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A Realization of Semisimple Symmetric Spaces and Construction of Boundary Value Maps

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Contents

§ 0. Introduction

A homogeneous space G/H is called a semisimple symmetric space if G is a real connected semisimple Lie group and there exists an involution of G such that H is an open subgroup of the fixed point group of the involution. The most fundamental problem on the harmonic analysis on G/H is to give an explicit decomposition of $L^2(G/H)$ into irreducible representations of G, that is, to get a Plancherel formula for $L^2(G/H)$. Here $L^2(G/H)$ is the space of square integrable functions on G/H with respect to the invariant measure. In [O3] I proposed a method to obtain the Plancherel formula. The method explained there works well for the most continuous spectra on $L^2(G/H)$ with respect to the ring $D(G/H)$ of invariant differential operators on G/H, and the Plancherel measure for the spectra is expressed by "c-function" for G/H which is explicitly calculated by the method in [O3, §8], where G^d/H^d should be corrected to G^d/K^d. Comparing to the continuous spectra, the discrete spectra (the discrete series for G/H) are not easy to analyse by the method mentioned in [O3, § 9] and it is hard to get the precise parametrization of the discrete series or to investigate its structure especially in the case when the symmetric space is not a K_ε-type. On the other hand, by using Flensted-Jensen's duality method, we can directly study the discrete series and in fact we get sufficient informations to analyse the discrete series ([F] and [MO] etc.). Applying the usual method of parabolic induction for repre-

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sentations of G to these extreme types of spectra, we can get all the spectra and finally the Plancherel formula for $L^2(G/H)$. We will explain and prove it in subsequent papers.

In this paper, we study and give a proof of the results explained in [O3, §3, §4 and §5]. In [OS2] we have already studied the structure of root spaces which are related to the symmetric space and it covers the statements in [O3, §2].

In this paper we assume that the center of G is finite for simplicity. This assumption is not serious in our arguments and even without the assumption almost all theorems here are still valid together with their proofs but some theorems should be modified. For example, the manifold $\tilde{X}$ constructed in Section 1 may be non-compact and Theorem 4.11 may have no natural meaning because the left hand side of (4.29) may be infinite sum (cf. Remark 4.16 in this case). These two are the only main differences.

In Section 1, we construct a compact G-manifold $\tilde{X}$ which has finite G-orbits and all the open G-orbits are isomorphic to G/H. The orbital structure of $\tilde{X}$ is of normally crossing type and every invariant differential operator on the open orbit can be analytically extended to an element of the ring $D(\tilde{X})$ of invariant differential operators on $\tilde{X}$. The method of the construction is same as that in [O1] (cf. [Sc]) which constructs $\tilde{X}$ when G/H is a riemannian symmetric space. Some cases are also considered in [Ko] and [Se]. A different realization for some series of semisimple symmetric spaces (which we call K_ε-type) is studied in [OS1] and in the complex category a similar compact G-manifold is constructed in [CP] by a little different method.

We identify G/H with an open G-orbit in $\tilde{X}$, fix a finite codimensional ideal J of $D(\tilde{X})$ and consider the space S of hyperfunctions on U which are killed by J, where U is the intersection of G/H and an open subset of $\tilde{X}$ containing a boundary point of G/H. Then in Section 3, uisng the results in [O4], we define boundary value maps of the space S to the spaces of hyperfunction-valued local sections of certain line bundles over the boundary components. Here the boundary components mean the G-orbits contained in the boundary of G/H in $\tilde{X}$ and the maps commute with the infinitesimal actions induced by G. If an element f of S is ideally analytic at a boundary point of G/H, f is expressed as a sum of convergentserieses (cf. (3.8)). If an element f of S is left K-finite, then f is automatically ideally analytic at any boundary point and this expression is studied by [H3] and [CM] in a group case and by [Ba] in a general case.

The images of the boundary value maps satisfy induced systems of invariant differential equations which are given in Section 2. The study of the images of the boundary value maps with respect to compact boundary components, which we call distinguished boundaries, lead us to the concept of (most continuous) principal series for G/H. We define it in Section 4 and study its properties. For example, it is proved that any irreducible subquotient of a Harish-Chandra module realized in a function space on G/H is imbedded in a suitable Harish-Chandra module belonging to the principal series for G/H (cf. [De]). A multiplicity free theorem (Theorem 4.5) was obtained after [O3] was written, which is clear if G is linear. This enables us to have a different definition of the principal series as in Theorem 4.10. This is the reason why I assumed that G is linear in [O3].

If G/H is a group manifold, the principal series in Section 4 is naturally identified with the usual principal series of the group which is defined by [Ha1] (cf. Remark 4.17). Some applications to this case will be discussed in other papers.

§ 1. Construction of a compact imbedding

Let G be a connected real semisimple Lie group with finite center and let σ be an involutive automorphism of G. Put $G^\sigma=\{g \in G; \sigma(g)=g\}$ and let H be a closed subgroup of G with $G_o^\sigma \subset H \subset G^\sigma$, where G_o^σ denotes the identity component of G^σ. In this section we construct a compact G-manifold $\tilde{X}$ without boundary such that an open G-orbit in $\tilde{X}$ is isomorphic to the semisimple symmetric space X defined by $X=G/H$.

First we give some notation concerning the symmetric pair (G, H) as in [OS2]. Let K be a σ-stable maximal compact subgroup of G and let θ denote the corresponding Cartan involution. The involutions of the Lie algebra $\mathfrak{g}$ of G induced by σ and θ are denoted by the same letters, respectively. Let $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ (resp. $\mathfrak{k}+\mathfrak{p}$) be the decompositions of $\mathfrak{g}$ into $+1$ and -1 eigenspaces for σ (resp. θ). Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}\cap\mathfrak{q}$ and let $\mathfrak{a}^*$ denote the dual space of $\mathfrak{a}$. For a $\lambda \in \mathfrak{a}^*$, put $\mathfrak{g}^\lambda=\{Y \in \mathfrak{g}; [Z, Y] =\lambda(Z)Y \text{ for any } Z \in \mathfrak{a}\}$. Then the set $\Sigma=\{\lambda \in \mathfrak{a}^*; \mathfrak{g}^\lambda \neq \{0\}, \lambda\neq 0\}$ defines a root system with the inner product induced by the Killing form $\langle\ ,\ \rangle$ of $\mathfrak{g}$, and the Weyl group W of Σ is identified with the normalizer $N_K(\mathfrak{a})$ of $\mathfrak{a}$ in K modulo the centralizer $Z_K(\mathfrak{a})$ of $\mathfrak{a}$ in K. (Cf. [Ro], [OS2], [Sc] etc.) On the other hand the normalizer $N_{K\cap H}(\mathfrak{a})$ of $\mathfrak{a}$ in $K\cap H$ modulo the centralizer $Z_{K\cap H}(\mathfrak{a})$ of $\mathfrak{a}$ in $K\cap H$ is denoted by $W(\mathfrak{a}; H)$, which is a subgroup of W. For each element w of W we fix a representative $\bar{w}$ in $N_K(\mathfrak{a})$ so that $\bar{w} \in N_{K\cap H}(\mathfrak{a})$ if $w \in W(\mathfrak{a}; H)$. Choose a fundamental system $\Psi=\{\alpha_1, \cdots, \alpha_l\}$ of Σ, where the number $l=\dim \mathfrak{a}$ is called the *split rank* of the symmetric space X, and let Σ^+ denote the corresponding set of all positive roots in Σ. Let P_σ denote the parabolic subgroup of G with the Langlands decomposition $P_\sigma=M_\sigma A_\sigma N_\sigma$ so that $M_\sigma A_\sigma$ is the centralizer of $\mathfrak{a}$ in G and the Lie algebra $\mathfrak{n}_\sigma$ of N_σ equals $\sum_{\alpha\in\Sigma^+}\mathfrak{g}^\alpha$.

Now we introduce a product manifold $\hat{X}=G\times \boldsymbol{R}^l\times W$. Let $x=(g, t, w)$ be an element of $\hat{X}$. Then we put $\operatorname{sgn} x=\operatorname{sgn} t$, which is an element of $\{-1, 0, 1\}^l$. Here $\operatorname{sgn} t=(\operatorname{sgn} t_1, \cdots, \operatorname{sgn} t_l)$ with $t=(t_1, \cdots, t_l) \in \boldsymbol{R}^l$ and for an $s\in \boldsymbol{R}$ we define $\operatorname{sgn} s=1$ (resp. $-1, 0$) if $s>0$ (resp. $s<0$, $s=0$). Moreover we put $\Theta_x=\{\alpha_j;\, t_j\neq 0\}$ and $\Sigma_x=(\sum_{\alpha\in\Theta_x}\boldsymbol{R}\alpha)\cap\Sigma$ and denote by W_x the subgroup of W generated by reflections with respect to α in Θ_x. Then we define a parabolic subalgebra

$$(1.1)\qquad \mathfrak{p}_x=(\mathfrak{m}_\sigma+\mathfrak{a}_\sigma+\textstyle\sum_{\alpha\in\Sigma_x}\mathfrak{g}^\alpha)+\sum_{\alpha\in\Sigma^+-\Sigma_x}\mathfrak{g}^\alpha$$

of $\mathfrak{g}$ and its Langlands decomposition $\mathfrak{p}_x=\mathfrak{m}_x+\mathfrak{a}_x+\mathfrak{n}_x$ so that $\mathfrak{a}_x\subset\mathfrak{a}_\sigma$. The corresponding parabolic subgroup of G is denoted by P_x and $P_x=M_xA_xN_x$ is the corresponding Langlands decomposition of P_x. Then it is clear that P_x is the closure of $P_\sigma W_x P_\sigma$ in G. Moreover we define a closed subgroup $P(x)$ of G by

$$(1.2)\qquad P(x)=(M_x\cap\bar{w}^{-1}H\bar{w})A_xN_x$$

and put

$$(1.3)\qquad a(x)=a(t)=\exp(-\textstyle\sum_{t_j\neq 0}\log|t_j|H_j).$$

Here and hereafter $\{H_1, \cdots, H_l\}$ denotes the dual basis of the fundamental system Ψ, that is, $H_j\in\mathfrak{a}$ and $\alpha_i(H_j)=\delta_{ij}$ for $i, j=1, \cdots, l$.

Definition 1.1. We say two elements $x=(g, t, w)$ and $x'=(g', t', w')$. of $\hat{X}$ is equivalent if and only if the following three conditions hold:

(i) $\operatorname{sgn} x=\operatorname{sgn} x'$.

(ii) $W(\mathfrak{a}; H)wW_x=W(\mathfrak{a}; H)w'W_{x'}$.

(iii) $ga(x)P(x)=g'a(x')(M_x\cap\bar{w}'^{-1}H\bar{w})A_xN_x$.

Lemma 1.2. *Let $x=(g, t, w)$ and $x'=(g', t', w')$ be elements of $\hat{X}$ with* $\operatorname{sgn} x=\operatorname{sgn} x'$. *Then we have*

$$(1.4)\qquad W_x=W_{x'}.$$

Suppose x and x' satisfy the above condition (ii). *Then there exist $u, u'\in W(\mathfrak{a}; H)$ and $v, v'\in W_x$ and $m, m'\in Z_K(\mathfrak{a})$ with $\bar{u}\bar{w}\bar{v}m=\bar{u}'\bar{w}'\bar{v}'m'$. In this case we have*

$$(1.5)\qquad (\bar{v}m)^{-1}P(x)\bar{v}m=(\bar{v}'m')^{-1}P(x')\bar{v}'m'$$

and the condition (iii) *in Definition* 1.1 *is equivalent to*

$$(1.6)\qquad ga(x)P(x)\bar{v}m=g'a(x')P(x')\bar{v}'m'.$$

Proof. The first statement (1.4) is clear by definition and therefore the existence of u, u', v, v', m and m' with the condition in the lemma is also clear. Since $\bar{v}, \bar{v}', m, m' \in M_x$ and $u, u' \in H$, we have $(\bar{v}m)^{-1}(M_x \cap \bar{w}^{-1}H\bar{w})\bar{v}m = M_x \cap (\bar{v}m)^{-1}\bar{w}^{-1}H\bar{w}\bar{v}m = M_x \cap (\bar{u}^{-1}\bar{u}'\bar{w}'\bar{v}'m')^{-1}H\bar{u}^{-1}\bar{u}'\bar{w}'\bar{v}'m' = M_x \cap (\bar{w}'\bar{v}'m')^{-1}H\bar{w}'\bar{v}'m' = (\bar{v}'m')^{-1}(M_x \cap \bar{w}'^{-1}H\bar{w}')\bar{v}'m'$, which means (1.5). In the same way we have $P(x')\bar{v}'m'(\bar{v}m)^{-1} = \bar{v}'m'(\bar{v}m)^{-1}P(x) = (M_x \cap \bar{v}'m' \cdot (\bar{v}m)^{-1}\bar{w}^{-1}H\bar{w})A_xN_x = (M_x \cap (u'\bar{w}')^{-1}\bar{u}H\bar{w})A_xN_x = (M_x \cap \bar{w}'^{-1}H\bar{w})A_xN_x$ and therefore the last statement in the lemma is clear. Q.E.D.

This lemma assures that Definition 1.1 really gives an equivalence relation, which we write $x \sim x'$. The quotient space of $\hat{X}$ by this equivalence relation is denoted by $\tilde{X}$ and becomes a topological space with the quotient topology. Let $\pi: \hat{X} \to \tilde{X}$ be the natural projection. Then an action of G on $\tilde{X}$ is defined by $g_o\pi(g, t, w) = \pi(g_og, t, w)$ for $g_o \in G$.

Remark 1.3. (i) The map π is factorized into the natural projection of $\hat{X}$ onto the product space $G \times \boldsymbol{R}^l \times (W(\mathfrak{a}; H)\backslash W)$ and a map $\bar{\pi}$ of this space onto $\tilde{X}$.

(ii) If X is a Riemannian symmetric space of the non-compact type, then $H = K$, $W(\mathfrak{a}; H) = W$ and therefore the G-space $\tilde{X}$ is isomorphic to the G-space constructed in [O1] (or in [Sc, Chapter 4]) but not isomorphic to the one in [OS1, Chapter 2].

To define an analytic structure on $\tilde{X}$ we prepare some notation. Let $\mathfrak{a}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}$ and let $\Sigma(\mathfrak{a}_\mathfrak{p})$ be the restricted root system corresponding to the pair $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$. Then the Weyl group $W(\mathfrak{a}_\mathfrak{p})$ of $\Sigma(\mathfrak{a}_\mathfrak{p})$ is isomorphic to the group $N_K(\mathfrak{a}_\mathfrak{p})/M$, where $N_K(\mathfrak{a}_\mathfrak{p})$ (resp. M) are the normalizer (resp. centralizer) of $\mathfrak{a}_\mathfrak{p}$ in K.

Lemma 1.4 ([OS2]). (i) *$\mathfrak{a}_\mathfrak{p}$ is σ-stable.*

(ii) $W(\mathfrak{a}) \simeq (N_K(\mathfrak{a}_\mathfrak{p}) \cap N_K(\mathfrak{a}))/(N_K(\mathfrak{a}_\mathfrak{p}) \cap Z_K(\mathfrak{a}))$.

(iii) $W(\mathfrak{a}; H) \simeq (N_{K\cap H}(\mathfrak{a}_\mathfrak{p}) \cap N_{K\cap H}(\mathfrak{a}))/(N_{K\cap H}(\mathfrak{a}_\mathfrak{p}) \cap Z_{K\cap H}(\mathfrak{a}))$,
$\simeq N_{K\cap H}(\mathfrak{a}_\mathfrak{p})/(N_{K\cap H}(\mathfrak{a}_\mathfrak{p}) \cap Z_{K\cap H}(\mathfrak{a}))$.

Proof. (i) Let $Y \in \mathfrak{a}_\mathfrak{p}$. Then $[\sigma Y, \mathfrak{a}] = \sigma[Y, \sigma(\mathfrak{a})] = \sigma[Y, \mathfrak{a}] = \{0\}$. Hence the element $\sigma Y - Y$ of $\mathfrak{p} \cap \mathfrak{q}$ centralizes $\mathfrak{a}$, which implies $\sigma Y - Y \in \mathfrak{a}$ because $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. Therefore $\sigma Y \in \mathfrak{a}_\mathfrak{p}$.

(ii) We will show that any $w \in W$ has a representative g in $N_K(\mathfrak{a}_\mathfrak{p}) \cap N_K(\mathfrak{a})$, which implies (ii). Put $\mathfrak{a}'_\mathfrak{p} = \mathrm{Ad}(\bar{w})(\mathfrak{a}_\mathfrak{p})$, $\mathfrak{p}' = \{Y \in \mathfrak{p};\ [Y, Z] = 0$ for all $Z \in \mathfrak{a}\}$ and $G' = \{g \in G;\ \mathrm{Ad}(g)(Z) = Z$ for all $Z \in \mathfrak{a}\}$. Since $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}'_\mathfrak{p}$ are maximal abelian subspaces of $\mathfrak{p}'$ and $G' = Z_K(\mathfrak{a}) \exp \mathfrak{p}'$ is a Cartan decomposition of the reductive group G', there exists an element z of $Z_K(\mathfrak{a})$ with $\mathrm{Ad}(z)(\mathfrak{a}'_\mathfrak{p}) = \mathfrak{a}_\mathfrak{p}$. Then $g = z\bar{w}$ is a required representative of w.

(iii) Replacing G and $\mathfrak{p}$ by H and $\mathfrak{h}\cap\mathfrak{p}$, we can prove the first isomorphism in the same way as in (ii). Let $g\in N_{K\cap H}(\mathfrak{a}_\mathfrak{p})$. Then $\mathfrak{a}_\mathfrak{p}=\mathrm{Ad}(g)(\mathfrak{a}_\mathfrak{p})=\mathrm{Ad}(g)(\mathfrak{a}_\mathfrak{p}\cap\mathfrak{h}+\mathfrak{a})$. Since $\mathrm{Ad}(g)(\mathfrak{a}_\mathfrak{p}\cap\mathfrak{h})\subset\mathfrak{h}$ and $\mathrm{Ad}(g)(\mathfrak{a})\subset\mathfrak{q}$, we have $\mathrm{Ad}(g)(\mathfrak{a})=\mathfrak{a}_\mathfrak{p}\cap\mathfrak{q}=\mathfrak{a}$. This means the second isomorphism in (iii).
Q.E.D.

Lemma 1.4 assures that we can assume that the representatives $\bar{w}$ of the elements w of W satisfy $\mathrm{Ad}(\bar{w})(\mathfrak{a}_\mathfrak{p})=\mathfrak{a}_\mathfrak{p}$. Let $\mathfrak{g}(\sigma)$ be the reductive Lie algebra generated by $\{g(\mathfrak{a}_\mathfrak{p};\lambda);\lambda\in\Sigma(\mathfrak{a}_\mathfrak{p})$ with $\lambda|\mathfrak{a}=0\}$, where $\mathfrak{g}(\mathfrak{a}_\mathfrak{p};\lambda)=\{Y\in\mathfrak{g};[Y,Z]=\lambda(Z)Y$ for all $Z\in\mathfrak{a}_\mathfrak{p}\}$, and put $\mathfrak{m}(\sigma)=\{X\in\mathfrak{m}_\sigma;[X,Y]=0$ for all $Y\in\mathfrak{g}(\sigma)\}$. We denote by $G(\sigma)$ and $M(\sigma)_o$ the analytic subgroups of G corresponding to $\mathfrak{g}(\sigma)$ and $\mathfrak{m}(\sigma)$, respectively, and put

$$M(\sigma)=M(\sigma)_o\,\mathrm{Ad}_G^{-1}(\mathrm{Ad}(K)\cap\exp(\mathrm{ad}(\sqrt{-1}\,\mathfrak{a}_\mathfrak{p}))).$$

Then the representative $\bar{w}$ normalizes $G(\sigma)$ and $M(\sigma)$ for any $w\in W$ and we have the following lemma from [OS2, Lemma 8.12] ($M(\sigma)=U(\sigma)M^\sigma T^\sigma Z^\sigma$ under the notation there).

Lemma 1.5. (i) $\mathfrak{m}_\sigma=\mathfrak{m}(\sigma)+\mathfrak{g}(\sigma)$
and this is a decomposition into the direct sum of ideals of $\mathfrak{m}_\sigma$.

(ii) $G(\sigma)\subset H$ *and* $M(\sigma)\subset M$.

(iii) $M_\sigma=M(\sigma)G(\sigma)$.

(iv) $M_\sigma/(M_\sigma\cap\bar{w}^{-1}H\bar{w})\xleftarrow{\sim}M/(M\cap\bar{w}^{-1}H\bar{w})\xleftarrow{\sim}M(\sigma)/(M(\sigma)\cap\bar{w}^{-1}H\bar{w})$.

We put

$$\mathfrak{a}^x=\sum_{\alpha_j\in\Theta_x}\boldsymbol{R}H_j,$$
$$\mathfrak{a}_x=(\mathfrak{a}_\sigma\cap\mathfrak{h})+\sum_{\alpha\in\Psi-\Theta_x}\boldsymbol{R}H_j,$$
$$\mathfrak{a}(x)=\{Y\in\mathfrak{a};\langle Y,Z\rangle=0 \text{ for all } Z\in\mathfrak{a}_x\},$$
$$\mathfrak{n}(x)=\sum_{\alpha\in\Sigma_x^+}\mathfrak{g}^\alpha,$$
$$\mathfrak{n}_\sigma^-=\theta(\mathfrak{n}_\sigma),\quad \mathfrak{n}_x^-=\theta(\mathfrak{n}_x)\quad\text{and}\quad \mathfrak{n}(x)^-=\theta(\mathfrak{n}(x)).$$

Then $\mathfrak{a}_x$ is the Lie algebra of A_x and the following

$$\mathfrak{a}_\sigma=\mathfrak{a}^x+\mathfrak{a}_x=\mathfrak{a}(x)+\mathfrak{a}_x, \tag{1.7}$$

$$\mathfrak{n}_\sigma=\mathfrak{n}(x)+\mathfrak{n}_x \tag{1.8}$$

are decomposition into direct sums and we have

$$\langle\mathfrak{a}_x,\mathfrak{a}(x)\rangle=0, \tag{1.9}$$

$$[\mathfrak{n}(x),\mathfrak{n}_x]\subset\mathfrak{n}_x. \tag{1.10}$$

Let A^x, $N(x)$, N_σ^-, N_x^- and $N(x)^-$ be the analytic subgroups of G corresponding to $\mathfrak{a}^x$, $\mathfrak{n}(x)$, $\mathfrak{n}_\sigma^-$, $\mathfrak{n}_x^-$ and $\mathfrak{n}(x)^-$, respectively. Then we have

Lemma 1.6. *Fix an element $x=(g,t,w)$ of $\hat{X}$ and consider the map*

$$p_x^v\colon N_\sigma^-\times(M(\sigma)/(M(\sigma)\cap\bar{v}^{-1}\bar{w}^{-1}H\bar{w}\bar{v})\times A^x)\longrightarrow G/P(x)$$
$$(n,m,a)\longmapsto gn\bar{m}a\bar{v}^{-1}P(x)$$

for any $v\in W_x$, where $\bar{m}\in M(\sigma)$ is a representative of m.

(i) *The map p_x^v is well-defined and gives an analytic diffeomorphism onto an open subset of $G/P(x)$.*

(ii) $\bigcup_{v\in W_x}\operatorname{Im}p_x^v$ *is dense in $G/P(x)$ and for $v, v'\in W_x$*

$$\begin{cases}\operatorname{Im}p_x^v=\operatorname{Im}p_x^{v'} & \text{if } wvv'^{-1}w^{-1}\in W(\mathfrak{a};H),\\ \operatorname{Im}p_x^v\cap\operatorname{Im}p_x^{v'}=\emptyset & \text{if } wvv'^{-1}w^{-1}\notin W(\mathfrak{a};H).\end{cases}$$

Proof. We may assume $g=e$ and moreover $w=e$ by replacing σ and H by σ' and $\bar{w}^{-1}H\bar{w}$, respectively, where $\sigma'(y)=\bar{w}^{-1}\sigma(\bar{w}y\bar{w}^{-1})\bar{w}$ for $y\in G$.

(i) By the same reason as above we may assume $\bar{v}=e$ because M_x, A_x and N_x are stable under the inner automorphism defined by $\bar{v}^{-1}$. It is clear that the map p_x^v is well-defined.

First we will prove that p_x^v is injective. So we suppose that $n_1m_1a_1h_1a_1'n_1'=n_2m_2a_2h_2a_2'n_2'$ for some $n_j\in N_\sigma^-$, $m_j\in M(\sigma)$, $a_j\in A^x$, $h_j\in M_x\cap H$, $a_j'\in A_x$ and $n_j'\in N_x$ $(j=1,2)$. Then there exist $u_j\in N(x)^-$ and $u_j'\in N_x^-$ with $n_j=u_j'u_j$ $(j=1,2)$ because $N_\sigma^-=N_x^-N(x)^-$ (cf. (1.8) and (1.10)). Since $h_ja_j'=a_j'h_j$ $(j=1,2)$ and $A_\sigma=A^xA_x=A(x)A_x$ (cf. (1.7)), we can find $b_j\in A(x)$ and $b_j'\in A_x$ which satisfy $a_ja_j'=b_jb_j'$ $(j=1,2)$. Thus we have $u_1'\cdot u_1b_1m_1h_1\cdot b_1'\cdot n_1'=u_2'\cdot u_2b_2m_2h_2\cdot b_2'\cdot n_2'$, which implies $u_1'=u_2'$, $u_1b_1m_1h_1=u_2b_2m_2h_2$, $b_1'=b_2'$ and $n_1'=n_2'$ because $u_jb_jm_jh_j\in M_x$ and because the natural map of $N_x^-\times M_x\times A_x\times N_x$ to G is injective. Now the following Lemma 1.7 means $b_1=b_2$, $u_1=u_2$ and $m_1h_1=m_2h_2$. Thus we have $n_1=u_1'u_1=u_2'u_2=n_2$ and $m_1=m_2(h_2h_1^{-1})$ with $h_2h_1^{-1}=m_2^{-1}m_1\in M(\sigma)\cap H$. Moreover since $a_1a_1'=b_1b_1'=b_2b_2'=a_2a_2'$, the decomposition (1.7) says $a_1=a_2$.

