\textit{-modules.}$$

Moreover, the orthogonal complement M_0 *of* $M(-1, 1)$ *in* $M^1_{\mathrm{cusp}}(X)$ *is isomorphic to* $\rho_{q+1,1}\oplus\rho_{q+1,1}$.

Proof. Under the above identification, the $+1$ (resp. -1)-eigenspace of T_2 corresponds to the space of symmetric (resp. skew-symmetric) bilinear forms $S^2(W^*)$ (resp. $\Lambda^2(W^*)$), which is stable under $\mathfrak{S}_{q+2}$. One knows that $S^2(W^*)$ contains the two $\mathfrak{S}_{q+2}$-invariant subspaces:

$$C\left(\sum_i x_i y_i\right) \simeq \rho_{q+2}, \quad \text{and} \quad \left\{\sum_i c_i x_i y_i;\ \sum_i c_i = 0\right\} \simeq \rho_{q+1,1}.$$

Since $S^2(W^*)$ and $\Lambda^2(W^*)$ are not stable under T_1, they must have irreducible $\mathfrak{S}_{q+2}$-subspaces which are mutually isomorphic. Now a simple argument of counting dimensions proves the assertions. Q.E.D.

From Proposition (5.7) and Schur's lemma, it follows that the representation of $C[T_1, T_2]$ induced on M_0 is the $q+1$ copies of a 2-dimensional irreducible one. The characteristic polynomial of T_1T_2 can be computed from the fact that $\operatorname{tr}(\rho^*(T_1T_2) \mid M^1(X^{(2)})) = 0$.

(9.20) Complete multipartite graph $K(s+1, \cdots, s+1)$.

This is a generalization of the complete graph. The vertices are divided into $l+1$ subsets $V_0, V_1, \cdots, V_l$ such that $\sharp(V_i) = s+1$ for $0 \leq i \leq l$. Two vertices $P \in V_i$, $Q \in V_j$ are adjacent if and only if $i \neq j$. If $s=0$, one obtains $K(1, \cdots, 1) = K(l+1)$. Observe that $X = K(s+1, \cdots, s+1)$ is a regular graph of valency $l(s+1)$. To get the spectrum of X, one can apply Proposition (8.34); in fact X can be obtained by the composition $K(l+1)[X_0, \cdots, X_0]$, where X_0 is the (disconnected) regular graph of valency 0 (i.e., $EX_0 = \phi$) with $s+1$ vertices. One obviously has $\operatorname{Spec}(X_0) = \{0, \cdots, 0\ (s+1 \text{ times})\}$, and applying (8.35), one obtains the following results:

(9.21) $\operatorname{Spec}(K(s+1, \cdots, s+1))$

$$= \{q+1, 0, \cdots, 0\ (s(l+1) \text{ times}), -s-1, \cdots, -s-1\ (l \text{ times})\},$$

(9.22) $Z_{K(s+1,\cdots,s+1)}(u)^{-1}$

$$= (1-u)^r (1+u)^{r-1} (1-qu)(1+qu^2)^{s(l+1)} (1+(s+1)u+qu^2)^l,$$

where we put $r = (l+1)(s+1)(q-1)/2+1$, $q = l(s+1)-1$. One can also prove them by a similar computation as in $K(q+2)$.

(9.23) Cube $Q(n)$.

$Q(n)$ is a regular graph of valency n and with 2^n vertices, which represents the cube in $\boldsymbol{R}^n$. Namely $VQ(n)$ is given by

$$VQ(n) = \{P = (p_1, \cdots, p_n) \in \boldsymbol{R}^n;\ p_i = 0, \text{ or } 1\ (1 \leq i \leq n)\},$$

and $P, Q \in VQ(n)$ are adjacent if and only if $|P-Q|=1$. Moreover, it is bipartite:

$$VQ(n)=V_1 \cup V_2, \quad V_i=\{P=(p_1, \cdots, p_n) \in VQ(n);\ \textstyle\sum_i p_i \equiv i \pmod 2\}.$$

