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Zeta Functions Associated to Cones and their Special Values

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Introduction

The purpose of this paper is to give a survey on zeta functions associated to (self-dual homogeneous) cones and their special values, including some recent results of ours on this subject.

In § 1 we summarize basic facts on self-dual homogeneous cones and the associated Γ-functions. § 2 is concerned with the zeta functions. Let V be a real vector space, $\mathscr{C}$ a self-dual homogeneous cone in V, and let G be the automorphism group $\mathrm{Aut}(V, \mathscr{C})^{\circ}$. We fix a $\boldsymbol{Q}$-simple $\boldsymbol{Q}$-structure on $(V, \mathscr{C})$. As is well-known, the pair (G, V) is a "prehomogeneous vector space" in the sense of Sato-Shintani [SS]. Following the general idea in [SS], we define a set of zeta functions $\{\xi_I\}$, each one of which is associated to a connected component V_I of $V^{\times}=V-S$, S denoting the singular set; in particular, $\xi_{(0)}=Z_{\mathscr{C}}$ is the zeta function associated to the cone $V_{(0)}=\mathscr{C}$. Then we give an explicit expression for the system of functional equations (Theorems 2.2.2, 2.3.3). Under the assumption that *d is even*, taking suitable linear combinations of these zeta functions, we define a new kind of L-functions L_I, which are shown to satisfy individually (or two in a pair, according to the cases) a functional equation of ordinary type (see (2.3.5)). We give some (new) results (Theorems 2.3.9, 2.4.1) on the residues and special values of these zeta and L-functions, where two extreme L-functions $L_{(0)}$ and $L_{(r_1)}$ play an essential role. These extreme L-functions, which generalize the (partial) Dedekind zeta function and the Shimizu L-function in the Hilbert modular case, seem to be of particular importance from the number-theoretic view point.

In § 3, we consider the corresponding (rational) symmetric tube domain $\mathscr{D}=V+\sqrt{-1}\mathscr{C}$ and, under an additional assumption that the ***Q*-rank *of G is one***, study the geometric invariants (χ_∞, τ_∞, etc.) associated to the cusp singularities appearing in the (standard) compactification of the arithmetic quotient space $\tilde{\Gamma}\backslash\mathscr{D}$ ([S3, 4]). A typical example is the Hilbert

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modular singularities, which were studied extensively by Hirzebruch and others ([H2], [HG]). In [H2] Hirzebruch gave a conjecture relating the "signature defects" of the cusps with the zero-values of the corresponding Shimizu L-functions, which was later proved by Atiyah-Donnelly-Singer and Müller ([ADS 1, 2], [M 4, 5]). In view of our results on these invariants and special values, we state in 3.3 some conjectures ((C1), (C2), (C3)) which may be regarded as a natural generalization of the Hirzebruch conjecture.

In § 4, we define a more general zeta function $Z_{\mathscr{C}}$ associated to a "Tsuchihashi singularity" and give a formula for the zero-value $Z_{\mathscr{C}}(0)$ (Theorem 4.2.5) by modifying a method due to Zagier [Z]. Recently, using this formula, Ishida [I3] proved the rationality of $Z_{\mathscr{C}}(0)$ in general. It is hoped that our approach might suggest a new possibility of attacking the generalized Hirzebruch conjecture.

Our study on this subject has been largely inspired by the fundamental works of Professor F. Hirzebruch, to whom this paper is respectfully dedicated. The paper was prepared during a stay at the MSRI, Berkeley in 1986–87, of the first-named author, who would like to thank the staffs of the Institute for superb service and hospitalities.

Notations. The symbols $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ are used in the usual sense, e.g. $\boldsymbol{Q}$ is the field of rational numbers. $\boldsymbol{H}$ is the Hamilton quaternion algebra. We use the symbol like $\boldsymbol{R}_{\geq 0}=\{\lambda \in \boldsymbol{R} \mid \lambda \geq 0\}$, and write $\boldsymbol{R}_+$ for $\boldsymbol{R}_{>0}$. For $\xi \in \boldsymbol{C}$, $\boldsymbol{e}(\xi)$ stands for $\exp(2\pi\sqrt{-1}\xi)$. Let V be a real vector space, $v_1, \cdots, v_r \in V$ and let S be a subset of $\boldsymbol{R}$. Then we write $\{v_1, \cdots, v_r\}_S$ for $\{\sum_{i=1}^r \lambda_i v_i \mid \lambda_i \in S\}$; e.g. $\{v_1, \cdots, v_r\}_{\boldsymbol{R}_{\geq 0}}$ is a closed polyhedral cone generated by $v_1, \cdots, v_r$. For a cone $\mathscr{C}$ and a lattice M in V, $\mathscr{C}^*$ and M^* denote, respectively, the dual cone and the dual lattice in the dual space V^*. For a topological group G, G° denotes the identity connected component of G. For a finite set S, $|S|$ denotes the cardinality of S.

Let F be a subfield of $\boldsymbol{R}$, $\mathscr{G}$ a (connected) algebraic group defined over F, and $G=\mathscr{G}(\boldsymbol{R})^\circ$. By an abuse of notations, we write G_F for $\mathscr{G}(F) \cap G$ and F-rk G for F-rk $\mathscr{G}$ (i.e. the dimension of maximal F-split tori in $\mathscr{G}$). If σ is an imbedding, $F \hookrightarrow \boldsymbol{R}$, then G^σ stands for $\mathscr{G}^\sigma(\boldsymbol{R})^\circ$ and, if F_0 is a subfield of F with $[F : F_0] < \infty$, then $R_{F/F_0}(G)$ stands for $R_{F/F_0}(\mathscr{G})(\boldsymbol{R})^\circ$. When $\mathscr{G}$ is reductive, $G=\mathscr{G}(\boldsymbol{R})^\circ$ is called "reductive", and we write G^s for $\mathscr{G}^s(\boldsymbol{R})^\circ$, $\mathscr{G}^s$ denoting the semisimple part of $\mathscr{G}$.

§ 1. Self-dual homogeneous cones ([BK], [S1], [V])

1.1. Let V be a real vector space of dimension $n>0$. By a *convex cone* in V, we mean a subset $\mathscr{C}$ of V with the following property:

$$x, y \in \mathscr{C},\ \lambda,\ \mu>0 \Longrightarrow \lambda x+\mu y \in \mathscr{C}.$$

The *dual* of $\mathscr{C}$ is defined by

$$\mathscr{C}^* = \{x^* \in V^* \mid \langle x, x^* \rangle > 0 \quad \text{for all } x \in \overline{\mathscr{C}} - \{0\}\}.$$

Then $\mathscr{C}^*$ is an open convex cone in the dual space V^*. It is clear that for a (non-empty) convex cone $\mathscr{C}$ the following three conditions are equivalent:

(i) $\mathscr{C}$ does not contain a line in V;

(ii) $\overline{\mathscr{C}} \cap (-\overline{\mathscr{C}}) = \{0\}$;

(iii) $\mathscr{C}^*$ is non-empty.

When these conditions are satisfied, $\mathscr{C}$ is called *non-degenerate*. In what follows, a non-degenerate open convex cone will simply be called a "cone". For a cone $\mathscr{C}$, one has $\mathscr{C}^{**} = \mathscr{C}$.

A cone $\mathscr{C}$ is called *self-dual* if there exists a linear isomorphism $S : (V, \mathscr{C}) \rightleftarrows (V^*, \mathscr{C}^*)$, which is symmetric and positive definite. A cone $\mathscr{C}$ is called *homogeneous* if the automorphism group

$$G = \mathrm{Aut}(V, \mathscr{C})^\circ = \{g \in GL(V) \mid g\mathscr{C} = \mathscr{C}\}^\circ.$$

($\circ$ denoting the identity connected component) is transitive on $\mathscr{C}$.

In §§ 1–3, unless otherwise specified, we always assume that $\mathscr{C}$ is self-dual and homogeneous, and fix a positive definite inner product $\langle \ \rangle$ on V defining an isomorphism S mentioned above. Then $(V, \mathscr{C})$ is identified with its dual $(V^*, \mathscr{C}^*)$. In this case, the automorphism group G is the identity connected component of a reductive algebraic group and for any $c_0 \in \mathscr{C}$ the isotropy subgroup

$$K = G_{c_0} = \{g \in G \mid gc_0 = c_0\}$$

is a maximal compact subgroup of G. Thus $\mathscr{C} \approx G/K$ has a structure of Riemannian symmetric space (with a flat part).

1.2. In 1957–58, M. Koecher made an observation that the category of self-dual homogeneous cones $(V, \mathscr{C})$ with a base point $c_0 \in \mathscr{C}$ is equivalent to that of "formally real" Jordan algebras by the correspondence given as follows ([BK], [S1]). Let $\mathscr{C}$ be a self-dual homogeneous cone in V with a base point c_0 and let G, K be as above. Let $\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{k} = \mathrm{Lie}\, K$ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition. Then by the homogeneity assumption there exists a unique linear isomorphism

$$V \ni x \longmapsto T_x \in \mathfrak{p}$$

such that $x = T_x c_0$. The Jordan product in V is then defined by

$$x \circ y = T_x y \quad (x, y \in V).$$

In particular, one has $T_{c0}=\mathrm{id}_V$, i.e. c_0 is the unit element of the Jordan algebra.

By virtue of this equivalence, the classification of self-dual homogeneous cones is reduced to that of formally real Jordan algebras, which was given (by a collaboration of phisicists) as early as in 1934 ([JNW]). A self-dual homogeneous cone $\mathscr{C}$ is decomposed uniquely into the direct product of the "irreducible" ones, for which one has $G=\boldsymbol{R}_+\times G^s$ with G^s $\boldsymbol{R}$-simple (or $=\{1\}$). The irreducible self-dual homogeneous cones are classified into the following five types:

$$\begin{cases}\mathscr{P}_1(\boldsymbol{R})=\boldsymbol{R}_+, \quad \mathscr{P}_r(\boldsymbol{F}) \quad (r\geq 2, \boldsymbol{F}=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}), \\ \mathscr{P}_3(\boldsymbol{O}) \quad (\boldsymbol{O} \text{ denotes the Cayley octonion algebra}), \\ \mathscr{P}(1, n-1)=\{(\xi_i)\in\boldsymbol{R}^n \mid \xi_1>0,\ \xi_1^2-\sum_{i=2}^n \xi_i^2>0\} \quad (n\geq 3),\end{cases}$$

where $\mathscr{P}_r(\boldsymbol{F})$ denotes the cone of positive definite hermitian matrices of size r with entries in $\boldsymbol{F}$. $\mathscr{P}_2(\boldsymbol{R})$, $\mathscr{P}_2(\boldsymbol{C})$, $\mathscr{P}_2(\boldsymbol{H})$ are isomorphic to the "quadratic cones" $\mathscr{P}(1, n-1)$ with $n=3, 4, 6$, respectively. For $\mathscr{C}=\mathscr{P}_3(\boldsymbol{O})$, G^s is an excetional group of type (E_6).