Next we will prove that p_x^v is a submersion. For the proof it is sufficient to show that the natural map of $N_\sigma^-\times M(\sigma)\times A^x\times P(x)$ to G is a submersion. By using [OS1, Lemma 1.8], this follows from the fact that $\mathfrak{a}^x+\mathfrak{p}(x)$ and $\mathfrak{m}(\sigma)+\mathfrak{a}^x+\mathfrak{p}(x)$ are subalgebras of $\mathfrak{g}$ and that $\mathfrak{n}_\sigma^-+\mathfrak{m}(\sigma)+\mathfrak{a}^x+\mathfrak{p}(x)=\mathfrak{g}$. Here the last equality holds because $\mathfrak{m}(\sigma)+\mathfrak{a}^x+\mathfrak{p}(x)\supset\mathfrak{m}(\sigma)+\mathfrak{a}^x+\mathfrak{g}(\sigma)+\mathfrak{a}_x=\mathfrak{m}_\sigma+\mathfrak{a}_\sigma$ and $\mathfrak{p}(x)+\mathfrak{n}_\sigma^-\supset\mathfrak{m}_x\cap\mathfrak{h}+\mathfrak{n}_x+\mathfrak{n}_\sigma^-\supset\mathfrak{n}_\sigma+\mathfrak{n}_\sigma^-$.

(ii) Put $U^v=N_\sigma^-M(\sigma)A^x\bar{v}^{-1}P(x)$, which is an open subset of G by the above argument. Also by the proof of (i) we can conclude

$$U^v = N_x^-(N(x)^- A(x) M(\sigma) \bar{v}^{-1}(M_x \cap H)) A_x N_x.$$

The following lemma 1.7 says that $\bigcup_{v \in W_x} N(x)^- A(x) M(\sigma) \bar{v}^{-1}(M_x \cap H)$ is dense in M_x and therefore $\bigcup_{v \in W_x} U^v$ is dense in G because the natural map of $N_x^- \times M_x \times A_x \times N_x$ to G defines an analytic homeomorphism onto an open dense subset of G. This proves that $\bigcup_{v \in W_x} \mathrm{Im}\, p_x^v$ is dense in $G/P(x)$.

If $vv'^{-1} \in W(\mathfrak{a}; H) \cap W_x$, then $\bar{v}\bar{v}'^{-1} \in M(\sigma)(M_x \cap H)$ and it is clear that $\mathrm{Im}\, p_x^v = \mathrm{Im}\, p_x^{v'}$. Now assume $\mathrm{Im}\, p_x^v \cap \mathrm{Im}\, p_x^{v'} \neq \emptyset$. Then by the argument in the proof of (i) shows the existence of $u_j \in N(x)^-$, $b_j \in A(x)$, $m_j \in M(\sigma)$ and $h_j \in M_x \cap H$ $(j=1, 2)$ satisfying $u_1 b_1 m_1 \bar{v}^{-1} h_1 = a_2 b_2 m_2 \bar{v}'^{-1} h_2$. Then by Lemma 1.7 we have $m_1 \bar{v}^{-1} h_1 = m_2 \bar{v}'^{-1} h_2$, which means $\bar{v} m_1^{-1} m_2 \bar{v}'^{-1} = h_1 h_2^{-1} \in H$ and hence $vv'^{-1} \in W(\mathfrak{a}; H)$. Q.E.D.

Lemma 1.7. (i) *Let $n_j \in N(x)$, $a_j \in A(x)$, $m_j \in M_x \cap N_K(\mathfrak{a})$ and $h_j \in M_x \cap H$ $(j=1, 2)$. If $n_1 a_1 m_1 h_1 = n_2 a_2 m_2 h_2$, then $n_1 = n_2$ and $a_1 = a_2$.*

(ii) *The set $\bigcup_{v \in W_x} N(x) A(x) M(\sigma) \bar{v}(M_x \cap H)$ is open dense in M_x.*

(iii) *The above statements* (i) *and* (ii) *hold even if we replace $N(x)$ by $N(x)^-$.*

Proof. (i) This is proved in the same way as in the proof of [OS1, Lemma 1.9]. Hence we omit the proof.

(ii) Since $\mathfrak{m}_x = \mathfrak{n}(x) + \mathfrak{a}(x) + \mathfrak{m}(\sigma) + (\mathfrak{m}_x \cap \mathfrak{h})$ and since $\mathfrak{n}(x) + \mathfrak{a}(x)$ and $\mathfrak{n}(x) + \mathfrak{a}(x) + \mathfrak{m}(\sigma)$ are subalgebras, the set $N(x) A(x) M(\sigma)(M_x \cap H)$ is open in M_x (cf. [OS1, Lemma 1.8]). Moreover since $\bar{v}$ normalizes $N(x) A(x) M(\sigma)$ for any $v \in W_x$, the set $N(x) A(x) M(\sigma) \bar{v}(M_x \cap H)$ is also open in M_x.

Put $\mathfrak{m}_x^s = [\mathfrak{m}_x, \mathfrak{m}_x]$, let M_x^s be the analytic subgroup of M_x corresponding to $\mathfrak{m}_x^s$ and let W_x' be the quotient group of the normalizer of $\mathfrak{a}$ in $M_x^s \cap K$ by the centralizer of $\mathfrak{a}$ in $M_x^s \cap K$, which is isomorphic to W_x. Then $P(x)^s = N(x) A(x)(M_\sigma \cap M_x^s)$ is a parabolic subgroup of M_x^s and therefore it follows from [Ma] that the set $U_x^s = \bigcup_{u \in W_x'} P(x)^s \bar{u}(M_x^s \cap H)$ is open dense in M_x^s, where $\bar{u}$ are representatives of u. For $v \in W_x$ we have

$$\begin{aligned} N(x) A(x) M(\sigma) \bar{v}(M_x \cap H) &= N(x) A(x) M(\sigma) \bar{v} G(\sigma)(M_x \cap H) \\ &= N(x) A(x) M_\sigma \bar{v}(M_x \cap H). \end{aligned}$$

Therefore the set $\bigcup_{v \in W_x} N(x) A(x) M(\sigma) \bar{v}(M_x \cap H)$ contains $M_\sigma U_x^s$. Since U_x^s is dense in M_x^s and $M_x = M_\sigma M_x^s$, the set $M_\sigma U_x^s$ is dense in M_x.

(iii) This is proved in the same way as above. Q.E.D.

For $\varepsilon \in \{-1, 0, 1\}^l$ and $t \in \boldsymbol{R}^l$, we put $\boldsymbol{R}^l_\varepsilon = \{s \in \boldsymbol{R}^l;\ \mathrm{sgn}\, s = \varepsilon\}$, $P(\varepsilon) = (M_x \cap H) A_x N_x$ and $a(t) = a(e, t, e)$ with $x = (e, \varepsilon, e) \in \hat{X}$ and $(e, t, e) \in \hat{X} = G \times \boldsymbol{R}^l \times W$. We introduce a map

$$(1.11)\qquad \begin{array}{ccc}\tilde{\pi}\colon N_\sigma^- \times M(\sigma) \times \boldsymbol{R}_\varepsilon^l & \longrightarrow & G/P(\varepsilon) \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ (n, m, t) & \longmapsto & nma(t)P(\varepsilon).\end{array}$$

Lemma 1.6 implies that the map $\tilde{\pi}$ is a submersion.

We fix a basis $\{X_1, \cdots, X_L\}$ of $\mathfrak{n}_\sigma$ so that $X_i \in \mathfrak{g}^{\alpha(i)}$ with some $\alpha(i) \in \Sigma^+$ for $i=1, \cdots, L$, where $L=\dim \mathfrak{n}_\sigma$. Also we fix a basis $\{Z_1, \cdots, Z_{L'}\}$ of $\mathfrak{m}_\sigma$ so that $\{Z_1, \cdots, Z_{L''}\}$ is a basis of $\mathfrak{m}(\sigma)$ and $\{Z_{L''+1}, \cdots, Z_{L'}\}$ is a basis of $\mathfrak{g}(\sigma)$, where $L'=\dim \mathfrak{m}_\sigma$ and $L''=\dim \mathfrak{m}(\sigma)$. Moreover we put $l''=\dim \mathfrak{a}_\sigma$ and choose $H_{l+1}, \cdots, H_{l''} \in \mathfrak{a}_\sigma \cap \mathfrak{h}$ so that $\{H_1, \cdots, H_l, H_{l+1}, \cdots, H_{l''}\}$ is a basis of $\mathfrak{a}_\sigma$. We put $X_{-i}=-\sigma(X_i)$. Then $\{X_{-1}, \cdots, X_{-L}\}$ is a basis of $\mathfrak{n}_\sigma^-$ and $\{X_1, \cdots, X_L, X_{-1}, \cdots, X_{-L}, Z_1, \cdots, Z_{L'}, H_1, \cdots, H_{l''}\}$ is a basis of $\mathfrak{g}$.

Lemma 1.8. *For $Y \in \mathfrak{g}$, $\varepsilon \in \{-1, 0, 1\}^l$ and $p=(n, m, t) \in N_\sigma^- \times M(\sigma) \times \boldsymbol{R}_\varepsilon^l$ the vector field Y_ε on $G/P(\varepsilon)$ corresponding to the action of $\exp sY (s \in \boldsymbol{R})$ satisfies*

$$(Y_\varepsilon)_{\tilde{\pi}(p)} = d\tilde{\pi}_p\Big(\Big(\sum_{i=1}^{L}(c_i^+(nm)t^{2\alpha(i)}+c_i^-(nm))\operatorname{Ad}(m)X_{-i} + \sum_{j=1}^{L''}(c_i^0(nm)Z_j - \sum_{k=1}^{l} c_k(nm)t_k\frac{\partial}{\partial t_k}\Big)_p\Big).$$

Here X_{-i} and Z_j are identified with left invariant vector fields on N_σ^- and $M(\sigma)$, respectively, and $t^\lambda = t_1^{\lambda(H_1)} \cdots t_l^{\lambda(H_l)}$ for an element λ of the complexification $\mathfrak{a}_c^*$ of $\mathfrak{a}^*$. We define $t_k^{\lambda(H_k)}=1$ if $\lambda(H_k)=0$. Moreover the analytic functions c_i^+, c_i^-, c_j^0 and c_k on G are defined by

$$(1.12)\qquad \operatorname{Ad}(g)^{-1}Y = \sum_{i=1}^{L}(c_i^+(g)X_i + c_i^-(g)X_{-i}) + \sum_{j=1}^{L'} c_j^0(g)Z_j + \sum_{k=1}^{l''} c_k(g)H_k$$

for $g \in G$.

Proof. We put $x=(e, \varepsilon, e) \in \hat{X}$ and denote by $\mathfrak{p}(\varepsilon)$ the Lie algebra of $P(\varepsilon)$, which equals $(\mathfrak{m}_x \cap \mathfrak{h}) + \mathfrak{a}_x + \mathfrak{n}_x$. For $s \in \boldsymbol{R}$ satisfying $|s| \ll 1$ we write

$$(1.13)\qquad \exp(sY)nma(t) \in n\exp N(s)\cdot m\exp M(s)\cdot a(t)\exp A(s)\cdot P(\varepsilon)$$

with $N(s) \in \mathfrak{n}_\sigma^-$, $M(s) \in \mathfrak{m}(\sigma) \cap \mathfrak{q}$ and $A(s) \in \mathfrak{a}^x$. We multiply the above equation from the left with $(nma(t))^{-1}$ and differentiate the expression with respect to s at $s=0$. Then we have

$$(1.14)\qquad \operatorname{Ad} a(t)^{-1}\operatorname{Ad}(nm)^{-1}Y \equiv \operatorname{Ad} a(t)^{-1}\operatorname{Ad}(m)^{-1}\frac{dN}{ds}(0) + \operatorname{Ad} a(t)^{-1}\frac{dM}{ds}(0) + \frac{dA}{ds}(0) \mod \mathfrak{p}(\varepsilon).$$

On the other hand, for $t \in \boldsymbol{R}_\varepsilon^l$ and $m \in M(\sigma)$ we have

$$\mathrm{Ad}\, a(t)^{-1}X_i \equiv t^{2\alpha(i)}\, \mathrm{Ad}\, a(t)^{-1}X_{-i} \mod \mathfrak{p}(\varepsilon). \tag{1.15}$$

In fact, if $X_i \in \mathfrak{n}_x$, then $\mathfrak{n}_x \subset \mathfrak{p}(\varepsilon)$ and $t^{2\alpha(i)}=0$. And if $X_i \notin \mathfrak{n}_x$, then $X_i \in \mathfrak{m}_x$ and $\sigma(\mathrm{Ad}\, a(t)^{-1}X_i) = \sigma(|t^{\alpha(i)}|X_i) = -|t^{\alpha(i)}|X_{-i} = -t^{2\alpha(i)}\, \mathrm{Ad}\, a(t)^{-1}X_{-i}$, which means (1.15).

Putting $g=nm$ and applying $\mathrm{Ad}\, a(t)^{-1}$ to (1.12), we have

$$\begin{aligned}\mathrm{Ad}\, a(t)^{-1}\, \mathrm{Ad}\,(nm)^{-1}Y = \mathrm{Ad}\, a(t)^{-1}\Big(&\sum_{i=1}^{L}(c_i^+(nm)X_i + c_i^-(nm)X_{-i}) \\ &+\sum_{j=1}^{L'} c_j^o(nm)Z_j + \sum_{k=1}^{l''} c_k(nm)H_k\Big) \\ \equiv \mathrm{Ad}\, a(t)^{-1}\Big(&\sum_{i=1}^{L}(c_i^+(nm)t^{2\alpha(i)}X_{-i} + c_i^-(nm)X_{-i}) \\ &+\sum_{j=1}^{L''} c_j^o(nm)Z_j + \sum_{k=1}^{l} c_k(nm)H_k\Big) \quad \mod \mathfrak{p}(\varepsilon).\end{aligned}$$

Comparing this with (1.14), we have

$$\frac{dN}{dt}(0) = \mathrm{Ad}\,(m)\Big(\sum_{i=1}^{L}(c_i^+(nm)t^{2\alpha(i)} + c_i^-(nm))X_{-i}\Big),$$

$$\frac{dM}{dt}(0) \equiv \sum_{j=1}^{L''} c_j^o(nm)Z_j \quad \mod \mathfrak{m}(\sigma)\cap\mathfrak{h},$$

$$\frac{dA}{dt}(0) \equiv \sum_{k=1}^{l} c_k(nm)H_k \quad \mod \mathfrak{a}_x.$$

Identifying $\boldsymbol{R}_\varepsilon^l$ with A^x by the map $a(t)$, the vector field on A^x defined by $H_k \in \mathfrak{a}^x$ corresponds to the operator $-t_k(\partial/\partial t_k)$. Hence we have the lemma.

Q.E.D.

For every $g \in G$ and $w \in W$, we put $U_g^W = \pi(gN_\sigma^- M(\sigma) \times \boldsymbol{R}^l \times \{w\})$. Then Lemma 1.6 shows the bijectivity of the continuous map

$$\begin{aligned}\phi_g^w\colon N_\sigma^- \times (M(\sigma)/(M(\sigma)\cap \bar{w}^{-1}H\bar{w})) \times \boldsymbol{R}^l &\xrightarrow{\ \sim\ } U_g^w (\subset \tilde{X}) \\ \rotatebox{90}{$\in$}\qquad\qquad &\qquad\ \rotatebox{90}{$\in$} \\ (n, m, t)\qquad\qquad &\longmapsto \pi(gn\bar{m}, t, w).\end{aligned} \tag{1.16}$$

For brevity we put $U^w = N_\sigma^- \times (M(\sigma)/(M(\sigma)\cap \bar{w}^{-1}H\bar{w})) \times \boldsymbol{R}^l$. Then we have

Lemma 1.9. *Fix $g, g' \in G$ and $w, w' \in W$.*

(i) *For an element Y of $\mathfrak{g}$ the local 1-parameter group of transformations $(\phi_g^w)^{-1} \circ \exp(sY) \circ \phi_g^w$ of U^w $(s \in \boldsymbol{R}, |s| \ll 1)$ defines an analytic* *vector field.*

(ii) *The map* $(\phi_{g'}^{w'})^{-1}\circ\phi_g^w$ *of* $(\phi_g^w)^{-1}(U_g^w\cap U_{g'}^{w'})$ *onto* $(\phi_{g'}^{w'})^{-1}(U_g^w\cap U_{g'}^{w'})$ *defines an analytic diffeomorphism between the open subset of* U^w *onto that of* $U^{w'}$.

(iii) ϕ_g^w *is a homeomorphism onto an open subset* U_g^w *of* $\tilde{X}$.

Proof. (i) We may assume $w=e$. It follows from Lemma 1.6 that the local 1-parameter group defines an analytic vector field $Y(\varepsilon)$ on $N_\sigma^-\times(M(\sigma)/(M(\sigma)\cap\bar{w}^{-1}H\bar{w}))\times\boldsymbol{R}_\varepsilon^l$ for any $\varepsilon\in\{-1,0,1\}^l$. Then Lemma 1.8 shows that these vector fields $Y(\varepsilon)$ piece together and define an analytic vector field on U_g^w.

(ii) We have only to show that $(\phi_{g'}^{w'})\circ\phi_g^w$ is analytic because the map is bijective and its inverse is of the same form. Moreover we may assume $g'=e$ because $(\phi_{g'}^{w'})^{-1}\circ\phi_g^w=(\phi_e^{w'})^{-1}\circ\phi_{g'^{-1}g}^w$. We fix an arbitrary point $p=(n_o, m_o, t_o)$ of the domain of the map $(\phi_e^{w'})^{-1}\circ\phi_g^w$ and put $x=(gn_o\bar{m}_o, t_o, w)\in\hat{X}$ and $p'=(n_o', m_o', t_o')=(\phi_e^{w'})^{-1}\circ\phi_g^w(p)\in U^{w'}$. We will show that the map is analytic in a neighborhood of p.

First we assume that $w'=w$ and $g\in N_\sigma^-M(\sigma)A^x$. Put $g=n_1m_1a_1$ with $n_1\in N_\sigma^-$, $m_1\in M(\sigma)$ and $a_1\in A^x$. Since $(gn\bar{m}, t, w)=(n_1m_1a_1n\bar{m}, t, w)\sim(n_1m_1a_1n(m_1a_1)^{-1}m_1\bar{m}, a_1t, w)$ for $n\in N_\sigma^-$, $\bar{m}\in M(\sigma)$, $t\in\boldsymbol{R}^l$ and $w\in W$, we have $(\phi_e^w)^{-1}\circ\phi_g^w(n, m, t)=(n_1m_1a_1n(m_1a_1)^{-1}, m_1m, a_1t)$, where

$$a_1t=(\exp\langle-\alpha_1, \log a_1\rangle t_1, \cdots, \exp\langle-\alpha_l, \log a_1\rangle t_l).$$

Hence the map is analytic.

Next we consider the case where $w'=w$ and $p'=p=(e, 1, \varepsilon)$ with an $\varepsilon\in\{-1,0,1\}^l$. Here 1 is the residue class of e in $M(\sigma)/(M(\sigma)\cap\bar{w}^{-1}H\bar{w})$. Then $g\in P(x)$. It follows from Lemma 1.9 (i) that there exist neighborhoods V of the origin in $\mathfrak{p}(x)$ and U_0 of p in U^w such that for any $Y\in V$ and $s\in[0,1]$, the map $(\phi_e^w)^{-1}\circ\exp(sY)\circ\phi_e^w$ defines an analytic diffeomorphism of U_0 onto a neighborhood of p. Since $(\phi_e^w)^{-1}\circ\exp(sY)\circ\phi_e^w\,|\,U_0=(\phi_e^w)^{-1}\circ\phi_{\exp(sY)}^w\,|\,U_0$, we have the claim if $g\in\exp V$. On the other hand any $g\in P(x)$ can be written in the form $g=g_0g_1\cdots g_k$ with $g_0\in M(\sigma)\cap\bar{w}^{-1}H\bar{w}$ and $g_j\in\exp V$ $(j=1,\cdots,k)$. Here k is a suitable positive integer. Then the relation

$$(\phi_e^w)^{-1}\circ\phi_g^w=((\phi_e^w)^{-1}\circ\phi_{g_0}^w)\circ((\phi_e^w)^{-1}\circ\phi_{g_1}^w)\circ\cdots\circ((\phi_e^w)^{-1}\circ\phi_{g_k}^w)$$

holds in the domain of the right hand side. Since $(\phi_e^w)^{-1}\circ\phi_{g_i}^w$ are analytic in some neighborhoods of p in U^w and map the point p to the same point for $i=0,\cdots,k$, we have the claim.

Now consider the case where $w'\neq w$, $g=e$ and $p=(e, 1, \varepsilon)$. Then under the notation in Lemma 1.2 we have $p'=(e, 1, \varepsilon)$ when $g'=\bar{v}'m'm^{-1}\bar{v}^{-1}$, which we will assume in this case. Put $\tilde{R}^l_\varepsilon=\{t\in R^l;\ \operatorname{sgn} t_i=\varepsilon_i \text{ if } \varepsilon_i\neq 0\}$. We will prove that $(\phi^{w'}_{g'})^{-1}\circ\phi^w_e$ is analytic in the set $U^w(\varepsilon)=N^-_\sigma\times(M(\sigma)/(M(\sigma)\cap\bar{w}^{-1}H\bar{w}))\times\tilde{R}^l_\varepsilon$. For any $(n, m, t)\in U^w(\varepsilon)$, we put $n'=g'^{-1}ng'$ and $a(t)'=g'^{-1}a(t)g'$. Let t' be the element of $R^l_{\operatorname{sgn} t}$ satisfying $a(t')=a(t)'$ and let m' be the residue class of $g'^{-1}\bar{m}g'$ in $M(\sigma)/(M(\sigma)\cap\bar{w}'^{-1}H\bar{w}')$ with a representative $\bar{m}$ of m. Then it follows from Lemma 1.2 that $(\phi^{w'}_{g'})^{-1}\circ\phi^w_e(n, m, t)=(n', m', t')$. Hence if the correspondence which maps t to t' as above is analytic on $\tilde{R}^l_\varepsilon$, we can conclude the claim. Put

$$I=\{i\in\{1,\cdots,l\};\ \varepsilon_i=0\} \text{ and } J=\{1,\cdots,l\}-I.$$

Then we have

$$v'v^{-1}\alpha_i=\alpha_i+\sum_{k\in J}m^k_i\alpha_k \qquad \text{if } i\in I,$$
$$v'v^{-1}\alpha_j=\sum_{k\in J}m^k_j\alpha_k \qquad \text{if } j\in J$$

because $v'v^{-1}$ belongs to W_x which is generated by the reflections with respect to α_j with $j\in J$. Here m^k_i and m^k_j are integers. Thus we have

$$\begin{aligned} t'_i&=\operatorname{sgn} t_i\cdot\exp\langle-\alpha_i, \log(g'^{-1}a(t)g')\rangle\\ &=\operatorname{sgn} t_i\cdot\exp\langle-\alpha_i, \operatorname{Ad}(g'^{-1})(\sum_{t_\nu\neq 0}-\log|t_\nu|H_\nu)\rangle\\ &=\operatorname{sgn} t_i\cdot\exp\langle v'v^{-1}\alpha_i, \sum_{t_\nu\neq 0}\log|t_\nu|H_\nu\rangle\\ &=\begin{cases} t_i|\prod_{k\in J}t_k^{m^k_i}| & \text{if } i\in I,\\ \operatorname{sgn} t_i|\prod_{k\in J}t_k^{m^k_i}| & \text{if } i\in J.\end{cases}\end{aligned}$$

Since $t_k\neq 0$ if $t\in\tilde{R}^l_\varepsilon$ and $k\in J$, we have the claim.

We consider the general case. We put $g_1=(n_0\bar{m}_0a(t_0))^{-1}$, $g_2=g_3^{-1}g_4^{-1}gg_1^{-1}$, $g_3=\bar{v}'m'm^{-1}\bar{v}^{-1}$ under the notation in Lemma 1.2, $g_4=n'_0\bar{m}'_0a(t'_0)$, $\psi_i=(\phi^w_e)^{-1}\circ(\phi^w_{g_i})$ for $i=1, 2$, $\psi_3=(\phi^{w'}_{g_3})^{-1}\circ(\phi^w_e)$ and $\psi_4=(\phi^{w'}_e)^{-1}\circ(\phi^{w'}_{g_4})$. Then $(\phi^{w'}_e)^{-1}\circ\phi^w_g=\psi_4\circ\psi_3\circ\psi_2\circ\psi_1$. It follows from what we have proved that this map is analytic in a neighborhood of p because $\psi_1(p)=(e, 1, \operatorname{sgn} x)$ and $g_2\in P(x)$.