To obtain its spectrum, one notes that $Q(1)=K(2)$, and $Q(n)$ is given inductively by the Cartesian product: $Q(n)=Q(n-1)\times K(2)$. Then it is easy to prove by induction and Proposition (8.16), that

$$\text{(9.24)} \qquad \operatorname{Spec}(Q(n))=\left\{n-2i\left(\binom{n}{i}\text{ times}\right);\ 0\leq i\leq n\right\}.$$

Hence, by (3.7), we have

$$\text{(9.25)} \qquad Z_{Q(n)}(u)^{-1}=(1-u^2)^{(n-1)2^{n-1}}\prod_{i=0}^{n}[1-(n-2i)u+qu^2]^{\binom{n}{i}},$$

with $q=n-1$. Noting that $Q(n)$ is regular bipartite, one can observe that this supplies (infinitely) many examples where the representation ρ^* on $M^1(Q(n))$ has an irreducible component φ of degree 2, of which the characteristic polynomial $p_\varphi(u)=\det(I_2-\varphi(T_1T_2)u)$ decomposes into a product of linear factors over $\boldsymbol{Z}$. In fact, for any positive integer i, put $n=4i+3$, $q=4i+2$. Then from (3.15), one sees that $Z_{Q(n),b}(u)^{-1}$ has the following factor corresponding to $\lambda=n-2i=2i+3 \in \operatorname{Spec}(Q(n))$:

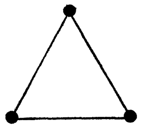
$$\begin{aligned} 1-(\lambda^2-2q)u+q^2u^2 &= 1-(4i^2+4i+5)u+4(2i+1)^2u^2 \\ &=(1-4u)(1-(2i+1)^2u). \end{aligned}$$

On the other hand, if one assumes that q is a prime number then all $p_\varphi(u)$'s corresponding to the 2-dimensional irreducible factors are irreducible over $\boldsymbol{Q}$.

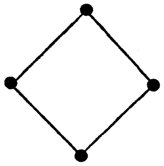
§ 10. Examples

In the following, we give a list of *all* graphs X, with their characteristic functions $\phi_X(z)$, zeta functions $Z_X(u)$, which have no end point and which satisfy $n=\#(VX)\leq 6$, $m=\#(EX)\leq 8$. Up to isomorphism, there are 30 such graphs. We added two graphs to complete the list for $n=5$ (cf. [Har]). $\phi_X(z)$ and $Z_X(u)^{-1}$ are decomposed into the products of *irreducible* factors over $\boldsymbol{Q}$. The symbol $X_k(n, m)$ indicates that X is the k-th graph among those satisfying $(\#(VX), \#(EX))=(n, m)$.

(1) $X_1(3,3)=\mathrm{Cir}_3: r=1$



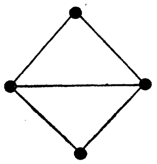
(2) $X_1(4,4)=\mathrm{Cir}_4: r=1$



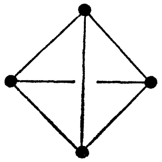
$$\phi_1(z)=(z+1)^2(z-2)$$
$$Z_1(u)^{-1}=(1-u)^2(1+u+u^2)^2$$

$$\phi_2(z)=z^2(z-2)(z+2)$$
$$Z_2(u)^{-1}=(1-u)^2(1+u)^2(1+u^2)^2$$

(3) $X_1(4,5): r=2$



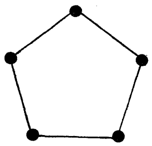
(4) $X_1(4,6)=K(4): r=3$



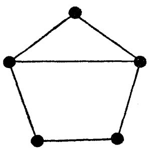
$$\phi_3(z)=z(z+1)(z^2-z-4)$$
$$Z_3(u)^{-1}=(1-u)^2(1+u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3)$$

