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Asymptotic Behavior of Spherical Functions on Semisimple Symmetric Spaces

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§ 0. Introduction

Let G be a connected real semisimple Lie group, σ an involution of G, and H an open subgroup of the fixed point group G^σ. Then the homogeneous space G/H is called a semisimple symmetric space. In this paper, a K-finite simultaneous eigenfunction of the invariant differential operators on G/H is called a spherical function, where K is a maximal compact subgroup of G modulo center. It is known that such a spherical function has an asymptotic expansion at infinity, which really converges, as is shown by [HC] and [CM] in the group case and by [Ba] and [O3] in general cases. In this paper, we will give the main non-vanishing terms in the expansion, that is, the growth order at infinity, by using some geometric interpretation. It plays an important role for the harmonic analysis on G/H.

The idea here is similar as in [MO], where we describe discrete series for G/H. But we get a better result here than [MO, Lemma 1] which is essential in [MO] and we can simplify the proof of the main theorem in [MO]. In fact we can omit complicated arguments according to the classification of root systems. The simpler proof is given in [Ma2]. Moreover for a given representation of G realized on a function space on G/H, we can tell in which principal series for G/H the representation is imbedded.

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In § 1, we will construct linearly independent eigenfunctions for Weyl group invariant differential operators with constant coefficients defined on a root space. For a fixed eigenvalue, they are well studied (cf. [St], [He2, Chapter 3.3]). We will construct the eigenfunctions which holomorphically depend on the eigenvalue. A nice characterization of them is given in Theorem 1.4. It is useful for the study of the expansions of spherical functions at infinity when the eigenvalue is singular.

In § 2, we will review on the boundary value maps for the eigenfunctions of invariant differential operators on riemannian symmetric spaces of the non-compact type, which are first introduced to prove Helgason's conjecture in [K-].

In § 3, we will study intertwining operators between locally defined sections of principal series for G/K. Combining the result in this section with boundary value maps and Flensted-Jensen's isomorphism, we have our main theorem, which we will explain:

Fix a Cartan involution θ of G with $\sigma\theta=\theta\sigma$. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}_c$ the complexification of $\mathfrak{g}$ and G_c the connected and simply connected Lie group with the Lie algebra $\mathfrak{g}_c$. The involutions of $\mathfrak{g}$, the complex linear involutions of $\mathfrak{g}_c$ and the complex analytic involutions of G_c which are induced by σ and θ are denoted by the same letters, respectively. Let $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ (resp. $\mathfrak{k}+\mathfrak{p}$) be the decomposition of $\mathfrak{g}$ into the $+1$ and -1 eigenspaces for σ (resp. θ). Let $\mathfrak{g}^d$, $\mathfrak{k}^d$ and $\mathfrak{h}^d$ be subalgebras of $\mathfrak{g}_c$ defined by

$$\mathfrak{g}^d=\mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}\,(\mathfrak{k}\cap\mathfrak{q})+\sqrt{-1}\,(\mathfrak{p}\cap\mathfrak{h})+(\mathfrak{p}\cap\mathfrak{q})$$

$$\mathfrak{k}^d=\mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}\,(\mathfrak{p}\cap\mathfrak{h}),\quad \mathfrak{h}^d=\mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}\,(\mathfrak{k}\cap\mathfrak{q}).$$

Fix maximal abelian subspaces $\mathfrak{a}$ of $\mathfrak{p}\cap\mathfrak{q}$ and $\mathfrak{a}_\mathfrak{p}^d$ of $\mathfrak{p}^d=\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q})+\mathfrak{p}\cap\mathfrak{q}$ so that $\mathfrak{a}_\mathfrak{p}^d\supset\mathfrak{a}$. Let $(\mathfrak{a}_\mathfrak{p}^d)_c^*$ be the dual of the complexfication $(\mathfrak{a}_\mathfrak{p}^d)_c$ of $\mathfrak{a}_\mathfrak{p}^d$ and let $\Sigma(\mathfrak{a})$ and $\Sigma(\mathfrak{a}_\mathfrak{p}^d)$ be the root systems corresponding to the pairs $(\mathfrak{g}, \mathfrak{a})$ and $(\mathfrak{g}^d, \mathfrak{a}_\mathfrak{p}^d)$, respectively. We fix compatible positive systems $\Sigma(\mathfrak{a})^+$ of $\Sigma(\mathfrak{a})$ and $\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ of $\Sigma(\mathfrak{a}_\mathfrak{p}^d)$. Let $\Psi(\mathfrak{a})$ and $\Psi(\mathfrak{a}_\mathfrak{p}^d)$ be the corresponding fundamental systems. Put $\Sigma(\mathfrak{a})^-=-\Sigma(\mathfrak{a})^+$ and $\Sigma(\mathfrak{a}_\mathfrak{p})^-=-\Sigma(\mathfrak{a}_\mathfrak{p})^+$. By the complexification $\langle\ ,\ \rangle$ of the Killing form of $\mathfrak{g}$, we will identify $(\mathfrak{a}_\mathfrak{p}^d)_c$ with $(\mathfrak{a}_\mathfrak{p}^d)_c^*$.

Let K be the analytic subgroup of G with the Lie algebra $\mathfrak{k}$ and let G^d, K^d and H^d be the analytic subgroups of G_c with the Lie algebras, $\mathfrak{g}^d$, $\mathfrak{k}^d$ and $\mathfrak{h}^d$, respectively. We call the homogeneous space G^d/K^d the non-compact riemannian form of G/H. Let $\hat{K}$ (resp. $\hat{H}^d$) denote the set of equivalence classes of finite-dimensional irreducible representations of K (resp. H^d) and for $\delta\in\hat{K}$ let $C_\delta^\infty(G/H)$ denote the linear span of all K-finite C^∞ functions of type δ. Let $C_K^\infty(G/H)=\bigoplus_{\delta\in\hat{K}}C_\delta^\infty(G/H)$ be the space of all K-finite C^∞ functions on G/H. Define $C_\delta^\infty(G^d/K^d)$ for $\delta \in \hat{H}^d$ and $C_{\hat{H}^d}^\infty(G^d/K^d)$ similarly.