(iii) Let V be an open subset of U^w and let (g', t', w') be a point of $\pi^{-1}(\phi^w_g(V))$. Put $\Omega=(\phi^{w'}_{g'})^{-1}(U^{w'}_{g'}\cap\phi^w_g(V))$. Then Ω is open in $U^{w'}$ and contains $(e, 1, t')$. Therefore we can find neighborhoods V_0 of the origin in $\mathfrak{g}$ and V_1 of t_0 in R^l so that $(\phi^{w'}_{g'})^{-1}\circ\exp(sY)\circ\phi^{w'}_{g'}(e, 1, t')$ is contained in Ω for any $s\in[0, 1]$, $Y\in V_0$ and $t\in V_1$, from which it follows that $\exp(V_0)\times V_1\times\{w'\}\subset\pi^{-1}(\phi^w_g(V))$. This implies that $\pi^{-1}(\phi^w_g(V))$ is open in $\hat{X}$ and hence $\phi^w_g(V)$ is open in $\tilde{X}$. We have proved that ϕ^w_g is an open map and we have the lemma. Q.E.D.

Lemma 1.9 shows that we can define an analytic structure on $\tilde{X}$ through the maps ϕ_g^w so that they define analytic diffeomorphisms onto open subsets U_g^w of $\tilde{X}$. Then we have

Theorem 1.10. (i) *$\tilde{X}$ is a connected compact real analytic manifold without boundaries and $\bigcup_{g\in G, w\in W} U_g^w$ is an open covering of $\tilde{X}$ such that ϕ_g^w are real analytic diffeomorphisms.*

(ii) *The action of G on $\tilde{X}$ is analytic and the G-orbital structure is of normal crossing type in the sense of* [O1, *Remark* 6].

(iii) *The orbit $G\pi(x)$ for a point x in $\hat{X}$ is isomorphic to $G/P(x)$ and the orbital decomposition of $\tilde{X}$ is of the form*

$$\tilde{X}=\bigcup_{\varepsilon\in\{-1,0,1\}^l, v\in W(\mathfrak{a};H)\backslash W/W_{(\varepsilon)}} G/P(e, \varepsilon, \bar{v}) \qquad (\textit{disjoint union}),$$

where $(\varepsilon)=(e, \varepsilon, e)\in\hat{X}$ and $\bar{v}$ is a representative of v.

(iv) *There are just 2^l open orbits and they are isomorphic to G/H. The number of compact orbits in $\tilde{X}$ equals that of the elements of the coset $W(\mathfrak{a}; H)\backslash W$ and the orbits are contained in the closure of every open orbit.*

Proof. The definition of $\tilde{X}$ and Lemma 1.9 prove (i), (ii) and (iii) except the connectedness, compactness and Hausdorff separation axiom for $\tilde{X}$. But the connectedness is clear because $\pi(G\times \boldsymbol{R}^l\times\{e\})$ is connected and contains any open orbit.

Put $U=\bigcup_{w\in W} U_e^w$. It follows from Lemma 1.6 (ii), (1.5) and (1.6) that the intersection of U and any G-orbit in $\tilde{X}$ is open dense in the orbit. Hence for $x_j\in\tilde{X}$ $(j=1, 2)$, the sets $V_j=\{g\in G; gx_j\in U\}$ are open dense in G and therefore we can choose an element g of G with $g^{-1}x_j\in U$ for $j=1, 2$. Then $x_j\in gU=\bigcup_{w\in W} U_g^w$. Since Lemma 1.6 (ii) also says that $U_g^w=U_g^{w'}$ if $w'\in W(\mathfrak{a}; H)w$ and $U_g^w\cap U_g^{w'}=\emptyset$ otherwise, the set gU satisfies the separation axiom. Since gU is open in $\tilde{X}$, two points x_1 and x_2 can be separated by their disjoint open neighborhoods.

Since $G=KAH$ (cf. [F]), $G=\bigcup_{w\in W} K\bar{A}_+\bar{w}H$. Here $\bar{A}_+=\{\exp X;\ X\in\mathfrak{a}$ with $\alpha(X)\geqq 0$ for all $\alpha\in\Sigma^+\}$, which equals $\{\exp(-\sum_j(\log t_j)H_j);\ (t_1, \cdots, t_l)\in(0, 1]^l\}$. Hence Definition 1.1, (1.5) and (1.6) prove that the compact set $\pi(K\times[-1, 1]^l\times W)$ contains all open G-orbits in $\tilde{X}$. This means the compact set is dense in $\tilde{X}$ and therefore it must coincide with $\tilde{X}$.

Let $x=(g, t, w)$ and $x'=(g', t', w')$ be elements in $\hat{X}$. Suppose $G\pi(x)$ and $G\pi(x')$ is open in $\tilde{X}$. Then sgn t and sgn t' belong to $\{-1, 1\}^l$ and if sgn $t=$ sgn t', then $G\pi(x)=G\pi(x')$. Hence the statement (iv) is clear from (iii). Q.E.D.

The orbits which are not open in $\tilde{X}$ are called boundary orbits in $\tilde{X}$.

The compact boundary orbits are called distinguished boundaries of the open orbits and the open orbits are isomorphic to G/H.

We give a lemma concerning the action of G on $\tilde{X}$ which will be used later. Let $g \in G$ and $w \in W$ and put $\Omega_g^w = (\phi_g^w)^{-1}(U_e^w \cap U_g^w)$. For any $(n, m, t) \in \Omega_g^w$ we define an element $H(g, n, m, t)$ of $\mathfrak{a}$ by

$$gn\bar{m}a(t) \in N_\sigma^- M(\sigma) a(t) \exp H(g, n, m, t)(M_x \cap \bar{w}^{-1} H \bar{w}) N_x, \tag{1.17}$$

where $x = (n\bar{m}, t, w) \in \hat{X}$. Moreover we put $(n'(g, n, m, t), m'(g, n, m, t), t'(g, n, m, t)) = (\phi_e^w)^{-1}(g(\phi_e^w(n, m, t)))$, which is denoted by $(n, m, t)^g$. Then

$$t_i' = t_i \exp\langle -\alpha_i, H(g, n, m, t)\rangle \qquad \text{for } i = 1, \cdots, l. \tag{1.18}$$

Lemma 1.11. *Let $(n, m, t) \in \Omega_g^w$ and assume $t_i = 0$. Then*

$$\frac{\partial t_i'}{\partial t_j}(g, n, m, t) = \begin{cases} \exp\langle -\alpha_i, H(g, n, m, t)\rangle & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases} \tag{1.19}$$

Proof. The following identity follows from the definition of $H(g, n, m, t)$:

$$H(g'g, n, m, t) = H(g, n, m, t) + H(g', (n, m, t)^g) \tag{1.20}$$

for $(n, m, t) \in \Omega_g^w \cap \Omega_{g'g}^w$.

If $g \in M_\sigma$, then (1.19) is clear because $t_i' = t_i$ and $H(g, n, m, t) = 0$. On the other hand it follows from (1.20) that the lemma holds for $g'g$ on $\Omega_g^w \cap \Omega_{g'g}^w$ if it holds for g' and g. Then by the same argument as in the proof of Lemma 1.9 (ii) we may assume that $g = \exp sY$ with $Y \in \mathfrak{g}$ and $|s| \ll 1$.

Let ν be the orthogonal projection of $\mathfrak{g}$ onto $\mathfrak{a}$ with respect to the Killing form. Then from (1.17) we have $(d/ds) H(\exp sY, n, m, t)|_{s=0} = \nu(\mathrm{Ad}\,(n\bar{m}a(t))^{-1} Y) = \nu(\mathrm{Ad}(n^{-1})Y)$. Combining this with (1.20), we see that $u = \exp\langle -\alpha_i, H(\exp sY, n, m, t)\rangle$ satisfies

$$\frac{du}{ds} = -\langle \alpha_i, \nu(\mathrm{Ad}(n'(\exp sY, n, m, t))^{-1} Y)\rangle u. \tag{1.21}$$

Suppose $t_1 > 0, \cdots, t_l > 0$. Then the above statement and (1.21) imply that $t_i'(\exp sY, n, m, t)$ also satisfies (1.21) and that

$$\frac{d}{ds}\left(\frac{\partial t_i'}{\partial t_j}\right) = -\langle \alpha_i, \nu(\mathrm{Ad}\,(n'\,(\exp sY, n, m, t))^{-1} Y)\rangle \frac{\partial t_i'}{\partial t_j}$$
$$-\left\langle \alpha_i, \frac{\partial}{\partial t_j} \nu(\mathrm{Ad}\,(n'(\exp sY, n, m, t))^{-1} Y)\right\rangle t_i'.$$

By analyticity this means that the function $(\partial t'_i/\partial t_j)(\exp sY, t, n, m)|_{t_i=0}$ also satisfies (1.21). Hence by putting $g=\exp sY$, the both sides of the equation (1.19) satisfy the same differential equation (1.21) and have the same initial value at $s=0$. This implies (1.19). Q.E.D.

§ 2. Invariant differential operators

For a real Lie algebra $\mathfrak{b}$ we denote by $\mathfrak{b}_c$ the complexification of $\mathfrak{b}$ and for a real or complex Lie subalgebra $\mathfrak{u}$ of $\mathfrak{g}_c$ we denote by $S(\mathfrak{u})$ and $U(\mathfrak{u})$ the symmetric algebra and the universal enveloping algebra of $\mathfrak{u}'$, respectively, where $\mathfrak{u}'$ is the complex Lie subalgebra of $\mathfrak{g}_c$ generated by $\mathfrak{u}$. For a non-negative integer m, we put $U^m(\mathfrak{u})=\Lambda(S^m(\mathfrak{u}))$, where $S^m(\mathfrak{u})$ is the set of homogeneous elements of degree m in $S(\mathfrak{u})$ and Λ is the symmetrization of $S(\mathfrak{u})$ onto $U(\mathfrak{u})$. Moreover we put $U_m(\mathfrak{u})=\oplus_{k=0}^{m} U^k(\mathfrak{u})$. Then $U_m(\mathfrak{u})/U_{m-1}(\mathfrak{u})$ is isomorphic to $S^m(\mathfrak{u})$. For a subset $\mathfrak{b}$ of $\mathfrak{g}_c$, $S(\mathfrak{u})^{\mathfrak{b}}$ (resp. $U(\mathfrak{u})^{\mathfrak{b}}$) the subalgebras of $\mathfrak{b}$-invariants of $S(\mathfrak{u})$ (resp. $U(\mathfrak{u})$).

Now retain the notation in Section 1. The complex linear extensions of the involution σ and θ on $\mathfrak{g}_c$ are also denoted by the same letters. Let $\mathfrak{j}$ be a maximal abelian subspace of $\mathfrak{q}$ containing $\mathfrak{a}$. By [OS2, Lemma 2.4] we have $[\mathfrak{j}, \mathfrak{a}_\mathfrak{p}]=0$ and we can choose a Cartan subalgebra $\tilde{\mathfrak{j}}$ of $\mathfrak{g}$ which contains both $\mathfrak{j}$ and $\mathfrak{a}_\mathfrak{p}$. Then the pairs $(\mathfrak{g}_c, \tilde{\mathfrak{j}}_c)$, $(\mathfrak{g}_c, \mathfrak{j}_c)$ and $(\mathfrak{g}_c, (\mathfrak{a}_\mathfrak{p})_c)$ define root systems, which we denote by $\Sigma(\tilde{\mathfrak{j}})$, $\Sigma(\mathfrak{j})$ and $\Sigma(\mathfrak{a}_\mathfrak{p})$, respectively, and we can define compatible orders for $\Sigma(\tilde{\mathfrak{j}})$, $\Sigma(\mathfrak{j})$, $\Sigma(\mathfrak{a}_\mathfrak{p})$ and Σ (cf. [OS2, §3.8]). We denote by $\Sigma(\tilde{\mathfrak{j}})^+$, $\Sigma(\mathfrak{j})^+$ and $\Sigma(\mathfrak{a}_\mathfrak{p})^+$ the corresponding sets of positive roots and by $W(\tilde{\mathfrak{j}})$, $W(\mathfrak{j})$ and $W(\mathfrak{a}_\mathfrak{p})$ the corresponding Weyl groups, respectively. Moreover we put $\rho=\frac{1}{2}\sum_{\alpha\in\Sigma(\tilde{\mathfrak{j}})^+}\alpha$. Then ρ is an element of the complexification $\tilde{\mathfrak{j}}_c^*$ of the dual space of $\tilde{\mathfrak{j}}$.

Let $\mathfrak{n}_c$ be the nilpotent subalgebra of $\mathfrak{g}_c$ corresponding to $\Sigma(\mathfrak{j})^+$ and put $\bar{\mathfrak{n}}_c=\sigma(\mathfrak{n}_c)$. From the Iwasawa decomposition $\mathfrak{g}_c=\bar{\mathfrak{n}}_c+\mathfrak{j}_c+\mathfrak{h}_c$ with respect to σ we have the decomposition into the direct sum

$$U(\mathfrak{g})=\bar{\mathfrak{n}}_c U(\bar{\mathfrak{n}}_c+\mathfrak{j}_c)\oplus U(\mathfrak{j})\oplus U(\mathfrak{g})\mathfrak{h}. \tag{2.1}$$

Let δ be the projection of $U(\mathfrak{g})$ to $U(\mathfrak{j})$ with respect to this decomposition and η the algebra automorphism of $U(\mathfrak{j})$ defined by $\eta(Y)=Y-\rho(Y)$ for $Y\in\mathfrak{j}$. Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$ and $U(\mathfrak{g})^{\mathfrak{h}}$ the centralizer of $\mathfrak{h}$ in $U(\mathfrak{g})$. Then the map $\gamma=\eta\circ\delta$ induces the Harish-Chandra isomorphism

$$\bar{\gamma}:\ U(\mathfrak{g})^{\mathfrak{h}}/(U(\mathfrak{g})^{\mathfrak{h}}\cap U(\mathfrak{g})\mathfrak{h})\xrightarrow{\sim} I(\mathfrak{j}), \tag{2.2}$$

where $I(\mathfrak{j})$ is the set of $W(\mathfrak{j})$-invariant elements of $U(\mathfrak{j})$. We remark here that $\gamma(U(\mathfrak{g})^{\mathfrak{h}}\cap U_m(\mathfrak{g}))=I(\mathfrak{j})\cap U_m(\mathfrak{j})$.

Lemma 2.1. *For any* $h\in H$, $\mathrm{Ad}(h)$ *acts trivially on the algebra*

$U(\mathfrak{g})^{\mathfrak{h}}/(U(\mathfrak{g})^{\mathfrak{h}} \cap U(\mathfrak{g})\mathfrak{h})$.

Proof. First note that $\mathrm{Ad}(h)$ really acts on the algebra because $\mathrm{Ad}(h)\mathfrak{h}=\mathfrak{h}$. Put $\kappa_h=\bar{\gamma}\circ\mathrm{Ad}(h)\circ\bar{\gamma}^{-1}$. By [Hel] the quotient field of $\gamma(Z(\mathfrak{g}))$ coincides with that of $I(\mathfrak{j})$. Hence for any $D\in I(\mathfrak{j})$ there exist D_1 and D_2 in $\gamma(Z(\mathfrak{g}))$ which satisfies $D_1D=D_2$ and $D_1\neq 0$. Then $D_2=\kappa_h(D_2)=\kappa_h(D_1D)=\kappa_h(D_1)\kappa_h(D)=D_1\kappa_h(D)$, which means $\kappa_h(D)=D$. This implies the lemma.
Q.E.D.

Let $D(G/H)$ (resp. $D(G/G_o^\sigma)$) denote the algebras of invariant differential operators on G/H (resp. G/G_o^σ). Then $D(G/G_o^\sigma)$ is naturally isomorphic to the algebra $U(\mathfrak{g})^{\mathfrak{h}}/(U(\mathfrak{g})^{\mathfrak{h}}\cap U(\mathfrak{g})\mathfrak{h})$ and Lemma 2.1 assures that $D(G/H)$ is also isomorphic to this. Hence by identifying these algebras we have the algebra isomorphism

(2.3) $$\bar{\gamma}\colon D(G/H)\xrightarrow{\sim} I(\mathfrak{j})$$

and the natural projection

(2.4) $$\Gamma\colon U(\mathfrak{g})^{\mathfrak{h}}\longrightarrow D(G/H)$$

which satisfy $\gamma=\bar{\gamma}\circ\Gamma$.

By the Killing form $\langle\ ,\ \rangle$ of $\mathfrak{g}$ we identify the complexification $\bar{\mathfrak{j}}_c^*$ of the dual space of $\bar{\mathfrak{j}}$ with $\bar{\mathfrak{j}}_c$ and then $\mathfrak{a}_c^*$ and $\mathfrak{j}_c^*$ are identified with subspaces of $\bar{\mathfrak{j}}_c$. Let $\Psi(\mathfrak{j})=\{\alpha'_1,\cdots,\alpha'_{l'}\}$ be the fundamental system of $\Sigma(\mathfrak{j})^+$, where l' is the rank of the symmetric space G/H. We put

$$\mathfrak{g}_c(\mathfrak{j};\lambda)=\{Y\in\mathfrak{g}_c;\ [Z,Y]=\lambda(Z)Y \text{ for all } Z\in\mathfrak{j}\}\qquad \text{for } \lambda\in\Sigma(\mathfrak{j})^+,$$
$$m(\mathfrak{j})=\{Y\in\mathfrak{h}_c;\ [Z,Y]=0 \text{ for all } Z\in\mathfrak{j}\}.$$

For each subset F of $\Psi(\mathfrak{j})$, we define (cf. [Sc, §3.2]):

$$\langle F\rangle=\Sigma(\mathfrak{j})\cap\textstyle\sum_{\alpha\in F}\boldsymbol{R}\alpha$$
$$\mathfrak{a}_F=\{Y\in\mathfrak{j}_c;\ \alpha(Y)=0 \text{ for all } \alpha\in F\}$$
$$\mathfrak{a}^F=\{Y\in\mathfrak{j}_c;\ \alpha(Y)=0 \text{ for all } \alpha\in\Psi(\mathfrak{j})-F\}$$
$$\mathfrak{a}(F)=\{Y\in\mathfrak{j}_c;\ \langle Z,Y\rangle=0 \text{ for all } Z\in\mathfrak{a}_F\}$$
$$\mathfrak{n}_F=\textstyle\sum_{\lambda\in\Sigma(\mathfrak{j})^+-\langle F\rangle}\mathfrak{g}_c(\mathfrak{j};\alpha) \quad\text{and}\quad \bar{\mathfrak{n}}_F=\sigma(\mathfrak{n}_F)$$
$$\mathfrak{n}(F)=\textstyle\sum_{\lambda\in\Sigma(\mathfrak{j})^+\cap\langle F\rangle}\mathfrak{g}_c(\mathfrak{j};\alpha) \quad\text{and}\quad \bar{\mathfrak{n}}(F)=\sigma(\mathfrak{n}(F))$$
$$\mathfrak{m}_F=\bar{\mathfrak{n}}(F)+\mathfrak{m}(\mathfrak{j})+\mathfrak{a}(F)+\mathfrak{n}(F)$$
$$\mathfrak{p}_F=\mathfrak{m}_F+\mathfrak{a}_F+\mathfrak{n}_F$$
$$W_F=\{w\in W(\mathfrak{j});\ wY=Y \text{ for all } Y\in\mathfrak{a}_F\}.$$

Let $\mathfrak{q}_F$ denote the orthocompliment of $\mathfrak{m}_F \cap \mathfrak{h}_c$ in $\mathfrak{h}_c$. Then $\mathfrak{g}_c$ decomposes as $\mathfrak{g}_c = \bar{\mathfrak{n}}_F + \mathfrak{m}_F + \mathfrak{a}_F + \mathfrak{q}_F$ and it follows that

$$(2.5) \qquad U(\mathfrak{g}) = \bar{\mathfrak{n}}_F U(\bar{\mathfrak{n}}_F + \mathfrak{a}_F + \mathfrak{m}_F) \oplus U(\mathfrak{m}_F + \mathfrak{a}_F) \oplus U(\mathfrak{g})\mathfrak{q}_F.$$

Let δ_F be the projection of $U(\mathfrak{g})$ to $U(\mathfrak{m}_F + \mathfrak{a}_F)$ with respect to this decomposition. Then δ_F maps $U(\mathfrak{g})^{\mathfrak{h}}$ (resp. $U(\mathfrak{g})\mathfrak{h}$) into $U(\mathfrak{m}_F + \mathfrak{a}_F)^{\mathfrak{m}_F \cap \mathfrak{h}_c}$ (resp. $U(\mathfrak{m}_F + \mathfrak{a}_F)(\mathfrak{m}_F \cap \mathfrak{h}_c)$) because $\mathfrak{m}_F \cap \mathfrak{h}_c$ normalizes $\bar{\mathfrak{n}}_F$ and $\mathfrak{q}_F$. Hence δ_F induces the map

$$(2.6) \quad U(\mathfrak{g})^{\mathfrak{h}}/(U(\mathfrak{g})^{\mathfrak{h}} \cap U(\mathfrak{g})\mathfrak{h}) \longrightarrow U(\mathfrak{m}_F + \mathfrak{a}_F)^{\mathfrak{m}_F \cap \mathfrak{h}_c}/(U(\mathfrak{m}_F + \mathfrak{a}_F)^{\mathfrak{m}_F \cap \mathfrak{h}_c} \cap U(\mathfrak{m}_F + \mathfrak{a}_F)(\mathfrak{m}_F \cap \mathfrak{h}_c)).$$

Let δ^F be the projection of $U(\mathfrak{m}_F + \mathfrak{a}_F)$ to $U(\mathfrak{j})$ with respect the decomposition

$$(2.7) \qquad U(\mathfrak{m}_F + \mathfrak{a}_F) = \bar{\mathfrak{n}}(F) U(\bar{\mathfrak{n}}(F) + \mathfrak{j}_c) \oplus U(\mathfrak{j}) \oplus U(\mathfrak{m}_F + \mathfrak{a}_F)(\mathfrak{m}_F \cap \mathfrak{h}_c).$$

Then $\delta = \delta^F \circ \delta_F$. Let η_F be the algebra automorphism of $U(\mathfrak{m}_F + \mathfrak{a}_F)$ defined by $\eta_F(Y+Z) = Y + Z - \rho(Z)$ for $Y \in \mathfrak{m}_F$ and $Z \in \mathfrak{a}_F$. Denoting $\rho_F(Z) = \frac{1}{2}\,\mathrm{trace}_C\,(ad(Z)\,|\,\mathfrak{n}(F))$ for $Z \in \mathfrak{j}$, we define an algebra automorphism η^F of $U(\mathfrak{j})$ so that $\eta^F(Z) = Z - \rho_F(Z)$. We put $\gamma_F = \eta_F \circ \delta_F$ and $\gamma^F = \eta^F \circ \delta^F$. Then

$$(2.8) \qquad \gamma = \gamma^F \circ \gamma_F.$$

Let $U(\mathfrak{j})^{W_F}$ denote the set of W_F-invariant elements of $U(\mathfrak{j})$. Then as in the case of $\bar{\gamma}$, the map γ^F induces the algebra isomorphism

$$(2.9) \quad \bar{\gamma}^F : U(\mathfrak{m}_F + \mathfrak{a}_F)^{\mathfrak{m}_F \cap \mathfrak{h}}/(U(\mathfrak{m}_F + \mathfrak{a}_F)^{\mathfrak{m}_F \cap \mathfrak{h}} \cap U(\mathfrak{m}_F + \mathfrak{a}_F)(\mathfrak{m}_F \cap \mathfrak{h})) \xrightarrow{\ \sim\ } U(\mathfrak{j})^{W_F}.$$

Lemma 2.2. *Let J be an ideal of $U(\mathfrak{j})$ which is generated by some homogeneous elements $p_1, \cdots, p_k$ in $I(\mathfrak{j})$. Then there exist finite elements $D_1, \cdots, D_n$ in $U(\mathfrak{g})^{\mathfrak{h}}$ which satisfy*

$$(2.10) \qquad \gamma(D_j) \in \sum_{i=1}^{k} I(\mathfrak{j}) p_i \qquad \textit{for } j = 1, \cdots, n$$

and moreover the following condition:

Let m be a non-negative integer and F a subset of $\Psi(\mathfrak{j})$. Then for any element q of $J \cap U_m(\mathfrak{j}) \cap U(\mathfrak{m}_F + \mathfrak{a}_F)^{\mathfrak{m}_F \cap \mathfrak{h}}$, there exist elements $Q_1, \cdots, Q_n$ of $U(\mathfrak{m}_F + \mathfrak{a}_F)^{\mathfrak{m}_F \cap \mathfrak{h}}$ such that

$$(2.11) \qquad \sum_{j=1}^{n} Q_j D_j - \eta^{-1}(q) \in \bar{\mathfrak{n}}_F U(\bar{\mathfrak{n}}_F + \mathfrak{m}_F + \mathfrak{a}_F) + U(\mathfrak{g})\mathfrak{h},$$

(2.12) $$Q_jD_j \in U_m(\mathfrak{g}) \qquad \textit{for } j=1, \cdots, n.$$

Proof. By Lemma 2.3 below, we have $d_1, \cdots, d_n \in I(\mathfrak{j})$ and $q_1, \cdots, q_n \in U(\mathfrak{j})$ such that $q=\sum_j q_jd_j$ and $q_jd_j \in U_m(\mathfrak{j})$. Replacing q_j by $(1/\sharp W_F)\cdot\sum_{w\in W_F} wq_j$, we may assume q_j are W_F-invariant. We can choose $D_j \in U(\mathfrak{g})^{\mathfrak{h}}$ and $Q_j \in U(\mathfrak{m}_F+\mathfrak{a}_F)^{\mathfrak{m}_F\cap\mathfrak{h}}$ with $\gamma(D_j)=d_j$ and $\gamma^F\circ\eta_F(Q_j)=q_j$ (cf. (2.3) and (2.9)). Moreover we can assume $Q_jD_j \in U_m(\mathfrak{g})$. Since $[Q_j, \bar{\mathfrak{n}}_F]\subset \bar{\mathfrak{n}}_F U(\mathfrak{m}_F+\mathfrak{a}_F)$, we have $\gamma_F(Q_jD_j)=\eta_F(Q_j)\gamma_F(D_j)$ by definition. Owing to the algebra isomorphism $\bar{\gamma}^F$, we have $\gamma^F\circ\gamma_F(Q_jD_j)=\gamma^F(\eta_F(Q_j)\gamma_F(D_j))=\gamma^F\circ\eta_F(Q_j)\cdot\gamma^F\circ\gamma_F(D_j)=q_jd_j$ because $\eta_F(Q_j)$ and $\gamma_F(D_j) \in U(\mathfrak{m}_F+\mathfrak{a}_F)^{\mathfrak{m}_F\cap\mathfrak{h}}$. Therefore $\gamma^F(\gamma_F(\sum_j Q_jD_j-\eta^{-1}(q)))=0$, which means

$$\gamma_F(\textstyle\sum_j Q_jD_j-\eta^{-1}(q)) \in U(\mathfrak{m}_F+\mathfrak{a}_F)(\mathfrak{m}_F\cap\mathfrak{h})$$

because of (2.9). On the other hand the definition of γ_F proves $\sum_j Q_jD_j -\eta^{-1}(q)-\gamma_F(\sum_j Q_jD_j-\eta^{-1}(q)) \in \bar{\mathfrak{n}}_F U(\bar{\mathfrak{n}}_F+\mathfrak{m}_F+\mathfrak{a}_F)+U(\mathfrak{g})\mathfrak{h}$. Hence the lemma holds. Q.E.D.

