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Vector Valued Modular Forms of Degree Two and their Application to Triple L-functions

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Abstract

We report some properties of vector valued Siegel modular forms of degree two and triple L-functions of different weight. Their relation is discussed in the section 3. This is motivated by an observation in the section 2. In both sections, certain differential operators are relevant. They are defined in the section 1.

§ 1. Differential operators

In this section we introduce some notations and construct differential operators. Let n be a positive integer and H_n the Siegel upper half space of degree n. Let ρ be a rational representation of $GL(n, \boldsymbol{C})$ and V_ρ the representation space of ρ. We denote by St the standard representation of $GL(n, \boldsymbol{C})$. A C^∞-Siegel modular form f of degree n and of 'weight' ρ is by definition a V_ρ-valued C^∞-function on H_n satisfying the equation

$$f(M\langle Z\rangle) = \rho(J(M, Z))f(Z)$$

for all $Z \in H_n$ and $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n = Sp(n, \boldsymbol{Z})$ where $M\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$ and $J(M, Z)=CZ+D$. We denote by $M_\rho^\infty(\Gamma_n)$ the $\boldsymbol{C}$-vector space of all such functions. We also denote by $M_\rho(\Gamma_n)$ and $S_\rho(\Gamma_n)$ the subspaces of $M_\rho^\infty(\Gamma_n)$ consisting of all holomorphic forms and of all holomorphic cusp forms, respectively. Instead of $M_\rho^\infty(\Gamma_n)$, we write $M_k^\infty(\Gamma_n)$ if $\rho=\det^k$ and $M_{k,r}^\infty(\Gamma_n)$ if $\rho=\det^k \otimes \mathrm{Sym}^r\,\mathrm{St}$ for simplicity. We define $M_k(\Gamma_n)$, $S_k(\Gamma_n)$, $M_{k,r}(\Gamma_n)$ and $S_{k,r}(\Gamma_n)$ similarly. If either $f \in M_k(\Gamma_1)$ or $g \in M_k(\Gamma_1)$ is a cusp form, we define the Petersson inner product by

$$\langle f, g\rangle_k = \int_{\Gamma_1 \backslash H_1} f(x+iy)\overline{g(x+iy)}\,y^{k-2}dxdy.$$

For a variable $Z=(z_{jl})$ on H_n, we put $\eta=(\eta_{jl})$ with $\eta_{jl}=-4\pi\,\mathrm{Im}(z_{jl})$ and

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$$\theta^{\pm}=\left(d_{jl}\frac{1}{2\pi i}\frac{\partial}{\partial z_{jl}^{\pm}}\right)_{1\leq j,l\leq n}$$

where $d_{jj}=1$, $d_{jl}=1/2$ for $j\neq l$, $z_{jl}^{+}=z_{jl}$, and $z_{jl}^{-}=\overline{z_{jl}}$. In what follows, we consider the degree two case. Let S_2 be the $\boldsymbol{C}$-vector space of all complex symmetric matrices of size two. The action of $GL(2,\boldsymbol{C})$ on S_2 defined by

$$A\longrightarrow GA{}^tG \quad (A\in S_2 \text{ and } G\in GL(2,\boldsymbol{C})) \tag{1.1}$$

is isomorphic to $\mathrm{Sym}^2\,\mathrm{St}$. We construct differential operators $\nabla^{+}=\nabla_k^{+}$ and $\nabla^{-}=\nabla_{k,r}^{-}$ acting on $M_k^{\infty}(\Gamma_2)$ and $M_{k,r}^{\infty}(\Gamma_2)$ respectively by

$$(\nabla^{+}f)(Z)=k\eta^{-1}f+\theta^{+}f$$

for $f\in M_k^{\infty}(\Gamma_2)$ and

$$(\nabla^{-}f)(Z)=(\mathrm{CFA}(\mathrm{Im}\,Z(\theta^{-}f_j)(Z)\,\mathrm{Im}\,Z))_{j=0}^{r}$$