A more general study on "homogeneous cones" was done by Vinberg [V] in the early 60's. In the study of general cones, the characteristic function plays an essential role. For any (non-degenerate, open convex) cone $\mathscr{C}$, the *characteristic function* $\phi(x)=\phi_{\mathscr{C}}(x)$ is defined by

$$\phi_{\mathscr{C}}(x)=\int_{\mathscr{C}^*} e^{-\langle x,x^*\rangle}\mathrm{d}x^*.$$

Clearly one has

$$\phi_{\mathscr{C}}(x)>0, \quad \phi_{\mathscr{C}}(gx)=\det(g)^{-1}\phi_{\mathscr{C}}(x) \qquad \text{for } x\in\mathscr{C},\ g\in G,$$

and $\log\phi_c(x)$ is a convex function, which tends to infinity when $x\in\mathscr{C}$ converges to a boundary point of $\mathscr{C}$. The characteristic function will be used later in §4.

1.3. Quasi-irreducible cones. Let $\mathscr{C}$ be a self-dual homogeneous cone in V. $\mathscr{C}$ is called *quasi-irreducible* if in its irreducible decomposition all irreducible components are isomorphic.

Lemma 1.3.1. *Suppose $(V, \mathscr{C})$ has a $\boldsymbol{Q}$-simple $\boldsymbol{Q}$-structure; this means that there is a $\boldsymbol{Q}$-vector space $V_{\boldsymbol{Q}}$ such that $V=V_{\boldsymbol{Q}}\otimes_{\boldsymbol{Q}}\boldsymbol{R}$, for which G is (the identity connected component of) an algebraic group defined over $\boldsymbol{Q}$ and that, if $(V, \mathscr{C})=\prod_{\mu=1}^m (V_\mu, \mathscr{C}_\mu)$ is the irreducible decomposition, no partial product of V_μ's is defined over $\boldsymbol{Q}$, or equivalently, that the center of G is of $\boldsymbol{Q}$-rank one. Then $\mathscr{C}$ is quasi-irreducible.*

In fact, under this assumption, there exists a totally real number field F_1 of degree m such that

$$G = G_{F_1/Q}(G_1) = \prod_{\mu=1}^{m} G_\mu, \quad G_\mu = G_1^{\sigma_\mu},$$

where $G_1 = \boldsymbol{R}_+ \times G_1^s$ with G_1^s $\boldsymbol{R}$-simple (which may reduce to $\{1\}$), defined over F_1, and $\{\sigma_\mu\ (1 \leq \mu \leq m)\}$ is the totality of the imbeddings $F_1 \hookrightarrow \boldsymbol{R}$. Then the G_μ^s's are all $\boldsymbol{C}$-isomorphic and hence, by the classification theory, are also $\boldsymbol{R}$-isomorphic except for the case when there exists an even integer r_1 such that every G_μ^s is isogeneous either to $SL(r_1, \boldsymbol{R})$ or to $SL(r_1/2, \boldsymbol{H})$ and when both types $SL(r_1, \boldsymbol{R})$ and $SL(r_1/2, \boldsymbol{H})$ occur in the G_μ^s's. But actually such a "mixed type" can not occur for the following reason. Since the $\boldsymbol{Q}$-rational points are dense in $\mathscr{C}$, one may take c_0 to be $\boldsymbol{Q}$-rational; then the maximal compact subgroup K is also defined over $\boldsymbol{Q}$. One then has the corresponding decomposition

$$K = R_{F_1/Q}(K_1) = \prod_{\mu=1}^{m} K_\mu, \quad K_\mu = K_1^{\sigma_\mu}$$

and hence all K_μ's are also $\boldsymbol{C}$-isomorphic. But the dimension of the maximal compact subgroups of $SL(r_1, \boldsymbol{R})$ and $SL(r_1/2, \boldsymbol{H})$ is equal to $\frac{1}{2}r_1(r_1-1)$, $\frac{1}{2}r_1(r_1+1)$, respectively. Therefore no mixture of these two types can occur, which proves our assertion.

1.4. The norm and trace. The rank of a self-dual homogeneous cone $\mathscr{C}$ is by definition the $\boldsymbol{R}$-rank of the Lie algebra $\mathfrak{g}$, which also coincides with the (absolute) rank of the formally real Jordan algebra (V, c_0). Let $n = \dim V$ and $r = \operatorname{rank} \mathscr{C}$. If $\mathscr{C}$ is irreducible, one has (from the Peirce decomposition of (V, c_0))

$$n = r + \frac{d}{2} r(r-1), \tag{1.4.1}$$

where d is a non-negative integer. For $\mathscr{C} = \boldsymbol{R}_+$, one puts $d=0$. For $\mathscr{C} = \mathscr{P}_r(\boldsymbol{F})$ $(r \geq 2)$, one has actually rank $\mathscr{C} = r$ and $d = \dim_{\boldsymbol{R}} \boldsymbol{F} = 1, 2, 4, 8$ according as $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}, \boldsymbol{O}$. For a quadratic cone $\mathscr{C}$, one has rank $\mathscr{C} = 2$ and $d = n-2$. Thus the pair (r, d) is a complete invariant for an isomorphism class of irreducible self-dual homogeneous cones.

In the Jordan algebra (V, c_0), one can define the (reduced) *norm* $N : V \to \boldsymbol{R}$ as the (unique) homogeneous polynomial function of degree r on V such that, for a "general element" x in V, $N(tc_0 - x)\,(\in \boldsymbol{R}[t])$ is the minimal polynomial for x in the usual sense. When $\mathscr{C}$ is irreducible, the norm is uniquely characterized by the property

(1.4.2) $$N(c_0)=1, \quad N(gx)=\det(g)^{r/n}N(x) \quad \text{for } g \in G,\ x \in V.$$

(Note that $\chi(g)=\det(g)^{r/n}$ is a rational character on G.) Hence one has the relation

(1.4.3) $$N(x)=(\phi_{\mathscr{C}}(c_0)^{-1}\phi_{\mathscr{C}}(x))^{-r/n} \quad (x \in \mathscr{C}).$$

The (reduced) trace $\mathrm{tr}(x)$ is defined by

$$N(tc_0-x)=t^r-\mathrm{tr}(x)t^{r-1}+\cdots+(-1)^r N(x).$$

The trace is K-invariant. It follows that, when $\mathscr{C}$ is irreducible, one has

(1.4.4) $$\mathrm{tr}\, x=\frac{r}{n}\mathrm{tr}(T_x),$$

where $T_x : y \mapsto xy$ is the multiplication in the Jordan algebra (V, c_0). (Note also, putting $P(x)=2T_x^2-T_{x^2}$, one has the relations $P(gx)=gP(x)^t g$, $\det(P(x))=N(x)^{2n/r}$.)

In what follows, we assume that $\mathscr{C}$ is quasi-irreducible. Let $(V, \mathscr{C}) = \prod_{\mu=1}^{m}(V_\mu, \mathscr{C}_\mu)$ be the irreducible decomposition,

$$G=\prod_{\mu=1}^{m} G_\mu, \quad G_\mu=\mathrm{Aut}(V_\mu, \mathscr{C}_\mu)^\circ,$$

and put $n_1=\dim V_1$, $r_1=\mathrm{rank}\,\mathscr{C}_1=\boldsymbol{R}\text{-rk}\, G_1$. Then one has $n=mn_1$, $r=mr_1$ and

$$\frac{n}{r}=\frac{n_1}{r_1}=1+\frac{d}{2}(r_1-1).$$

Hence the formulae (1.4.2–4) remain valid. We normalize the inner product on V in such a way that $\langle c_{0,\mu}, c_{0,\mu}\rangle=r_1$ $(1\le\mu\le m)$, where $c_0=(c_{0,\mu})$. Then one has

(1.4.5) $$\langle x, y\rangle=\mathrm{tr}(xy).$$

The Euclidean (i.e. self-dual) measure on V for this $\langle\ \rangle$ will be denoted as $\mathrm{d}x$. A G-invariant measure on $\mathscr{C}$ is then given by $N(x)^{-n/r}\mathrm{d}x$.

1.5. The Γ-function. Let $\mathscr{C}$ be a quasi-irreducible self-dual homogeneous cone in V. The "Γ-function" of $\mathscr{C}$ (introduced by Koecher) is defined by the integral

(1.5.1) $$\Gamma_{\mathscr{C}}(s)=\int_{\mathscr{C}} e^{-\mathrm{tr}(x)}N(x)^{s-(n/r)}\mathrm{d}x \quad (s \in \boldsymbol{C}),$$

which converges absolutely for $\operatorname{Re} s > n/r - 1$. By a change of variable, one gets

$$N(x)^{-s}\Gamma_{\mathscr{C}}(s)=\int_{\mathscr{C}} e^{-\langle x,y\rangle}N(y)^{s-n/r}dy \quad (x \in \mathscr{C}). \tag{1.5.2}$$

On V one can define a (unique) differential operator of degree r, denoted as $N(\nabla_x)$, with the property

$$N(\nabla_x)e^{\langle x,y\rangle}=N(y)e^{\langle x,y\rangle}$$

(cf. [R], [SS]). Then $N(\nabla_x)$ is relatively invariant in the sense that one has

$$L_g^{-1}N(\nabla_x)L_g=\det(g)^{-r/n}N(\nabla_x) \quad (g \in G,\ x \in V),$$

where $(L_g f)(x)=f(g^{-1}x)$ for any function f on V. The associated "b-function" is defined by

$$N(\nabla_x)N(x)^s=b(s)N(x)^{s-1} \quad (x \in \mathscr{C},\ s \in \boldsymbol{C})$$

(cf. [SS]).*⁾ Then, applying $N(\nabla_x)$ on the both sides of (1.5.2), one gets

$$b(s)=(-1)^r\frac{\Gamma_{\mathscr{C}}(1-s)}{\Gamma_{\mathscr{C}}(-s)}.$$

By a direct computation from (1.5.1) (see e.g. [S2]), one obtains

$$\Gamma_{\mathscr{C}}(s)=(2\pi)^{(n-r)/2}\left(\prod_{i=1}^{r_1}\Gamma\left(s-\frac{d}{2}(i-1)\right)\right)^m, \tag{1.5.3}$$

$$b(s)=\left(\prod_{i=1}^{r_1}\left(s+\frac{d}{2}(i-1)\right)\right)^m. \tag{1.5.4}$$

(For the Γ-function of a more general cone, see [G].)