Lemma 2.3. *Let J be an ideal of $U(\mathfrak{j})$ generated by some elements $p_1, \cdots, p_k$ in $I(\mathfrak{j})$. Then there exist finite elements $d_1, \cdots, d_n$ of $\sum_i I(\mathfrak{j})p_i$ which satisfy the following*:

For any element q of $J\cap U_m(\mathfrak{j})$, there exist $q_1, \cdots, q_n \in U(\mathfrak{j})$ such that $q=\sum_j q_jd_j$ and $q_id_i \in U_m(\mathfrak{j})$ for $i=1, \cdots, n$.

Proof. Let $H(\mathfrak{j})$ be the set of harmonic polynomials in $U(\mathfrak{j})$ corresponding to $W(\mathfrak{j})$. Then $U(\mathfrak{j})=H(\mathfrak{j})\otimes I(\mathfrak{j})$, which implies $J=H(\mathfrak{j})\otimes(\sum_i I(\mathfrak{j})p_i)$. Put $J_\nu=U_\nu(\mathfrak{j})\cap\sum_i I(\mathfrak{j})p_i$. Since $U_m(\mathfrak{j})=\sum_\nu(H(\mathfrak{j})\cap U_{m-\nu}(\mathfrak{j}))\otimes(I(\mathfrak{j})\cap U_\nu(\mathfrak{j}))$, we have

(2.13) $$J\cap U_m(\mathfrak{j})=\sum_{\nu=0}^{m}(H(\mathfrak{j})\cap U_{m-\nu}(\mathfrak{j}))\otimes J_\nu.$$

Let $S(\mathfrak{j})^{W(\mathfrak{j})}$ be the set of $W(\mathfrak{j})$-invariant elements of $S(\mathfrak{j})$ and put $\bar{J}=\oplus_{\nu=0}^{\infty}J_\nu/J_{\nu-1}$. Then $\bar{J}$ defines an ideal of $S(\mathfrak{j})^{W(\mathfrak{j})}$. Since $S(\mathfrak{j})^{W(\mathfrak{j})}$ is noetherian, there exist homogeneous generators $\bar{d}_1, \cdots, \bar{d}_n$ of $\bar{J}$. Let $d_1, \cdots, d_n$ be the elements of $\sum_i I(\mathfrak{j})p_i$ whose residue classes equal $\bar{d}_1, \cdots, \bar{d}_n$, respectively. Then for any $q \in J_\nu$, we can find $q_1, \cdots, q_n \in I(\mathfrak{j})$ by the induction on ν so that $q=\sum_j q_jd_j$ and $q_jd_j \in U_\nu(\mathfrak{j})$. Combining this with (2.13), we have the claim in the lemma. Q.E.D.

Now we want to study G-invariant differential operators on the G-manifold $\tilde{X}$ constructed in Section 1. Let $\{w_1, \cdots, w_r\}$ be a complete set of representatives of the coset $W(\mathfrak{a}; H)\backslash W$, where

(2.14) $$r=[W: W(\mathfrak{a}; H)].$$

Then [OS2, Corollray 7.10] assures that we may assume $\bar{w}_1=e$ and

$$\text{(2.15)}\qquad \mathrm{Ad}(\bar{w}_j)\tilde{\mathfrak{j}}=\tilde{\mathfrak{j}},\quad \mathrm{Ad}(\bar{w}_j)\mathfrak{j}=\mathfrak{j},\quad \mathrm{Ad}(\bar{w}_j)\mathfrak{a}_{\mathfrak{p}}=\mathfrak{a}_{\mathfrak{p}}$$
$$\text{and } w_j(\Sigma(\mathfrak{j})_\theta^+)=\Sigma(\mathfrak{j})_\theta^+ \quad \text{for } j=1,\cdots,r$$

where $\Sigma(\mathfrak{j})_\theta^+=\{\alpha\in\Sigma(\mathfrak{j})^+;\ \alpha\,|\,\mathfrak{a}=0\}$. Using the decomposition $U(\mathfrak{g})=\bar{\mathfrak{n}}_c U(\bar{\mathfrak{n}}_c+\mathfrak{j}_c)\oplus U(\mathfrak{j})\oplus U(\mathfrak{g})\,\mathrm{Ad}(\bar{w}_j^{-1})\mathfrak{h}_c$ in place of (2.1), we can define an isomorphism $\bar{\gamma}^j$ of $D(G/\bar{w}_j^{-1}H\bar{w}_j)$ onto $I(\mathfrak{j})$ in the same way as $\bar{\gamma}$. For each $\varepsilon\in\{-1,1\}^l$ we put $X_\varepsilon=\pi(G\times \boldsymbol{R}_\varepsilon^l\times W)$. Then X_ε is isomorphic to G/H. Moreover for each $w\in W$ we define the map

$$\text{(2.16)}\qquad \begin{array}{rcl}\iota_\varepsilon^w:\ G/\bar{w}^{-1}H\bar{w} & \xrightarrow{\ \sim\ } & X_\varepsilon\subset\tilde{X}\\ \cup\!\!\!\!\!\!\;\; & & \cup\\ g\bar{w}^{-1}H\bar{w} & \longmapsto & \pi(g,\varepsilon,w)\quad \text{for } g\in G.\end{array}$$

Since $\mathrm{Ad}(\bar{w}^{-1})$ defines an isomorphism of $U(\mathfrak{g})^{\mathfrak{h}}$ onto $U(\mathfrak{g})^{\mathrm{Ad}(\bar{w}^{-1})\mathfrak{h}}$, it induces an isomorphism of $D(G/H)$ onto $D(G/\bar{w}^{-1}H\bar{w})$, which is also denoted by $\mathrm{Ad}(\bar{w}^{-1})$. Let τ_ε be the automorphism of $U(\mathfrak{g})$ defined by

$$\text{(2.17)}\qquad \tau_\varepsilon(Y)=\prod_{i=1}^{l}\varepsilon_i^{\alpha(H_i)}Y\quad \text{for } \alpha\in\Sigma\cup\{0\}\ \text{ and } Y\in\mathfrak{g}^\alpha.$$

Since τ_ε preserves $\mathrm{Ad}(\bar{w}_j^{-1})\mathfrak{h}$, it induces an automorphism of $D(G/\bar{w}_j^{-1}H\bar{w}_j)$, which is also denoted by τ_ε.

Lemma 2.4. (i) *$\tau_\varepsilon(D)=D$ for any $D\in D(G/\bar{w}_j^{-1}H\bar{w}_j)$.*
(ii) *$\bar{\gamma}^j\circ\mathrm{Ad}(\bar{w}_j^{-1})(D)=\bar{\gamma}(D)$ for any $D\in D(G/H)$.*

Proof. (i) Since τ_ε preserves $\mathrm{Ad}(\bar{w}_j^{-1})\mathfrak{h}_c$ and $\bar{\mathfrak{n}}_c$, and is trivial on $\mathfrak{j}_c$, we have (i) from the definition of the isomorphism $\bar{\gamma}^j$.

(ii) Let G^d be the analytic subgroup of the adjoint group of $\mathfrak{g}_c$ corresponding to $(\mathfrak{k}\cap\mathfrak{h})+\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q})+\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h})+(\mathfrak{p}\cap\mathfrak{q})$ and let K^d be the analytic subgroup of G^d corresponding to $(\mathfrak{k}\cap\mathfrak{h})+\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h})$. Then K^d is a maximal compact subgroup of G^d and the Weyl group $W(\mathfrak{j})$ is naturally identified with the normalizer of $\mathfrak{a}_{\mathfrak{p}}^d$ in K^d modulo centralizer of $\mathfrak{a}_{\mathfrak{p}}^d$ in K^d, where $\mathfrak{a}_{\mathfrak{p}}^d=\sqrt{-1}(\mathfrak{j}\cap\mathfrak{k})+(\mathfrak{j}\cap\mathfrak{p})$. Since $\mathrm{Ad}(\bar{w}_j)$ defines an element of $W(\tilde{\mathfrak{j}})$, [*W*, Proposition 1.1.3.3] assures that there exists an element k in K^d such that $\mathrm{Ad}(\bar{w}_j)\,|\,\mathfrak{j}_c=k\,|\,\mathfrak{j}_c$.

Let u be an element of $Z(\mathfrak{g})$ and u' an element of $U(\mathfrak{j})$ defined by $u-u'\in\bar{\mathfrak{n}}_cU(\mathfrak{g})+U(\mathfrak{g})\mathfrak{h}_c$. Then $u-k(u')\in k(\bar{\mathfrak{n}}_cU(\mathfrak{g}))+U(\mathfrak{g})\mathfrak{h}_c$ because $k(u)=u$ and $k(\mathfrak{h}_c)=\mathfrak{h}_c$. Applying $\mathrm{Ad}(\bar{w}_j^{-1})$ to this relation, we have $u-u'\in\bar{\mathfrak{n}}_cU(\mathfrak{g})+U(\mathfrak{g})\,\mathrm{Ad}(\bar{w}_j^{-1})\mathfrak{h}_c$. This means that the map $\bar{\gamma}^j\circ\mathrm{Ad}(\bar{w}_j^{-1})\circ\bar{\gamma}^{-1}$ is trivial on $\bar{\gamma}(Z(\mathfrak{g}))$. Since the map is an algebra automorphism of $I(\mathfrak{j})$ and the quotient field of $\bar{\gamma}(Z(\mathfrak{g}))$ coincide with that of $I(\mathfrak{j})$, we can conclude that the map is trivial on $I(\mathfrak{j})$. Q.E.D.

Let $D(\tilde{X})$ denote the ring of G-invariant differential operators on $\tilde{X}$ whose coefficients are real analytic functions. Then we have

Theorem 2.5. *There exists a surjective algebra isomorphism*

$$\bar{\gamma}:\ D(\tilde{X}) \longrightarrow I(\mathfrak{j})$$
$$\quad\ \ \cup\!\!\!\!\!\!\!\!\quad \qquad\quad \cup\!\!\!\!\!\!\!\!\quad$$
$$D \longmapsto \bar{\gamma}^j \circ \tau_\varepsilon \circ (\iota_\varepsilon^{w_j})^{-1}(D\,|\,X_\varepsilon),$$

which does not depend on $\varepsilon \in \{-1, 1\}^l$ *and* $j=1, \cdots, r$.

Proof. Fix an element $u \in I(\mathfrak{j})$. Then the differential operator $D_\varepsilon^j = \iota_\varepsilon^{w_j} \circ \tau_\varepsilon \circ (\bar{\gamma}^j)^{-1}(u)$ on X_ε is G-invariant and it follows from Lemma 2.4 that $D_\varepsilon^j = \iota_\varepsilon^{w_j} \circ (\bar{\gamma}^j)^{-1}(u) = \iota_\varepsilon^{w_j} \circ (\bar{\gamma}^j)^{-1} \circ \bar{\gamma} \circ (\bar{\gamma}^{-1}(u)) = \iota_\varepsilon^{w_j} \circ \mathrm{Ad}\,(\bar{w}_j^{-1})(\bar{\gamma}^{-1}(u))$. Comparing this with (2.16) and Lemma 1.2 and the definition of $\tilde{X}$, we see that D_ε^j does not depend on j, so we define a G-invariant differential operator D_U on $U = \bigcup_{\varepsilon \in \{-1,1\}^l} X_\varepsilon$ by

$$D_U\,|\,X_\varepsilon = \iota_\varepsilon^{w_j} \circ \tau_\varepsilon \circ (\bar{\gamma}^j)^{-1}(u) \tag{2.18}$$

for $\varepsilon \in \{-1, 1\}^l$ and $j=1, \cdots, l$.

To get the theorem we have only to show that D_U has an analytic extension on $\tilde{X}$ because U is open dense in $\tilde{X}$. Since $\tilde{X} = \bigcup_{g \in G, 1 \le j \le r} U_g^{w_j}$ (cf. Theorem 1.10 (i)), the proposition follows if $D_U\,|\,U \cap U_g^{w_j}$ has an analytic extension on $U_g^{w_j}$ for any $g \in G$ and any $j=1, \cdots, l$.

Let p_g^ε denote the submersion of $N_\sigma^- M(\sigma) A$ onto the open subset $U \cap U_g^{w_j}$ defined by $p_g^\varepsilon(nma) = \pi(gnm, (\varepsilon_1 a^{-\alpha_1}, \cdots, \varepsilon_l a^{-\alpha_l}), w_j)$ for $n \in N_\sigma^-$, $m \in M(\sigma)$ and $a \in A$. Let u_j be an element of $U(\mathfrak{g})^{\mathrm{Ad}(\bar{w}_j^{-1})\mathfrak{h}}$ which corresponds to $(\bar{\gamma}^j)^{-1}(u)$ and let u'_j be an element of $U(\mathfrak{n}_\sigma^- + \mathfrak{m}(\sigma) + \mathfrak{a})$ with $u_j - u'_j \in U(\mathfrak{g})\,\mathrm{Ad}\,(\bar{w}_j^{-1})\mathfrak{h}$. Then for any C^∞-function ϕ on X_ε, we have

$$u'_j(\phi \circ p_g^\varepsilon) = (D_U \phi) \circ p_g^\varepsilon \tag{2.19}$$

where u'_j acts on $\phi \circ p_g^\varepsilon$ from the right.

Let

$$Y = \sum_{i=1}^{L} C_i X_{-i} + \sum_{j=1}^{L''} C''_j Z_j + \sum_{k=1}^{l} C'_k H_k \tag{2.20}$$

be an element of $\mathfrak{n}_\sigma^- + \mathfrak{m}(\sigma) + \mathfrak{a}$ under the notation just before Lemma 1.8. Since $\mathrm{Ad}\,(a) X_{-i} = a^{-\alpha(i)} X_{-i}$ for $a \in A$, the action of Y on the Lie group $N_\sigma^- M(\sigma) A$ from the right is expressed as the following vector field on $N_\sigma^- \times M(\sigma) \times A$:

$$(2.21)\qquad \sum_{i=1}^{L} C_i a^{-\alpha(i)} \operatorname{Ad}(m) X_{-i} + \sum_{j=1}^{L''} C''_j Z_j + \sum_{k=1}^{l} C'_k H_k.$$

Identifying $\boldsymbol{R}^l_\varepsilon$ with A by the map $\boldsymbol{R}^l_\varepsilon \ni t \mapsto a(t) = \exp(-\sum_k \log|t_k| H_k) \in A$, we have the corresponding expression

$$(2.22)\qquad \sum_{i=1}^{L} C_i \,|t_1^{\langle\alpha(i), H_1\rangle} \cdots t_l^{\langle\alpha(i), H_l\rangle}| \operatorname{Ad}(m) X_{-i} + \sum_{j=1}^{L''} C''_j Z_j - \sum_{k=1}^{l} C'_k t_k \frac{\partial}{\partial t_k}$$

on $N^-_\sigma \times M(\sigma) \times \boldsymbol{R}^l_\varepsilon$. Applying τ_ε to Y, we have the expression for $\tau_\varepsilon(Y)$

$$(2.23)\qquad \sum_{i=1}^{L} C_i t^{\alpha(i)} \operatorname{Ad}(m) X_{-i} + \sum_{j=1}^{L''} C''_j Z_j - \sum_{k=1}^{l} C'_k t_k \frac{\partial}{\partial t_k}$$

in place of Y. This vector field has analytic extension on $N^-_\sigma \times M(\sigma) \times \boldsymbol{R}^l$ and is moreover independent of ε.

The above statement holds for any $Y \in \mathfrak{n}^-_\sigma + \mathfrak{m}(\sigma) + \mathfrak{a}$. Hence the similar statement holds for $u' \in U(\mathfrak{n}^-_\sigma + \mathfrak{m}(\sigma) + \mathfrak{a})$ and therefore the operator $D_U \mid U \cap U^{w_j}_g$ has an analytic extension on U (cf. (2.19)). Q.E.D.

Now we review systems of differential equations with regular singularities defined in [O4]. Let M be an $(l+n)$-dimensional real analytic manifold with a local coordinate system $(t, x) = (t_1, \cdots, t_l, x_1, \cdots, x_n)$ and let $N_1, \cdots, N_l$ be hypersurfaces of M such that each N_j is defined by the equation $t_j = 0$. We put $N = N_1 \cap \cdots \cap N_l$, $D_x = (\partial/\partial x_1, \cdots, \partial/\partial x_n)$, $D_t = (\partial/\partial t_1, \cdots, \partial/\partial t_l)$, $\vartheta_j = t_j \partial/\partial t_j$, $\vartheta = (\vartheta_1, \cdots, \vartheta_l)$ and $tD_x = (t_1\partial/\partial x_1, t_1\partial/\partial x_2, \cdots, t_l\partial/\partial x_n)$. Let Z be an open subset of $\boldsymbol{C}^N$, let $\mathcal{O}_Z$ be the sheaf of holomorphic functions on Z and let ${}_Z\mathcal{A}_M$ (resp. ${}_Z\mathcal{A}_N$) be the sheaves of real analytic functions on M (resp. N) with holomorphic parameter λ in Z. Moreover let ${}_Z\mathcal{D}_M$ be the sheaf of differential operators on X whose coefficients are sections of ${}_Z\mathcal{A}_M$. We consider the system of differential equations

$$\mathcal{M}:\ P_i(\lambda, x, t, D_x, D_t)u = 0 \qquad (i = 1, \cdots, L)$$

with one unknown function u, where P_i are sections of ${}_Z\mathcal{D}_M$. We remark that if there is no holomorphic parameter we write $\mathcal{A}_M$, $\mathcal{D}_M$, etc. in place of ${}_Z\mathcal{A}_M$, ${}_Z\mathcal{D}_M$, etc., respectively.

Let ${}_Z\mathcal{D}^r_M$ denote the subRing of ${}_Z\mathcal{D}_M$ whose sections P are of the form

$$(2.24)\qquad P = \sum_{\alpha \in N^l} a_\alpha(\lambda, x)\vartheta^\alpha + \sum_{j=1}^{l} \sum_{\alpha \in N^l, \beta \in N^n} t_j b_{j\,\alpha,\beta}(\lambda, t, x)\vartheta^\alpha D_x^\beta$$

where $a_\alpha \in {}_Z\mathcal{A}_N$, $b_{j,\alpha,\beta} \in {}_Z\mathcal{A}_M$, $\vartheta^\alpha = \vartheta_1^{\alpha_1} \cdots \vartheta_l^{\alpha_l}$ and $D_x^\beta = (\partial/\partial x_1)^{\beta_1} \cdots (\partial/\partial x_n)^{\beta_n}$.

Let $\boldsymbol{C}[s]$ denote the polynomial ring over $\boldsymbol{C}$ with l indeterminants $s_1, \cdots, s_l$ and put ${}_Z\mathscr{A}_N[s]={}_Z\mathscr{A}_N\otimes\boldsymbol{C}[s]$ and $\mathscr{O}_Z[s]=\mathscr{O}_Z\otimes\boldsymbol{C}[s]$. Then we have an algebra homomorphism σ_* of ${}_Z\mathscr{D}^r_M$ onto ${}_Z\mathscr{A}_N[s]$ defined by $\sigma_*(P)(\lambda, x, s)=\sum_\alpha a_\alpha(\lambda, x)s^\alpha$ for P of the form (2.24).

For any point of N we assume that there exist sections $Q_1, \cdots, Q_{n'}$ of ${}_Z\mathscr{D}^r_M\cap\sum_i {}_Z\mathscr{D}_M P_i$ over a neighborhood of the point which satisfy the following three conditions:

$$\text{Put } q_j=\sigma_*(Q_j) \quad \text{and} \quad \bar{J}=\textstyle\sum_j \mathscr{O}_Z[s]q_j.$$

Then

(2.25) $\qquad q_j$ do not depend on x for $j=1, \cdots, n'$.

(2.26) $\mathscr{O}_Z[s]/\bar{J}$ is a free $\mathscr{O}_Z$-Module of rank r, where r is a certain non-negative integer.

(2.27) For any $q\in\bar{J}$ there exist a section Q of ${}_Z\mathscr{D}^r_M\cap\sum_i {}_Z\mathscr{D}_M P_i$ so that $\sigma_*(Q)=q$ and $\operatorname{ord} Q=\deg q$. Here $\deg f$ means the degree of $f\in\mathscr{O}_z[s]$ with respect to s and $\operatorname{ord} Q$ means the order of a differential operator Q.

Then the system $\mathscr{M}$ is said to have regular singularities in the weak sense along the set of walls $\{N_1, \cdots, N_l\}$ with the edge N. The solutions $s_\nu(\lambda)=(s_{\nu,1}(\lambda), \cdots, s_{\nu,l}(\lambda))$ of the indicial equation

(2.28) $$\bar{\mathscr{M}}:\ q_j(\lambda, s)=0 \qquad \text{for } j=1, \cdots, n'$$

are called characteristic exponents. They are indexed by $\nu\in\{1, \cdots, r\}$ because $\bar{\mathscr{M}}$ has r solutions including their multiplicities. For simplicity we moreover assume

(2.29) $s_{\nu,i}(\lambda)$ are holomorphic with respect to λ for $\nu=1, \cdots, r$ and $i=1, \cdots, l$.

Remark 2.6. By the coordinate transformation $t_i\mapsto t_i^k$ $(1\leq i\leq l)$ with a large positive integer k, the operators $Q_j\in{}_Z\mathscr{D}^r_M$ change into the form $Q_j(\lambda, t, x, \vartheta, tD_x)$. In [O4] we defined systems of differential equations with regular singularities and assumed there that Q_j are of the form $Q_j(\lambda, t, x, \vartheta, tD_x)$. Hence here we say that the system $\mathscr{M}$ has regular singularities "in the weak sense". We note that each characteristic exponent $s_\nu(\lambda)$ changes into $ks_\nu(\lambda)$ by the coordinate transformation. The other assumptions are little stronger than the ones given in [O4]. But the assumptions here are sufficient for our purpose.

To define the boundary value map for solutions of $\mathcal{M}$ we assume the following condition:

(2.30) For any point of each wall N_j we can find a section R_j of $\sum_{iZ}\mathscr{D}_M P_i$ over a neighborhood of the point such that R_j is of the form

$$R_j = \sum_{\alpha\in N^l} a^j_\alpha(\lambda, x)\vartheta^\alpha + \sum_{\alpha\in N^l,\beta\in N^n} t_j b^j_{\alpha,\beta}(\lambda, t, x)\vartheta^\alpha D^\beta_x$$

and the coefficient $a^j_{(0,\cdots,m_j,0,\cdots,0)}(\lambda, x)$ of $\vartheta_j^{m_j}$ does not vanish for any λ and x, where $m_j = \operatorname{ord} R_j$.

Now we consider the system of differential equations

$$(2.31) \qquad \mathcal{N}\colon\ P_i u = 0 \qquad \text{for } i=1, \cdots, L$$

on the manifold $\tilde{X}$, where $P_i \in D(\tilde{X})$. We say that $\mathcal{N}$ is $D(\tilde{X})$-finite if the dimension of $D(\tilde{X})/\sum_i D(\tilde{X})P_i$ is finite. It is equivalent to say that there exist $l'(=\operatorname{rank} G/H)$ algebraically independent elements in $\{P_1, \cdots, P_L\}$.

We fix an $\varepsilon \in \{-1, 0, 1\}^l$ and a $w \in W$ so that $\varepsilon \notin \{-1, 1\}^l$. Then the set $X_{\varepsilon,w} = \pi(G\times\{\varepsilon\}\times\{w\})$ is one of the boundary orbits in $\tilde{X}$. Put $I(\varepsilon) = \{i \in \{1, \cdots, l\};\ \varepsilon_i = 0\}$ and $X^i_{\varepsilon,w} = \{\pi(g, \varepsilon', w);\ g \in G,\ \varepsilon'_i = 0 \text{ and } \varepsilon'_j = \varepsilon_j \text{ if } \varepsilon_j \neq 0\}$ for $i \in I(\varepsilon)$.