for $f=(f_j)_{j=0}^{r}\in M_{k,r}^{\infty}(\Gamma_2)$, where $\mathrm{CFA}(A)$ is the cofactor matrix of A. Using Shimura [15, (4.2) and (4.4)] and their anti-holomorphic version we see that $\nabla^{\pm}$ define maps

$$\nabla^{+}\colon M_k^{\infty}(\Gamma_2)\longrightarrow M_{k,2}^{\infty}(\Gamma_2)$$

and

$$\nabla^{-}\colon M_{k,r}^{\infty}(\Gamma_2)\longrightarrow M^{\infty}_{\det^{k-2}\otimes\mathrm{Sym}^r\mathrm{St}\otimes\mathrm{Sym}^2\mathrm{St}}(\Gamma_2). \tag{1.2}$$

We note that degree n analogue of (1.2) does not hold since $A\to\mathrm{CFA}(A)$ is a linear map only for A of size two. These operators have a rather bad property on holomorphy. Holomorphic modular forms are, in general, mapped to non-holomorphic ones by ∇^{+} and annihilated by ∇^{-}. If $r\geq 2$, we have a decomposition

$$\begin{aligned}&\det{}^k\otimes\mathrm{Sym}^r\mathrm{St}\otimes\mathrm{Sym}^2\mathrm{St}\\ &\qquad\cong\det{}^k\otimes\mathrm{Sym}^{r+2}\mathrm{St}\oplus\det{}^{k+1}\otimes\mathrm{Sym}^r\mathrm{St}\oplus\det{}^{k+2}\otimes\mathrm{Sym}^{r-2}\mathrm{St}.\end{aligned} \tag{1.3}$$

In application, we must frequently show the non-vanishing of a desired component. Generally this is not trivial. If $r=0$, there is no such problem. Moreover, in the section 2, we construct holomorphic modular forms using ∇^{+}. The operator ∇^{-} has a nice action on vector valued non-holomorphic Klingen type Eisenstein series. This is used for proving the meromorphy of triple L-functions in the section 3.

§ 2. Construction of certain vector valued modular forms

To cancel the non-holomorphic term, we put

$$[f, g]=\frac{1}{j}f\nabla^{+}g-\frac{1}{k}g\nabla^{+}f$$

for $f\in M_k(\Gamma_2)$ and $g\in M_j(\Gamma_2)$. Then $[f, g]\in M_{k+j,2}(\Gamma_2)$ since it is holomorphic. We show the structure theorem of $M_{k,2}(\Gamma_2)$ and $S_{k,2}(\Gamma_2)$ by using this differential operator.

Recall that the graded $\boldsymbol{C}$-algebra $\bigoplus_k M_k(\Gamma_2)$ where k runs over even integers is generated over $\boldsymbol{C}$ by four algebraically independent elements. (We understand that $M_k(\Gamma_2)=\{0\}$ for a negative k.) They are $\varphi_4\in M_4(\Gamma_2)$, $\varphi_6\in M_6(\Gamma_2)$, $\chi_{10}\in S_{10}(\Gamma_2)$ and $\chi_{12}\in S_{12}(\Gamma_2)$. (See Igusa [4] and Maass [9].)

Theorem 1. *For each even integer k, we have (as a $\boldsymbol{C}$-vector space)*

$$\begin{aligned} M_{k,2}(\Gamma_2)=&M_{k-10}(\Gamma_2)[\varphi_4, \varphi_6]\oplus M_{k-14}(\Gamma_2)[\varphi_4, \chi_{10}]\\ &\oplus M_{k-16}(\Gamma_2)[\varphi_4, \chi_{12}]\oplus V_{k-16}(\Gamma_2)[\varphi_6, \chi_{10}]\\ &\oplus V_{k-18}(\Gamma_2)[\varphi_6, \chi_{12}]\oplus W_{k-22}(\Gamma_2)[\chi_{10}, \chi_{12}] \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} S_{k,2}(\Gamma_2)=&S_{k-10}(\Gamma_2)[\varphi_4, \varphi_6]\oplus M_{k-14}(\Gamma_2)[\varphi_4, \chi_{10}]\\ &\oplus M_{k-16}(\Gamma_2)[\varphi_4, \chi_{12}]\oplus V_{k-16}(\Gamma_2)[\varphi_6, \chi_{10}]\\ &\oplus V_{k-18}(\Gamma_2)[\varphi_6, \chi_{12}]\oplus W_{k-22}(\Gamma_2)[\chi_{10}, \chi_{12}] \end{aligned} \tag{2.2}$$