Let $D(G/H)$ and $D(G^d/K^d)$ be the algebras of invariant differential operators on G/H and G^d/K^d, respectively. Then $D(G/H)$ and $D(G^d/K^d)$ are naturally isomorphic through analytic continuation (cf. [O3, Lemma 2.1]). Let λ be an element of $(\mathfrak{a}_\mathfrak{p}^d)_c^*$. Then the algebra homomorphisms χ_λ of $D(G/H) \simeq D(G^d/K^d)$ to C is defined by the Harish-Chandra isomorphism $D(G^d/K^d) \simeq I(W(\mathfrak{a}_\mathfrak{p}^d))$. Here $W(\mathfrak{a}_\mathfrak{p}^d)$ is the Weyl group of $\Sigma(\mathfrak{a}_\mathfrak{p}^d)$ and $I(W(\mathfrak{a}_\mathfrak{p}^d))$ is the algebra of $W(\mathfrak{a}_\mathfrak{p}^d)$-invariant elements in the symmetric algebra of $(\mathfrak{a}_\mathfrak{p}^d)_c$.

Now we define the spaces

$$\mathscr{A}_K(G/H; \mathscr{M}_\lambda) = \{u \in C_K^\infty(G/H);\ Du = \chi_\lambda(D)u \text{ for all } D \in D(G/H)\}$$

and

$$\mathscr{A}_{\hat{H}^d}(G^d/K^d; \mathscr{M}_\lambda) = \{u \in C_{\hat{H}^d}^\infty(G^d/K^d);\ Du = \chi_\lambda(D)u \text{ for all } D \in D(G^d/K^d)\}.$$

Then there exists an injective $\mathfrak{g}$-homomorphism

$$\eta: \mathscr{A}_K(G/H; \mathscr{M}_\lambda) \longrightarrow \mathscr{A}_{\hat{H}^d}(G^d/K^d; \mathscr{M}_\lambda) \tag{0.1}$$

which we call the Flensted-Jensen isomorphism ([FJ1]). In fact, η is an isomorphism if G/H is simply connected. The map η is defined through the analytic continuation.

Since $\chi_{w\lambda} = \chi_\lambda$ for $w \in W$, we may assume

$$\operatorname{Re}\langle \lambda, \alpha \rangle \geqq 0 \qquad \text{for all } \alpha \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+ \tag{0.2}$$

and there exists a subset Θ of $\Psi(\mathfrak{a}_\mathfrak{p}^d)$ so that

$$\Sigma \cap \lambda^\perp = \Sigma \cap \textstyle\sum_{\alpha \in \Theta} C\alpha. \tag{0.3}$$

Here $\lambda^\perp = \{\mu \in (\mathfrak{a}_\mathfrak{p}^d)_c^*;\ \langle \mu, \lambda \rangle = 0\}$.

Put $\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}; \alpha) = \{X \in \mathfrak{g}^d;\ [Y, X] = \alpha(Y) \text{ for all } Y \in \mathfrak{a}_\mathfrak{p}\}$, $\mathfrak{n}^d = \sum_{\alpha \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+} \mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}; \alpha)$ and $\bar{\mathfrak{n}}^d = \sigma(\mathfrak{n}^d)$. Let N^d, $\bar{N}^d$, $A_\mathfrak{p}^d$ and A be the analytic subgroup of G^d with the Lie algebras $\mathfrak{n}^d$, $\bar{\mathfrak{n}}^d$, $\mathfrak{a}_\mathfrak{p}^d$ and $\mathfrak{a}$, respectively, and let M^d be the centralizer of $A_\mathfrak{p}^d$ in K^d. Then the subgroup $P^d = M^d A_\mathfrak{p}^d N^d$ is a minimal parabolic subgroup of G^d. For $\mu \in (\mathfrak{a}_\mathfrak{p}^d)_c^*$, we put

$$\begin{aligned}\mathscr{B}_{\hat{H}^d}(G^d/P^d; L_\mu) = \textstyle\bigoplus_{\delta \in \hat{H}^d} \{&f \in \mathscr{B}(G);\ f(gman) = f(g)a^{\mu-\rho} \\ &\text{for } (g, m, a, n) \in G^d \times M^d \times A_\mathfrak{p}^d \times N^d \text{ and } f \text{ is of type } \delta\}.\end{aligned} \tag{0.4}$$

Here $\rho \in (\mathfrak{a}_\mathfrak{p}^d)_c^*$ is defined by $\rho(Y) = \operatorname{trace} \operatorname{ad}(Y)|_{\mathfrak{n}^d}$ for $Y \in \mathfrak{a}_\mathfrak{p}^d$ and $\mathscr{B}(G)$ is the space of hyperfunctions on G. It is known that any element of $\mathscr{B}_{H^d}(G^d/P^d; L_\mu)$ is a distribution ([FJ2, IV Corollary 10]).

We define the Poisson transform

$$(0.5)\qquad \mathscr{P}_\mu\colon \mathscr{B}_{H^d}(G^d/P^d; L_\mu) \longrightarrow \mathscr{A}_{H^d}(G^d/K^d; \mathscr{M}_\mu)$$

by the formula

$$(0.6)\qquad (\mathscr{P}_\mu f)(x) = \int_{K^d} f(xk)dk.$$

Here dk is the normalized Haar measure on K^d.

Suppose $\lambda \in (\mathfrak{a}_\mathfrak{p}^d)_c^*$ satisfies (0.2) and (0.3). Then we can define a boundary value map

$$(0.7)\qquad \beta_\lambda\colon \mathscr{A}_{H^d}(G^d/K^d; \mathscr{M}_\lambda) \longrightarrow \mathscr{B}_{H^d}(G^d/P^d; L_\lambda)$$

and it follows from the main result in [K-] that $\mathscr{P}_\lambda$ and β_λ are bijective and $\beta_\lambda \circ \mathscr{P}_\lambda$ is a constant multiple of the identity map.