Theorem 2.7. *Assume the system $\mathcal{N}$ on $\tilde{X}$ is $D(\tilde{X})$-finite. Then for any $\varepsilon \in \{-1, 0, 1\}^l - \{-1, 1\}^l$ and any $w \in W$, $\mathcal{N}$ has regular singularities in the weak sense along the set of walls $\{X^i_{\varepsilon,w};\ i \in I(\varepsilon)\}$ with the edge $X_{\varepsilon,w}$ and satisfies the condition* (2.30). *The indicial equation equals*

$$\bar{\mathcal{N}}\colon\ (\rho - \sum_{k\in I(\varepsilon)} s_k\alpha_k)(q) = 0 \qquad \text{for all } q \in U(\mathfrak{a}_\varepsilon) \cap \sum_{j=1}^{L} U(\mathfrak{j})\bar{\gamma}(P_j)$$

with $\mathfrak{a}_\varepsilon = \sum_{k\in I(\varepsilon)} \boldsymbol{R}H_k$. Here $(\rho - \sum_{i\in I(\varepsilon)} s_i\alpha_i)$ is the algebra homomorphism of $U(\mathfrak{j})$ to the polynomial ring of s_i ($i \in I(\varepsilon)$) induced by

$$\mathfrak{j} \ni Y \longmapsto (\rho - \sum_{i\in I(\varepsilon)} s_i\alpha_i)(Y).$$

Proof. We may assume $w = w_j$ with a suitable $j \in \{1, \cdots, r\}$. We will prove in the case where $w = e$. The proof in the general case is also obtained in the following argument by replacing $\mathfrak{h}$ and H by $\operatorname{Ad}(\bar{w}_j^{-1})\mathfrak{h}$ and $\bar{w}_j^{-1}H\bar{w}_j$, respectively.

Fix any $g \in G$. By the map ϕ^e_g (cf. (1.16)) we identify $N^-_\sigma \times (M(\sigma)/(M(\sigma) \cap H)) \times \boldsymbol{R}^l$ with the open subset U^e_g of $\tilde{X}$. As we have seen in the proof of Theorem 2.5, any element v of $U(\mathfrak{n}^-_\sigma) \otimes U(m(\sigma) + \mathfrak{a})^{\mathfrak{m}(\sigma)\cap\mathfrak{h}}$ defines a differential operator on U_g^e through the map ϕ_g^e and the expression (2.23) for $Y \in \mathfrak{n}_\sigma^- + \mathfrak{m}(\sigma) + \mathfrak{a}$.

Put $p_i = \bar{\gamma}(P_i)$ and $J = \sum_i U(\mathfrak{j})p_i$. Since the assumption is equivalent to the condition that $U(\mathfrak{j})/J$ is of finite dimension, the dimension of the space $U(\mathfrak{a}_\varepsilon)/(J \cap U(\mathfrak{a}_\varepsilon))$ is also finite. Let q be an element of $J \cap U(\mathfrak{a}_\varepsilon)$. Put $F = \{\alpha \in \Psi(\mathfrak{j});\ \alpha\,|\,\mathfrak{a}_\varepsilon = 0\}$. Then $\bar{\mathfrak{n}}_F \subset (\mathfrak{n}_\sigma^-)_c$ and $\bar{\mathfrak{n}}_F + \mathfrak{m}_F + \mathfrak{a}_F \supset (\mathfrak{n}_\sigma^- + \mathfrak{m}_\sigma + \mathfrak{a}_\sigma)_c$. Then Lemma 2.2 says that there exist $D_1, \cdots, D_n \in U(\mathfrak{g})^{\mathfrak{h}}$ and $S_1, \cdots, S_n \in U(\mathfrak{m}_F + \mathfrak{a}_F)^{\mathfrak{m}_F \cap \mathfrak{h}}$ so that $\gamma(D_j) \in J$, $S_j D_j \in U_{\deg q}(\mathfrak{g})$ and

$$(2.32) \qquad \sum_{j=1}^{n} S_j D_j - \eta^{-1}(q) \in \bar{\mathfrak{n}}_F U(\bar{\mathfrak{n}}_F + \mathfrak{m}_F + \mathfrak{a}_F) + U(\mathfrak{g})\mathfrak{h}.$$

Let S_j' and D_j' be elements of $U(\mathfrak{n}_\sigma^-) \otimes U(\mathfrak{m}(\sigma) + \mathfrak{a})^{\mathfrak{m}(\sigma) \cap \mathfrak{h}}$ so that $S_j - S_j'$ and $D_j - D_j'$ belong to $U(\mathfrak{g})(\mathfrak{h} \cap \mathfrak{m}(\sigma)^\perp)$. Then we have

$$(2.33) \qquad \sum_{j=1}^{n} S_j' D_j' - \eta^{-1}(q) \in \bar{\mathfrak{n}}_F U(\mathfrak{n}_\sigma^-) \otimes U(\mathfrak{m}(\sigma) + \mathfrak{a})^{\mathfrak{m}(\sigma) \cap \mathfrak{h}} + U(\mathfrak{g})\mathfrak{h}$$

because $(S_j - S_j')D_j \in U(\mathfrak{g})\mathfrak{h}$. Let Q be the differential operator on U_g^e corresponding to $\sum_j S_j' D_j'$. Then clearly $Q \in \sum_i \mathscr{D}_{\tilde{X}} P_i$ and ord $Q = \deg q$. Since $t^{\alpha(i)} \in \sum_{i \in I(\varepsilon)} \mathscr{A}_{\tilde{X}} t_i$ if $X_{-\alpha(i)} \in \bar{\mathfrak{n}}_F$, it follows from the expression (2.23) that the operator Q is of the form (2.24) and $\sigma_*(Q)(s) = -(\sum_{i \in I(\varepsilon)} s_i \alpha_i)\eta^{-1}(q)$. (We remark that $\mathcal{O}_Z = C$ and that t_i ($i \in I(\varepsilon)$) correspond to $t_1, \cdots, t_l$ in (2.24). Hence if $\varepsilon = (0, \cdots, 0)$, the correspondence is straightforward.)

Let $\{q_1, \cdots, q_n\}$ be a basis of $J \cap U(\mathfrak{a}_\varepsilon)$ and let Q_j be the differential operators on U_g^e which are constructed as above corresponding to q_j, respectively. Then it is clear that $Q_1, \cdots, Q_n$ satisfy the conditions (2.25), (2.26) and (2.27) and that the indicial equation equals $\mathscr{N}$ given in Theorem 2.7.

If we consider the case where $X_{\varepsilon,w}$ is a hypersurface, it is also clear that the system $\mathscr{N}$ satisfies the condition (2.30). Q.E.D.

Let $\lambda \in \mathfrak{j}_c^*$. We define an algebra homomorphism χ_λ of $D(\tilde{X})$ to $\boldsymbol{C}$ by $\chi_\lambda(D) = \lambda(\bar{\gamma}(D))$. Then $\chi_\lambda = \chi_{\lambda'}$ if and only if $\lambda' \in W(\mathfrak{j})\lambda$. In most cases we only consider the system

$$(2.34) \qquad \mathscr{M}_\lambda\colon\ (D - \chi_\lambda(D))u = 0 \qquad \text{for all } D \in D(\tilde{X}).$$

The following proposition gives the characteristic exponents of $\mathscr{M}_\lambda$.

Proposition 2.8. *Retain the notation in Thorem 2.7 and put* $W(\varepsilon) = \{w \in W(\mathfrak{j});\ w\,|\,\mathfrak{a}_\varepsilon = \mathrm{id}\}$, $J = \sum_{p \in I(\mathfrak{j})} U(\mathfrak{j})(p - \lambda(p))$ *and* $\bar{J} = U(\mathfrak{a}_\varepsilon) \cap J$. *Then any solution of the equation*

$$\bar{\mathscr{M}}_\lambda\colon\ (\rho-\sum_{i\in I(\varepsilon)} s_i\alpha_i)(q)=0 \qquad \textit{for all } q\in\bar{J}.$$

is equal to $s_w(\lambda)=(s_{w,i}(\lambda))_{i\in I(\varepsilon)}$ *with a suitable* $w\in W(\mathfrak{j})$, *where* $s_{w,i}(\lambda)=\langle\rho-w\lambda, H_i\rangle$. *Moreover the multiplicity of the solution* $s_w(\lambda)$ *is a positive integer which is not larger than* $\#(W(\varepsilon)\backslash\{v\in W(\mathfrak{j});\ s_{vw}(\lambda)=s_w(\lambda)\})$.

Proof. Replacing s_i by $s_i+\rho(H_i)$, we may assume $\rho=0$. Put $J'=\{q\in U(\mathfrak{j});\ (w\lambda)(q)=0 \text{ for } w\in W(\mathfrak{j})\}$. Then clearly $J'\supset J$.

First consider the case where the stabilizer of λ in $W(\mathfrak{j})$ is trivial. Then $\dim U(\mathfrak{j})/J'=\#W(\mathfrak{j})$. Let $H(\mathfrak{j})$ be the space of harmonic polynomials in $U(\mathfrak{j})$ corresponding to $W(\mathfrak{j})$. Then $U(\mathfrak{j})=H(\mathfrak{j})\otimes I(\mathfrak{j})$ and $\dim H(\mathfrak{j})=\#W(\mathfrak{j})$. This means $\dim U(\mathfrak{j})/J=\#W(\mathfrak{j})(=\dim U(\mathfrak{j})/J')$ and therefore $J=J'$ because $J'\supset J$. Hence $\bar{J}=\{q\in U(\mathfrak{a}_\varepsilon);\ (w\lambda)(q)=0 \text{ for } w\in W(\mathfrak{j})\}$, which implies the lemma and every root of $\bar{\mathscr{M}}_\lambda$ is simple.

Consider the general case. Then there exist a positive number m so that the condition $q\in\bar{J}$ implies $q^m\in J'$. Therefore the condition $q\in\bar{J}$ implies $q^m\in J'\cap U(\mathfrak{a}_\varepsilon)$. Hence $s_w(\lambda)$ are roots of $\bar{\mathscr{M}}_\lambda$ and every root of $\bar{\mathscr{M}}_\lambda$ equals $s_w(\lambda)$ with a suitable $w\in W(\mathfrak{j})$. The remaining problem is to estimate the multiplicities of roots.

We have only to estimate the multiplicity of the root $s_e(\lambda)$ by replacing λ by $w\lambda$. We choose $\mu\in\mathfrak{j}_c^*$ so that the stabilizer of $\lambda+\mu t$ in $W(\mathfrak{j})$ is trivial for any $t\in\boldsymbol{C}$ with $0<|t|<1$. Put $J_t=\sum_{p\in I(\mathfrak{j})}U(\mathfrak{j})(p-(\lambda+\mu t)(p))$ and $J'_t=\{q\in U(\mathfrak{j});\ (w(\lambda+\mu t))(q)=0 \text{ for } w\in W(\mathfrak{j})\}$. We also choose $Z\in\mathfrak{a}_\varepsilon$ so that for every $w\in W(\mathfrak{j})$ the condition $\langle w\lambda, Z\rangle=\langle\lambda, Z\rangle$ implies $s_w(\lambda)=s_e(\lambda)$ and put $p_0(t)=\prod_{w\in W(\mathfrak{j})}(Z-\langle w(\lambda+\mu t), Z\rangle)$. Then $p_0(t)$ is contained in J'_t. Since $J'_t=J_t$ if $0<|t|<1$, $p_0(t)\in J_t$ for any $t\in\boldsymbol{C}$ with $|t|<1$. Put $W_1=\{w\in W(\mathfrak{j});\ \langle w\lambda, Z\rangle=\langle\lambda, Z\rangle\}$, $W_2=W(\mathfrak{j})-W_1$ and $p_i(t)=\prod_{w\in W_i}(Z-\langle w(\lambda+\mu t), Z\rangle)$ for $i=1, 2$. Let $\mathcal{O}_t[Z]$ denote the ring of polynomials of Z with coefficients in the ring $\mathcal{O}_t$ of convergent power series of t. Then there exist $q_1(t)$ and $q_2(t)$ in $\mathcal{O}_t[Z]$ such that $p_1(t)q_1(t)+p_2(t)q_2(t)=1$. Put $r_i(t)=p_i(t)q_i(t)$. Then the elements $r_1(t)r_2(t)$, $r_1(t)^2-r_1(t)$ and $r_2(t)^2-r_2(t)$ are contained in J_t for $|t|\ll 1$.

For any $p\in U(\mathfrak{j})$ we put $\phi_t(p)=\sum_i(\lambda+\mu t)(f_i)h_i$ by using the expression $p=\sum_i f_ih_i$ with $f_i\in I(\mathfrak{j})$ and $h_i\in H(\mathfrak{j})$. Then for any t, ϕ_t defines a map of $U(\mathfrak{j})$ onto $H(\mathfrak{j})$ with the kernel J_t, and $\dim\phi_t(U(\mathfrak{a}_\varepsilon))$ equals $\dim U(\mathfrak{a}_\varepsilon)/(J_t\cap U(\mathfrak{a}_\varepsilon))$, which also equals the number of roots of $\bar{\mathscr{M}}_{\lambda+\mu t}$ including their multiplicities. Putting $\phi_t^i(p)=\phi_t(r_i(t)p)$ for $i=1, 2$, $p\in U(\mathfrak{j})$ and $t\in\boldsymbol{C}$, we have $\phi_t=\phi_t^1+\phi_t^2$, $\phi_t^1\circ\phi_t^1=\phi_t^1$, $\phi_t^2\circ\phi_t^2=\phi_t^2$ and $\phi_t^1\circ\phi_t^2=\phi_t^2\circ\phi_t^1=0$. Hence the multiplicities of the root $s_e(\lambda)$ is given $\dim\phi_0(r_2(0)U(\mathfrak{a}_\varepsilon))$. On the other hand if $0<|t|\ll 1$, then $\dim\phi_t(r_2(t)U(\mathfrak{a}_\varepsilon))=\#\{s_w(\lambda+\mu t);\ w\in W_1\}$ because all roots of $\bar{\mathscr{M}}_{\lambda+\mu t}$ are simple in this case. Since $s_w(\lambda+\mu t)=s_{w'}(\lambda+\mu t)$ if $w'\in W(\varepsilon)w$ and since $\dim\phi_0(r_2(0)U(\mathfrak{a}_\varepsilon))\leq\dim\phi_t(r_2(t)U(\mathfrak{a}_\varepsilon))$

if $|t| \ll 1$, we have $\dim \phi_0(r_2(0)U(\mathfrak{a}_\varepsilon)) \leq \sharp(W(\varepsilon)\setminus W_1)$. Q.E.D.

In the first step of the proof of Proposition 2.8 we have obtained.

Corollary 2.9. *If an element λ of $\mathfrak{j}_c^*$ is regular (i.e. $w\lambda \neq \lambda$ for any $w \in W(\mathfrak{j})$ with $w \neq e$), then all solutions of $\mathcal{M}_\lambda$ are simple.*

§ 3. Boundary value maps

First we define G-modules attached to G-orbits in $\tilde{X}$. Retain the notation in Section 1 and Section 2. Recall $\{w_1, \cdots, w_r\}$ is a complete set of representatives of the coset $W(\mathfrak{a}; H)\setminus W$. For any $\varepsilon \in \{0, 1\}^l$ and $j \in \{1, \cdots, r\}$ we put $X_\varepsilon^j = \pi(G \times \{\varepsilon\} \times \{w_j\})$ $(\subset \tilde{X})$. Then X_ε^j is isomorphic to $G/P(x)$ with $x = (e, \varepsilon, w_j) \in \hat{X}$. In (1.2) the subgroup $P(x)$ is defined by

$$P(x) = (M_x \cap \bar{w}_j^{-1} H \bar{w}_j) A_x N_x.$$

For simplicity we put $P_\varepsilon^j = P(x)$, $M_\varepsilon = M_x$, $M_\varepsilon^j = M_x \cap \bar{w}_j^{-1} H \bar{w}_j$, $A_\varepsilon = A_x$, $N_\varepsilon = N_x$ and $N_\varepsilon^- = N_x^-$ in the case $x = (e, \varepsilon, w_j)$ and denote by $\mathfrak{p}_\varepsilon^j$, $\mathfrak{m}_\varepsilon$, $\mathfrak{m}_\varepsilon^j$, $\mathfrak{a}_\varepsilon$, $\mathfrak{n}_\varepsilon$ and $\mathfrak{n}_\varepsilon^-$ the corresponding Lie algebras. Moreover we put $\mathfrak{j}_\varepsilon = \mathfrak{a}_\varepsilon \cap \mathfrak{j}$. Note that any G-orbit in $\tilde{X}$ is isomorphic to G/P_ε^j with suitable ε and j.

The space $\mathcal{B}(G)$ of hyperfunctions on G is naturally left G-module by

$$\begin{array}{ccc} G \times \mathcal{B}(G) & \longrightarrow & \mathcal{B}(G) \\ \cup & & \cup \\ (g, f(x)) & \longmapsto & (\pi_g f)(x) = f(g^{-1}x). \end{array}$$

The induced action of $Y \in \mathfrak{g}$ on $\mathcal{B}(G)$ is also denoted by π_Y. For any $\mu \in (\mathfrak{j}_\varepsilon)_c^*$ let $\mathcal{B}(X_\varepsilon^j; L_\mu)$ be the space of hyperfunctions f on G satisfying

$$f(gman) = a^{\mu-\rho} f(g) \tag{3.1}$$

for all $g \in G$, $m \in M_\varepsilon^j$, $a \in A_\varepsilon$ and $n \in N_\varepsilon$. Then $\mathcal{B}(X_\varepsilon^j; L_\mu)$ is a G-submodule of $\mathcal{B}(G)$ and canonically identified with the space of hyperfunction valued sections of the line bundle L_μ on G/P_ε^j associated with the character τ_μ^ε of P_ε^j given by

$$\tau_\mu^\varepsilon(man) = a^{\rho-\mu}. \tag{3.2}$$

Here we remark

Lemma 3.1. $\quad \rho(Y) = 0 \qquad$ *for any* $Y \in \mathfrak{a}_\sigma \cap \mathfrak{h}$

and

$$\rho(Z) = \frac{1}{2} \operatorname{tr}_C \operatorname{ad}(Z) \mid \mathfrak{n}_c \cap \mathfrak{m}(\sigma)_c \qquad \textit{for any } Z \in \mathfrak{j} \cap \mathfrak{k}.$$

Proof. Let $Y \in \mathfrak{a}_\sigma$. Since $[Y, \mathfrak{n}(\sigma)]=\{0\}$, $\rho(Y)=\frac{1}{2}\operatorname{tr}\operatorname{ad}(Y)\,|\,\mathfrak{n}_\sigma+\mathfrak{n}(\sigma)$ $=\frac{1}{2}\operatorname{tr}\operatorname{ad}(Y)\,|\,\mathfrak{n}_\sigma$. Combining this with $\sigma(\mathfrak{n}_\sigma)=\bar{\mathfrak{n}}_\sigma$, we have $\rho(\sigma(Y))=$ $-\rho(Y)$. Hence if $Y \in \mathfrak{a}_\sigma \cap \mathfrak{h}$, $\rho(Y)=-\rho(Y)$ and therefore $\rho(Y)=0$. Let $Z \in \mathfrak{j} \cap \mathfrak{k}$. Since $\theta(\mathfrak{n}_\sigma)=\bar{\mathfrak{n}}_\sigma$ and $\theta(Z)=Z$ (resp. $\sigma(\mathfrak{n}_c \cap \mathfrak{g}(\sigma)_c)=\mathfrak{n}_c \cap \mathfrak{g}(\sigma)_c$ and $\sigma(Z)=-Z$), we have similarly $\operatorname{tr}\operatorname{ad}(Z)\,|\,\mathfrak{n}_\sigma=0$ (resp. $\operatorname{tr}_C \operatorname{ad}(Z)\,|\,\mathfrak{n}_c \cap \mathfrak{g}(\sigma)_c$ $=0$). Hence owing to the direct sum decomposition $\mathfrak{n}_c=(\mathfrak{n}_c \cap \mathfrak{m}(\sigma)_c)+$ $(\mathfrak{n}_c \cap \mathfrak{g}(\sigma)_c)+(\mathfrak{n}_\sigma)_c$, we have $\rho(Z)=\frac{1}{2}\operatorname{tr}_C \operatorname{ad}(Z)\,|\,\mathfrak{n}_c=\frac{1}{2}\operatorname{tr}_C \operatorname{ad}(Z)\,|\,\mathfrak{n}_c \cap \mathfrak{m}(\sigma)_c$.

Q.E.D.

For an open subset U of X^j_ε we denote by $\mathscr{B}(U; L_\mu)$ the space of hyperfunction sections of the line bundle L_μ over U.

Let $D(X^j_\varepsilon)$ denote the algebra of M^j_ε-invariant differential operators on the symmetric space $M_\varepsilon/M^j_\varepsilon$. Since $\operatorname{Ad}(M_\varepsilon)\,|\,\mathfrak{m}_\varepsilon=\operatorname{Int}(\mathfrak{m}_\varepsilon)$, we have the isomorphism

$$\bar{\gamma}^j_\varepsilon \colon D(X^j_\varepsilon) \xrightarrow{\sim} I(\mathfrak{j}(\varepsilon)) \tag{3.3}$$

as in the case where $\varepsilon=(1, \cdots, 1)$ (cf. Lemma 2.1 and (2.3)). Here $\mathfrak{j}(\varepsilon)=$ $\{Y \in \mathfrak{j};\ \langle Y, Z\rangle=0 \text{ for all } Z \in \mathfrak{j}_\varepsilon\}$ and $I(\mathfrak{j}(\varepsilon))$ is the set of W_F-invariants in $U(\mathfrak{j}(\varepsilon))$ with $F=\{\alpha \in \Psi(\mathfrak{j});\ \alpha\,|\,\mathfrak{j}_\varepsilon=0\}$. (If $j=1$, the map $\bar{\gamma}^j_\varepsilon$ is defined by the restriction of $\bar{\gamma}^F$ in (2.9).) Since M_ε normalizes N_ε and centralizes A_ε, the action of M^j_ε-invariant elements of $U(\mathfrak{m}_\varepsilon)$ on $\mathscr{B}(G)$ from the right leaves the space $\mathscr{B}(X^j_\varepsilon; L_\mu)$ invariant. Hence the elements of $D(X^j_\varepsilon)$ define differential operators on $\mathscr{B}(X^j_\varepsilon; L_\mu)$ which commute with the left action of G. For any ideal J' of $D(X^j_\varepsilon)$ we define a system of differential equations

$$\mathscr{M}' \colon Du=0 \qquad \text{for all } D \in J' \tag{3.4}$$

on $\mathscr{B}(X^j_\varepsilon; L_\mu)$ and denote by $\mathscr{B}(X^j_\varepsilon; L_\mu; \mathscr{M}')$ the G-submodule of $\mathscr{B}(X^j_\varepsilon; L_\mu)$ consisting of the solutions of the system $\mathscr{M}'$. Similarly we define a $\mathfrak{g}$-submodule $\mathscr{B}(U; L_\mu\colon \mathscr{M}')$.

We identify the symmetric space $X=G/H$ with the open G-orbit $\pi(G\times(1, \cdots, 1)\times W)$ of $\tilde{X}$. Let $\tilde{J}$ be an ideal of finite codimension in $D(\tilde{X})$ and let $\mathscr{B}(X; \mathscr{N})$ denote the space of hyperfunction solutions of the system

$$\mathscr{N} \colon Du=0 \qquad \text{for all } D \in \tilde{J} \tag{3.5}$$

defined on the symmetric space X. We remark that $\mathscr{N}$ can be written in the form (2.31) with suitable $P_1, \cdots, P_L$ because $\tilde{J}$ is finitely generated. For any $\mu \in (\mathfrak{j}_\varepsilon)^*_c$ we have an algebra homomorphism $\bar{\iota}_\mu$ of $U(\mathfrak{j})$ onto $U(\mathfrak{j}(\varepsilon))$ which satisfies $\bar{\iota}_\mu(Y)$ equals $\mu(Y)$ if $Y \in \mathfrak{j}_\varepsilon$ and Y if $Y \in \mathfrak{j}(\varepsilon)$. Then the map $\iota_\mu=(\bar{\gamma}^j_\varepsilon)^{-1}\circ\bar{\iota}_\mu\circ\bar{\gamma}$ defines an algebra homomorphism of $D(\tilde{X})$ to

$D(X_\varepsilon^j)$. Let $\tilde{J}_\mu$ denote the ideal of $D(X_\varepsilon^j)$ generated by $\iota_\mu(\tilde{J})$. Then we can define the induced system

$$(3.6) \qquad \mathcal{N}_\mu\colon Du=0 \qquad \text{for all } D \in \tilde{J}_\mu$$

on X_ε^j associated to μ. We note that $\tilde{J}_\mu$ is also of finite codimension in $D(X_\varepsilon^j)$.

Owing to Theorem 2.7 we can define boundary value map of $\mathscr{B}(X; \mathcal{N})$. Let $\lambda_1, \cdots, \lambda_m$ be the solutions of the indicial equation $\bar{\mathcal{N}}$ in Theorem 2.7 including their multiplicities. Here $\lambda_\nu=(\lambda_{\nu,i}) \in C^n$ by putting $n=\sharp I(\varepsilon)$ and λ_ν are called the characteristic exponents. Then for any point p of X_ε^j we have a boundary value map

$$(3.7) \qquad \beta_\nu\colon \mathscr{B}(X; \mathcal{N}) \longrightarrow \mathscr{B}(U_p)$$

attached to each characteristic exponent λ_ν, where U_p is an open neighborhood of p in X_ε^j and $\mathscr{B}(U_p)$ is the space of hyperfunctions on U_p. The definition of β_ν is given in [O4, §4]. We give some properties of β_ν.

There exist a neighborhood V of U_p in $\tilde{X}$ so that if $u \in \mathscr{B}(X; \mathcal{N})$ satisfies $\beta_\nu(u)=0$ for $\nu=1, \cdots, m$, then u is identically zero on $V \cap X$ (cf. [O4, Thorem 4.4]).