§ 2. Zeta functions associated to a self-dual homogeneous cone

2.1. We assume in this section that $(V, \mathscr{C})$ is endowed with a $\boldsymbol{Q}$-simple $\boldsymbol{Q}$-structure in the sense stated in Lemma 1.3.1. Then G^s is $\boldsymbol{Q}$-simple (or reduces to $\{1\}$), the center of G is of $\boldsymbol{Q}$-rank one and $\mathscr{C}$ is quasi-irreducible. The $\boldsymbol{Q}$-rank of G, which we denote by r_0, is a divisor of $r_1=\boldsymbol{R}\text{-rk}\ G_1$: hence we set $\delta=r_1/r_0$. We fix a base point c_0 in $\mathscr{C}\cap V_{\boldsymbol{Q}}$; then the norm, trace and the (normalized) inner product $\langle\ \rangle$ are all defined over $\boldsymbol{Q}$. We choose a lattice M in $V_{\boldsymbol{Q}}$ and an arithmetic subgroup Γ of G such

*⁾ Note that in some recent literature (e.g. [I1]) it has become more customary to denote our $b(s)$ by $b(s-1)$.

that $\Gamma M=M$. We define a zeta function by

(2.1.1) $$Z_{\mathscr{C},c_0}(\Gamma, M; s)=\sum_{x:\Gamma\backslash\mathscr{C}\cap M}|\Gamma_x|^{-1}N(x)^{-s} \quad (s\in \boldsymbol{C}),$$

where $\Gamma_x=\{\gamma\in\Gamma\,|\,\gamma x\in x\}$ (which is finite) and the summation is taken over a complete set of representatives of the Γ-orbits in $\mathscr{C}\cap M$. When c_0 is kept fixed, we write $Z_{\mathscr{C}}$ for $Z_{\mathscr{C},c_0}$. It is known that the series on the right hand side of (2.1.1) is absolutely convergent for $\operatorname{Re}s>n/r$ and has an analytic continuation to a meromorphic function on the whole plane $\boldsymbol{C}$. It is clear that, if Γ' is a subgroup of Γ of finite index, then one has

$$Z_{\mathscr{C}}(\Gamma', M; s)=[\Gamma:\Gamma']Z_{\mathscr{C}}(\Gamma, M; s).$$

Hence it suffices to consider the zeta function for the full stabilizer $\Gamma_M=\{\gamma\in G\,|\,\gamma M=M\}$. In that case, we write $Z_{\mathscr{C}}(M; s)$ for $Z_{\mathscr{C}}(\Gamma_M, M; s)$.

In the simplest case where G^s reduces to $\{1\}$, one obtains essentially the (partial) Dedekind zeta function of the totally real number field F_1 (see the Example 2.1.2 below). The case where $\mathscr{C}$ is a quadratic cone was studied by Siegel [S9]. Our zeta function is a special case of the zeta function associated to a (real) "prehomogeneous vector space" in the sense of Sato-Shintani [SS], who treated as examples the cases of $\mathscr{P}_r(\boldsymbol{R})$, $\mathscr{P}_r(\boldsymbol{C})$ and the quadratic cones (see [S7], [SS], pp. 160–168, pp. 155–157). For other cases, see [M3] (cf. also [SF]).

Example 2.1.2. Let F_1 be a totally real number field of degree m and let $V=F_1\otimes_{\boldsymbol{Q}}\boldsymbol{R}\cong\boldsymbol{R}^m$. Then the "angular domain" $\mathscr{C}=\boldsymbol{R}_+^m$ is a self-dual homogeneous cone with respect to the standard inner product in $\boldsymbol{R}^m$. G is identified with the multiplicative group $\boldsymbol{R}_+^m$ and G^s and K reduce to the identity. If one takes c_0 to be 1 (the unit element of F_1), then the norm and the trace are given by

$$N(x)=\prod\xi_\mu,\quad \operatorname{tr}(x)=\sum\xi_\mu \qquad \text{for } x=(\xi_\mu)\in V,$$

and the standard inner product in $\boldsymbol{R}^m$ is normalized. V has a natural $\boldsymbol{Q}$-structure for which $V_{\boldsymbol{Q}}=F_1$, G is defined over $\boldsymbol{Q}$, of $\boldsymbol{Q}$-rank 1, and $G_{\boldsymbol{Q}}=\{\alpha\in F_1^\times\,|\,\alpha^{\sigma_\mu}>0\,(1\le\mu\le m)\}$. Hence the above assumptions are all satisfied and one has $n_1=r_1=r_0=1$, $n=r=m$. Let $\mathcal{O}_{F_1}$ be the ring of integers in F_1 and choose M to be an ideal $\mathfrak{a}_1$ in $\mathcal{O}_{F_1}$. Then Γ_M is the group of totally positive units of $\mathcal{O}_{F_1}$ and one has

$$Z_{\mathscr{C}}(M; s)=\sum_{x:\Gamma_M\backslash\mathscr{C}\cap M}N(x)^{-s}=N(\mathfrak{a}_1)^{-s}\sum_{\mathfrak{a}\sim\mathfrak{a}_1^{-1}}N(\mathfrak{a})^{-s},$$

where the summation in the last expression is taken over all integral ideals $\mathfrak{a}$ "equivalent" to $\mathfrak{a}_1^{-1}$ in the narrow sense. Thus essentially $Z_{\mathscr{C}}(M; s)$ is nothing but a "parital" Dedekind zeta function of F_1 corresponding to the "ray class' of $\mathfrak{a}_1^{-1}$.

2.2. Functional equations. According to the general theory of Sato-Shintani, the functional equations for $Z_{\mathscr{C}}(M;s)$ are obtained as follows. Let

$$V^{\times}=\{x \in V \mid N(x)\neq 0\}=\prod_{\mu=1}^{m} V_{\mu}^{\times}$$

and let

$$V_{\mu}^{\times}=\coprod_{i=0}^{r_1} V_{\mu,i}$$

be the decomposition of $V_{\mu}^{\times}$ into the disjoint union of the connected components, or what amounts to the same thing, into that of the G_{μ}-orbits. (If $c_0=(c_{0,\mu})$ and if $c_{0,\mu}=\sum_{i=1}^{r_1} e_i^{(\mu)}$ is primitive decomposition, then $V_{\mu,i}$ is defined to be the G_{μ}-orbit of $-\sum_{k=1}^{i} e_k^{(\mu)}+\sum_{k=i+1}^{r_1} e_k^{(\mu)}$.) Thus one has

$$V^{\times}=\coprod_{I\in\mathscr{I}_{r_1}^m} V_I,$$

where $\mathscr{I}_{r_1}^m$ denotes the set of all m-tuples $I=(i_1, \cdots, i_m)$ with $0\leq i_{\mu}\leq r_1$ $(1\leq\mu\leq m)$ and for $I=(i_{\mu})$, one sets $V_I=\prod_{\mu=1}^{m} V_{\mu,i_{\mu}}$. Hence $V^{\times}$ consists of $(r_1+1)^m$ connected components. We write (k) for $(k, \cdots, k) \in \mathscr{I}_{r_1}^m$; then $V_{(0)}=\mathscr{C}$ and $V_{(r_1)}=-\mathscr{C}$.

For each $I \in \mathscr{I}_{r_1}^m$, we define a zeta function

$$\xi_I(M;s)=\sum_{x:\Gamma_M\backslash V_I\cap M}\frac{\mu(x)}{|N(x)|^s}, \tag{2.2.1}$$

where the summation is taken over a complete set of representatives of Γ_M-orbits in $V_I\cap M$ and $\mu(x)$ is a "density" defined as follows. For $x \in V_I$, let U_x be a relatively compact neighbourhood of x in V_I and let

$$W_x=\{g \in G \mid gx \in U_x\},$$
$$G_x=\{g \in G \mid gx=x\}, \quad \Gamma_x=G_x\cap\Gamma.$$

Then one has

$$\mu(x)=\int_{\Gamma_x\backslash W_x} dg \Big/ \int_{U_x} |N(x)|^{-n/r} dx,$$

where dg is a Haar measure on G normalized in such a way that for any non-negative continuous function f on $\mathscr{C}$ one has

$$\int_G f(gc_0)\,dg = \int_{\mathscr{C}} f(x) N(x)^{-n/r}\,dx.$$

Then, except for the case $r_1 = r_0 = 2$, $d = 1$ (treated in [S7], [S9]), $\mu(x)$ is finite and coincides with the volume of $\Gamma_x \backslash G_x$ with respect to a suitably normalized Haar measure on G_x ([SS], Lemma 2.4) and hence depends only on the Γ-equivalence class of x. In what follows, we omit the above-mentioned exceptional case. Then the series (2.2.1) is absolutely convergent for $\mathrm{Re}\, s > n/r$ and has an analytic continuation to a meromorphic function on $\boldsymbol{C}$ ([SS], [S7]). Clearly one has $\xi_{(0)} = \xi_{(r_1)} = Z_{\mathscr{C}}$. For $I = (i_\mu) \in \mathscr{I}^m_{r_1}$, we set $I^* = (r_1 - i_\mu)$. Then it is clear that

$$V_{I^*} = -V_I \quad \text{and} \quad \xi_{I^*} = \xi_I.$$

Thus essentially we get $[(r_1+1)^m/2]$ zeta functions.