$$\phi_4(z)=(z+1)^3(z-3)$$
$$Z_4(u)^{-1}=(1-u)^3(1+u)^2(1-2u)(1+u+2u^2)^3$$

(5) $X_1(5,5)=\mathrm{Cir}_5: r=1$



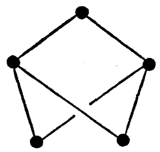
(6) $X_1(5,6): r=2$



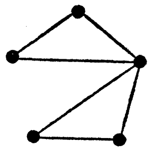
$$\phi_5(z)=(z-2)(z^2+z-1)^2$$
$$Z_5(u)^{-1}=(1-u)^2(1+u+u^2+u^3+u^4)^2$$

$$\phi_6(z)=z(z+2)(z^3-2z^2-2z+2)$$
$$Z_6(u)^{-1}=(1-u)^2(1+u)(1+2u+3u^2+3u^3+2u^4)$$
$$\times(1-u+u^2-2u^3+u^4-2u^5)$$

(7) $X_2(5,6)\colon r=2$



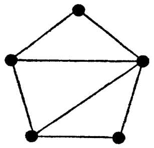
(8) $X_3(5,6)\colon r=2$



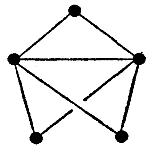
$$\phi_7(z)=z^3(z^2-6)$$
$$Z_7(u)^{-1}=(1-u)^2(1+u)^2(1+u^2)^2(1-2u^2)(1+2u^2)$$

$$\phi_8(z)=(z-1)(z+1)^2(z^2-z-4)$$
$$Z_8(u)^{-1}=(1-u)^2(1+u)(1+u+u^2)^2(1-u+u^2)(1-3u^3)$$

(9) $X_1(5,7)\colon r=3$



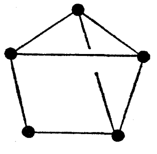
(10) $X_2(5,7)\colon r=3$



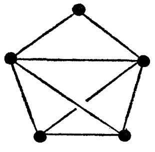
$$\phi_9(z)=(z^2+z-1)(z^3-z^2-5z-2)$$
$$Z_9(u)^{-1}=(1-u)^3(1+u)^2(1+u+2u^2)(1-u-3u^3)\times(1+u+2u^2+u^3+2u^4)$$

$$\phi_{10}(z)=z^2(z+1)(z+2)(z-3)$$
$$Z_{10}(u)^{-1}=(1-u)^3(1+u)^2(1+u^2)^2(1+u+3u^2)(1-2u^2-3u^3)$$

(11) $X_3(5,7)\colon r=3$



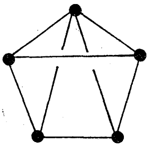
(12) $X_1(5,8)\colon r=4$



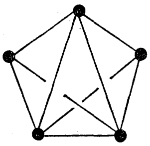
$$\phi_{11}(z)=z(z+1)(z^3-z^2-6z+2)$$
$$Z_{11}(u)^{-1}=(1-u)^3(1+u)^2(1+u+2u^2)(1+2u^2)(1-u^2-2u^3-2u^4-4u^5)$$

$$\phi_{12}(z)=(z+1)^2(z^3-2z^2-5z+2)$$
$$Z_{12}(u)^{-1}=(1-u)^4(1+u)^3(1+u+2u^2)(1+u+3u^2)\times(1-u-5u^3-u^4-6u^5)$$

(13) $X_2(5, 8)\colon r=4$



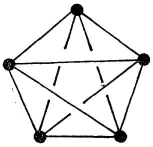
(14) $X_1(5, 9)\colon r=5$



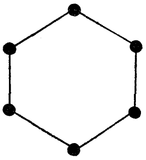
$$\phi_{13}(z)=z^2(z+2)(z^2-2z-4)$$
$$Z_{13}(u)^{-1}=(1-u)^4(1+u)^3(1+2u^2)^2(1+2u+2u^2)(1-u-6u^3)$$

$$\phi_{14}(z)=z(z+1)^2(z^2-2z-6)$$
$$Z_{14}(u)^{-1}=(1-u)^5(1+u)^4(1+2u^2)(1+u+3u^2)^2(1-u-2u^2-6u^3)$$