Assume $\beta_\nu(u)$ are analytic for $\nu=1, \cdots, m$. Then u has an expression

$$(3.8) \qquad u(t, x)=\sum_{\nu=1}^{m} a_\nu(t, x) t^{\lambda_\nu} q_\nu(\log t)$$

in $V \cap X$ and is called ideally analytic at p (cf. [O4, Theorem 5.3]). Here (t, x) is a local coordinate system with $t=(t_i)_{i \in I(\varepsilon)}$ so that X is defined by $t_i>0$ for all $i \in I(\varepsilon)$, X_ε^j is defined by $t_i=0$ for all $i \in I(\varepsilon)$, $a_\nu(t, x)$ are real analytic functions on V and $q_\nu(\log t)$ are polynomials of $\log t_i$ $(i \in I(\varepsilon))$. Moreover if

$$(3.9) \qquad \frac{1}{2}(\lambda_\nu-\lambda_{\nu'}) \notin N^n \qquad \text{for any } \nu \text{ and } \nu' \text{ with } \nu \neq \nu',$$

then $q_\nu=1$ and $\beta_\nu(u)(x)=a_\nu(0, x)$. Here we put $N=\{0, 1, 2, \cdots\}$. The expression (3.8) is the same one given in [Ha] or [CM] but here we obtain (3.8) under much weaker assumption (cf. [O4, Theorem 5.2]).

We may assume $I(\varepsilon)=\{1, \cdots, n\}$ without loss of generarity. Then we can define a semi-order for $\beta_1, \cdots, \beta_m$ which has the following properties (cf. [O4, Theorem 4.5]):

$$(3.10) \qquad \text{If } \beta_\nu>\beta_{\nu'}, \text{ then } \frac{1}{2}(\lambda_\nu-\lambda_{\nu'}) \in N^n.$$

(3.11) Define a line bundle

$$L(\lambda_\nu)=(T^*_{X^1_{\varepsilon,w_j}}\tilde{X})^{\otimes\lambda_{\nu,1}} \underset{X^j_\varepsilon}{\otimes} \cdots \underset{X^j_\varepsilon}{\otimes} (T^*_{X^n_{\varepsilon,w_j}}\tilde{X})^{\otimes\lambda_{\nu,n}}$$

over X^j_ε under the notation in Theorem 2.7. For an open subset U of X^j_ε, let $\mathscr{B}(U; L(\lambda_\nu))$ denote the space of hyperfunction sections of $L(\lambda_\nu)$ over U. Moreover we define

$$\mathscr{B}(U, X; \mathscr{N})_\nu=\{u \in \mathscr{B}(X; \mathscr{N});\ \beta_{\nu'}(u)=0 \text{ on a neighborhood of every point of } U \text{ for any } \nu' \text{ with } \beta_{\nu'}<\beta_\nu\}.$$

Then for any $u \in \mathscr{B}(U, X; \mathscr{N})_\nu$, the definition of the hyperfunction section

$$\beta_\nu(u)(dt)^{\lambda_\nu}=\beta_\nu(u)(dt_1)^{\lambda_{\nu,1}}\cdots(dt_n)^{\lambda_{\nu,n}}$$

does not depend on the choice of local coordinate systems. In other words we can define the following map

$$\beta_\nu\colon\ \mathscr{B}(U, X; \mathscr{N})_\nu \longrightarrow \mathscr{B}(U; L(\lambda_\nu)).$$

Put $t'_i=t_i^{k/2}$ for $i=1, \cdots, n$ with a positive integer k and consider the coordinate system $(t', x)=(t'_1, \cdots, t'_n, x)$. Then Lemma 1.8 and the proof of Lemma 1.9 (i) say that the vector field $\tilde{Y}$ on X corresponding to an element Y of $\mathfrak{g}$ defines an analytic vector field under the coordinate system (t', x). Moreover Lemma 2.4 (i) and the proof of Proposition 2.5 (cf. (2.16) and (2.23)) say that any $P \in D(X)$ defines an analytic differential operator under the coordinate system (t', x) If k is sufficiently large, the system $\mathscr{N}$ has regular singularities along $X^j_{\varepsilon,w}$ and the characteristic exponents are $(k/2)\lambda_1, \cdots, (k/2)\lambda_m$ under the coordinate system (cf. Remark 2.6). Here we can also choose k so that $(k/2)\lambda_\nu-(k/2)\lambda_{\nu'} \notin \boldsymbol{N}^n$ if $\frac{1}{2}(\lambda_\nu-\lambda_{\nu'}) \notin \boldsymbol{N}^n$ because there are only finite λ_ν. This corresponds to the conditions (3.9) and (3.10). Applying the results in [O4] to $\mathscr{N}$ under this coordinate system, we define the map β_ν mentioned above.

Since G acts on $\tilde{X}$ and preserves each X^i_{ε,w_j} there is a natural action of G on $L(\lambda_\nu)$ given by

$$y(dt_1)^{\lambda_{\nu,1}}\cdots(dt_n)^{\lambda_{\nu,n}}=(dt'_1)^{\lambda_{\nu,1}}\cdots(dt'_n)^{\lambda_{\nu,n}} \tag{3.12}$$

under the coordinate $\phi^w_g(n, m, t)$ and $\phi^w_{g'}(n', m', t')$, respectively, where $w=w_j$ and $g, g', y \in G$ with $g'=yg$ and $(n, m, t), (n', m', t') \in N^-_\sigma \times (M(\sigma)/(M(\sigma)\cap \overline{w}^{-1}H\overline{w}))\times \boldsymbol{R}^l$ (cf. Theorem 1.10 (i)). Thus for any $g \in G$ we have a commutative diagram

(3.13)
$$\begin{array}{ccc} \mathscr{B}(U, X; \mathscr{N})_\nu & \xrightarrow{\beta_\nu} & \mathscr{B}(U; L(\lambda_\nu)) \\ \pi_g \downarrow & & \pi_g \downarrow \\ \mathscr{B}_\nu(gU, X; \mathscr{N})_\nu & \xrightarrow{\beta_\nu} & \mathscr{B}(gU; L(\lambda_\nu)) \end{array}$$

and the map β_ν in (3.13) is a $\mathfrak{g}$-homomorphism.

Lemma 3.2. *Define* $\mu \in (\mathfrak{j}_\varepsilon)_c^*$ *by*

$$\mu = \left(\rho - \sum_{i=1}^{n} \lambda_{\nu,i}\alpha_i\right) | \mathfrak{j}_\varepsilon. \tag{3.14}$$

Then the line bundles $L(\lambda_\nu)$ *and* L_μ *on* X_ε^j *are isomorphic and the isomorphism is G-equivariant.*

Proof. Retain the notation just before the lemma. Then the relation $\phi_g^w(n, m, t) = \phi_{g'}^w(n', m', t')$ is equivalent to $(n', m', t') = (\phi_e^w)^{-1} \circ \phi_{g''}^w(n, m, t)$ with $g'' = g'^{-1}g$. It follows from Lemma 1.11 that

$$dt'_i = \exp\langle -\alpha_i, H(g'', n, m, t)\rangle dt_i \qquad \text{on } X_\varepsilon^j$$

for $i=1, \cdots, n$ under the notation there. Hence

$$(dt'_i)^{\lambda_\nu} = \exp\left(-\sum_{i=1}^{n} \lambda_{\nu,i}\langle \alpha_i, H(g'', n, m, t)\rangle\right)(dt_i)^{\lambda_\nu} \qquad \text{on } X_\varepsilon^j.$$

The definition of $H(g'', n, m, t)$ means

$$g''n\bar{m}a(t) \in n'\bar{m}'a(t) \exp H(g'', n, m, t) M_\varepsilon^j N_\varepsilon.$$

Since $\log a(t)$ and $\log a(t')$ are killed by $\alpha_1, \cdots, \alpha_n$ and since

$$\tau_\mu^\varepsilon(a(t)^{-1}\bar{m}'^{-1}n'^{-1}g''n\bar{m}a(t)) = \exp\langle \rho - \mu, H(g'', n, m, t)\rangle,$$

$$(dt')^{\lambda_\nu} = \tau_\mu^\varepsilon(a(t')^{-1}\bar{m}'^{-1}n'^{-1}g'^{-1}gn\bar{m}a(t))^{-1}(dt)^{\lambda_\nu} \tag{3.15}$$

if $\mu = (\rho - \sum_i \lambda_{\nu,i}\alpha_i) | \mathfrak{j}_\varepsilon$.

On the other hand, by definition L_μ is $G \times \boldsymbol{C}$ modulo the equivalence relation $(xb, c) \sim (x, \tau_\mu^\varepsilon(b)c)$ for $x \in G$, $b \in P_\varepsilon^j$ and $c \in \boldsymbol{C}$. Therefore

$$(g'n'\bar{m}'a(t'), 1) \sim (gnma(t), \tau_\mu^\varepsilon(a(t')^{-1}\bar{m}'^{-1}n'^{-1}g'^{-1}gnma(t))). \tag{3.16}$$

Thus the lemma follows from (3.12), (3.15) and (3.16). Q.E.D.

By Lemma 3.2 we identify $L(\lambda_\nu)$ with L_μ, where μ is given by (3.14). Therefore we may replace $L(\lambda_\nu)$ by L_μ in the commutative diagram (3.13).

Next we consider the system of differential equations which are satisfied by the boundary values, which is studied in [O4, §6]. Use the local coordinate system $(t, x)=(t_1, \cdots, t_n, x_1, \cdots, x_{n'})$ as before. Then [O4, Theorem 6.1 (ii)] says that if $u(t, x) \in \mathscr{B}(U, X; \mathscr{N})_\nu$ satisfies a differential equation

$$P(t, x, t_1\partial/t_1, \cdots, t_n\partial/\partial t_n, \partial/\partial x_1, \cdots, \partial/\partial x_{n'})u(t, x)=0,$$

then the boundary value $\beta_\nu(u)(x)$ satisfies the induced equation

$$P(0, x, \lambda_{\nu,1}, \cdots, \lambda_{\nu,n}, \partial/\partial x_1, \cdots, \partial/\partial x_{n'})\beta_\nu(u)(x)=0.$$

We apply the above statement to all P in $\tilde{J}$. Let $\tilde{P}$ be an $\mathrm{Ad}(\bar{w}_j^{-1})\mathfrak{h}$-invariant element of $U(\mathfrak{g})$ corresponding to P. Then the expression of P in the local coordinate system is obtained from (2.23). In fact P is a sum of the products of vector fields of the form (2.23). The induced equation is given by changing $t^{\alpha(i)}$ to $t^{\alpha(i)}|_{t_1=\cdots=t_n=0}$ and $t_k\partial/t_k$ to $\lambda_{\nu,k}$ for $k=1, \cdots, n$ in the expression (2.23). Then the term X_{-i} vanishes if $X_{-i} \in \mathfrak{n}_\varepsilon^-$.

Put $F=\{\alpha \in \Psi(\mathfrak{j}); \alpha|\mathfrak{j}_\varepsilon \neq 0\}$ and suppose $w_j=e$. Then the above statement says that the induced equation coincides with the differential operator corresponding to $\bar{\iota}_\nu \circ \delta_F(\tilde{P})$ (cf. (2.5)). Here $\bar{\iota}_\nu$ denotes an algebra homomorphism of $U(\mathfrak{m}_F+\mathfrak{a}_F)$ to $U(\mathfrak{m}_F)$ which is identity on $U(\mathfrak{m}_F)$ and satisfies $\bar{\iota}_\nu(H_k)=-\lambda_{\nu,k}$ for $k=1, \cdots, n$. This implies $\gamma^F \circ \bar{\iota}_\nu \circ \delta_F(\tilde{P})= \bar{\iota}_\nu \circ \gamma^F \circ \delta_F(\tilde{P})=\bar{\iota}_\nu \circ \eta_F^{-1} \circ \gamma(\tilde{P})=\bar{\iota}_\mu \circ \gamma(\tilde{P})$. Since $\bar{\iota}_\nu \circ \delta_F(\tilde{P})$ is a $(\mathfrak{m}_F \cap \mathfrak{h})$-invariant element of $U(\mathfrak{m}_F)$ and since the map γ^F induces an algebra isomorphism $\bar{\gamma}^F$ of $D(X_\varepsilon^1)$ onto $U(\mathfrak{j}(\varepsilon))^{W_F}$ (cf. (2.9)) the induced equation corresponds to $\iota_\mu(P)$. Hence the boundary value $\beta_\nu(u)$ satisfies the induced system (3.6). The above argument is also valid in the case where $w_j \neq e$. Thus we have the main theorem in this section.

Theorem 3.3. *Let X_ε^j be a boundary component contained in the closure of X in $\tilde{X}$ and let U be an open subset of X_ε^j. Let $\mathscr{N}$ be the system* (3.5) *of differential equations defined by an ideal $\tilde{J}$ of finite codimension in $D(\tilde{X})$. Then the boundary map β_ν defines a commutative diagram*

$$\text{(3.16)} \qquad \begin{array}{ccc} \mathscr{B}(U, X; \mathscr{N})_\nu & \xrightarrow{\beta_\nu} & \mathscr{B}(U; L_\mu; \mathscr{N}_\mu) \\ \pi_g \downarrow & & \downarrow \pi_g \\ \mathscr{B}(gU, X; \mathscr{N})_\nu & \xrightarrow{\beta_\nu} & \mathscr{B}(gU; L_\mu; \mathscr{N}_\mu) \end{array}$$

for $g \in G$ and the map β_ν in (3.16) *is $U(\mathfrak{g})$-equivariant. Here μ is defined by* (3.14) *and $\mathscr{N}_\mu$ is the induced system* (3.6) *associated to μ.*

Remark 3.4. (i) If $U=X_\varepsilon^j$, then we have a G-equivariant map β_ν: $\mathscr{B}(X_\varepsilon^j, X; \mathscr{N})_\nu \to \mathscr{B}(X_\varepsilon^j; L_\mu)$.

(ii) Fix ν. Then if $\frac{1}{2}(\lambda_{\nu'}-\lambda_\nu)\notin N^n$ for $\nu'\in\{1,\cdots,m\}-\nu$, $\mathscr{B}(X_\varepsilon^j, X;\mathscr{N})_\nu=\mathscr{B}(X;\mathscr{N})$.

(iii) For any non-zero $u\in\mathscr{B}(X;\mathscr{N})$ there exist at least one $\nu\in\{1,\cdots,m\}$ such that $u\in\mathscr{B}(X_\varepsilon^j, X;\mathscr{N})_\nu$ and $\beta_\nu(u)\neq 0$. This is proved as follows:

Suppose $\beta_\nu(u)=0$ for all β_ν. Then there exist an open neighborhood V of X_ε^j in $\tilde{X}$ so that $u\,|\,\tilde{U}\cap X=0$. Since V is independent of u (cf. [O4, Theorem 4.4]) and since $\beta_\nu(\pi_g(u))=\pi_g(\beta_\nu(u))=0$, we have $\pi_g(u)\,|\,V\cap X=0$ for all $g\in G$ and therefore $u=0$. Hence if $u\neq 0$, there exist β_ν with $\beta_\nu(u)\neq 0$. Then any minimal element of $\{\beta_\nu;\,\beta_\nu(u)\neq 0\}$ with respect to the semi-order is the required one.

§ 4. Principal series

In the preceding section we construct some G-modules attached to boundary components of X. The G-modules attached to the distinguished boundaries are most important, which lead us a concept of (most continuous) principal series for X. When X is a group manifold, this coincides with the usual principal series defined in [Hal]. First we give a proposition which will be useful for the study of principal series for X.

Proposition 4.1. *We can choose the complete set $\{w_1,\cdots,w_r\}$ of representatives of $W(\mathfrak{a};H)\backslash W(\mathfrak{a})$ and the representatives $\bar{w}_j\in N_K(\mathfrak{a})$ of w_j for $j=1,\cdots,r$ with $\bar{w}_1=e$ such that they satisfy both* (2.15) *and*

$$(4.1)\qquad \mathfrak{m}(\sigma)\cap\mathrm{Ad}\,(\bar{w}_j^{-1})\mathfrak{h}=\mathfrak{m}(\sigma)\cap\mathfrak{h}\qquad \textit{for } j=1,\cdots,r.$$

Proof. We can assume that w_j and $\bar{w}_j$ satisfy (2.15) (cf. [OS2, Corollary 7.10]). Put $\mathfrak{m}(\sigma)_r=[\mathfrak{m}(\sigma),\mathfrak{m}(\sigma)]$. Since $\mathrm{Ad}\,(\bar{w}_j)\mathfrak{m}(\sigma)=\mathfrak{m}(\sigma)$, $\mathrm{Ad}\,(\bar{w}_j)^{-1}\sigma\,\mathrm{Ad}\,(\bar{w}_j)$ induce involutive automorphisms of $\mathfrak{m}(\sigma)_r$. Applying Lemma 4.2 below to these involutions, we find $Z_j\in\mathfrak{j}\cap\mathfrak{m}(\sigma)_r$ such that $\mathrm{Ad}\,(\exp Z_j)^{-1}\,\mathrm{Ad}\,(\bar{w}_j)^{-1}\sigma\,\mathrm{Ad}\,(\bar{w}_j)\,\mathrm{Ad}\,(\exp Z_j)X=\sigma X$ for all $X\in\mathfrak{m}(\sigma)_r$. Replacing $\bar{w}_j$ by $\bar{w}_j\exp Z_j$, we can moreover assume $\mathrm{Ad}\,(\bar{w}_j)^{-1}\sigma\,\mathrm{Ad}\,(\bar{w}_j)X=\sigma X$ for $X\in\mathfrak{m}(\sigma)_r$. Hence if $X\in\mathrm{Ad}\,(\bar{w}_j)^{-1}(\mathfrak{m}(\sigma)_r\cap\mathfrak{h})$, we have $\sigma X=\mathrm{Ad}\,(\bar{w}_j)^{-1}\sigma\,\mathrm{Ad}\,(\bar{w}_j)X=X$ and so $\mathrm{Ad}\,(\bar{w}_j)^{-1}(\mathfrak{m}(\sigma)_r\cap\mathfrak{h})=\mathfrak{m}(\sigma)_r\cap\mathfrak{h}$. Since $\tilde{\mathfrak{j}}\cap\mathfrak{m}(\sigma)\cap\mathfrak{h}$ is the orthogonal complement of $\mathfrak{j}$ in $\tilde{\mathfrak{j}}+\mathfrak{a}_p$ with respect to the Killing form, it is stable under the map $\mathrm{Ad}\,(\bar{w}_j)^{-1}$ (cf. (2.15)). Hence $\mathfrak{m}(\sigma)\cap\mathrm{Ad}\,(\bar{w}_j)^{-1}\mathfrak{h}=\mathrm{Ad}\,(\bar{w}_j)^{-1}(\mathfrak{m}(\sigma)\cap\mathfrak{h})=\mathrm{Ad}\,(\bar{w}_j)^{-1}(\mathfrak{m}(\sigma)_r\cap\mathfrak{h}+\tilde{\mathfrak{j}}\cap\mathfrak{m}(\sigma)\cap\mathfrak{h})=\mathfrak{m}(\sigma)_r\cap\mathfrak{h}+\tilde{\mathfrak{j}}\cap\mathfrak{m}(\sigma)\cap\mathfrak{h}=\mathfrak{m}(\sigma)\cap\mathfrak{h}$. Q.E.D.

Lemma 4.2. *Let $\mathfrak{u}$ be a compact semisimple Lie algebra and let σ_1 and σ_2 be two involutive automorphisms of $\mathfrak{u}$. Put $\mathfrak{q}_j=\{X\in\mathfrak{u};\,\sigma_j(X)=-X\}$ for $j=1$ and 2. Suppose there exists an abelian subalgebra $\mathfrak{t}$ of $\mathfrak{u}$ such that*

$\mathfrak{t}$ is a maximal abelian subspace of $\mathfrak{q}_1$ and also that of $\mathfrak{q}_2$. Then there exists an element Z of $\mathfrak{t}$ such that $(\exp \mathrm{ad}(Z))\sigma_1 = \sigma_2(\exp \mathrm{ad}(Z))$.

Proof. We extend σ_1 and σ_2 to complex linear involutions of the complexification $\mathfrak{u}_c$ of $\mathfrak{u}$. Let $\tilde{\mathfrak{t}}$ be a maximal abelian subalgebra of $\mathfrak{u}$ which contains $\mathfrak{t}$. We remark that $\sigma_1 | \tilde{\mathfrak{t}}_c = \sigma_2 | \tilde{\mathfrak{t}}_c$. If $\tilde{\mathfrak{t}} = \mathfrak{t}$, the lemma coincides with [He, Ch. IX, Theorem 3.4]. We will proceed in the same way as in the proof of the theorem.

Let Δ be the set of non-zero roots of $\mathfrak{u}_c$ with respect to $\tilde{\mathfrak{t}}_c$ and suppose Δ is ordered so that if $\alpha \in \Delta^+$ and $\alpha | \mathfrak{t} \neq 0$, then $-\sigma_1(\alpha) \in \Delta^+$. Let $\{X_\alpha; \alpha \in \Delta\}$ be a Weyl basis of $\mathfrak{u}_c$ mod $\tilde{\mathfrak{t}}_c$ with respect to $\mathfrak{u}$. Then

$$\sigma_j(X_\alpha) = a_{\alpha,j} X_{\sigma_j(\alpha)}$$

where $a_{\alpha,j} a_{\alpha,-j} = 1$ and $|a_{\alpha,j}| = 1$ for $\alpha \in \Delta$ and $j = 1, 2$ (cf. [He, Ch. IX, Corollary 2.4]) Let Z be an element of $\tilde{\mathfrak{t}}_c$ such that

$$a_{\alpha,1} = a_{\alpha,2} \exp 2\langle \alpha, Z \rangle \tag{4.2}$$

for any simple root α for Δ^+. Then $Z \in \tilde{\mathfrak{t}}$ because $|a_{\alpha,j}| = 1$.

Suppose α, β, $\alpha+\beta \in \Delta$. Then $a_{\alpha+\beta,j} X_{\sigma_j(\alpha+\beta)} = \sigma_j[X_\alpha, X_\beta] = [\sigma_j(X_\alpha), \sigma_j(X_\beta)] = [a_{\alpha,j} X_{\sigma_j(\alpha)}, a_{\beta,j} X_{\sigma_j(\beta)}] = a_{\alpha,j} a_{\beta,j} [X_{\sigma_j(\alpha)}, X_{\sigma_j(\beta)}]$. Since $\sigma_1(\gamma) = \sigma_2(\gamma)$ for all $\gamma \in \Delta$, we have

$$a_{\alpha+\beta,1}/a_{\alpha+\beta,2} = (a_{\alpha,1}/a_{\alpha,2})(a_{\beta,1}/a_{\beta,2}). \tag{4.3}$$

This implies (4.2) for all $\alpha \in \Delta^+$ by induction. Since $a_{-\alpha,j} a_{\alpha,j} = 1$, we have (4.2) for all $\alpha \in \Delta$.

Suppose $\gamma \in \Delta$ satisfies $\gamma | \mathfrak{t} = 0$. Since $\sigma_j(\gamma) = \gamma$ and $\sigma_j^2 = 1$, it is clear that $\sigma_j(X_\gamma) = X_\gamma$ or $\sigma_j(X_\gamma) = -X_\gamma$. If $\sigma_j(X_\gamma) = -X_\gamma$, then $X_\gamma \in (\mathfrak{q}_j)_c$ and $[Y, X_\gamma] = \gamma(Y) X_\gamma = 0$ for $Y \in \mathfrak{t}_c$. which contradicts to the fact that $\mathfrak{t}_c$ is a maximal abelian subspace of $(\mathfrak{q}_j)_c$. Hence $\sigma_j(X_\gamma) = X_\gamma$ and so $a_{\gamma,j} = 1$. Next suppose α is a simple root in Δ^+ with $\alpha | \mathfrak{t} \neq 0$. Then $-\sigma_j(\alpha) = \beta + \sum_i m_i \gamma_i$, where m_i are non-negative integers and β and γ_i are simple roots in Δ^+ with $\beta | \mathfrak{t} \neq 0$ and $\gamma_i | \mathfrak{t} = 0$. Since $a_{\gamma_i,j} = 1$, we have $a_{\alpha,1}/a_{\alpha,2} = a_{\beta,1}/a_{\beta,2}$ by using (4.3). Hence if α, β and γ are simple roots for Δ^+ with $\alpha | \mathfrak{t} = \beta | \mathfrak{t}$ and $\gamma | \mathfrak{t} = 0$, then $a_{\alpha,1}/a_{\alpha,2} = a_{\beta,1}/a_{\beta,2}$ and $a_{\gamma,1}/a_{\gamma,2} = 1$. This assures that we can choose $Z \in \mathfrak{t}$. Then for any $\alpha \in \Delta$, $\sigma_2(\exp \mathrm{ad}(Z)) X_\alpha = \sigma_2(\exp\langle \alpha, Z \rangle) X_\alpha = a_{\alpha,2}(\exp\langle \alpha, Z \rangle) X_{\sigma_2(\alpha)} = a_{\alpha,1}(\exp -\langle \alpha, Z \rangle) X_{\sigma_1(\alpha)} = (\exp \mathrm{ad}(Z)) a_{\alpha,1} X_{\sigma_1(\alpha)} = (\exp \mathrm{ad}(Z)) \sigma_1 X_\alpha$. This implies the lemma. Q.E.D.