Theorem 2.2.2. *The functions $\xi_I(M; s)$ satisfy the functional equations of the following form:*

$$\xi_J\left(M^*; \frac{n}{r} - s\right) = v(M)(2\pi)^{-rs} \Gamma_{\mathscr{C}}(s)\, \boldsymbol{e}\left(\frac{rs}{4}\right) \sum_{I \in \mathscr{I}^m_{r_1}} \xi_I(M; s) u_{IJ}(s),$$

where M^* is the dual lattice of M, $v(M) = \mathrm{vol}(V/M)$ and, for $I = (i_\mu)$, $J = (j_\mu)$, $u_{IJ}(s) = \prod_{\mu=1}^m u_{i_\mu, j_\mu}(s)$, u_{ij} $(0 \le i, j \le r_1)$ being integral polynomials in $\boldsymbol{e}(-s/2)$ of degree $\le r_1$.

For an explicit expression of u_{ij}, see [SF]*).

2.3. To obtain more precise results, we assume in the rest of this section that $r \ge 2$ *and d is even.* (Note that, if $r = 1$, $Z_{\mathscr{C}}$ is essentially the Riemann zeta function. If d is odd, then one has either $r_1 = 2$ (quadratic cones) or $d = 1$ ($\mathscr{P}_{r_1}(\boldsymbol{R})^m (r_1 \ge 2)$).)

Under this assumption, n/r is an integer and there are two cases:

(a) $d \equiv 0 \pmod 4$, or $d \equiv 2 \pmod 4$ and r_1 is odd. In this case, n/r is odd.

(a′) $d \equiv 2 \pmod 4$ and r_1 is even. In this case, n/r is even.

Applying the methods in [SS] and [SF], one obtains

Theorem 2.3.1. *Under the above assumption, the function $\xi_I(M; s)$ has at most r_0 simple poles at $s = n/r - (d/2)\rho$ for $0 \le \rho \le r_1 - 1$, $\delta \mid \rho$ $(\delta = r_1/r_0)$ and one has*

*) Note that $u_{ij}(s)$ in [SS] is in our notation (and in [SF]) given by $c(2\pi)^{(n-r)/2} u_{r-i,r-j}(s)$, if the measure on V in [SS] is equal to $c\,dx$ with dx self-dual. For instance, in the case $V = \mathrm{Her}_r(\boldsymbol{C})$ ([SS], pp. 160–168), one has $c = 2^{-(n-r)/2}$.

$$\operatorname{Res}_{s=n/r-(d/2)\rho}\xi_I(M;s)=v(M^*)\left((2\pi)^{-(d/2)(\rho(\rho+1)/2)}\prod_{k=1}^{\rho}\Gamma\left(\frac{d}{2}k\right)\right)^m$$

$$\times(-1)^{(d/2)\rho|I|}\sqrt{-1}^{(d/2)\rho r}\sum_{J\in\mathcal{I}_\rho^m}(-1)^{(d/2)(r_1-\rho-1)|J|}\binom{\rho}{J}\kappa_J^{(\rho)}(M^*),$$

where M^* is the dual lattice of M and, for $J=(j_\mu)\in\mathcal{I}_\rho^m$, we set

$$|J|=\sum_{\mu=1}^m j_\mu,\quad \binom{\rho}{J}=\prod_{\mu=1}^m\binom{\rho}{j_\mu},$$

$$\kappa_J^{(\rho)}(M^*)=\sum_{x:{}^t\Gamma\backslash S_J^{(\rho)}\cap M^*}\int_{G_x/{}^t\Gamma_x}\mathrm{d}\nu_x.$$

$S_J^{(\rho)}$ is the G-orbit in $S=V-V^\times$ containing $\sum_{\mu=1}^m(-\sum_{k=1}^{j_\mu}e_k^{(\mu)}+\sum_{k=j_\mu+1}^{\rho}e_k^{(\mu)})$ and, for $x\in S_J^{(\rho)}$, $\mathrm{d}\nu_x$ is a suitably normalized Haar measure on G_x.

In the special cases, this result is due to [S9], [SS], [S7] and [M3]. It is possible to give a unified proof along the line of [SF] (see [S5a]).

Corollary 2.3.2. *Put $\nu=n/r-(d/2)\rho$ with $0\leq\rho\leq r_1-1$, $\delta\,|\,\rho$. For a fixed ν, the $(r_1+1)^m$-tuple $(\operatorname{Res}_{s=\nu}\xi_I(M;s))_{I\in\mathcal{I}_{r_1}^m}$ is proportional to $((-1)^{(d/2)\rho|I|})_{I\in\mathcal{I}_{r_1}^m}$. For $\nu\equiv1$ (mod 2), one has*

$$\operatorname{Res}_{s=\nu}\xi_I(M;s)\neq0.$$

Also, for $\nu=n/r$, one has

$$\operatorname{Res}_{s=n/r}\xi_I(M;s)=v(M^*)\int_{\Gamma\backslash\mathscr{C}^1}\mathrm{d}x^1>0,$$

where $\mathscr{C}^1=\{x\in\mathscr{C}\,|\,N(x)=1\}$ and $\mathrm{d}x^1$ is the invariant measure on $\mathscr{C}^1$ induced from $N(x)^{-n/r}\mathrm{d}x$.

Let P_{r_1} denote the symmetric tensor representation of $GL(2,\boldsymbol{C})$ defined by

$$(1,y,\cdots,y^{r_1})\mathrm{P}_{r_1}\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\right)=((a+cy)^{r_1},(a+cy)^{r_1-1}(b+dy),$$
$$\cdots,(b+dy)^{r_1}).$$

For $C=(c_{ij})_{0\leq i,j\leq r_1}\in GL(r_1+1,\boldsymbol{C})$, we write

$$C^{\otimes m}=(c_{IJ})_{I,J\in\mathcal{I}_{r_1}^m}\in GL((r_1+1)^m,\boldsymbol{C}),\quad c_{IJ}=\prod_{\mu=1}^m c_{i_\mu,j_\mu}.$$

In these notations, we have

Theorem 2.3.3. *The polynomials $u_{IJ}(s)$ in Theorem 2.2.2 are given as follows:*

$$(u_{IJ}(s))_{I,J\in\mathcal{I}^m_{r_1}}=\begin{cases}\mathrm{P}_{r_1}\left(\begin{pmatrix}1 & x\\ x & 1\end{pmatrix}\right)^{\otimes m} & \text{in Case (a),}\\ \mathrm{P}_{r_1}\left(\begin{pmatrix}1 & x\\ -x & -1\end{pmatrix}\right)^{\otimes m} & \text{in Case (a'),}\end{cases}$$

where $x=\boldsymbol{e}(-s/2)$.

This follows from [SF], Theorem 2. We put

$$U^{(r_1,m)}(x)=(u_{IJ}(s)),\quad A^{(r_1,m)}=(a_{IJ})=\mathrm{P}_{r_1}\left(\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\right)^{\otimes m}.$$

Then the matrix $U^{(r_1,m)}(x)$ can be diagonalized as follows:

$$A^{(r_1,m)-1}U^{(r_1,m)}(x)A^{(r_1,m)}=\begin{cases}((1+x)^{r-|I|}(1-x)^{|I|}\delta_{IJ}) & \text{in Case (a),}\\ ((1+x)^{r-|I^*|}(1-x)^{|I^*|}\delta_{I^*,J}) & \text{in Case (a').}\end{cases}$$

Hence, putting

$$L_J(M;s)=\sum_{I\in\mathcal{I}^m_{r_1}}\xi_I(M;s)a_{IJ}, \tag{2.3.4}$$

one has

$$\begin{aligned}&L_I\left(M^*;\frac{n}{r}-s\right)=v(M)(2\pi)^{-rs}\Gamma_{\mathscr{C}}(s)\\ &\times\left(\boldsymbol{e}\left(\frac{s}{4}\right)+\boldsymbol{e}\left(-\frac{s}{4}\right)\right)^{r-|I|}\left(\boldsymbol{e}\left(\frac{s}{4}\right)-\boldsymbol{e}\left(-\frac{s}{4}\right)\right)^{|I|}\begin{cases}L_I(M;s),\\ L_{I^*}(M;s),\end{cases}\end{aligned} \tag{2.3.5}$$

according to the case (a) and (a').*)

It is easy to see that the matrix $A^{(r_1,m)}=(a_{IJ})$ has the following properties:

$$\begin{cases}a_{(0),J}=1, & a_{I,(0)}=\binom{r_1}{I},\\ a_{I^*,J}=(-1)^{|J|}a_{IJ}, & a_{I,J^*}=(-1)^{|I|}a_{IJ},\\ \sum_{I\in\mathcal{I}^m_{r_1}}a_{IJ}=\delta_{(0),J}2^r, & \sum_{I\in\mathcal{I}^m_{r_1}}(-1)^{|I|}a_{IJ}=\delta_{(r_1),J}2^r.\end{cases} \tag{2.3.6}$$

---

*) This corresponds to the formula (30) in [SF], which was printed erroneously. It should read

$$\begin{aligned}&\Phi_i'(\hat{f},s-n/r)\ (\text{resp. } \Phi'_{r-i}(\hat{f},s-n/r))\\ &\quad=(2\pi)^{-rs}\Gamma_{\Omega_0}(s)(e(s/4)+e(-s/4))^{i-r}(e(s/4)-e(-s/4))^{i}\Phi_i'(f,-s)\end{aligned}$$

in Case (a) (resp. (a')).

We use the last relations in the following form.

Lemma 2.3.7. *For* $(x_I) \in C^{(r_1+1)^m}$ *and* $b \in C$, *one has*

$$\sum_{I \in \mathscr{I}^m_{r_1}} x_I a_{IJ} = \delta_{(0),J} b \quad (\textit{resp. } \delta_{(r_1),J} b)$$

if and only if

$$x_I = 2^{-r} b \quad (\textit{resp. } (-1)^{|I|} 2^{-r} b) \qquad \textit{for } I \in \mathscr{I}^m_{r_1}.$$

As another consequence of (2.3.6), one has

Corollary 2.3.8. *If* $|I|$ *is odd, one has* $L_I(M; s) \equiv 0$. *In particular, if* r *is odd, one has* $L_{(r_1)}(M; s) \equiv 0$.