where

$$V_k(\Gamma_2)=M_k(\Gamma_2)\cap\boldsymbol{C}[\varphi_6, \chi_{10}, \chi_{12}] \quad \textit{and} \quad W_k(\Gamma_2)=M_k(\Gamma_2)\cap\boldsymbol{C}[\chi_{10}, \chi_{12}].$$

Corollary 2. *Let k be an even integer. Then $M_{k,2}(\Gamma_2)$ and $S_{k,2}(\Gamma_2)$ have a $\boldsymbol{C}$-basis consisting of modular forms whose all Fourier coefficients lie in $\boldsymbol{Z}$.*

The proof proceed as follows. The inclusion $\supset$ is clear in both (2.1) and (2.2). First, we show that subspaces appearing in the right hand side of (2.1) are mutually linearly independent. This is shown by the following lemma.

Lemma ([13, Lemma2.1]). *Let k be an integer. For $j=4, 6, 10$ and 12, let $f_j\in M_{k-j}(\Gamma_2)$. If*

$$f_4\theta^{+}\varphi_4+f_6\theta^{+}\varphi_6+f_{10}\theta^{+}\chi_{10}+f_{12}\theta^{+}\chi_{12}=0,$$

then we have

$$f_4 = f_6 = f_{10} = f_{12} = 0.$$

The dimension formula of $S_{k,r}(\Gamma_2)$ for $r=0$ and $k \geq 4$ or $r \geq 1$ and $k \geq 5$ is obtained by Tsushima [16, 17]. For $k \leq 6$, we have dim $S_{k,2}(\Gamma_2)=0$. This is proved by a method similar to Maass [8, pp. 189–196]. Comparison of dimension shows equality. For details, see [13, §§ 1–2]. Corollary follows from Theorem 1 and Igusa [5].

Our method is so explicit that we can obtain some interesting numerical values. For $f \in S_{k+r}(\Gamma_1)$, we denote by $[f]_r \in M_{k,r}(\Gamma_2)$ the Klingen type Eisenstein series attached to f. (Precise definition is given in (3.2) below.) As an example, let $f=\Delta_{16} \in S_{16}(\Gamma_1)$ be the normalized eigenform. By Theorem 1, we have

$$M_{14,2}(\Gamma_2) = \boldsymbol{C}[\Delta_{16}]_2 \oplus \boldsymbol{C}[\chi_{10}, \varphi_4].$$

Using Resnikoff and Saldaña [12, Tables III–V], we have

$$\frac{1}{144}[\varphi_6, \varphi_4^2] = [\Delta_{16}]_2 - \frac{403200}{373}[\chi_{10}, \varphi_4].$$

On the other hand, let $L_2(s, \Delta_{16})$ be the second L-function of Δ_{16}. By the method of Zagier [18], we obtain

$$\frac{L_2(28, \Delta_{16})(2\pi)^{-41}\Gamma(28)}{\langle \Delta_{16}, \Delta_{16} \rangle_{16}} = \frac{2^{10} \cdot 373}{3 \cdot 5^2 \cdot 7^2 \cdot 11}.$$

Here we note $28 = 2(k+r)-2-r$ with $k=14$ and $r=2$. Let $f \in S_{k+r}(\Gamma_1)$ be a normalized eigenform. The above fact suggests that $L_2(2(k+r)-2-r, f)$ appears in the denominator of Fourier coefficients of $[f]_r$. We note that the case $r=0$ is proved in Mizumoto [10]. (Cf. Kurokawa [6].) In the next section, we show a modified statement: the special value $L_2(2(k+r)-2-r, f)$ appears in the denominator of the pullback of $[f]_r$.