Definition 0.1. For a function ψ in $\mathscr{A}_K(G/H; \mathscr{M}_\lambda)$ we define an H^d-invariant closed subset $\mathrm{FBI}_\lambda(\psi)$ of G^d/P^d by

$$(0.8)\qquad \mathrm{FBI}_\lambda(\psi) = \operatorname{supp} \beta_\lambda \circ \eta(\psi).$$

For simplicity we will sometimes write $\mathrm{FBI}(\psi)$ in place of $\mathrm{FBI}_\lambda(\psi)$.

If the set $\lambda^\perp \cap \Sigma(\mathfrak{a}_\mathfrak{p}^d)$ equals $\{\alpha \in \Sigma(\mathfrak{a}_\mathfrak{p}^d);\ \operatorname{Re}\langle\lambda, \alpha\rangle = 0\}$, then any element of $W(\mathfrak{a}_\mathfrak{p}^d)\lambda$ except λ does not satisfy at least one of the conditions (0.2) and (0.3). Hence in this case we have no confusion even if we write $\mathrm{FBI}(\psi)$ in place of $\mathrm{FBI}_\lambda(\psi)$. Now we can state one of our results which is a special case of Theorem 4.1:

Theorem 0.2. *Consider a non-zero function* ψ *in* $\mathscr{A}_K(G/H; \mathscr{M}_\lambda)$. *Suppose*

$$(0.9)\qquad \operatorname{Re}\langle\lambda, \alpha\rangle > 0 \qquad \textit{for all } \alpha \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+.$$

Put $V = \{w \in W(\mathfrak{a}_\mathfrak{p}^d);\ \mathrm{FBI}(\psi)\mathrm{Cl}(P^d w^{-1} P^d) \supset P^d\}$, $\partial V = \{w \in V;\ \{v \in V;\ v < w\} = \emptyset\}$ *and* $\Lambda = \{w\lambda|_\mathfrak{a};\ w \in \partial V\}$. *Then for any* $\mu \in \Lambda$ *there exists a non-zero analytic function* $c_\mu(k)$ *on* K *such that*

$$(0.10)\qquad \psi(kah) = \sum_{\mu\in\Lambda} c_\mu(k) a^{\mu-\rho} + b(k, a) \sum_{\mu\in\Lambda, \alpha\in\Psi(\mathfrak{a})} |a^{\mu-\rho-\varepsilon\alpha}|$$

for $(k, a, h) \in K \times A \times H$. *Here* ε *is a suitable positive number and the real analytic function* $b(k, a)$ *on* $K \times A$ *is bounded on the set* $\{(k, a) \in K \times A;\ a^{-\alpha} < \delta \textit{ for all } \alpha \in \Psi(\mathfrak{a})\}$ *for any positive number* δ.

In the above theorem, $v<w$ means that v is smaller than w with respect to the Bruhat ordering and in general, $\mathrm{Cl}(Y)$ means the closure of Y for a subset Y of a topological space. In § 4, we will give a similar exact result as above in general cases where, for example, λ does not satisfy the regularity condition (0.9) or a tends to infinity along another direction. Thus the problem to get spherical functions with a required growth condition at infinity is reduced to the problem to get functions in $\mathscr{B}_{H^d}(G^d/P^d; L_\lambda)$ with a certain support property. We remark that the support of any element in $\mathscr{B}_{H^d}(G^d/P^d; L_\lambda)$ is an H^d-invariant compact subset of G^d/P^d and that the double coset decomposition $H^d\backslash G^d/P^d$ is studied well in [Ma1], especially the number of cosets in $H^d\backslash G^d/P^d$ is finite.

For example, regard a semisimple Lie group as a semisimple symmetric space and suppose that ψ is a matrix element of an irreducible Harish-Chandra module for the group. Then $\mathrm{FBI}(\psi)$ coincides with the support of the $\mathscr{D}$-module realized in a flag manifold through Beilinson-Bernstein's correspondence (cf. [BB], [V]). We remark that out result in § 4 covers the case where λ is singular and in this case the support of the corresponding $\mathscr{D}$-module is generally not unique.

In this paper we will use the standard notation $\boldsymbol{N}$, $\boldsymbol{Z}$, $\boldsymbol{R}$, $\boldsymbol{R}_+$ and $\boldsymbol{C}$. Here $\boldsymbol{N}$ is the set of non-negative integers and $\boldsymbol{R}_+$ is the set of positive real numbers. For a real vector space E, let E^* denote the dual space of E and let E_c and E_c^* denote the complexification of E and E^*, respectively. In this paper, a manifold always means a real analytic manifold and a differential operator always means a linear differential operator whose coefficients are analytic functions. For a manifold M, we denote by $\mathscr{A}(M)$, $C^\infty(M)$, $\mathscr{D}'(M)$ and $\mathscr{B}(M)$ the spaces of real analytic functions, infinitely differentiable functions, distributions and hyperfunctions defined on M, respectively. If M is a complex manifold, we denote by $\mathscr{O}(M)$ the space of holomorphic functions on M.

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§ 1. Eigenfunctions of Weyl group invariant operators

Let E be an l-dimensional vector space over $\boldsymbol{R}$, Σ a reduced root system in the dual space E^* of E and W its Weyl group. Fix a positive non-degenerate bilinear form $\langle\ ,\ \rangle$ on E^* which is invariant under the Weyl group and identify E and E^* by this bilinear form. Choose a fundamental system $\Psi=\{\alpha_1, \cdots, \alpha_l\}$ for Σ. Let Σ^+ be the corresponding positive system and put $\Sigma^-=-\Sigma^+$. For a root $\alpha \in \Sigma$ we denote by s_α the reflection with respect to α and by $\alpha^\vee$ the co-root associated with α. Then we have by definition

$$(1.1) \qquad \alpha^\vee=\frac{2\alpha}{\langle\alpha, \alpha\rangle}.$$

Let $w=w_k\cdots w_1$ be a minimal expression for an element w in W as a product of reflections with respect to simple roots. We will simply call this a minimal expression for w and the number k will be called the length of w and denoted by $l(w)$. Let $\alpha(j)$ be simple roots with $w_j=s_{\alpha(j)}$ for $j=1, \cdots, k$ and put

$$(1.2) \qquad \Sigma(w)=\Sigma^+\cap w^{-1}\Sigma^-.$$

Then

$$(1.3) \qquad \Sigma(w)=\{w_1\cdots w_{k-1}\alpha(k), \cdots, w_1\alpha(2), \alpha(1)\}$$

and $l(w)=\sharp\Sigma(w)$. In this paper we will write $w\geqq w'$ or $w'\leqq w$ for elements w and w' in W if and only if the following equivalent conditions (1.4) and (1.5) are satisfied (cf. [De]). This defines a certain ordering in W, which is called the Bruhat ordering.