In fact, if $|J|$ is odd, one has by (2.3.4), (2.3.6)

$$\begin{aligned} L_J(s) &= \sum_I \xi_I(s) a_{IJ} = \sum_I \xi_{I^*}(s) a_{I^*,J} \\ &= (-1)^{|J|} \sum_I \xi_I(s) a_{IJ} = -L_J(s), \end{aligned}$$

whence follows that $L_J(s) \equiv 0$.

Theorem 2.3.9. *Let* $\nu = n/r - (d/2)\rho$ *with* $0 \le \rho \le r_1 - 1$, $\delta \mid \rho$. *Then*

$$\operatorname{Res}_{s=\nu} L_I(M; s) = \begin{cases} 2^r \operatorname{Res}_{s=\nu} Z_{\mathscr{C}}(M; s) & \textit{if } \nu \equiv \frac{n}{r}\ (2) \textit{ and } I = (0), \\ & \textit{or if } \nu \equiv \frac{n}{r} - 1(2) \textit{ and } I = (r_1), \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, $L_I(M; s)$ *with* $I \ne (0), (r_1)$ *is entire. If* $(d/2)\delta$ *is even,* $L_{(r_1)}(M; s)$ *is also entire.*

This follows from Corollary 2.3.2 and Lemma 2.3.7.

Corollary 2.3.10. *If* r *is odd, one has*

$$\operatorname{Res}_{s=\nu} Z_{\mathscr{C}}(M; s) = 0 \qquad \textit{for } \nu \equiv \frac{n}{r} - 1(2).$$

This follows from Theorem 2.3.9 and Corollary 2.3.8. In view of the formula in Theorem 2.3.1, this implies that the residue of the zeta functions $\xi_I(M; s)$ is always *real*.

2.4. Special values of zeta and L-functions. We are interested in the special values of the functions $\xi_I(M; s)$ and $L_I(M; s)$ at $s=\nu \in \boldsymbol{Z}$, $\nu \leq 0$. For simplicity, we write $Z_{\mathscr{C}}(s)$, $\xi_I(s)$, $L_I(s)$, $\xi_I^*(s)$ and $L_I^*(s)$ for $Z_{\mathscr{C}}(M; s)$, $\xi_I(M; s)$, $L_I(M; s)$, $\xi_I(M^*, s)$ and $L_I(M^*, s)$, respectively. Putting $s=n/r-\nu$ and replacing M by M^* in (2.3.5), one has for $\nu \equiv n/r-1$ (2) and ≤ 0

$$\operatorname{ord}_{s=\nu} L_I = r-|I|+\begin{cases}\operatorname{ord}_{s=n/r-\nu} L_I^* & \text{(Case (a))},\\ \operatorname{ord}_{s=n/r-\nu} L_{I^*}^* & \text{(Case (a'))},\end{cases}$$

whence follows (noting $r \geq 2$) that

$$L_I(\nu)=0 \qquad \text{for } I \neq (r_1).$$

Similarly, for $\nu \equiv n/r$ (2) and ≤ 0, one has

$$L_I(\nu)=0 \qquad \text{for } I \neq (0).$$

By Lemma 2.3.7, this implies the following

Theorem 2.4.1. *For $\nu \equiv n/r-1$ (2) and ≤ 0, one has*

$$\xi_I(M; \nu) = (-1)^{|I|} Z_{\mathscr{C}}(M; \nu),$$

$$L_I(M; \nu) = \begin{cases} 2^r Z_{\mathscr{C}}(M; \nu) & \text{for } I=(r_1),\\ 0 & \text{for } I \neq (r_1).\end{cases}$$

In particular, if r is odd, one has $Z_{\mathscr{C}}(M; \nu)=0$ for all $\nu \equiv 0$ (2) and ≤ 0.
For $\nu \equiv n/r$ (2) and ≤ 0, one has

$$\xi_I(M; \nu) = Z_{\mathscr{C}}(M; \nu),$$

$$L_I(M; \nu) = \begin{cases} 2^r Z_{\mathscr{C}}(M; \nu) & \text{for } I=(0),\\ 0 & \text{for } I \neq (0).\end{cases}$$

It has been conjectured in general (including the case d odd) that the special values $Z_{\mathscr{C}}(M; \nu) = \xi_{(0)}(M; \nu)$ ($\nu \in \boldsymbol{Z}$, $\nu \leq 0$) are rational. There are a few evidences supporting this conjecture.

(2.4.2) In the case explained in Example 2.1.2, where one has $r_0=r_1=1$, $r=m$ and $d=0$, the zeta function $Z_{\mathscr{C}}(M; s)$ and the L-function $L_{(0)}=\sum_I' \xi_I$ are essentially the (partial) Dedekind zeta functions of the totally real number field F_1, and the L-function $L_{(1)}=\sum (-1)^{|I|} \xi_I$ coincides with the "Shimizu L-function" (which is a special case of Hecke's zeta function with "Grössencharaktere"). The value $L_{(1)}(M; 1)$, which is related to $L_{(1)}(M^*; 0)$ by the functional equation, appears in the dimension formula for the space of Hilbert modular forms (cf. [S6]). The relation in Theorem 2.4.1 for $\nu=0$, $L_{(1)}(M;0)=2^r Z_{\mathscr{C}}(M;0)$, has been known*). In this case, the rationality of the special values $Z_{\mathscr{C}}(M;\nu)$ ($\nu\in \boldsymbol{Z}$, $\nu\leq 0$) is well-known (see e.g. [S8]).

(2.4.3) In the case where $\mathscr{C}=\mathscr{P}_r(\boldsymbol{R})$ ($r\geq 2$), $M=\mathrm{Sym}_r(\boldsymbol{Z})$ and Γ is a congruence subgroup of $Sp_r(\boldsymbol{Z})$, the special value $Z_{\mathscr{C}}(M^*;\nu)$ appears in the cusp contribution in the dimension formula for the space of Siegel cusp forms for the corresponding congruence subgroup of $Sp_{r-\nu}(\boldsymbol{Z})$ (cf. [M1], [S7]). In this case, one has $F_1=\boldsymbol{Q}$, $r_0=r_1=r$ and $d=1$. (So this is a case excluded here.) For the case $r_1=r=2$, $r_0=1$ and $d=1$ with a $\boldsymbol{Q}$-structure defined by an indefinite quaternion algebra over $\boldsymbol{Q}$, a similar observation was made by Arakawa [A].

(2.4.4) In the case of quadratic cones with $r_0=1$, $r_1=r=2$, Kurihara [K] showed that the values $Z_{\mathscr{C}}(\nu)$ ($\nu\in\boldsymbol{Z}$, $\nu\leq 0$) are rational for $n\leq 3$ and gave an example with $n=4$ for which the rationality holds.

(2.4.5) For the case $r_0=1$, $m=1$, one of the authors ([S2]) has made the following observation, generalizing Shintani's method in [S8]. Suppose Γ is torsion-free and let

$$\mathscr{C}=\coprod_{\substack{\gamma\in\Gamma\\ \alpha\in A}}\gamma\,\mathrm{Int}(\sigma_\alpha)$$

be a Γ-invariant non-singular "r.p.p. decomposition" in the sense to be explained in 4.2. Here $\sigma_\alpha=\{v_1^{(\alpha)},\cdots,v_{l_\alpha}^{(\alpha)}\}_{\boldsymbol{R}\geq 0}$ is a simplicial cone generated by $v_i^{(\alpha)}$ ($1\leq i\leq l_\alpha$) such that $\{v_i^{(\alpha)}\ (1\leq i\leq l_\alpha)\}$ can be extended to a $\boldsymbol{Z}$-basis of M.

Following Ishida [I], we put

$$f(u)=\sum_{\alpha\in A}\prod_{j=1}^{l_\alpha}\frac{1}{e^{\langle v_j{}^{(\alpha)},u\rangle}-1}\qquad (u\in\mathscr{C}),$$

$$f^K(t_1,\cdots,t_r)=\int_K f\Big(k\sum_{i=1}^r t_i e_i\Big)\mathrm{d}k\qquad (t_i\in\boldsymbol{R}_+),$$

where $c_0=\sum_{i=1}^r e_i$ is a primitive decomposition of the unit element. Then, it was shown in [S2] that the special values $Z_{\mathscr{C}}(\nu)$ ($\nu\in\boldsymbol{Z}$, $\nu\leq 0$) are a $\boldsymbol{Q}$-linear combination of the Laurent coefficients of $f^K(t_1,\cdots,t_r)$ after a suitable change of variables and hence are a $\boldsymbol{Q}$-linear combination of the integrals of the form

*) We are grateful to R. Sczech for informing us of this relation along with a sketch of proof.

$$\int_K \prod_{i,j} \langle v_j^{(\alpha)}, ke_i \rangle^{\nu_{ij}^{(\alpha)}} \mathrm{d}k,$$

where $\alpha \in A$ and, for a given ν, $(\nu_{ij}^{(\alpha)})$ ranges over a certain finite set of $r \times l_\alpha$ integral matrices.

(2.4.6) In the case of $r_0=1$, the zero-value $Z_{\mathscr{C}}(0)$, which coincides with $2^{-r}L_{(0)}(0)$ or $2^{-r}L_{(r_1)}(0)$ according as $n/r \equiv 0$ or $1 \pmod 2$, seems to have a close connection with some geometric invariants of the corresponding cusp singularity, as we shall explain in the next section. The rationality of $Z_{\mathscr{C}}(0)$ in a more general context of "Tsuchihashi singularities" was proved by Ishida [I3] (see 4.2),

§ 3. Geometric invariants of cusp singularities

3.1. Rational symmetric tube domains. Let V, $\mathscr{C}$, G, $\cdots$ be as before. We consider the tube domain $\mathscr{D}=V+\sqrt{-1}\mathscr{C}$ in $V_{\boldsymbol{C}} \cong \boldsymbol{C}^n$ and let $\tilde{G}=(\mathrm{Hol}\,\mathscr{D})^\circ$ denote the identity connected component of the group of holomorphic automorphisms of $\mathscr{D}$. As is well-known, $\mathscr{D}$ is a symmetric domain (hermitian symmetric space of non-compact type) and $\tilde{G}$ is a semi-simple Lie group of hermitian type with center reduced to the identity, of $\boldsymbol{R}$-rank equal to $r=\boldsymbol{R}$-rk G. The group of affine automorphisms of $\mathscr{D}$, $P=(\mathrm{Aff}\,\mathscr{D})^\circ$, which may naturally be identified with the semi-direct product $G \cdot V$, is a parabolic subgroup of $\tilde{G}$ corresponding to a point boundary component of $\mathscr{D}$, which we denote symbolically by $\sqrt{-1}\infty$. The given $\boldsymbol{Q}$-structure on V determines uniquely a $\boldsymbol{Q}$-structure on $\tilde{G}$ such that P and $G(\subset P)$ are subgroups defined over $\boldsymbol{Q}$; then the $\boldsymbol{Q}$-rank of $\tilde{G}$ is equal to $r_0=\boldsymbol{Q}$-rk G. ($\tilde{\mathfrak{g}}=\mathrm{Lie}\,\tilde{G}$ is the so-called "superstructure algebra" of the Jordan algebra (V, c_0), see [S1].) A symmetric tube domain $\mathscr{D}$ with a $\boldsymbol{Q}$-structure on $\tilde{G}$ determined in this manner is called a "rational symmetric tube domain".