§ 3. Triple L-functions

We study relation between vector valued Klingen type Eisenstein series of degree two and triple L-functions of different weight. Properties of triple L-functions of the same weight are investigated by Garrett [3], where scalar valued degree three Siegel's Eisenstein series are used. For a non-negative integer r, let $V^{(r)} \cong \boldsymbol{C}^{r+1}$ be the representation space of $\rho = \det^k \otimes \mathrm{Sym}^r \mathrm{St}$. We realize a representation ρ on $V^{(r)}$ as

$$\rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix}_r = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}^k \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right)_r \tag{3.1}$$

where $\begin{pmatrix} x \\ y \end{pmatrix}_r = {}^t(x^r, x^{r-1}y, \cdots, y^r)$. For an integer q with $0 \leq q \leq r$, we put $v_q^{(r)} = {}^t(0, \cdots, 0, 1, 0, \cdots, 0)$ where 1 lies in the $(q+1)$-th column. This is compatible with the definition of v_0 in Arakawa [1, (0.2)]. We omit superscript (r) if there is no ambiguity. By (1.1) and (3.1) we identify S_2 with $V^{(2)}$ as a representation space of $\mathrm{Sym}^2\mathrm{St}$. Especially, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in S_2$ corresponds to $v_0^{(2)}$. We use $\{v_q \mid 0 \leq q \leq r\}$ as a base of $V^{(r)}$. Set $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = a$. For an integer t with $0 \leq t \leq n$, we define $\Gamma_{n,t}$ as the subgroup of Γ_n consisting of the elements whose entries in the first $n+t$ columns and the last $n-t$ rows are zero. For $\rho = \det^k \otimes \mathrm{Sym}^r \mathrm{St}$ and $f \in S_{k+r}(\Gamma_1)$ and $s \in \boldsymbol{C}$, we define the (vector valued non-holomorphic) Klingen type Eisenstein series attached to f by

$$(3.2) \quad [f]_r(Z, s) = \sum_{M \in \Gamma_{2,1} \backslash \Gamma_2} \left(\frac{\det(\mathrm{Im}\, M\langle Z \rangle)}{\mathrm{Im}\, M\langle Z \rangle^*} \right)^s f(M\langle Z \rangle^*) \rho(J(M, Z)^{-1}) v_0.$$

By a method similar to Arakawa [1, Proposition 1.2], we see that $[f]_r(Z, s)$ converges absolutely if $\mathrm{Re}(k+2s) > 4$, and it belongs to $M_\rho^\infty(\Gamma_2)$. For Siegel modular forms $f_1, \cdots, f_m$ (of various degree) and a field K, we denote by $K(f_1, \cdots, f_m)$ the field generated by all the Fourier coefficients of $f_1, \cdots, f_m$ over K. For simplicity, we put $[f]_r(Z) = [f]_r(Z, 0)$ and $\boldsymbol{Q}([f]_r) = \boldsymbol{Q}([f]_r(Z, 0))$. We note that under our normalization of v_0,

$$(3.3) \qquad \boldsymbol{Q}(f) \subset \boldsymbol{Q}([f]_r)$$

by Arakawa [1, Proposition 1.2]. For even integers $r \geq 0$ and k and $f \in S_{k+r}(\Gamma_1)$, we have

$$(3.4) \qquad \nabla^- [f]_r(Z, s) = \frac{s}{4\pi} [f]_{r+2}(Z, s+1).$$

for $\mathrm{Re}(k+2s) > 4$ ([14, Proposition 1.1]). Here the right hand side of (3.4) means that the second and the third components in the decomposition (1.3) are zero. By (3.4), the functional equations of scalar valued Klingen type Eisenstein series, which is obtained by Böcherer [2, (16) and (22)], are generalized for vector valued ones as follows.