(15) $X_1(5, 10)=K(5)\colon r=6$



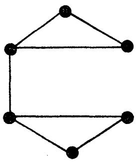
(16) $X_1(6, 6)=\mathrm{Cir}_6\colon r=1$



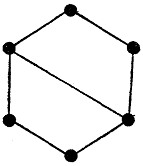
$$\phi_{15}(z)=(z+1)^4(z-4)$$
$$Z_{15}(u)^{-1}=(1-u)^6(1+u)^5(1-3u)(1+u+3u^2)^4$$

$$\phi_{16}(z)=(z-1)^2(z+1)^2(z-2)(z+2)$$
$$Z_{16}(u)^{-1}=(1-u)^2(1+u)^2(1+u+u^2)^2(1-u+u^2)^2$$

(17) $X_1(6, 7)\colon r=2$



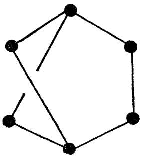
(18) $X_2(6, 7)\colon r=2$



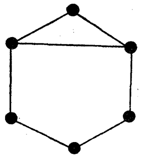
$$\phi_{17}(z)=(z+1)^2(z^2-3)(z^2-2z-1)$$
$$Z_{17}(u)^{-1}=(1-u)^2(1+u)(1+u+u^2)^2(1-u+u^2-2u^3)(1-u^3+2u^4)$$

$$\phi_{18}(z)=(z-1)(z+1)(z^2+2z-1)(z^2-2z-1)$$
$$Z_{18}(u)^{-1}=(1-u)^2(1+u)^2(1+u+u^2)(1-u+u^2)(1+u+u^2+2u^3)\times(1-u+u^2-2u^3)$$

(19) $X_3(6, 7)$: $r=2$

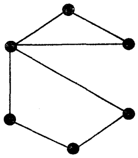


(20) $X_4(6, 7)$: $r=2$

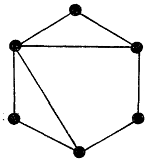


$$\phi_{19}(z)=z(z^2+z-1)(z^3-z^2-5z+4)$$
$$Z_{19}(u)^{-1}=(1-u)^2(1+u)(1+u^2)(1+u+2u^2+2u^3+2u^4)(1-u^2-2u^5)$$
$$\phi_{20}(z)=(z^2+z-1)(z^4-z^3-5z^2+2z+4)$$
$$Z_{20}(u)^{-1}=(1-u)^2(1+u)(1+u+2u^2+u^3+2u^4)(1-u^3-u^5-u^6-2u^7)$$

(21) $X_5(6, 7)$: $r=2$

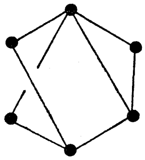


(22) $X_1(6, 8)$: $r=3$

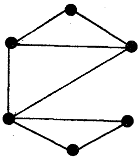


$$\phi_{21}(z)=z(z+1)(z^4-z^3-6z^2+4z+4)$$
$$Z_{21}(u)^{-1}=(1-u)^2(1+u)(1+u^2)(1+u+u^2)(1-u^3-u^4-3u^7)$$
$$\phi_{22}(z)=(z-1)(z+1)(z+2)(z^3-2z^2-3z+2)$$
$$Z_{22}(u)^{-1}=(1-u)^3(1+u)^2(1+2u^2+2u^4)$$
$$\times(1+u+u^2-3u^3-4u^4-8u^5-6u^6-6u^7)$$