(1.4) Let $w=w_k\cdots w_1$ be a minimal expression for w. Then there exist indices $j_1, \cdots, j_r$ so that $w'=w_{j_r}\cdots w_{j_1}$ and $1\leqq j_1<\cdots<j_r\leqq k$.

(1.5) There exist non-negative integer r and elements $w^{(0)}, \cdots, w^{(r)}$ in W so that $w=w^{(0)}$, $w'=w^{(r)}$, $l(w^{(j)})<l(w^{(j-1)})$ and $w^{(j)}(w^{(j-1)})^{-1}$ is a reflection with respect to a root in Σ for $j=1, \cdots, r$.

We remark that hereafter in this section all the statements and proves are also valid even if we replace the Bruhat ordering in W by the following one in W:

$$w<w' \text{ if and only if } l(w)<l(w').$$

For a subset Θ of Ψ we put $E^*_\Theta=E^*\cap\sum_{\alpha\in\Theta}\boldsymbol{R}\alpha$, $\Sigma_\Theta=\Sigma\cap E^*_\Theta$, $\Sigma^+_\Theta=\Sigma^+\cap\Sigma_\Theta$, $\Sigma^-_\Theta=-\Sigma^+_\Theta$ and

$$(1.6) \qquad W(\Theta)=\{w\in W;\ w\Sigma^+_\Theta\subset\Sigma^+\}.$$

Let W_Θ be the subgroup of W generated by reflections with respect to simple roots in Θ. Then W_Θ is the Weyl group of the root system Σ_Θ in

E_Θ^*. Since the map $W(\Theta)\times W_\Theta \ni (w'', w')\mapsto w''w'$ to W is bijective (cf. [War, Chapter 1.1.2]), for an element w in W we can define two elements $w(\Theta)$ in $W(\Theta)$ and w_Θ in W_Θ so that $w=w(\Theta)w_\Theta$. Then $l(w)=l(w(\Theta))+l(w_\Theta)$. Let s^* denote the unique element in W_Θ satisfying $s^*\Sigma^+=\Sigma^-$. Then $s_\Theta^*\Sigma_\Theta^+=\Sigma_\Theta^-$.

Let E_c be the complexification of E, $S(E)$ the symmetric algebra of E_c, $P(E)$ the algebra of polynomial functions on E_c and $R(E)$ the field of rational functions on E_c in the complex category. Replacing E by E^*, we similarly define E_c^*, $S(E^*)$, $P(E^*)$ and $R(E^*)$. Then naturally $S(E)\simeq P(E^*)$ and $S(E^*)\simeq P(E)$. Extending $\langle\ ,\ \rangle$ to a complex bilinear form on E_c^*, we identify E_c^* with E_c. In fact for $\lambda\in E_c^*$ the corresponding element $H_\lambda\in E_c$ is defined so that $\langle\lambda, \mu\rangle=\mu(H_\lambda)$ for any $\mu\in E_c^*$. Moreover any polynomial function on a subspace of E_c is extended to an element of $P(E)$ through the orthogonal projection of E_c onto the subspace with respect to $\langle\ ,\ \rangle$. Given $Y\in E$, let ∂_Y denote the differential operator on E_c defined by

$$(1.7)\qquad (\partial_Y f)(X)=\left(\frac{d}{dt}f(X+tY)\right)\Big|_{t=0}\quad \text{for } X\in E, f\in\mathcal{O}(E_c) \text{ and } t\in \boldsymbol{C}.$$

The map $E\ni Y\mapsto\partial_Y$ can be uniquely extended to an algebra isomorphism of $S(E)$ onto the algebra $\boldsymbol{C}[\partial]$ of holomorphic differential operators on E_c with constant coefficients. For an element p in $S(E)$, let ∂_p denote the corresponding differential operator. Let $I(W)$ denote the algebra of W-invariant elements in $S(E)$, which is generated by l algebraically independent homogeneous elements.

Given $\lambda\in E_c^*$, we will consider the space

$$(1.8)\qquad H(\lambda)=\{u\in\mathcal{O}(E_c);\ \partial_p u=p(\lambda)u \text{ for all } p\in I(W)\}.$$

Especially the element of $H(0)$ is called a W-harmonic polynomial and it is known that

$$(1.9)\qquad S(E)=I(W)\underset{\boldsymbol{C}}{\otimes} H(0).$$

Since $\dim \boldsymbol{C}[\partial]/\sum_{p\in I(W)}\boldsymbol{C}[\partial](\partial_p-p(\lambda))=\sharp W$, the dimension of the space $H(\lambda)$ equals $\sharp W$. We will construct functions in $\mathcal{O}(E_c^*\times E_c)$ which form a basis of $H(\lambda)$ for any $\lambda\in E_c^*$. Since $\partial_p\exp\langle\lambda, X\rangle=p(\lambda)\exp\langle\lambda, X\rangle$, the functions $\exp\langle w\lambda, X\rangle$ are elements of (1.8) for all w in W. If $\langle\lambda, \alpha\rangle\neq 0$ for any $\alpha\in\Sigma$, then obviously $H(\lambda)=\sum_{w\in W}\boldsymbol{C}\exp\langle w\lambda, X\rangle$.

In this section we will usually use the notation (λ, X) for the variable of functions on $E_c^*\times E_c$ and at the same time we will sometimes regard λ as a holomorphic parameter of functions on E_c with the variable X.