Proposition 3. *Let k and r be even integers and let $f \in S_{k+r}(\Gamma_1)$. Put*

$$K(s, f) = \pi^{-3s} 2^{-2s} \frac{\Gamma_3\left(s + \dfrac{k+r}{2}\right) \Gamma(s) \Gamma(2s-1)}{\Gamma_3(s) \left(s + \dfrac{k+r}{2} - 1\right)} L_2(2s-2+k+r, f)$$

and

$$\boldsymbol{E}_r(Z, s, f) = \prod_{j=1}^{r/2} (s-j) K(s+(k/2), f)[f]_r(Z, s)$$

where $\Gamma_3(s) = \prod_{j=0}^{2} \Gamma(s-(j/2))$ *and* $L_2(s, f)$ *is the second L-function of* f. *Then* $[f]_r(Z, s)$ *extends meromorphically to the whole s-plane and satisfies the functional equation*

$$\boldsymbol{E}_r(Z, s, f) = \boldsymbol{E}_r(Z, 2-k-s, f). \tag{3.5}$$

We write Fourier expansion of $f \in S_k(\Gamma_1)$ as $f(z) = \sum_{n=1}^{\infty} a(n, f) e^{2\pi i n z}$. Let I_n be the identity matrix of size n. If $f \in S_k(\Gamma_1)$ is a normalized Hecke eigenform and p is a prime, we define semi-simple $M_p(f) \in GL(2, \boldsymbol{C})$ (up to the conjugacy class) by $\det(I_2 - tM_p(f)) = 1 - a(p, f)t + p^{k-1}t^2$. For normalized eigen cuspforms f, g and h, define the 'triple L-function' $L(s; f, g, h)$ by

$$L(s; f, g, h) = \prod_{p:\mathrm{prime}} \det(I_8 - p^{-s} M_p(f) \otimes M_p(g) \otimes M_p(h))^{-1}.$$

Theorem 4. *Let* f, g *and* h *be normalized eigen cuspforms of degree one and of weight* $\nu(f)$, $\nu(g)$ *and* $\nu(h)$ *respectively satisfying* $\nu(f) \geq \nu(g) \geq \nu(h)$. *Put*

$$\begin{aligned}&\tilde{L}(s; f, g, h)\\&\quad = \Gamma_{\boldsymbol{C}}(s)\Gamma_{\boldsymbol{C}}(s-\nu(f)+1)\Gamma_{\boldsymbol{C}}(s-\nu(g)+1)\Gamma_{\boldsymbol{C}}(s-\nu(h)+1)L(s; f, g, h)\end{aligned}$$

where $\Gamma_{\boldsymbol{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. *Then* $\tilde{L}(s; f, g, h)$ *extends meromorphically to the whole s-plane and satisfies the functional equation*

$$\tilde{L}(s; f, g, h) = -\tilde{L}(\nu(f)+\nu(g)+\nu(h)-2-s; f, g, h). \tag{3.6}$$

Moreover, if $\nu(g)+\nu(h)-\nu(f)>0$ *and* $L(s; f, g, h)$ *is holomorphic at* $s = \dfrac{\nu(f)+\nu(g)+\nu(h)}{2} - 1$, *then*

$$L\left(\frac{\nu(f)+\nu(g)+\nu(h)}{2} - 1; f, g, h\right) = 0, \tag{3.7}$$

and if $\nu(g)+\nu(h)-\nu(f)>4$, *then*

$$\frac{\pi^{5+\nu(f)-3\nu(g)-3\nu(h)} L(\nu(g)+\nu(h)-2; f, g, h)}{\langle f, f\rangle_{\nu(f)} \langle g, g\rangle_{\nu(g)} \langle h, h\rangle_{\nu(h)}} \in \boldsymbol{Q}([f]_{2\nu(f)-\nu(g)-\nu(h)}, g, h). \tag{3.8}$$

Corollary 5. *Let* $f \in S_k(\Gamma_1)$ *be a normalized Hecke eigenform. Let*

$$L_3(s,f)=\prod_{p:\mathrm{prime}} \det(I_4-p^{-s}\,\mathrm{Sym}^3 M_p(f))^{-1}$$

be the third L-function of f. Put

$$\tilde{L}_3(s,f)=\Gamma_C(s)\Gamma_C(s-k+1)L_3(s,f).$$

Then $\tilde{L}_3(s,f)$ satisfies the functional equation

$$\tilde{L}_3(s,f)=-\tilde{L}_3(3k-2-s,f).$$

In particular we have $L_3((3k/2)-1,f)=0$.