For $k = 1, \cdots, r$ let Π_k be the set of equivalence classes of finite dimensional irreducible representations of P_σ with $(P_\sigma \cap \overline{w}_k^{-1} H \overline{w}_k)$-fixed vectors and put $\Pi = \cup_{k=1}^r \Pi_k$. Let (τ, E_τ) be a representation of P_σ belonging to a class in Π_k. Then there exist an element $\mu \in \mathfrak{a}_c^*$ and a finite dimensional irreducible representation ξ of $M(\sigma)$ with a non-zero $(M(\sigma) \cap \bar{w}_k^{-1} H \bar{w}_k)$-fixed vector such that

$$\tau(mxan) = a^{\rho-\mu}\xi(m) \tag{4.4}$$

for any $m \in M(\sigma)$, $x \in G(\sigma)$, $a \in A_\sigma$ and $n \in N_\sigma$ (cf. Lemma 3.1 and [W, Lemma 5.5.1.3]).

Let $M(\sigma)_k^\wedge$ denote the set of equivalence classes of irreducible unitary representations of $M(\sigma)$ with non-zero $(M(\sigma) \cap \bar{w}_k^{-1} H \bar{w}_k)$-fixed vectors. Then

$$\Pi_k \simeq M(\sigma)_k^\wedge \times \mathfrak{a}_c^* \tag{4.5}$$

by the correspondence (4.4).

Here and hereafter in this section the surfix k and the superfix k always mean a positive integer between 1 and r.

Definition 4.3. Let (τ, E_τ) be a representation of P_σ belonging to a class in Π and let V_τ be a vector bundle over G/P_σ associated to τ. Then the G-module $\mathscr{B}(G/P_\sigma; V_\tau)$ of hyperfunction sections of V_τ is called the space (of hyperfunction sections of a representation of G) belonging to the (most continuous) principal series for G/H. The isomorphic class of G-modules containing $\mathscr{B}(G/P_\sigma; V_\tau)$ will be called a member of the principal series for G/H and it is uniquely defined by the class in Π containing τ.

Let $\xi \in M(\sigma)_k^\wedge$ and $\mu \in \mathfrak{a}_c^*$ such that they correspond to τ through (4.5). Then we write $V_\tau = V_{\xi,\mu}$ in Definition 4.3. Let $d(\xi)$ be the dimension of E_τ and let $m \mapsto \xi(m) = (a_{ij}(m))$ $(i, j = 1, \cdots, d(\xi))$ be a unitary matrix representation of $M(\sigma)$ corresponding to ξ. Let $\mathscr{B}(G)^{d(\xi)}$ be the space of column vectors of hyperfunctions on G with length $d(\xi)$. Then the space $\mathscr{B}(G/P_\sigma; V_{\xi,\mu})$ can be regarded as the space of functions f in $\mathscr{B}(G)^{d(\xi)}$ which satisfy

$$f(gmxan) = a^{\mu-\rho}\xi(m^{-1})f(g) \tag{4.6}$$

for any $g \in G$, $m \in M(\sigma)$, $x \in G(\sigma)$, $a \in A_\sigma$ and $n \in N_\sigma$.

Next we will study the set Π. For a Lie group G' we denote by $\hat{G}'$ or $G'^\wedge$ the set of equivalence classes of finite dimensional irreducible representations of G'. Put

$$Z(A_\mathfrak{p}) = \mathrm{Ad}_G^{-1}(\mathrm{Ad}_G(K) \cap \exp \mathrm{ad}(\sqrt{-1}\,\mathfrak{a}_\mathfrak{p})). \tag{4.7}$$

Then the group $\tilde{J} = Z(A_\mathfrak{p}) \exp \tilde{\mathfrak{j}}$ is the centralizer of $\tilde{\mathfrak{j}}$ in G and called the Cartan subgroup of G with the Lie algebra $\tilde{\mathfrak{j}}$. Let $(\xi, \mu) \in M(\sigma)_k^\wedge \times \mathfrak{a}_c^*$ and let $\tau \in \Pi_k$ which is related to (ξ, μ) as above. Let $\tilde{\xi}$ be an equivalent class of irreducible representations of $Z(A_\mathfrak{p})\times M(\sigma)_o$ which corresponds to ξ through the natural map of $Z(A_\mathfrak{p})\times M(\sigma)_o$ onto $M(\sigma)$. Since the finite group $Z(A_\mathfrak{p})$ centralizes $M(\sigma)_o$, it follows from the following lemma 4.4 that $\hat{M}(\sigma)$ is identified with the subset of $\hat{Z}(A_\mathfrak{p})\times \hat{M}(\sigma)_o$ whose members have non-zero vectors fixed by $\{(g, g^{-1}); g \in Z(A_\mathfrak{p})\cap M(\sigma)\}$. Thus we have

$$(4.8)\qquad \Pi_k \simeq \{\tau \in (Z(A_\mathfrak{p})\times M(\sigma)_o\times A)^\wedge;\ [\tau \,|\, H'_k\colon 1]>0\} \quad \text{with}$$
$$H'_k=\{(g_1, g_2, e) \in Z(A_\mathfrak{p})\times M(\sigma)_o\times A;\ g_1g_2 \in \bar{w}_k^{-1}H\bar{w}_k\}.$$

Lemma 4.4. *Let (π, E) be a finite dimensional irreducible representation of a Lie group G', H' a normal subgroup of G' and E_o a minimal H'-invariant subspace of E with $\dim E_o>0$. If $(\pi \,|\, H', E_o)$ is a trivial representation, all the elements of E are fixed by H' and therefore π is regarded as an irreducible representation of the quotient group G'/H'. On the other hand, if $Z_{G'}(H')H'=G'$, $(\pi \,|\, H', E)$ is isomorphic to the direct sum of finite copies of $(\pi \,|\, H', E_o)$. Here $Z_{G'}(H')=\{g \in G'; gh=hg$ for all $h \in H'\}$.*

Proof. Let E' be the union of H'-invariant subspaces E'' of E such that (π, E'') is isomorphic to (π, E_o). It is esay to see that E' is G'-invariant. Hence the lemma. Q.E.D.

Let $\delta_o \in \hat{Z}(A_\mathfrak{p})$ and $d\xi \in \hat{M}(\sigma)_o$ such that they correspond to $\tau \in \Pi_k$ through (4.8). We identify $d\xi$ with the highest weight of the correspooding representation of $\mathfrak{m}(\sigma)$ with respect to $\Sigma(\mathfrak{j})^+_\theta$. Since $(\mathfrak{m}(\sigma), \mathfrak{m}(\sigma)\cap\mathfrak{h})$ is a symmetric pair and $\Sigma(\mathfrak{j})^+_\theta$ is a corresponding positive restricted root system, the element $d\xi$ of $(\tilde{\mathfrak{j}}\cap\mathfrak{m}(\sigma))^*_c$ belongs to $\mathfrak{t}^*_c$ by denoting

$$(4.9)\qquad \mathfrak{t}=\mathfrak{j}\cap\mathfrak{k}$$

(cf. Theorem 4.5). Let E^o_τ be the corresponding subspace of highest weight vectors in E_τ. Then $\dim E^o_\tau=\dim\delta_o$ and $(\tau \,|\, \tilde{J}, E^o_\tau)$ is an irreducible representation of $\tilde{J}$ and we have

Theorem 4.5. (i) *The above correspondence gives the bijection*

$$(4.10)\qquad \begin{array}{ccc} \omega^k\colon \Pi_k & \xrightarrow{\sim} & \Pi'_k, \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ \tau & \longmapsto & (\tau \,|\, \tilde{J}, E^o_\tau) \end{array}$$

with

$$\Pi'_k=\{\zeta \in \tilde{J}^\wedge;\ [\zeta \,|\, \tilde{J}\cap \bar{w}_k^{-1}H\bar{w}_k\colon 1]>0 \text{ and } \langle d\zeta, \alpha\rangle\geq 0 \text{ for any } \alpha \in \Sigma(\mathfrak{j})^+_\theta\}$$

and $d\zeta$ is the element of $\tilde{\mathfrak{j}}^*_c$ which satisfies $\pi_{\exp Z}=\exp\langle d\zeta, Z\rangle$ for the *representation* π *belonging to* $\zeta \in \tilde{J}^{\wedge}$.

(ii) $[\tau \mid M(\sigma) \cap \bar{w}_k^{-1} H \bar{w}_k : 1] = [\omega^k(\tau) \mid \tilde{J} \cap \bar{w}_k^{-1} H \bar{w}_k : 1] = 1$ *for all* $\tau \in \Pi_k$.

Now we prepare two lemmas to prove the theorem.

Lemma 4.6. *Let* $(M(\sigma) \cap H)_o$ *be the identity component of* $M(\sigma) \cap H$. *Then*

$$M(\sigma) \cap H = (M(\sigma) \cap H \cap Z(A_\mathfrak{p}) \exp \mathfrak{t})(M(\sigma) \cap H)_o. \tag{4.11}$$

Proof. Fix an arbitrary $m \in M(\sigma) \cap H$. Put $M' = M(\sigma)/Z(A_\mathfrak{p})$ and denote by $\bar{m}$ the residue class of m in M'. We remark that M' is a compact connected Lie group with the Lie algebra $\mathfrak{m}(\sigma)$. Since $M(\sigma)$ and $Z(A_\mathfrak{p})$ are σ-stable, σ induces an involution σ' on M'. Let Z' be the identity component of the centralizer of $\bar{m}$ in M'. Then $\bar{m} \in Z'$ because $\bar{m}$ is contained in a maximal torus in M'. Since $\sigma'(\bar{m}) = \bar{m}$, Z' is σ'-stable. Let $\mathfrak{t}'$ be a σ'-stable maximal abelian subspace of the Lie algebra of Z'. Then the maximal abelian subgroup $\exp \mathfrak{t}'$ of Z' contains $\bar{m}$. Hence there exist $X_1 \in \mathfrak{m}(\sigma) \cap \mathfrak{h}$ and $X_2 \in \mathfrak{m}(\sigma) \cap \mathfrak{q}$ such that $[X_1, X_2] = 0$ and $\bar{m} = \exp(X_1 + X_2)$ in M'. Moreover since maximal abelian subspaces of $\mathfrak{m}(\sigma) \cap \mathfrak{q}$ are conjugate under the action of Ad exp $(\mathfrak{m}(\sigma) \cap \mathfrak{h})$, we can find a X_0 in $\mathfrak{m}(\sigma) \cap \mathfrak{h}$ such that $\mathrm{Ad} \exp (X_0) X_2 \in \mathfrak{t}$. Hence $\exp (X_0) \bar{m} \exp (-X_1) \exp (-X_0) \in \exp \mathfrak{t}$ in M' and the element $z = \exp (X_0) m \exp (-X_1) \exp (-X_0)$ of $M(\sigma)$ is contained in $H \cap Z(A_\mathfrak{p}) \exp \mathfrak{t}$. Therefore we have $m \in (M(\sigma) \cap H)_o z \times (M(\sigma) \cap H)_o$, which equals $z(M(\sigma) \cap H)_o$ because $(M(\sigma) \cap H)_o$ is a normal subgroup of $M(\sigma) \cap H$. Thus we can conclude that the left hand side of (4.11) is contained in the right hand side of (4.11). The converse inclusion relation is obvious. Q.E.D.

The author obtained the following lemma and Proposition 4.8 in cooperate with J. Sekiguchi and H. Midorikawa. Hence the author expresses his gratitude to them.

Lemma 4.7. *If* G *is a simple Lie group, then the dimension of any irreducible representation belonging to* Π_k *equals one.*

Proof. We assume $k = 1$ without loss of generality. First we remark that if G is a real form of a complex Lie group G_c, the lemma is clear because $\tilde{J}$ is abelian. Hence we may assume that G is not isomorphic to a real form of a complex Lie group.

Let $\tilde{G}$ be the universal covering group of G and π the natural projection of $\tilde{G}$ onto G. Let Z_s be the subgroup of the center of $\tilde{G}$ such that the Lie group $G_s = \tilde{G}/Z_s$ is isomorphic to a real form of a simply connected complex Lie group. The involution σ of the Lie algebra $\mathfrak{g}$ defines involutions on $\tilde{G}$ and G_s, which are denoted by the same letter σ. Moreover the fixed point group $\tilde{G}^\sigma$ of $\tilde{G}$ with respect to σ is known to be connected.

Now consider the case where $\tilde{G}$ has a finite center. Then it is known that $Z_s \simeq Z_2$. Put $Z_s = \{e, z\}$. Since Z_s is σ-stable, $\sigma(z) = z$. Let ζ be an irreducible representation of $\tilde{J}$ belonging to a class in Π_1. Then ζ naturally defines an irreducible representation of $\pi^{-1}(\tilde{J})$, which will be denoted by $\tilde{\zeta}$. Since $\pi^{-1}(H)$ contains the analytic subgroup of $\tilde{G}$ with the Lie algebra $\mathfrak{h}$, $\pi^{-1}(\tilde{J} \cap H)$ contains Z_s and therefore $\tilde{\zeta}$ has a non-zero vector fixed by Z_s. This means that $\tilde{\zeta} | Z_s = \mathrm{id}$ and that $\tilde{\zeta}$ induces an irreducible representation of the abelian group $\tilde{J}/Z_s$. Hence $\tilde{\zeta}$ and ζ are one-dimensional representations.

Next consider the case that $\tilde{G}$ has an infinite center. Put $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$ and let $\mathfrak{c}_{\mathfrak{k}}$ denote the center of $\mathfrak{k}$. Then it is known that the analytic subgroup K' of G_s with the Lie algebra $\mathfrak{k}'$ is simply connected and therefore K' is identified with the analytic subgroup of $\tilde{G}$. Let π_s be the natural projection of $\tilde{G}$ onto G_s.Let z_1 and z_2 be arbitrary elements of $\pi^{-1}(Z(A_{\mathfrak{p}}))$. Then there exist $X_j \in \mathfrak{c}_{\mathfrak{k}}$ and $k_j \in K'$ such that $z_j = k_j \exp X_j$ in $\tilde{G}$ for $j = 1$ and 2. Since $\pi_s(\pi^{-1}(Z(A_{\mathfrak{p}})))$ is abelian, $k_1 k_2 \pi_s (\exp X_1) \pi_s (\exp X_2) = \pi_s(k_1 \exp X_1) \pi_s \cdot (k_2 \exp X_2) = \pi_s(k_2 \exp X_2) \pi_s(k_1 \exp X_1) = k_2 k_1 \pi_s (\exp X_2) \pi_s(\exp X_1)$ and therefore $k_1 k_2 = k_2 k_1$ and $z_1 z_2 = z_2 z_1$. Thus we see that $\pi^{-1}(Z(A_{\mathfrak{p}}))$ is abelian and so is $Z(A_{\mathfrak{p}})$. Since $\tilde{J} = Z(A_{\mathfrak{p}}) \exp \tilde{\mathfrak{j}}$, $\tilde{J}$ is also abelian and the lemma is clear.

Q.E.D.

As a corollary of the proof of Lemma 4.7 we have

Proposition 4.8. *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a symmetric pair with a real simple Lie algebra* $\mathfrak{g}$ *and let* σ *be the corresponding involution. Suppose the center of a* σ*-stable maximal compact Lie algebra* $\mathfrak{k}$ *is zero or contained in* $\mathfrak{h}$. *Let* G_c *be a simply connected Lie group with the Lie algebra* $\mathfrak{g}$ *and let* G *and* H *be the analytic subgroups of* G_c *with the Lie algebra* $\mathfrak{g}$ *and* $\mathfrak{h}$, *respectively. Then* G/H *is simply conneted.*

Proof. Use the notation in the proof of Lemma 4.7. Then $\tilde{G}/\tilde{G}^\sigma$ is simply connected and $\tilde{G}^\sigma$ is connected. First suppose $\mathfrak{k}$ is semisimple. If $\tilde{G}$ is isomorphic to G, we have nothing to prove. Therefore we suppose moreover that $\tilde{G}$ is not isomorphic to G. Then $\tilde{G}/Z_s \simeq G$ and $Z_s \subset \tilde{G}^\sigma$. This implies $\tilde{G}^\sigma/Z_s \simeq H$ and $G/H \simeq \tilde{G}/\tilde{G}^\sigma$, which is simply connected. Next suppose the cener $\mathfrak{c}_{\mathfrak{k}}$ of $\mathfrak{k}$ is not zero and $\mathfrak{c}_{\mathfrak{k}} \subset \mathfrak{h}$. Then $K/(K \cap H) \simeq K'/(K' \cap H)$. Since K' is simply connected, so is $K'/(K')^\sigma$ and the group $(K')^\sigma = \{g \in K'; \sigma(g) = g\}$ is connected. Hence $K' \cap H = (K')^\sigma$. On the other hand $K/(K \cap H)$ is homotopic to G/H (cf. [B]), it is also simply connected.

Q.E.D.

Proof of Theorem 4.5. We may assume $k=1$ without loss of generality. Let (τ, E_τ) be an irreducible representation of $M(\sigma)$ belonging to a class in Π_1. Let E_τ^H be the space of vectors in E_τ fixed by $M(\sigma)\cap H$ and let p_o be the projection of E_τ onto the space E_τ^o of highest weight vectors of the representation of $M(\sigma)_o$. Since the map p_o commutes with the action of $\tau(\tilde{J})$, each element of $p_o(E_\tau^H)$ is fixed by $\tau(\tilde{J}\cap H)$.

Put $E_\tau=E_\tau^1\oplus\cdots\oplus E_\tau^n$. Here E_τ^i are $M(\sigma)_o$-invariant minimal non-trivial subspaces, $\tau|E_\tau^i$ are equivalent to $d\xi$ and $n=\dim\delta_o$ under the notation just before the theorem. Then $p_o(E_\tau^i)\subset E_\tau^i$. Let (,) be the inner product on E_τ so that $\tau|M(\sigma)$ is unitary. If u^i is a non-zero vector fixed by $(M(\sigma)\cap H)_o$ and v^i is a non-zero vector in $E_\tau^i\cap E_\tau^o$, then u^i and v^i are unique to within scalar factors. Moreover we have $(u^i, v^i)\neq 0$ (cf. the proof of [W, Theorem 3.3.1.1]). Hence $p_o(u^i)=C_i v^i$ with suitable non-zero $C_i\in\boldsymbol{C}$ for $i=1,\cdots,n$. Thus $p_o|E_\tau^H$ is injective and the map ω^k in the theorem is well-defined. On the other hand it is clear that the map ω^k is injective because $(\tau|\tilde{J}, E_\tau^o)$ determines the representations δ_o and $d\xi$.

Let (ζ, E) be an irreducible representation of $\tilde{J}$ which belongs to a class in Π_k'. Since $\tilde{J}=Z(A_\mathfrak{p})\exp\tilde{\mathfrak{j}}$, we can find an irreducible representation $(\tilde{\tau}, \tilde{E})$ of $Z(A_\mathfrak{p})\times M(\sigma)_o\exp\tilde{\mathfrak{j}}$ such that the space of highest weight vectors of $(\tilde{\tau}|M(\sigma)_o, \tilde{E})$ coincides with E and that $\tilde{\tau}_g(v)=\zeta_g(v)$ for any $g\in Z(A_\mathfrak{p})\cup\exp\tilde{\mathfrak{j}}$ and any $v\in E$. Let $\delta\in Z(A_\mathfrak{p})^\wedge$ and $d\xi\in M(\sigma)_o^\wedge$ such that $\tilde{\tau}$ belongs to the class $(\delta, d\xi)\in Z(A_\mathfrak{p})^\wedge\times M(\sigma)_o^\wedge\simeq(Z(A_\mathfrak{p})\times M(\sigma)_o)^\wedge$. Put $Z'=\{(g_1, g_2)\in Z(A_\mathfrak{p})\times\exp\tilde{\mathfrak{j}};\ g_1g_2=e\}$. Since the element of E is fixed by $\tilde{\tau}_g$ with any $g\in Z'$, any element of $\tilde{E}$ is fixed by Z' (cf. Lemma 4.4). Hence $(\tilde{\tau}, \tilde{E})$ induces an irreducible representation of $M(\sigma)$, which we will denote by $(\tau, \tilde{E})$. Next we will prove that $(\tau, \tilde{E})$ has a non-zero $(M(\sigma)\cap H)$-fixed vector. The proof of [W, Theorem 3.3.1.1] says that there the representation of $M(\sigma)_o$ belonging to $d\xi$ has a non-zero $(M(\sigma)\cap H)_o$-fixed vector unique up to a scalar factor. By the same reason for the injectivity of p_o the map

$$p_H:\ E\longrightarrow\tilde{E},\qquad E\ni v\longmapsto\int_{(M(\sigma)\cap H)_o}\tau_h(v)dh\in\tilde{E}\tag{4.12}$$

is also injective, where dh is the normalized Haar measure of $(M(\sigma)\cap H)_o$. Let $z\in H\cap Z(A_\mathfrak{p})\exp\tilde{\mathfrak{j}}$ and let v be a non-zero element of E fixed by $\tilde{J}\cap H$. Since the map $(M(\sigma)\cap H)_o\ni h\mapsto h'=zhz^{-1}$ defines an automorphism of $(M(\sigma)\cap H)_o$ and $dh=dh'$, we have

$$\tau_z(p_H(v))=\int\tau_z\tau_h(v)dh=\int\tau_{h'}\tau_z(v)dh'=\int\tau_{h'}(v)dh'=p_H(v).$$

Moreover since $p_H(v)$ is also fixed by $(M(\sigma)\cap H)_o$, it follows from Lemma 4.6 that $\tilde{E}$ has a non-zero vector fixed by $M(\sigma)\cap H$.

Thus we have proved that the map ω^k is bijective and moreover that $[\tau\,|\,M(\sigma)\cap H\colon 1]=[\omega^k(\tau)\,|\,\tilde{J}\cap H\colon 1]$ for any $\tau\in\Pi_k$. To complete the proof of the theorem, it is sufficient to show

$$(4.13)\qquad [\zeta\,|\,\tilde{J}\cap H\colon 1]<2 \qquad \text{for any } \zeta\in\tilde{J}^{\wedge}.$$

Let $\tilde{G}$ be a covering group of G such that σ can be lifted to an involution of $\tilde{G}$ and let π be the projection of $\tilde{G}$ onto G. Since any $\zeta\in\tilde{J}^{\wedge}$ naturally defines an element $\tilde{\zeta}\in\pi^{-1}(\tilde{J})^{\wedge}$ and $[\zeta\,|\,\tilde{J}\cap H\colon 1]=[\tilde{\zeta}\,|\,\pi^{-1}(\tilde{J}\cap H)\colon 1]$, we have only to show that $[\zeta'\,|\,\pi^{-1}(\tilde{J})\cap\tilde{G}_o^\sigma\colon 1]<2$ for any $\zeta'\in(\pi^{-1}(\tilde{J}))^{\wedge}$ where $\tilde{G}_o^\sigma$ is the analytic subgroup of $\tilde{G}$ with the Lie algebra $\mathfrak{h}$. Let $\mathfrak{g}=\mathfrak{g}_1+\cdots+\mathfrak{g}_N$ be the decomposition of $\mathfrak{g}$ into simple Lie algebras and let G_i be the analytic subgroup of G with the Lie algebra $\mathfrak{g}_i$ for any $i=1,\cdots,N$. Putting $\tilde{G}=G_1\times\cdots\times G_N$, we can reduce the proof of (4.13) in the case when $(\mathfrak{g},\mathfrak{h})$ is an irreducible symmetric pair. If G is simple, then Lemma 4.7 gives the proof. Hence assume (G, H) is irreducible and G is not simple. Then $G=G'\times G'$ and we can assume $\sigma(g_1\,g_2)=(g_2, g_1)$ for $(g_1, g_2)\in G'\times G'$. Thus $\tilde{J}=\tilde{J}'\times\tilde{J}'$, $\tilde{J}\cap H=\{(g,g);\,g\in J'\}$ with a Cartan subgroup J' of G' and therefore (4.13) is clear. Q.E.D.

Put $\tilde{J}_K=\tilde{J}\cap K$ and

$$(4.14)\qquad \Pi_{K,k}=\{\zeta\in\tilde{J}_K^{\wedge};\,[\zeta\,|\,\tilde{J}_K\cap\bar{w}_k^{-1}H\bar{w}_k\colon 1]>0 \text{ and } \langle d\zeta,\alpha\rangle\geq 0 \text{ for all } \alpha\in\Sigma(\mathfrak{j})_\theta^+\}.$$

Then we have the natural isomorphism $\Pi'_k\simeq\Pi_{K,k}\times\mathfrak{a}_c^*$ because $\tilde{J}=\tilde{J}_K\exp\mathfrak{a}_\mathfrak{p}$. Here $d\zeta$ is the element of $(\tilde{\mathfrak{j}}\cap\mathfrak{k})_c^*$ corresponding to ζ in (4.14). Now we have

Theorem 4.9. (i) *Let m be a positive integer. Then the restrictions of representations to subgroups induce the bijective maps between the following sets of equivalence classes of representations:*

$$(4.15)\qquad \{\delta\in\hat{M}_\sigma;\,[\delta\,|\,M_\sigma\cap H\colon 1]=m\}.$$

$$(4.16)\qquad \{\delta\in\hat{Z}_K(\mathfrak{a});\,[\delta\,|\,Z_K(\mathfrak{a})\cap H\colon 1]=m\}.$$

$$(4.17)\qquad \{\delta\in\hat{M};\,[\delta\,|\,M\cap H\colon 1]=m\}.$$

$$(4.18)\qquad \{\delta\in\hat{M}(\sigma)\colon[\delta\,|\,M(\sigma)\cap H\colon 1]=m\}.$$

Moreover restricting to the space of highest weight vectors with respect to $\Sigma(\tilde{\mathfrak{j}})_\theta^+$, the above sets are isomorphic to

(4.19) $$\{\delta \in \tilde{J}_{\hat{K}};\ [\delta | \tilde{J}_K \cap H: 1] = m\}.$$

(ii) *The sets in* (i) *are empty if* $m>1$.