Definition 1.1. Let $R(E^*)^W$ denote the group algebra of W over $R(E^*)$. For every element w in W with $w\neq e$, fix a simple root $\alpha(w)$ in Ψ so that $l(ws_{\alpha(w)})<l(w)$. Then inductively define elements

$$\delta_w=\sum_{v\in W}a(w,v;\lambda)v \tag{1.10}$$

in $R(E^*)^W$ as follows. Here $w\in W$ and $a(w,v;\lambda)\in R(E^*)$.

$$\begin{cases}\delta_e=e,\\ \delta_w=\langle\lambda,\alpha(w)^\vee\rangle^{-1}\sum_{v\in W}a(ws_{\alpha(w)},v;\lambda)v-a(ws_{\alpha(w)},v;s_{\alpha(w)}\lambda)vs_{\alpha(w)}\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{if } w\neq e.\end{cases} \tag{1.11}$$

Lemma 1.2. *If $a(w,v;\lambda)\neq 0$, then $w\geqq v$ and $a(w,v;\lambda)$ is homogeneous of degree $-l(w)$ with respect to λ. Especially*

$$a(w,w;\lambda)=(-1)^{l(w)}\prod_{\alpha\in\Sigma(w)}\langle\lambda,\alpha^\vee\rangle^{-1}.$$

Proof. We will prove the lemma by the induction on $l(w)$. We may suppose $w\neq e$. Put $w'=ws_{\alpha(w)}$. Definition 1.1 implies

$$a(w,v;\lambda)=\langle\lambda,\alpha(w)^\vee\rangle^{-1}(a(w',v;\lambda)-a(w',vs_{\alpha(w)};s_{\alpha(w)}\lambda)). \tag{1.12}$$

If $w'=w_k\cdots w_1$ is a minimal expression for w', then $w=w_k\cdots w_1s_{\alpha(w)}$ is a minimal expression for w. Hence if $w'\geqq vs_{\alpha(w)}$ or $w'\geqq v$, then $w\geqq v$ (cf. (1.4)). Combining this with the hypothesis of the induction, we have $w\geqq v$ if $a(w,v;\lambda)\neq 0$. It is clear that $a(w,v;\lambda)$ is homogeneous of degree $-l(w)$. Also by the hypothesis of the induction we have $a(w,w;\lambda)=-\langle\lambda,\alpha(w)^\vee\rangle^{-1}(-1)^{l(w')}\prod_{\alpha\in\Sigma(w')}\langle s_{\alpha(w)}\lambda,\alpha^\vee\rangle^{-1}=(-1)^{l(w)}\prod_{\alpha\in\Sigma(w)}\langle\lambda,\alpha^\vee\rangle^{-1}$ because $\Sigma(w)=\{\alpha(w)\}\cup s_{\alpha(w)}\Sigma(w')$ (cf. (1.3)). Q.E.D.

Theorem 1.3. *For $\Theta\subset\Psi$ and $w\in W$, put*

$$\pi_\Theta(\lambda)=\prod_{\alpha\in\Sigma_\Theta^+}\langle\lambda,\alpha^\vee\rangle$$

and

$$\psi(w,\lambda;X)=\sum_{v\in W}a(w,v;\lambda)\exp\langle v\lambda,X\rangle. \tag{1.13}$$

Then we have the following:

i) $a(w,v;\lambda)$ *are well-defined in the sense that they do not depend on the choice of* $\alpha(w)$ *in Definition* 1.1.

ii) $\pi_\Theta(\lambda)a(w,v;\lambda)\in P(E^*)$ *if* $w\in W_\Theta$.

iii) $\psi(w,\lambda;X)\in\mathcal{O}(E_c^*\times E_c)$ *for* $w\in W$.

iv) $\psi(s_\Theta^*, \lambda; X)=\pi_\Theta(\lambda)^{-1}\sum_{w\in W_\Theta}(-1)^{l(w)}\exp\langle w\lambda, X\rangle.$

v) $(\partial_p-p(w\lambda))\psi(w, \lambda; X)\in\sum_{v<w}P(E^*)\psi(v, \lambda; X)$ *for* $p\in S(E)$.

vi) *Put* $h_w(X)=\psi(w, 0; X)$. *Then the matrix*

$$(\partial_{h_v}\psi(w, \lambda; X)|_{X=0})_{v,w\in W_\Theta}$$

with components in $P(E^*)$ *is invertible for any* $\lambda\in E_c^*$.

vii) *Denoting*

$$E(\Theta)'=\{\lambda\in E_c^*; \langle\lambda, \alpha\rangle\neq 0 \text{ for } \alpha\notin\Sigma_\Theta\} \tag{1.14}$$

and

$$\phi_\Theta(w, \lambda; X)=\psi(w_\Theta, \lambda; w(\Theta)^{-1}X), \tag{1.15}$$

we have

$$H(\lambda)=\sum_{w\in W}C\phi_\Theta(w, \lambda; X) \qquad \text{for } \lambda\in E(\Theta)'. \tag{1.16}$$

Here we remark $E(\Psi)'=E_c^*$, $w_\Psi=w$ *and* $w(\Psi)=e$.

viii) *If* $w\in W_\Theta$, $h_w(X)$ *is a homogeneous* W_Θ*-harmonic polynomial of degree* $l(w)$ *and*

$$\psi(w, \lambda_0; X)=h_w(X)\exp\langle\lambda_0, X\rangle \tag{1.17}$$

for any $\lambda_0\in E_c^*$ *satisfying* $\langle\lambda_0, \alpha\rangle=0$ *for all* $\alpha\in\Theta$.