Let $\tilde{\Gamma}$ be a neat arithmetic subgroup of $\tilde{G}$. Here $\tilde{\Gamma}$ being "neat" means that for $\gamma \in \tilde{\Gamma}$ if γ^ν is unipotent for some positive integer ν then γ itself is unipotent; in particular, $\tilde{\Gamma}$ is torsion-free. Then $M=\tilde{\Gamma} \cap V$ is a lattice in V and $\Gamma=(\tilde{\Gamma} \cap P)/(\tilde{\Gamma} \cap V)$ may be regarded as a (torsion-free) arithmetic subgroup of G. We assume that our M and Γ (in § 2) are obtained in this manner. Thus one has an exact sequence

$$1 \longrightarrow M \longrightarrow \tilde{\Gamma} \cap P \longrightarrow \Gamma \longrightarrow 1.$$

In general, this group extension may not split, but $\tilde{\Gamma} \cap P$ is a subgroup of finite index in the semi-direct product $\Gamma_M \cdot M$.

In what follows, we restrict ourselves to the *case* $r_0=1$. Then it is well-known that the quotient space $Y=\tilde{\Gamma} \backslash \mathscr{D}$ can be compactified to a normal projective variety Y^* by adding a finitely many points p_ν ($1 \leq \nu \leq h$), called "cusps":

$$Y^* = (\tilde{\Gamma} \backslash \mathscr{D}) \cup \{p_1, \cdots, p_h\}.$$

This is so-to-speak the minimal compactification. Each cusp corresponds to a $\tilde{\Gamma}$-equivalence class of (point) rational boundary components of $\mathscr{D}$, or equivalently to a $\tilde{\Gamma}$-conjugacy class of (proper) $\boldsymbol{Q}$-parabolic subgroups of $\tilde{G}$. We assume that p_ν corresponds to the class of a $\boldsymbol{Q}$-parabolic subgroup P_ν and, in particular, $P_1 = P$, i.e. p_1 is the class of $\sqrt{-1}\infty$.

By the classification theory, it can be seen that the rational symmetric tube domain with $r_0 = 1$ occurs only in the following cases

	$\mathscr{C}$	$\mathscr{D}$	$\tilde{G}_{\boldsymbol{Q}}$
(Case 1)	$\boldsymbol{R}_+^m$	$(\mathrm{III}_1)^m$	$SL_2(F_1)/\{\pm 1\}$
(Case 2)	$\mathscr{P}_2(\boldsymbol{R})^m$	$(\mathrm{III}_2)^m$	$SU_2(\boldsymbol{D}_1/F_1)/\{\pm 1\}$
(Case 3)	$\mathscr{P}_\delta(\boldsymbol{C})^m$	$(\mathrm{I}_{\delta,\delta})^m$	$SU_2(\boldsymbol{D}_1'/F_1'/F_1)/\{\pm 1\}$

(Case 1) is usually referred to as the "Hilbert modular case". In this list, F_1 is a totally real number field of degree m, $\boldsymbol{D}_1$ is a totally indefinite quaternion algebra over F_1, F_1' is a totally imaginary quadratic extension of F_1, $\boldsymbol{D}_1'$ is a central division algebra over F_1' with involution of the second kind relative to F_1'/F_1, and SU_2 denotes the special unitary group for an "isotropic" hermitian form of 2 variables (i.e. the hermitian form with matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$) in $\boldsymbol{D}_1$ or $\boldsymbol{D}_1'$.

3.2. Geometric invariants of cusps. In general, the cusp p_ν is a singular point on Y^*. A neighborhood of p_ν in Y^* is analytically isomorphic to a neighbourhood of p_ν in the local compactification

$$(\tilde{\Gamma} \cap P_\nu \backslash \mathscr{D}) \cup \{p_\nu\}.$$

(Note that the P_ν are all $G_{\boldsymbol{Q}}$-conjugate to $P_1 = G \cdot V$.) Resolving these cusp singularities by the method of toroidal embeddings [AMRT], one obtains a smooth compactification:

$$X \xrightarrow{\pi} Y^*,$$
$$X = (\tilde{\Gamma} \backslash \mathscr{D}) \cup D, \quad D = \sum_{\nu=1}^{h} D^{(\nu)},$$

where D is a divisor with simple normal crossings and $D^{(\nu)} = \pi^{-1}(p_\nu)$ is a connected component of D. Let $D = \sum_{i \in I} D_i$ be the irreducible decomposition of D and put

$$I^{(\nu)}=\{i\in I\,|\,\pi(D_i)=p_\nu\};$$

then $D^{(\nu)}=\sum_{i\in I^{(\nu)}} D_i$. From the construction, $D^{(\nu)}$ is a "toric divisor" (see [E], [S4, 5]); in particular, each D_i is a toric variety.

We denote by γ_i, $\bar{\gamma}_i$ $(1\leq i\leq n)$ the Chern roots of X and the "logarithmic Chern roots" of X relative to D, respectively (see [H1], [S5]). Let δ_i $(i\in I)$ denotes the 2-cohomology class on X defined by D_i. Then in the cohomology ring $H^*(X, \boldsymbol{Z})$ one has the relation

$$\prod_{i=1}^{n}(1+\gamma_i)=\prod_{i=1}^{n}(1+\bar{\gamma}_i)\prod_{i\in I}(1+\delta_i) \tag{3.2.1}$$

and, since D is toric,

$$\bar{\chi}_i\cdot\delta_j=0 \quad (1\leq i\leq n, j\in I) \tag{3.2.2}$$

([S4], Lemma 2). The arithmetic genus of X and the "logarithmic arithmetic genus" of X relative to D are defined, respectively, by

$$\begin{aligned}\chi(X)&=\left(\prod_{i=1}^{n}\frac{\gamma_i}{(1-e^{-\gamma_i})}\right)_n[X],\\ \bar{\gamma}(X, D)&=\left(\prod_{i=1}^{n}\frac{\bar{\gamma}_i}{(1-e^{-\bar{\gamma}_i})}\right)_n[X],\end{aligned} \tag{3.2.3}$$

where $(\cdots)_n$ denotes the homogeneous part of degree $2n$ in $H^*(X, \boldsymbol{Z})$ and $(\cdots)_n[X]$ is its evaluation on the fundamental $2n$-cycle $[X]$. One defines the "cusp contribution" to χ at p_ν by

$$\chi_\infty(p_\nu)=\left(\prod_{i\in I^{(\nu)}}\frac{\delta_i}{1-e^{-\delta_i}}\right)_n[X]. \tag{3.2.4}$$

Then from (3.2.1)–(3.2.4) one obtains the relation

$$\chi(X)=\bar{\chi}(X, D)+\sum_{\nu=1}^{h}\chi_\infty(p_\nu). \tag{3.2.5}$$

Note that, in our case, $\bar{\chi}(X, D)$ can easily be computed by the "proportionality theorem" of Hirzebruch-Mumford ([M2]); in particular, it is independent of the toroidal compactification.

Let $\mathcal{N}(D^{(\nu)})$ denote the nerve (or "dual graph") of $D^{(\nu)}$:

$$\mathcal{N}(D^{(\nu)})=\{J\subset I^{(\nu)}\,|\,J\neq\emptyset,\ X_J=\textstyle\bigcap_{i\in J}D_i\neq\emptyset\}. \tag{3.2.6}$$

Then $\mathcal{N}(D^{(\nu)})$ is a simplicial complex of dimension $n-1$, which, for a toric divisor $D^{(\nu)}$, is an "Euler complex" ([S4], Lemma 3). It follows that, if one puts

$$q^{(\nu)}(t) = \sum_{J \in \mathcal{N}(D^{(\nu)})} t^{n-|J|},$$

then $q^{(\nu)}(t)$ is a polynomial of degree $n-1$ in $\boldsymbol{Z}[t]$ satisfying the functional equation

$$t^n q^{(\nu)}\left(\frac{1}{t}-1\right) = q^{(\nu)}(t-1) + q^{(\nu)}(-1)(t-1)^n; \tag{3.2.7}$$

in particular, one has

$$q^{(\nu)}(-1) = (1-(-1)^n)2^{-n} q^{(\nu)}(-2). \tag{3.2.8}$$

It is easy to see that $q^{(1)}(-1)$ coincides with the Euler number $e(\Gamma \backslash \mathscr{C}^1)$, which vanishes for a self-dual homogeneous cone $\mathscr{C}$ except for (Cases 1, 2) with $m=1$,

Theorem 3.2.9. *When n is odd, one has*

$$\chi_\infty(p_1) = \frac{1}{2} q^{(1)}(-1)\left(= \frac{1}{2} e(\Gamma \backslash \mathscr{C}^1)\right),$$

where $\mathscr{C}^1 = \{x \in \mathscr{C} \mid N(x) = 1\} \approx \mathscr{C}/\boldsymbol{R}_+$.

(This is essentially due to Ehlers [E], who proved it in the Hilbert modular case. The general case is given in [S4].)