(iii) *The statements in* (i) *and* (ii) *also hold even if we replace* H *by* $\bar{w}_k^{-1} H \bar{w}_k$ *for* $k=1, \cdots, r$.

Proof. Note that $M_\sigma = M(\sigma)G(\sigma)$, $M(\sigma) \subset M \subset Z_K(\mathfrak{a})$, $G(\sigma) \subset H$ and $G(\sigma)$ is a normal subgroup of M_σ. Then the theorem follows from Lemma 4.4 and Theorem 4.5. Q.E.D.

We are going to define another realization of principal series for G/H as G-submodules of $\mathscr{B}(G)$ which we will relate to G-modules constructed in Section 3. Let (τ, E_τ) be a representation of P_σ belonging to a class in Π_k and χ_τ the corresponding character. We fix a Hermitian inner product $(\ ,\ )$ on E_τ by which $\tau | M(\sigma)$ is unitary. Let (τ^*, E_τ) be the contragradient representation of (τ, E_τ). Then

(4.20) $$\mathscr{B}(G/P_\sigma;\ V_\tau) = \{\sum_{\text{finite sum}} f_i \otimes v_i;\ f_i \in \mathscr{B}(G),\ v_i \in E_\tau \text{ and } \sum f_i(g) \otimes v_i = \sum f_i(gx) \otimes \tau_x(v_i) \text{ for any } g \in G \text{ and } x \in P_\sigma\}$$

by definition. We fix an $(M(\sigma) \cap \bar{w}_k^{-1} H \bar{w}_k)$-fixed vector u_τ so normalized that $(u_\tau, u_\tau) = 1$ and we define the G-homomorphism

(4.21) $$p_\tau^k:\ \mathscr{B}(G/P_\sigma;\ V_\tau) \longrightarrow \mathscr{B}(G), \qquad \sum f_i \otimes v_i \longmapsto \sum (v_i, u_\tau) f_i.$$

Put $Q_k = (M_\sigma \cap \bar{w}_k^{-1} H \bar{w}_k) A_\sigma N_\sigma$ and define a G-submodule of $\mathscr{B}(G)$:

(4.22) $$\mathscr{B}(G/Q_k;\ L_\tau) = \{f \in \mathscr{B}(G);\ f(g) = \chi_\tau(e) \int_{M(\sigma)} f(gm) \chi_\tau(m) dm \text{ and } f(ghan) = f(g) a^{\mu - \rho} \text{ for any } h \in M_\sigma \cap \bar{w}_k^{-1} H \bar{w}_k,\ a \in A_\sigma \text{ and } n \in N_\sigma\},$$

where the element $\mu \in \mathfrak{a}_c^*$ is determined by τ through (4.5). Then we have

Theorem 4.10. *The map* p_τ^k *induces the isomorphism between G-modules*:

(4.23) $$p_\tau^k:\ \mathscr{B}(G/P_\sigma;\ V_\tau) \xrightarrow{\sim} \mathscr{B}(G/Q_k;\ L_\tau)$$

Its inverse is given by

(4.24) $$q_\tau^k:\ \mathscr{B}(G/Q_k;\ L_\tau) \longrightarrow \mathscr{B}(G/P_\sigma;\ V_\tau), \qquad f(g) \longmapsto \chi_\tau(e) \int_{M(\sigma)} f(gm) \otimes \tau_m(u_\tau) dm.$$

Here dm is the Haar measure on $M(\sigma)$ with $\int dm=1$.

Proof. Let $\sum f_i \otimes v_i \in \mathscr{B}(G/P_\sigma; V_\tau)$. Then for any $g \in G$, $h \in M(\sigma) \cap \bar{w}_k^{-1} H \bar{w}_k$, $x \in G(\sigma)$, $a \in A_\sigma$ and $n \in N_\sigma$, we have $\sum (v_i, u_\tau) f_i(ghxan) = \sum (\tau_{(hxan)^{-1}}(v_i), u_\tau) f_i(g) = \sum (\tau_{h(hxan)^{-1}}(v_i), \tau_h(u_\tau)) f_i(g) = a^{\mu-\rho} \sum (v_i, u_\tau) f_i(g)$. Moreover we have

$$\int \sum (v_i, u_\tau) f_i(gm) \chi_\tau(m) dm = \int \sum (\tau_{m^{-1}}(v_i), u_\tau) f_i(g) \chi_\tau(m) dm$$

$$= \sum \left(\left(\int \chi_\tau(m) \tau_{m^{-1}}(v_i) dm,\ u_\tau \right) f_i(g) \right)$$

$$= \chi_\tau(e)^{-1} \sum (v_i, u_\tau) f_i(g).$$

Hence we see that the image of p_τ^k is contained in $\mathscr{B}(G/Q_k; L_\tau)$.

Fix an orthonormal basis $u_1, \cdots, u_d$ of E_τ with $u_1 = u_\tau$. Here $d = \chi_\tau(e)$. Put $M_k = M(\sigma) \cap \bar{w}_k^{-1} H \bar{w}_k$ for simplicity and put $a_{ij}(m) = (\tau_m(u_i), u_j)$ for $m \in M(\sigma)$ and $i, j = 1, \cdots, d$. Then by the Peter-Weyl theory we have

$$\overline{a_{ij}(m)} = a_{ji}(m^{-1}),$$

$$\int a_{ij}(m) \overline{a_{st}(m)} dm = \begin{cases} d^{-1} & \text{if } (i, j) = (s, t), \\ 0 & \text{if } (i, j) \neq (s, t) \end{cases}$$

and since it follows from Theorem 4.5 that any $(M(\sigma) \cap \bar{w}_k^{-1} H \bar{w}_k)$-fixed element of E_τ is a scalar multiple of u_1, we have

$$\int_{M_k} a_{ij}(mh) dh = \int_{M_k} (\tau_{mh}(u_i), u_j) dh = \left(\tau_m \int_{M_k} \tau_h(u_i) dh,\ u_j \right) = 0$$

for $i = 2, \cdots, d$. Here dh is the Haar measure on M_k with $\int dh = 1$.

Let $\sum f_i(g) \otimes u_i$ be any element of $\mathscr{B}(G/P_\sigma; V_\tau)$. Then

$$q_\tau^k \circ p_\tau^k \left(\sum f_i(g) \otimes u_i \right) = d \int \sum (u_i, u_1) f_i(gm) \otimes \tau_m(u_1) dm$$

$$= d \sum f_i(g) \otimes \int (\tau_{m^{-1}}(u_i), u_1) \tau_m(u_1) dm$$

$$= d \sum_i f_i(g) \otimes \int a_{i1}(m^{-1}) \sum_j a_{1j} u_j dm$$

$$= d \sum_{i,j} f_i(g) \otimes \left(\int \overline{a_{1i}(m)} a_{1j}(m) dm \right) u_j$$

$$= \sum f_i(g) \otimes u_i.$$

On the other hand for any $f(g) \in \mathscr{B}(G/Q_k, L_\tau)$ and $1<i\leq d$, we have

$$\begin{aligned}\int_{M(\sigma)} f(gm)\, a_{ii}(m)\, dm &= \int_{M_k}\int_{M(\sigma)} f(gmh) a_{ii}(mh) dm dh \\ &= \int_{M(\sigma)} f(gm)\, dm \int_{M_k} a_{ii}(mh) dh \\ &= 0\end{aligned}$$

and therefore

$$\begin{aligned}p_\tau^k \circ q_\tau^k(f) &= p_\tau^k\left(d\int f(gm) \otimes \tau_m(u_1)\, dm\right) \\ &= d\int f(gm)(\tau_m(u_1), u_1) dm \\ &= d\int f(gm) a_{11}(m) dm \\ &= d\int f(gm)\,(a_{11}(m)+\cdots+a_{dd}(m)) dm \\ &= d\int f(gm) \chi_\tau(m) dm \\ &= f(g).\end{aligned}$$

Q.E.D.

Since the G-module $\mathscr{B}(G/Q_k; L_\tau)$ is determined by the class $\bar{\tau} \in \Pi_k$ containing τ, we sometimes write $\mathscr{B}(G/Q_k; L_{\bar{\tau}})$ or $\mathscr{B}(G/Q_k; L_\zeta)$ in place of $\mathscr{B}(G/Q_k; L_\tau)$, where $\zeta = \omega_k(\bar{\tau}) \in \Pi'_k$. We sometimes fix a representation belonging to each class in Π_k and identify them. For any $\zeta \in \Pi'_k$ let $d\zeta$ be the element of $\tilde{\mathfrak{j}}_c^*$ under the notation in Theorem 4.5. Then clearly $d\zeta \in \mathfrak{j}_c^*$ and we can define the map

(4.25)
$$\begin{array}{ccl}\Pi'_k & \longrightarrow & \mathfrak{j}_c^* \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ \zeta & \longmapsto & d'\zeta = -d\zeta - 2\rho|\mathfrak{t}_c + \rho.\end{array}$$

Now recall the notation in Section 3. Here we are concerned with compact boundary components of G/H. Hence we put $\varepsilon = 0 = (0, \cdots, 0) \in \{-1, 0, 1\}^l$. Then $M_\varepsilon = M_\sigma$, $M_\varepsilon^k = M_\sigma \cap \bar{w}_k^{-1} H \bar{w}_k$, $A_\varepsilon = A_\sigma$, $N_\varepsilon = N_\sigma$, $P_\varepsilon^k = Q_k$, $\mathfrak{j}_\varepsilon = \mathfrak{a}$, $\mathfrak{j}(\varepsilon) = \mathfrak{t}$ and $X_\varepsilon^k = G/Q_k$. In this case the isomorphism (3.3) becomes

(4.26)
$$\bar{\tau}_0^k \colon D(X_0^k) \xrightarrow{\sim} I(\mathfrak{t}).$$

Let J' be any finite codimensional ideal of $D(X_0^k)$ and let μ be any element of $\mathfrak{a}_c^*$. We consider the system of differential equations

$$(4.27) \qquad \mathscr{M}': Du=0 \qquad \text{for all } D \in J'$$

on $\mathscr{B}(X_0^k; L_\mu)$ as in Section 3. We put

$$(4.28) \qquad V(\mathscr{M}')=\{\xi \in \mathfrak{t}_c^*; \tau_0^k(D)(\xi)=0 \text{ for any } D \in J'\}.$$

Then $V(\mathscr{M}')\times\{\mu\}$ is a finite subset of $\mathfrak{t}_c^*\times\mathfrak{a}_c^*\simeq j_c^*$ and we have

Theorem 4.11. *Under the above notation the G-homomorphism*

$$(4.29) \qquad \begin{array}{ccc} \bigoplus_{\zeta\in\Pi'_k,\, d'\zeta\in V(\mathscr{M}')\times\{\mu\}} \mathscr{B}(G/Q_k; L_\zeta) & \xrightarrow{\sim} & \mathscr{B}(X_0^k; L_\mu; \mathscr{M}') \\ \cup\!\!\!\!\!\!\;\; \text{ⱷ} & & \text{ⱷ} \\ (f_\zeta) & \longmapsto & f=\sum f_\zeta \end{array}$$

is well-defined and bijective. The inverse map is obtained by

$$\begin{array}{ccc} r_\zeta^k\colon \mathscr{B}(X_0^k; L_\mu; \mathscr{M}') & \longrightarrow & \mathscr{B}(G/Q_k; L_\zeta) \\ \text{ⱷ} & & \text{ⱷ} \\ f(g) & \longmapsto & (r_\zeta^k(f))(g)=\chi_\zeta(e)\int_{M(\sigma)} f(gm)\chi_\zeta(m)dm. \end{array}$$

Here χ_ζ is the character of the representation of P_σ which corresponds to $\zeta \in \Pi'_k\simeq\Pi_k$ by Theorem 4.5.

Proof. The space $\mathscr{B}(K)$ of hyperfunctions on K has a natural topology as the dual of the space $\mathscr{A}(K)$ of real analytic functions on K and any $\psi \in \mathscr{B}(K)$ has the expansion

$$\psi(g)=\sum_{\delta\in M(\sigma)^\wedge}\chi_\delta(e)\int_{M(\sigma)}\psi(gm)\chi_\delta(m^{-1})dm$$

in this topology. Here χ_δ denotes the character corresponding to $\delta \in M(\sigma)^\wedge$. Considering the restriction on K, $\mathscr{B}(X_0^k; L_\mu; \mathscr{M}')$ is identified with a closed subspace of $\mathscr{B}(K)$.

Let $f \in \mathscr{B}(X_0^k; L_\mu; \mathscr{M}')$. Then

$$(4.30) \qquad \begin{cases} f=\sum_{\delta\in M(\sigma)^\wedge} f_\delta \\ f_\delta(g)=\chi_\delta(e)\int_{M(\sigma)} f(gm)\chi_\delta(m^{-1})dm. \end{cases}$$

Since $\int f(gmh)\chi_\delta(m^{-1})dm=\int f(gm)\chi_\delta(hm^{-1})dm=\int f(gm)\chi_\delta(m^{-1}h)dm$ for any $h \in M(\sigma)\cap \bar{w}_k^{-1}H\bar{w}_k$, we have

$$f_\delta(g)=\chi_\delta(e)\int_{M(\sigma)} f(gm)\phi_\delta(m^{-1})dm$$

with

$$\phi_\delta(m)=\int_{M(\sigma)\cap\bar{w}_k^{-1}H\bar{w}_k}\chi_\delta(mh)dh$$

and $f_\delta=\phi_\delta=0$ if $[\delta|M(\sigma)\cap\bar{w}_k^{-1}H\bar{w}_k\colon 1]=0$. Moreover for any $D\in U(\mathfrak{m}(\sigma))^{\mathfrak{m}(\sigma)\cap\mathfrak{h}}$ we have

$$(Df_\delta)(g)=\chi_\delta(e)\int_{M(\sigma)}f(gm)(D\phi_\delta)(m^{-1})\,dm.$$

Soppose $f_\delta\neq 0$. Now we apply Lemma 4.12 to $D\phi_\delta$. Lemma 4.12 can be obviously extended to the case where G is a compact connected Lie group and moreover to the case where the symmetric pair in the lemma equals $(M(\sigma), M(\sigma)\cap\bar{w}_k^{-1}H\bar{w}_k)$ because $M(\sigma)=Z(A_\mathfrak{p})M(\sigma)_o$ and $Z(A_\mathfrak{p})$ centralizes $M(\sigma)_o$. Owing to Lemma 1.5, Lemma 2.1 and Lemma 3.1, we have $D\phi_\delta=\bar{\tau}_0^k(D)(d\delta+\rho|\mathfrak{t}_c)\phi_\delta$ for any

$$D\in U(\mathfrak{m}(\sigma))^{\mathfrak{m}(\sigma)\cap\mathfrak{h}}/(U(\mathfrak{m}(\sigma))^{\mathfrak{m}(\sigma)\cap\mathfrak{h}}\cap U(\mathfrak{m}(\sigma))(\mathfrak{m}(\sigma)\cap\mathfrak{h}))\simeq D(X_0^\varepsilon),$$

where $d\delta\in\mathfrak{t}_c^*$ is the highest weight corresponding to δ. Hence $Df=\sum\bar{\tau}_0^k(D)(d\delta+\rho|\mathfrak{t}_c)f_\delta$ for any $D\in D(X_0^k)$.

By the uniqueness of the expansion (4.30), we can conclude that $f_\delta=0$ if $[\delta|M(\sigma)\cap\bar{w}_k^{-1}H\bar{w}_k\colon 1]=0$ or $d\delta+\rho|\mathfrak{t}_c\notin V(\mathcal{M}')$. Now if $\zeta\in\Pi_k'$ $(\simeq\Pi_k\simeq M(\sigma)_k^\wedge\times\mathfrak{a}_c^*)$ corresponds to $(\delta,\mu)\in M(\sigma)^\wedge\times\mathfrak{a}_c^*$, then $\chi_\zeta(m)=\chi_\delta(m)$ for any $m\in M(\sigma)$ and therefore $r_\zeta^k(f)=f_{\delta*}$. Moreover it is easy to see that $r_\zeta^k\circ r_\zeta^k=r_\zeta^k$ and $r_\zeta^k\circ r_{\zeta'}^k=0$ if $\zeta\neq\zeta'$. Hence the theorem is clear Q.E.D.

The following lemma used in the above proof is well-known. But we will give its proof in order to formulate it in our notation.

Lemma 4.12. *Suppose that G is compact and G/H is a compact symmetric space. Let $\delta\in\hat{G}$ with $[\delta|H\colon 1]\neq 0$ and let χ_δ be the corresponding character. Then the zonal spherical function $\phi_\delta(g)=\int\chi_\delta(gh)dh$ satisfies*

(4.31) $$D\phi_\delta=\bar{\gamma}(D)(d\delta+\rho|\mathfrak{t}_c)\phi_\delta \qquad \textit{for any } D\in D(G/H).$$

Here dh is the Haar measure on H with $\int dh=1$ and $d\delta\in\mathfrak{j}_c^*$ is the highest weight corresponding to δ.

Proof. First we remark $\mathfrak{t}=\mathfrak{j}$ and $d\delta\in\sqrt{-1}\,\mathfrak{t}^*$. Let (π, E) be a representation of G belonging to δ and fix a Hermitian inner product $(\ ,\ )$ on E such that (π, E) is unitary. Let $u_1,\cdots,u_N$ be an orthonormal basis of E such that u_1 is a H-fixed vector and let $u_0 \in E$ be the highest weight vector with respect to $\Sigma(\tilde{\mathfrak{j}})^+$. Let G_c be a complex Lie group with the Lie algebra $\mathfrak{g}_c$ such that G is a maximal compact subgroup of G_c. Recall that $\mathfrak{n}_c$ is a nilpotent subalgebra of $\mathfrak{g}_c$ corresponding to $\Sigma(\mathfrak{j})^+$ and $\bar{\mathfrak{n}}_c=\sigma(\mathfrak{n}_c)$ (cf. §2). Moreover we remark that if $X_1+\sqrt{-1}X_2 \in \mathfrak{n}_c$ with elements X_1 and X_2 of $\mathfrak{g}$, then $X_1-\sqrt{-1}X_2 \in \bar{\mathfrak{n}}_c$.

Put $\psi_\delta(g)=(\pi_g(u_1), u_0)$. Then $\psi_\delta(g)$ is extended to a holomorphic function on G_c, which is denoted by the same letter, and we have

$$\psi_\delta(\exp(X)\exp(Y)\exp(Z))=\exp d\delta(U)$$

for any $X \in \bar{\mathfrak{n}}_c$, $Y \in \mathfrak{t}_c$ and $Z \in \mathfrak{h}_c$ because $\overline{d\delta(-U)}=d\delta(U)$ if $U \in \mathfrak{t}$. Hence for $D \in U(\mathfrak{g})$ we have $(D\psi_\delta)(\exp(X)\exp(Y))=\gamma(D)(d'\delta)\psi_\delta(\exp(X)\exp(Y))$ by putting $d'\delta=d\delta+\rho|\mathfrak{t}_c$. Therefore if $D \in U(\mathfrak{g})^{\mathfrak{h}}$, we have $(D\phi_\delta)(z)=\gamma(D)(d\delta)\psi_\delta(z)$ for any $z \in \{\exp(X)\exp(Y)\exp(Z);\ X \in \bar{\mathfrak{n}}_c,\ Y \in \mathfrak{t}_c$ and $Z \in \mathfrak{h}_c\}$. Hence $D\psi_\delta=\bar{\gamma}(D)(d'\delta)\psi_\delta$ for any $D \in D(G/H)$ because it holds on a neighborhood of the identity in G_c.

Now we remark that $\phi_\delta(g)=(\pi_g(u_1), u_1)$. Since $u_1 \in \sum_{x \in G} C\pi_x u_0$, $\phi_\delta(g)$ belongs to the C-linear span of left translations of $\psi_\delta(g)$ by $x \in G$. Hence $D\phi_\delta=\bar{\gamma}(D)(d'\delta)\phi_\delta$ for any $D \in D(G/H)$. Q.E.D.

We have a direct consequence of Theorem 4.11:

Corollary 4.13. *For an element* $(\eta, \mu) \in \mathfrak{t}_c^* \times \mathfrak{a}_c^*$ *we define the system*

$$\mathcal{N}'_\eta:\ Du=\bar{\tau}_0^k(D)(\eta, \mu)u \qquad \text{for any } D \in D(X_0^k)$$

of differential equations on $\mathscr{B}(X_0^k; L_\mu)$. *Moreover putting*

$$\mathfrak{W}(\mathfrak{t})_k^+=\{\eta \in \sqrt{-1}\mathfrak{t}^*;\ \langle\eta, \alpha\rangle>0 \text{ for any } \alpha \in \Sigma(\mathfrak{j})^+ \text{ and } \exp\langle\eta-\rho, Y\rangle=0$$
$$\text{for any } Y \in \mathfrak{t} \text{ satisfying } \exp Y \in \bar{w}_k^{-1}H\bar{w}_k\},$$

we have

$$\bigoplus_{\eta \in V(\mathcal{M}') \cap \mathfrak{W}(\mathfrak{t})_k^+} \mathscr{B}(X_0^k; L_\mu; \mathcal{N}'_\eta) \xrightarrow{\sim} \mathscr{B}(X_0^k; L_\mu; \mathcal{M}').$$

Especially $\mathscr{B}(X_0^k; L_\mu; \mathcal{M}')=\{0\}$ *if* $V(\mathcal{M}') \cap \mathfrak{W}(\mathfrak{t})_k^+=\varnothing$.

Remark 4.14. By a theorem due to E. Cartan and S. Helgason (cf. [*W*, Theorem 3.3.1.1]) the condition $\eta \in V(\mathcal{M}') \cap \mathfrak{W}(\mathfrak{t})_k^+$ implies that $\langle\eta-\rho, \alpha\rangle/\langle\alpha, \alpha\rangle$ is a non-negative integer for any $\alpha \in \Sigma(\mathfrak{j})_\theta^+$.

Finally we give the following theorem which was announced in [O1] and [O2].

Theorem 4.15. *Let $\tilde{J}$ be a finite codimensional ideal of the ring $D(G/H)$ of the invariant differential operators on G/H and let $\mathscr{B}(G/H; \mathscr{N})$ be the space of hyperfunction solutions of the system*

$$\mathscr{N}: Du=0 \qquad \text{for all } D \in \tilde{J}$$

on G/H. Then there exist finite elements $\tau_1, \cdots, \tau_N \in \Pi_k$ and G-invariant subspaces $V(0), \cdots, V(n)$ of $\mathscr{B}(G/H; \mathscr{N})$ such that $V(0)=\{0\}$, $V(N)=\mathscr{B}(G/H; \mathscr{N})$ and that the quotient space $V(i)/V(i-1)$ is G-isomorphic to a G-invariant subspace of the principal series $\mathscr{B}(G/P_\sigma; V_{\tau_i})$ for any $i \in \{1, \cdots, N\}$.

The above theorem is easily obtained by combining Remark 3.4, Theorem 4.10, Theorem 4.11 and Corollary 4.13. Much more explicit imbedding theorem will be given in [O5].

Remark 4.16. When we consider the space of K-finite solutions of $\mathscr{N}$ in Theorem 4.15, we conclude that the space is a Harish-Chandra module and every irreducible subquotient of the Harish-Chandra module is a submodule of the Harish-Chandra module corresponding to a certain principal series for G/H. This is clear because the principal series for G/H is a G-invariant subspace of a usual principal series for G.

In the case when the center C of G is infinite and $\sharp(C/C\cap H)$ is also infinite, the space of K-finite solutions of $\mathscr{N}$ in the above statement should be replaced by $U(\mathfrak{g})$-module V generated by a K-finite solution of $\mathscr{N}$. In fact, in this case, the left hand side of (4.29) is an infinite direct sum. But since the K-type of V is discrete, only the finite summands are related to V and we have the same statement for V even if C is infinite.

Remark 4.17. Let G' be a connected real semisimple Lie group. Put $G=G'\times G'$, $\sigma(g_1, g_2)=(g_2, g_1)$ and $H=\{(g, g) \in G; g \in G'\}$. Then the symmetric space G/H is naturally identified with the group manifold G'. We call this case a group case.

Then $M(\sigma)=M'\times M'$, $G(\sigma)=\{e\}$, $A_\sigma=\{(a, a^{-1}) \in G; a \in A_\mathfrak{p}\}$ and $N_\sigma=\{(n, \theta'(n)) \in G; n \in N'\}$. Here $G'=K'A_\mathfrak{p}N'$ is an Iwasawa decomposition of G', θ' is the corresponding Cartan involution of G' etc. We denote an object for G' by the symbol with dash of the corresponding object for G. An irreducible representation of M with a non-zero $(M(\sigma)\cap H)$-fixed vector is a direct tensor product of an irreducible representation ξ of M' and its contragradient representation ξ^*. Hence the representation of G belonging to the principal series for G/H is identified with the representation of $G'\times G'$ induced from the representation

$$\tau((man, m'a'n') \in \xi(m)a^{\lambda-\rho'}\xi^*(m')a'^{-w^*\lambda-\rho'}$$
$$\text{for } (m, a, n, m', a', n') \in M'\times A_{\mathfrak{p}}\times N'\times M'\times A_{\mathfrak{p}}\times N'$$

of $P'\times P'$. Here ξ is an irreducible representation of M', λ is an element of $(\mathfrak{a}'_{\mathfrak{p}})^*_c$ and w^* is the longest element of the Weyl group of the symmetric space G'/K'. Hence the representation is the direct tensor product of a representation belonging to the principal series of G' and its contragradient. We remark that we consider both left and right actions of G' on the functions on G'.

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