Proof. Suppose $w\neq e$. By putting $w'=ws_{\alpha(w)}$ we have

$$\psi(w, \lambda; X)=\langle\lambda, \alpha(w)^\vee\rangle^{-1}(\psi(w', \lambda; X)-\psi(w', s_{\alpha(w)}\lambda; X)). \tag{1.18}$$

Suppose $\psi(w', \lambda; X)\in\mathcal{O}(E_c^*\times E_c)$. Since $\psi(w', s_{\alpha(w)}\lambda; X)=\psi(w', \lambda; X)$ if $\langle\lambda, \alpha(w)^\vee\rangle=0$, we have $\psi(w, \lambda; X)\in\mathcal{O}(E_c^*\times E_c)$ by (1.18). Hence Theorem 1.3. iii) is obtained by the induction on $l(w)$.

Put $\lambda^\perp=\{\nu\in E_c^*; \langle\lambda, \nu\rangle=0\}$ for $\lambda\in E_c^*$, put

$$U(m)=\{\lambda\in E_c^*; \sharp(\lambda^\perp\cap\Sigma^+)=m\}$$

for a non-negative integer m and moreover put

$$H'(\lambda)=\sum_{w\in W}C\phi_\Theta(w, \lambda; X).$$

Since $a(w, v; \lambda)\in\mathcal{O}(U(0))$ and therefore $\psi(w, \lambda; w'X)\in H(\lambda)$ for $(w, w', \lambda)\in W\times W\times U(0)$, we have $H'(\lambda)\subset H(\lambda)$ for any $\lambda\in E_c^*$ by the analytic continuation for the parameter $\lambda\in E_c^*$. We will show $\dim H'(\lambda)=\sharp W$ for any $\lambda\in E(\Theta)'$, which implies (1.16). When $\lambda\in U(0)$, it follows from

(1.13) and Lemma 1.1 that the functions $\phi_\Theta(w, \lambda; X)$ on E_c are linearly independent for $w \in W$ and therefore dim $H'(\lambda)=\# W$.

Let $\beta \in \Sigma^+$ and $\mu \in \beta^\perp \cap U(1)$. First we remark that the condition $w\mu = w'\mu$ for any elements w and w' in W means $w=w'$ or $w=w's_\beta$. Put $W'=\{w \in W; \Sigma(w) \ni \beta\}$ and $W''=W-W'$. It is clear that $v \in W'$ if and only if $vs_\beta \in W''$. Put $H''(\mu)=\sum_{w\in W} \boldsymbol{C} \exp\langle w\mu, X\rangle$. Then dim $H''(\mu)=\# W/2$ and $H''(\mu)=\sum_{w\in W'} \boldsymbol{C} \exp\langle w\mu, X\rangle$. It is clear that $\pi(\lambda)^N a(w, v; \lambda) \in P(E^*)$ for a suitable integer N. Fix $w \in W$. When $a(w, v; \lambda)$ is not identically zero, we put $a(w, v; \beta t+\mu)=t^{-m(v)}c(v; t)$ with a suitable $m(v) \in \boldsymbol{Z}$. Here $c(v; t)$ is a holomorphic function defined in a neighborhood of the origin in $\boldsymbol{C}$ with $c(v; 0)\neq 0$. Put $m(v)=-\infty$ when $a(w, v; \lambda)\equiv 0$. Let m be the largest integer in $\{m(v); v \in W\}$. Assume $m\geqq 1$. Put $\Lambda=\{v \in W; m(v)=m\}$ and $\Lambda'=\{v \in W; m(v)=m-1\}$. Since

$$t^m \psi(w, \mu+\beta t; X)=\sum_{v\in W} c(v; t)t^{m-m(v)} \exp\langle v\mu+v\beta t, X\rangle,$$

we have $\sum_{v\in\Lambda} c(v; 0) \exp\langle v\mu, X\rangle=0$, which implies that if $v \in \Lambda$, then $vs_\beta \in \Lambda$ and $c(v; 0)+c(vs_\beta; 0)=0$. Putting $d(v; t)=t^{-1}(c(v; t)+c(vs_\beta; t))$ for $v \in \Lambda$, we have $c(v; t)\exp\langle v\mu+v\beta t, X\rangle+c(vs_\beta; t)\exp\langle vs_\beta\mu+vs_\beta\beta t, X\rangle = c(v; t) \exp\langle v\mu+v\beta t, X\rangle(1-\exp\langle -2v\beta t, X\rangle)+td(v; t)\exp\langle vs_\beta\mu+vs_\beta\beta t, X\rangle$ and therefore the function

$$\sum_{v\in\Lambda\cap W'} (c(v; 0)\langle 2v\beta, X\rangle+d(v; 0)) \exp\langle v\mu; X\rangle + \sum_{v\in\Lambda'} c(v; 0) \exp\langle v\mu, X\rangle \tag{1.19}$$

is $\psi(w, \mu; X)$ if $m=1$ and 0 otherwise. Hence $m=1$. Thus we have $\pi_\Psi(\lambda)a(w, v; \lambda) \in \mathcal{O}(U(0)\cap U(1))$. Moreover $(\langle\lambda, \beta\rangle a(w, v; \lambda))|_{\beta^\perp}$ is a well-defined meromorphic function on $\beta^\perp$ and satisfies

$$(\langle\lambda, \beta\rangle a(w, v; \lambda))|_{\beta^\perp}+(\langle\lambda, \beta\rangle a(w, vs_\beta; \lambda))|_{\beta^\perp}=0. \tag{1.20}$$

Since the codimension of the compliment of $U(0)\cap U(1)$ is larger than one, we have $\pi_\Psi(\lambda)a(w, v; \lambda) \in \mathcal{O}(E_c^*)$. On the other hand, if $w \in W_\Theta$, it follows from (1.12) that there exists an integer N so that $\pi_\Theta(\lambda)^N a(w, v; \lambda) \in P(E^*)$ for $(w, v) \in W'\times W$ because Σ_Θ is a root system with the Weyl group W_Θ. Hence $\pi_\Theta(\lambda)a(w, v; \lambda) \in P(E^*)$ if $w \in W_\Theta$.