When n is even, the signature $\tau(X)$, the logarithmic signature $\bar{\tau}(X, D)$ and the cusp contribution to the signature $\tau_\infty(p_\nu)$ are defined similarly to the above; for instance,

$$\tau_\infty(p_\nu) = 2^n \left(\prod_{i \in I^{(\nu)}} \frac{1+e^{-\delta_i}}{1-e^{-\delta_i}} \frac{\delta_i}{2} \right)_n [X]. \tag{3.2.10}$$

Then from (3.2.1), (3.2.2) and from the fact that D is toric one has

$$\tau(X) = \bar{\tau}(X, D) + \sum_{\nu=1}^{h} \tau_\infty(p_\nu), \tag{3.2.11}$$

$$\tau_\infty(p_\nu) = 2^n \chi_\infty(p_\nu) + q^{(\nu)}(-2) \tag{3.2.12}$$

(cf. [S4, 5]).

3.3. Generalized Hirzebruch conjecture. Here we assume that *n is even.* In the Hilbert modular case, the cusp p_1 is "rationally pararelizable" in the sense that p_1 has a compact neighbourhood U in Y^* such that $U - \{p_1\}$ does not contain any other singularities and is retractable to ∂U, and that all Chern classes of the tangent bundle $T(U - \{p_1\})|_{\partial U}$ (restricted to ∂U) in $H^*(\partial U, \boldsymbol{Q})$ vanish. In that case, the "signature defect" $\delta(p_1)$ was defined by Hirzebruch [H2], which in our notation is equal to $\tau_\infty(p_1) - \tau(\tilde{U})$, where $\tilde{U} \to U$ is a desingularization of (U, p_1) and the signature $\tau(\tilde{U})$ is computed in $H^*(\tilde{U}, \partial\tilde{U}, \boldsymbol{Q})$. In the general case, we define the signature defect by $\delta(p_1) = \tau_\infty(p_1) - \tau(\tilde{U})$. (For a direct generalization of the definition, see Looijenga [L]). Then it seems likely that one has

(C1) $$\delta(p_1) = 2^n \chi_\infty(p_1),$$

or equivalently in view of (3.2.12),

(C1′) $$\tau(\tilde{U}) = q^{(1)}(-2).$$

In the Hilbert modular case, Hirzebruch ([H2], p. 230) conjectured that

$(C2)_{r_1=1}$ $$\delta(p_1) = L_{(1)}({}^t\Gamma, M^*; 0),$$

which was proved by Atiyah-Donnelly-Singer [ADS1, 2] and Müller [M4, 5]. (In the case $n=2$, this relation and (C1) were already proved in [H2].) In general (at least for (Case 1, 3)), in view of Theorem 2.4.1 and (C1), it seems natural to conjecture

(C2) $$\delta(p_1) = 2^{n-r} L_{I_1}({}^t\Gamma, M^*; 0),$$

where $I_1 = (0)$ or (r_1) according as $n/r \equiv 0$ or $1 \pmod 2$, and

(C3) $$\chi_\infty(p_1) = Z_{\mathscr{C}}({}^t\Gamma, M^*; 0),$$

or equivalently

(C3′) $$\chi_\infty(p_1) = 2^{-r} L_{I_1}({}^t\Gamma, M^*; 0).$$

The relation similar to (C3) or (C3′) for n odd, where $\chi_\infty(p_1)$ should be replaced by $-\chi_\infty(p_1)$, was proved by Ogata [O], as we shall see in the next section (Theorem 4.2.3, note that, in (Cases 1, 3) with n odd >1, one has $Z_{\mathscr{C}}(0) = \chi_\infty(p_1) = 0$ by Theorems 2.4.1 and 3.2.9). In the Hilbert modular case, the conjecture $(C3')_{r_1=1}$ was mentioned in [E] and [HG] (p. 95). In this case, comparing the cusp contribution in the dimension formulae for the space of Hilbert cusp forms obtained by Selberg trace formula and by Riemann-Roch-Hirzebruch Theorem ([H1]), one obtains a "weaker form" of $(C3')_{r_1=1}$:

$$\sum_{\nu=1}^{h} \chi_\infty(p_\nu) = 2^{-n} \sum_{\nu=1}^{h} L_{(1)}({}^t\Gamma_\nu, M_\nu^*; 0)$$

([S6], [F], [S3]; cf. also [A] for (Case 2)).

The relation between these conjectures is shown in the following diagram:

$$\begin{array}{ccc} \chi_\infty(p_1) & \overset{\text{(C3)}}{=\!=\!=} & Z_{\mathscr{C}}({}^t\Gamma, M^*; 0) \\ \text{(C1)}\Big\| & \overset{\text{(C3')}}{\diagdown\!\diagdown} & \Big\|\text{(Th. 2.4.1)} \\ 2^{-n}\delta(p_1) & \underset{\text{(C2)}}{=\!=\!=} & 2^{-r}L_{I_1}({}^t\Gamma, M^*; 0) \end{array}$$

Since the proofs of $(\mathrm{C2})_{r_1=1}$ given in [ADS1, 2] and [M4, 5] are both rather complicated, depending on differential geometry and hard analysis, it seems desirable to give more direct proofs for (C1) and (C3) or (C3′)[*].

§ 4. Zeta functions associated to Tsuchihashi singularities

4.1. Tsuchihashi singularities. We consider a normal isolated singularity called "Tsuchihashi cusp" and define the cusp contribution χ_∞ for this kind of singularities.

As before, let V be a real vector space of dimension n and M a lattice (of rank n) in V. Consider a pair $(\mathscr{C}, \Gamma)$ consisting of a (non-degenerate, open convex) cone $\mathscr{C}$ in V, which may not be self-dual nor homogeneous, and a subgroup Γ of $GL(V)$ satisfying the following conditions:

(i) Γ leaves M invariant and is torsion-free;

(ii) Γ leaves $\mathscr{C}$ invariant;

(iii) the quotient space $\Gamma\backslash\mathscr{C}/\boldsymbol{R}_+$ is compact.

Let $\mathscr{D}=V+\sqrt{-1}\mathscr{C}$ be the corresponding (not necessarily symmetric) tube domain. Tsuchihashi [T] constructed a normal isolated singularity associated to the pair $(\mathscr{C}, \Gamma)$, which is the singularity at "infinity" p_1 of $(\Gamma M\backslash\mathscr{D})\cup\{p_1\}$. Let U be a suitable (open) neighbourhood of p_1 and π: $\tilde{U}\to U$ a toroidal desingularization. Let $\pi^{-1}(p_1)=D^{(1)}=\sum_{i\in I^{(1)}} D_i$ be the decomposition of the exceptional set into the union of irreducible components, and let δ_i be the cohomology class determined by D_i in $H_c^2(\tilde{U}, \boldsymbol{Z})$ (the integral 2-cohomology group with compact support) and $[\tilde{U}]$ the fundamental class of $\tilde{U}$. We define $\chi_\infty(p_1)$ by

$$\chi_\infty(p_1)=\left(\prod_{i\in I^{(1)}}\frac{\delta_i}{1-e^{-\delta_i}}\right)_n[\tilde{U}].$$

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[*] Some statements in [S4] and [O] on Hirzebruch conjecture were incorrect or misleading. In [S4], p. 366, l. 18, "χ_∞" should read "$4\chi_\infty$" and the "Hirzebruch conjecture" there should be understood in a weaker sense that the sums of each side of (C2) over all cusps are equal. The "conjecture" on [O], p. 370, l. 14 is correct for $n=2$ but should be rectified by (C3) for $n\geq 4$. Note also that our invariants δ and χ_∞ are written in [E] as $2^n\phi$ and ψ.

Then, Theorem 3.2.9 and (3.2.12) remain true, $\tau_\infty(p_1)$ being defined similarly to (3.2.10). We want to relate this cusp contribution to a special value of a zeta function associated to $(\mathscr{C}, \Gamma)$.

4.2. Zeta function associated to $(\mathscr{C}, \Gamma)$. Let $(\mathscr{C}, \Gamma)$ be a pair satisfying (i)–(iii). Let $\mathscr{C}^*$ be the dual cone of $\mathscr{C}$ and M^* the dual lattice of M inthe dual space V^*. $\phi_{\mathscr{C}}(x)$ denotes the characteristic function of the cone $\mathscr{C}$ defined in 1.2. We define the zeta function associated to $(\mathscr{C}, \Gamma)$ by

$$Z_{\mathscr{C}}(\Gamma, M; s)=\sum_{u:\Gamma\backslash\mathscr{C}\cap M}\phi_{\mathscr{C}}(u)^s \quad (\mathrm{Re}\, s>1). \tag{4.2.1}$$

Note that, when $\mathscr{C}$ is self-dual and homogeneous, we have by (1.4.3)

$$Z_{\mathscr{C},c_0}(\Gamma, M; s)=\phi_{\mathscr{C}}(c_0)^{-(r/n)s}Z_{\mathscr{C}}\left(\Gamma, M; \frac{r}{n}s\right).$$

Theorem 4.2.2. *The function $Z_{\mathscr{C}}(\Gamma, M; s)$ admits a meromorphic continuation to the whole plane and is holomorphic at $s=0$.*

Theorem 4.2.3. *When n is odd, one has*

$$Z_{\mathscr{C}}(\Gamma, M; 0)=-\frac{1}{2}e(\Gamma\backslash\mathscr{C}/\boldsymbol{R}_+)=-\chi_\infty(p_1).$$

A sketch of proofs of these theorems will be given in 4.3.

First, in order to describe the zero-value of $Z_{\mathscr{C}}$, we need the notion of "rational partial polyhedral decomposition" (r.p.p. decomposition, for short) of $\mathscr{C}\cup\{0\}$.

Definition 4.2.4. A (non-empty) collection Σ of closed rational polyhedral cones in V is called an *r.p.p. decomposition* of $\mathscr{C}\cup\{0\}$ if it satisfies the following conditions:

(1) If $\sigma\in\Sigma$ and $\tau\prec\sigma$ (i.e. τ is a face of σ), then $\tau\in\Sigma$. In particular, one has $\{0\}\in\Sigma$. We set $\Sigma^\times=\Sigma-\{0\}$.

(2) If $\sigma, \tau\in\Sigma$, then $\sigma\cap\tau\prec\sigma$.

(3) One has $\mathscr{C}=\coprod_{\sigma\in\Sigma^\times}\mathrm{Int}(\sigma)$.

(4) For any compact subset K contained in $\mathscr{C}$, the set $\{\sigma\in\Sigma\,|\,\sigma\cap K\neq\emptyset\}$ is finite.