We sketch the proof. For details, see [14, § 2]. Put

$$\begin{aligned} k&=\nu(g)+\nu(h)-\nu(f),\\ q&=\nu(f)-\nu(g),\\ r&=2\nu(f)-\nu(g)-\nu(h). \end{aligned}$$

Then k, q and r are even integers satisfying $0\le q\le r$ and $k+r=\nu(f)\ge 12$. Let $F_q(z,w,s)$ be the component at v_q of $[f]_r\left(\begin{pmatrix} z & 0\\ 0 & w\end{pmatrix}, s\right)$. After some computation, we obtain

$$\begin{aligned} &\langle\langle F_q(z,w,s), g(z)\rangle_{k+r-q}, h(w)\rangle_{k+q}\\ (3.9)\quad &=2(4\pi)^{2-s-2k-r}\frac{\Gamma(2k+r-2+s)\Gamma(k+r-q-1+s)\Gamma(k+q-1+s)}{\Gamma(2k+r-2+2s)}\\ &\quad\times\frac{L(s+2k+r-2;f,g,h)}{L_2(2s+2k+r-2,f)}, \end{aligned}$$

where $L_2(s,f)$ is the second L-function of f. Hence by Proposition 3,

$$\begin{aligned} &\left\langle\left\langle K\left(s+\frac{k}{2}\right)F_q(z,w,s), g(z)\right\rangle_{k+r-q}, h(w)\right\rangle_{k+q}\\ &=\pi^{4-4s-r-(7k/2)}2^{9-4s-2k-(3r/2)}\left(s+\frac{k}{2}-1\right)\Gamma(s+k-1)\Gamma(s+k+r-q-1)\\ &\quad\times\Gamma(s+k+q-1)\Gamma(2k+r-2+s)L(s+2k+r-2;f,g,h) \end{aligned}$$

is invariant under $s\leftrightarrow 2-k-s$. This yields (3.6). Using properties of Eisenstein series, $L(s;f,g,h)$ is shown to be meromorphically extended to whole s-plane. Noting a minus sign in (3.6), we have (3.7) under the holomorphy of $L(s;f,g,h)$ at $s=(k/2)+r-1$. By (3.9) and the same argument as in Garrett [3, § 6], we have

$$\frac{\pi^{2-2k-r}L(2k+r-2;f,g,h)}{L_2(2k+r-2,f)\langle g,g\rangle_{k+q-r}\langle h,h\rangle_{k+q}} \in \boldsymbol{Q}([f]_r,g,h). \tag{3.10}$$

Now (3.8) follows from (3.3), (3.10) and

$$L_2(2k+r-2,f)\pi^{-(3k+r-3)}\langle f,f\rangle_{k+r}^{-1} \in \boldsymbol{Q}(f)$$

due to Zagier [18, Corollary to Theorem 2]. Corollary 5 is a direct consequence of

$$L(s;f,f,f)=L_3(s,f)L_1(s-k+1,f)^2$$

and Moreno and Shahidi [11, Lemma 2].

Remark 6. One can expect that $L(s;f,g,h)$ extends holomorphically to the whole s-plane. At least, this holds for $L(s;f,f,f)$ where $f \in S_k(\Gamma_1)$ with $k \leq 50$ due to Moreno and Shahidi [11, Remark 1].

Remark 7. In fact $\boldsymbol{Q}([f]_r)=\boldsymbol{Q}(f)$ for $r=0$ and $r=2$. For $r=0$, see e.g. Kurokawa [7, Theorem 3] or Mizumoto [10, Theorem 2]. For $r=2$, this is a consequence of Corollary 2.

Note added in proof. I. Piatetski-Shapiro and S. Rallis (Rankin triple L functions: Compositio Math., 64 (1987), 311–115) and T. Ikeda (private communication) obtained information on where poles of triple L functions locate and when they are holomorphic respectively. By virtue of these results, we see that triple L functions are holomorphic at $s=(\nu(f)+\nu(g)+\nu(h))/2-1$ in our full modular case. Hence, the holomorphy condition in Theorem 4 is unnecessary.

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