Suppose $\mu \in E(\Theta)'\cap U(1)$. Then the element β in Σ^+ with $\beta^\perp \ni \mu$ is contained in Σ_Θ^+. Suppose $v \in W'$ and put $v'=vs_\beta$. Lemma 1.2 says that $a(v, v; \mu+\beta t)$ has a pole of order 1 at the origin. Hence $v>v'$ as we have seen. In general we have proved

$$ws_\beta<w \quad \text{if and only if} \quad \beta \in \Sigma(w) \tag{1.21}$$

(cf. [De, Proposition 2.3]). The preceding argument (cf. (1.19)) assures the existence of non-zero constants C_w so that

$$(1.22)\quad \begin{aligned}&\psi(w,\mu;X)-C_w\psi_\beta(w,\mu;X)\in \textstyle\sum_{w'<w}\boldsymbol{C}\psi_\beta(w',\mu;X) \quad \text{with}\\ &\psi_\beta(w,\mu;X)=\begin{cases}\exp\langle w\mu, X\rangle & \text{if } w\in W'',\\ \langle w\beta, X\rangle \exp\langle w\mu, X\rangle & \text{if } w\in W'.\end{cases}\end{aligned}$$

If $w'<w$ and $w\in W_\Theta$, then $w'\in W_\Theta$. Hence by the induction on $l(w_\Theta)$ we have $\psi_\beta(w_\Theta,\mu;W(\Theta)^{-1}X)\in H(\mu)$ for $w\in W$ and therefore $\dim H'(\mu)=\#W$.

Next put $\pi_\Theta(\lambda)\psi(s_\Theta^*,\lambda;X)=\sum_{w\in W_\Theta} b(w;\lambda)\exp\langle w\lambda;X\rangle$. Here the elements $b(w;\lambda)$ in $P(E^*)$ are homogeneous of degree 0 with respect to λ because $a(s_\Theta^*,v;\lambda)$ are homogeneous of degree $-l(s_\Theta^*)$. Hence $b(w;\lambda)$ are constant functions and therefore it follows from (1.20) that $b(w;\lambda)+b(ws_\beta;\lambda)=0$ for any $\beta\in\Sigma_\Theta^+$. Since $b(s_\Theta^*;\lambda)=(-1)^{l(s_\Theta^*)}$ by Lemma 1.2, we have Theorem 1.3. iv).

Let $\{q(v)\}$ be a set of polynomials in $H(0)$ so that $H(0)=\sum_{v\in W}\boldsymbol{C}q(v)$. Put $A(\lambda;X)=(\partial_{q(v)}\phi_\Theta(w,\lambda;X))_{v,w\in W}$. It follows from (1.8) and (1.9) that for any fixed $(\lambda,X)\in E_c^*\times E_c$, if a function u in $H(\lambda)$ satisfies $(\partial_{q(v)}u)(X)=0$ for any $v\in W$, then $u=0$ because the Taylor expansion of u at X vanishes. Hence $\dim H'(\lambda)=\operatorname{rank} A(\lambda;X)$ for any $X\in E_c$. Since $\dim H'(\lambda)=\#W$ for $\lambda\in E(\Theta)'\cap(U(0)\cup U(1))$ as we have shown, the holomorphic function $\det A(\lambda;0)$ never vanishes on $E(\Theta)'\cap(U(0)\cup U(1))$. Since the codimension of the compliment of the set $U(0)\cup U(1)$ is larger than one, we have $\det A(\lambda;0)\neq 0$ for any $\lambda\in E(\Theta)'$ and we obtain (1.16). The above argument also proves Theorem 1.3. vii) in the case when $\Theta=\Psi$.

It follows from (1.13) and Lemma 1.2 that $(\partial_p-p(w\lambda))\psi(w,\lambda;X)\in\mathcal{O}(E_c^*\times E_c)\cap\sum_{v<w}R(E^*)\psi(v,\lambda;X)$ for $w\in W$ and $p\in S(E)$. Moreover since $\boldsymbol{C}[\partial]H(\lambda)\subset H(\lambda)$ and since $\psi(w,\lambda;X)$ $(w\in W)$ form a basis of $H(\lambda)$ for any $\lambda\in E_c^*$, we have Theorem 1.3. v).

Let $f_1,\cdots,f_r$ be a basis of the space of W_Θ-harmonic polynomials. Here $r=\#W_\Theta$. Fix an element ν in E_c^* satisfying $\langle\nu,\alpha\rangle\neq 0$ for $\alpha\in\Sigma_\Theta$. Then replacing W by W_Θ, the equality (1.16) assures the existence of the functions $f_j'(t,X)$ in $\mathcal{O}(\boldsymbol{C}\times E_c)$ which satisfy

$$\begin{cases}\sum_{j=1}^r \boldsymbol{C}f_j'(t,X)=\sum_{w\in W_\Theta}\boldsymbol{C}\exp\langle w\nu t, X\rangle & \text{if } t\neq 0,\\ f_j'(0,X)=f_j(X)\end{cases}$$

for $j=1,\cdots,r$. Hence

$$(1.23)\qquad \sum_{j=1}^r \boldsymbol{C}f_j'(t,X)=\sum_{w\in W_\Theta}\boldsymbol{C}\psi(w,\nu t;X)$$

for $t\neq 0$. We can put $t=0$ in (1.23) by the analytic continuation because $f'_j(t, X)$ $(j=1, \cdots, r)$ are linearly independent for any t and so are $\psi(w, \nu t; X)$ $(w\in W_\Theta)$. Hence $h_w(X)$ is a W_Θ-harmonic polynomial for any $w\in W_\Theta$. Let $\lambda_\circ\in E_c^*$. Suppose $\langle\lambda_\circ, \alpha\rangle=0$ for $\alpha\in\Theta$. Then $\psi(w, \nu t+\lambda_\circ; X)=\psi(w, \nu t; X)\exp\langle\lambda_\circ, X\rangle$ if $w\in W_\Theta$. This means (1.17). Now we remark that Lemma 1.2 implies

$$(1.24)\qquad \psi(w, t^{-1}\lambda; tX)=t^{l(w)}\psi(w, \lambda; X)\qquad \text{for } t\in \boldsymbol{C}-\{0\}.$$

This proves that $h_w(X)$ is a homogeneous polynomial of degree $l(w)$.