In what follows, we further assume that Σ is "Γ-invariant" and "non-singular", i.e. the following additional conditions are satisfied:

(5) Γ leaves Σ invariant, and acts freely on $\Sigma^\times$.

(6) For every σ in Σ, there exists a $\boldsymbol{Z}$-basis $\{u_1, \cdots, u_n\}$ of M and r with $0\leq r\leq n$ such that $\sigma=\{u_1, \cdots, u_r\}_{\boldsymbol{R}_{\geq 0}}$.

We now assume that the toroidal desingularization $\pi:\tilde{U}\to U$ is defined by a Γ-invariant and non-singular r.p.p. decomposition Σ of $\mathscr{C}\cup\{0\}$. As before, let $\mathscr{N}(D^{(1)})$ be the nerve of $D^{(1)}$. Then there exists an injective map

$$\mathscr{N}(D^{(1)})\ni J\longmapsto\sigma_J\in\Sigma^\times$$

such that $\{\sigma_J\,|\,J\in\mathscr{N}(D^{(1)})\}$ is a complete set of representatives of $\Gamma\backslash\Sigma^\times$, $\dim\sigma_J=|J|$, and that for any $J, J'\in\mathscr{N}(D^{(1)})$ one has $J\subset J'$ if and only if $\gamma(\sigma_J)\subset\sigma_{J'}$ for some $\gamma\in\Gamma$. For $\sigma\in\Sigma^\times$, we put

$$\sigma(1)=\{\tau\in\Sigma\,|\,\dim\tau=1\text{ and }\tau\prec\sigma\}.$$

We denote by $\mathrm{d}x_\sigma$ the Lebesgue measure on the linear subspace $\sigma+(-\sigma)$ of V normalized so that for a $\boldsymbol{Z}$-basis $\{u_1,\cdots,u_n\}$ of M with $\sigma=\{u_1,\cdots,u_r\}_{\boldsymbol{R}\geq 0}$ the volume of the parallelotope spanned by $\{u_1,\cdots,u_r\}$ is one. For $\rho\in\Sigma$ with $\dim\rho=1$, the symbol ∂_ρ denotes the derivation

$$(\partial_\rho F)(x)=\lim_{t\to 0}\frac{1}{t}\{F(x+tu)-F(x)\}$$

for any differentiable function $F(x)$ on V, where u is the unique primitive element in $\rho\cap M$.

In these notations, we have

Theorem 4.2.5. *For any integer $\nu\geq 2$, one has*

$$\begin{aligned}Z_{\mathscr{C}}(\Gamma,M;0)&=\sum_{\tau:\Gamma\backslash\Sigma^\times}\int_\tau\Bigg[\prod_{\rho\in\tau(1)}\frac{\partial_\rho}{1-e^{-\partial_\rho}}\Bigg]_{\dim\tau}G_\nu(x)\mathrm{d}x_\tau\\&=(-1)^n2^{-n}q^{(1)}(-2)+\sum_{J\in\mathscr{N}(D^{(1)})}(-2)^{-(n-|J|)}\tau(X_J)\\&\quad\times\int_{\sigma_J}\Bigg[\prod_{\rho\in\sigma_J(1)}\frac{\partial_\rho}{2}\frac{1+e^{-\partial_\rho}}{1-e^{-\partial_\rho}}\Bigg]_{|J|}G_\nu(x)\mathrm{d}x_{\sigma_J},\end{aligned}$$

where $G_\nu(x)=\exp(-\phi_{\mathscr{C}}(x)^{-\nu})$ and, for any rational function $\Phi(t)\in\boldsymbol{Q}(t)$, $[\prod_{\rho\in\sigma(1)}\Phi(\partial_\rho)]_k$ is the differential operator of degree k on V obtained as the homogeneous part of total degree k in the formal power series expansion of $\prod_{\rho\in\sigma(1)}\Phi(\partial_\rho)$.

Ishida's proof ([I3]) of the rationality of $Z_{\mathscr{C}}(\Gamma,M;0)$ is based on the above formula. We note a remarkable similarity of it to (3.2.4) and (3.2.12).

4.3. Outline of proofs ([O]). For $\sigma\in\Sigma^\times$ let

$$Z(\sigma,M;s)=\sum_{u\in M\cap\mathrm{Int}(\sigma)}\phi_{\mathscr{C}}(u)^s,\quad\mathrm{Re}\,s>1.$$

We prove that $Z(\sigma, M; s)$ can be continued to a meromorphic function on $\boldsymbol{C}$ and calculate the value $Z(\sigma, M; 0)$.

Let $\dim \sigma = r \geq 1$ and $u_1, \cdots, u_r$ a part of a $\boldsymbol{Z}$-basis of M with $\sigma = \{u_1, \cdots, u_r\}_{\boldsymbol{R}_{\geq 0}}$. Then we may write as

$$Z(\sigma, M; s) = \sum_{(l_1, \cdots, l_r) \in (\boldsymbol{Z}_{>0})^r} \phi_{\mathscr{C}}(l_1 u_1 + \cdots + l_r u_r)^s.$$

We employ the method of Zagier [Z] who calculated the values of the zeta functions of real quadratic fields. For simplicity we consider the case $r=2$. The following proposition is well-known.

Proposition 4.3.1. *Let $\psi(s) = \sum_{k=0}^{\infty} a_k \lambda_k^{-s}$ with $\lambda_k > 0$ be a Dirichlet series absolutely convergent for* Re $s > 1$. *Assume that the function $h(t) = \sum_{k=0}^{\infty} a_k \exp(-\lambda_k t)$ has an asymptotic expansion of the form*

$$h(t) \sim b_{-1} \cdot \frac{1}{t} + b_0 + b_1 t + \cdots \quad (t \to 0).$$

Then $\psi(s)$ admits a meromorphic continuation to $\boldsymbol{C}$ and is holomorphic at $s=0$ with the value $\psi(0) = b_0$.

In order to apply this proposition we need an asymptotic expansion of $h(t)$. It is easy to derive the following proposition from the Euler-Maclaurin summation formula.

Proposition 4.3.2. *Let $f(t)$ be a real-valued C^∞-function on $[0, \infty)$. Assume that $\int_0^\infty f(t)\,\mathrm{d}t$ is finite. Then $g(t) = \sum_{l=1}^{\infty} f(tl)$ has an asymptotic expansion of the form*

$$g(t) \sim \frac{1}{t} \cdot \int_0^\infty f(t)\,\mathrm{d}t + \sum_{l=0}^{\infty} t^l \beta_l f^{(l)}(0) \quad (t \to 0),$$

where β_l's are the coefficients in the expansion $t/(1-e^{-t}) = \sum_{l=0}^{\infty} \beta_l t^l$.

Applying this to a function to a function $F(x, y)$ of two variables, we have under a certain condition

$$\sum_{m,n \in \boldsymbol{Z}_{>0}} F(mt, nt) \sim \frac{1}{t^2}\left(\int_0^\infty \int_0^\infty F(x, y)\,\mathrm{d}x\mathrm{d}y\right)$$
$$+ \sum_{i=1}^{\infty} \beta_i t^{i-1} \int_0^\infty F^{(i,0)}(0, y)\,\mathrm{d}y + \sum_{j=1}^{\infty} \beta_j t^{j-1} \int_0^\infty F^{(0,j)}(x, 0)\,\mathrm{d}x$$
$$+ \sum_{i,j \in \boldsymbol{Z}_{>0}} \beta_i \beta_j F^{(i,j)}(0, 0) t^{i+j} \quad (t \to 0),$$

where $F^{(i,j)}(x, y) = (\partial^{i+j}/\partial x^i \partial y^j) F(x, y)$. Unfortunately this can not be applied directly to $F(x, y)=\exp(-\phi_{\mathscr{C}}(x\cdot u_1+y\cdot u_2)^{-1})$, because the derivatives of $\exp(-\phi_{\mathscr{C}}(x)^{-1})$ behave badly near $x=0$. In order to surmount this difficulty, we need the following

Lemma 4.3.3. *For a positive integer ν and an r-dimensional cone $\sigma=\{u_1, \cdots, u_r\}_{\boldsymbol{R}_{\geq 0}}$ contained in $\mathscr{C}$, the function*

$$G_{\sigma,\nu}(x_1, \cdots, x_r)=\exp(-\phi_{\mathscr{C}}(x_1u_1+\cdots+x_ru_r)^{-\nu})$$

and its partial derivatives of total order up to $n\nu-1$ have limits at the origin and the partial derivatives of total order $n\nu$ are bounded.

Modifying Zagier's method and using this lemma, we can prove that the function $Z(\sigma, M; \nu s)$ can be continued to the half plane $\operatorname{Re} s>-1+1/\nu$. Thus $Z(\sigma, M; s)$ can be continued to $\operatorname{Re} s>-\nu+1$, and hence to the whole complex plane. And we get the value of $Z(\sigma, M; s)$ at $s=0$:

Proposition 4.3.4. *For any integer $\nu\geq 2$ we have*

$$Z(\sigma, M; 0)=\sum_{|k|=r}\int_{(\boldsymbol{R}_{\geq 0})^r}\beta_k\left(\frac{\partial}{\partial x}\right)^k G_{\sigma,\nu}(x)\,\mathrm{d}x,$$

where $k=(k_1, \cdots, k_r)\in(\boldsymbol{Z}_{\geq 0})^r$, $\beta_k=\beta_{k_1}\cdots\beta_{k_r}$, and $(\partial/\partial x)^k=(\partial/\partial x_1)^{k_1}\cdots(\partial/\partial x_r)^{k_r}$.

We rewrite this as

$$Z(\sigma, M; 0)=\int_\sigma\left[\prod_{\rho\in\sigma(1)}\frac{\partial_\rho}{1-e^{-\partial_\rho}}\right]_{\dim\sigma}G_\nu(x)\,\mathrm{d}x_\sigma,$$

where $G_\nu(x)=\exp(-\phi_{\mathscr{C}}(x)^{-\nu})$. By summing this equality side by side over $\sigma\in\Sigma^\times \bmod \Gamma$ we get the first equality in Theorem 4.2.5. By a calculation similar to that leading to (3.2.12) we get the second expression in Theorem 4.2.5, whence follows Theorem 4.2.3.

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