(23) $X_2(6, 8)$: $r=3$

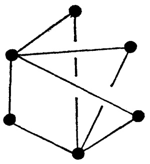


(24) $X_3(6, 8)$: $r=3$

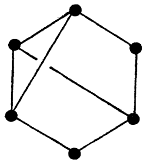


$$\phi_{23}(z)=z^2(z^4-8z^2-2z+7)$$
$$Z_{23}(u)^{-1}=(1-u)^3(1+u)^2(1+u^2)$$
$$\times(1+u+2u^2-4u^4-10u^5-13u^6-17u^7-12u^8-12u^9)$$
$$\phi_{24}(z)=(z+1)^2(z^4-2z^3-5z^2+6z+4)$$
$$Z_{24}(u)^{-1}=(1-u)^3(1+u)^2(1+u+u^2)(1+u+2u^2)$$
$$\times(1-u+u^2-4u^3+3u^4-5u^5+3u^6-6u^7)$$

(25) $X_4(6, 8)=K(2, 4)$: $r=3$

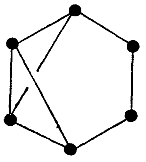


(26) $X_5(6, 8)$: $r=3$

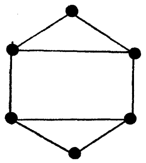


$$\phi_{25}(z)=z^4(z^2-8)$$
$$Z_{25}(u)^{-1}=(1-u)^3(1+u)^3(1-3u^2)(1+3u^2)(1+u^2)^3$$
$$\phi_{26}(z)=(z^2+z-1)(z^4-z^3-6z^2+3z+1)$$
$$Z_{26}(u)^{-1}=(1-u)^3(1+u)^2(1+u+2u^2+u^3+2u^4) \times(1+u^2-u^3-3u^4-6u^5-4u^6-8u^7)$$

(27) $X_6(6, 8)$: $r=3$

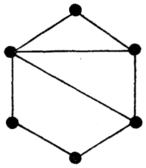


(28) $X_7(6, 8)$: $r=3$

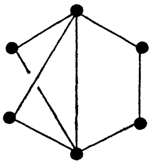


$$\phi_{27}(z)=(z-1)^2(z^4-2z^3-5z^2+8z+1)$$
$$Z_{27}(u)^{-1}=(1-u)^3(1+u)^2(1+u+2u^2)(1+u+2u^2+2u^3+2u^4) \times(1-u-2u^3+2u^4-4u^5)$$
$$\phi_{28}(z)=z(z+2)(z^2-2)(z^2-2z-2)$$
$$Z_{28}(u)^{-1}=(1-u)^3(1+u)^2(1+2u^2)(1+2u+2u^2)(1-u-2u^3)(1+u^2+2u^4)$$

(29) $X_8(6, 8)$: $r=3$

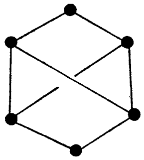


(30) $X_9(6, 8)$: $r=3$

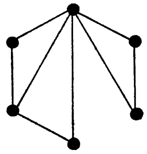


$$\phi_{29}(z)=z^6-8z^4-4z^3+9z^2+4z-1$$
$$Z_{29}(u)^{-1}=(1-u)^3(1+u)^2(1+u+2u^2) \times(1+u^2-2u^3-2u^4-6u^5-4u^6-8u^7-3u^8-6u^9)$$
$$\phi_{30}(z)=z^2(z+2)(z^3-2z^2-4z+4)$$
$$Z_{30}(u)^{-1}=(1-u)^3(1+u)^2(1+u^2)(1+2u+4u^2+4u^3+3u^4) \times(1-u-2u^3+u^4-3u^5)$$

(31) $X_{10}(6, 8)$: $r=3$



(32) $X_{11}(6, 8)$: $r=3$



$$\phi_{31}(z)=z^2(z^2+2z-2)(z^2-2z-2)$$
$$Z_{31}(u)^{-1}=(1-u)^3(1+u)^3(1+2u^2)^2(1+u+2u^3)(1-u-2u^3)$$

$$\phi_{32}(z)=z(z+1)(z^4-z^3-7z^2+z+8)$$
$$Z_{32}(u)^{-1}=(1-u)^3(1+u)^2(1+u^2)(1+u+u^2)(1+u^2-5u^3-5u^5-8u^7)$$

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Department of Mathematics
School of Science and Engineering
Waseda University
Tokyo, 160 Japan