Let $\psi'(w, \lambda; X)$ be functions given in (1.13) by using other simple roots $\alpha'(w)$ in Definition 1.1 which satisfy $l(ws_{\alpha'(w)})<l(w)$. Then Lemma 1.2 proves

$$(1.25)\qquad \psi(v, \lambda; X)-\psi'(v, \lambda; X)=\textstyle\sum_{v'<v} r(v'; \lambda)\psi(v', \lambda; X)$$

for some $r(v'; \lambda)\in R(E^*)$. Since the left hand side of (1.25) belongs to $\mathcal{O}(E_c^*\times E_c)$ and since $\psi(w, \lambda; X)$ $(w\in W)$ are linearly independent for any $\lambda\in E_c^*$, we have $r(v'; \lambda)\in P(E^*)$. Owing to (1.24), we see that $r(v'; \lambda)$ is homogeneous of degree $l(v')-l(v)$. Hence we can conclude $r(v'; \lambda)=0$ because $l(v')<l(v)$ if $v'<v$. Thus we have Theorem 1.3. i).

Put $f_{v,w}(\lambda)=\partial_{h_v}\psi(w, \lambda; X)|_{X=0}$. Since h_v is homogeneous of degree $l(v)$, it follows from Lemma 1.2 that $f_{v,w}(\lambda)$ is a homogeneous polynomial with degree $l(v)-l(w)$. For any permutation σ of W_Θ, $\sum_{w\in W_\Theta}(l(w)-l(\sigma w))=0$. This means $\det(f_{v,w}(\lambda))_{v,w\in W_\Theta}$ is constant, which we denote by f. On the other hand, we see from Theorem 1.3. viii) that h_w $(w\in W_\Theta)$ are basis of W_Θ-harmonic polynomials. Moreover we have $\det(\partial_{h_v}h_w|_{X=0})_{v,w\in W_\Theta}\neq 0$ because we have proved the same statement for the basis of W-harmonic polynomials. Hence $f\neq 0$. Q.E.D.

Now we have the following characterization of $\psi(w, \lambda; X)$ or $a(w, v; \lambda)$:

Theorem 1.4. *Fix an element w in W. Then any function $f_w(\lambda, X)$ in $\mathcal{O}(E_c^*\times E_c)$ satisfying the following two conditions is a constant multiple of $\psi(w, \lambda; X)$ defined in Theorem* 1.3.

i) $f_w(t^{-1}\lambda, tX)=t^{l(w)}f_w(\lambda, X)$ *for any* $t\in \boldsymbol{C}-\{0\}$.

ii) $f_w(\lambda_0, X)\in \boldsymbol{C}\exp\langle w\lambda_0, X\rangle+\sum_{l(v)<l(w), v\in W}\boldsymbol{C}\exp\langle v\lambda_0, X\rangle$ *for any fixed* $\lambda_0\in(E_c^*)'$.

Proof. Put $f_w(\lambda, X)=\sum_{v\in W}a_v(\lambda)\exp\langle v\lambda, X\rangle$. Here $a_v(\lambda)$ are certain meromorphic functions on E_c^* with the homogeneous degree $-l(w)$.

Then the proof of (1.20) and (1.21) implies that $a_w(\lambda)\prod_{\alpha\in\Sigma(w)}\langle\lambda,\alpha\rangle\in\mathcal{O}(E_c^*)$, whence $a_w(\lambda)=Ca(w,w;\lambda)$ with a suitable $C\in\boldsymbol{C}$. Considering the function $f_w(\lambda,X)-C\psi(w,\lambda;X)$, the same proof as that of Theorem 1.3. i) gives $f_w(\lambda,X)=C\psi(w,\lambda;X)$. Q.E.D.

Put $P_\Theta(E^*)=R(E^*)\cap\mathcal{O}(E(\Theta)')$. The map $\partial\colon S(E)\ni p\mapsto\partial_p$ is extended to the $P_\Theta(E^*)$-linear map of $P_\Theta(E^*)\otimes S(E)$, which will be denoted similarly. Then we have

Proposition 1.5. *Retain the notation in Theorem* 1.3. *Given* $\Theta\subset\Psi$ *and* $v\in W$, *there exists* $p\in P_\Theta(E^*)\otimes H(0)$ *which satisfies the conditions* (1.26)–(1.30) *for any* $v'\in W$:

(1.26) $$\partial_p\phi_\Theta(v',\lambda;X)=0 \qquad \textit{if } v'(\Theta)\neq v(\Theta).$$

(1.27) $$\partial_p\phi_\Theta(v(\Theta)s_\Theta^*,\lambda;X)=\phi_\Theta(v,\lambda;X).$$

(1.28) $$\partial_p\phi_\Theta(v's_\Theta^*,s_\Theta^*\lambda;X)\neq 0 \textit{ implies } v'(\Theta)=v(\Theta) \textit{ and } v'_\Theta\geqq v_\Theta.$$

(1.29) $$\partial_p\phi_\Theta(vs_\Theta^*,s_\Theta^*\lambda;X)=\exp\langle v\lambda,X\rangle.$$

(1.30) $$\partial_p\exp\langle v\lambda,X\rangle=\Big(\prod_{\alpha\in\Sigma_\Theta^+-\Sigma(v_\Theta)}\langle\lambda,\alpha^\vee\rangle\Big)\exp\langle v\lambda,X\rangle.$$

Proof. For simplicity we denote s_Θ^* by u. Since $\phi_\Theta(w,\lambda;X)=\psi(w_\Theta,\lambda;w(\Theta)^{-1}X)$, we can easily reduce the proposition in the case when $v(\Theta)=e$. Therefore we assume $v\in W_\Theta$. Use the notation in Theorem 1.3 and put $B(\lambda)=(\partial_{h_w}\partial_{h_{w'}}\psi(u,\lambda;X)|_{X=0})_{w,w'\in W_\Theta}$. Since $a(u,w;\lambda)$ are homogeneous of degree $-l(u)$ with respect to λ, $\partial_q\psi(u,\lambda;X)|_{X=0}$ is homogeneous of degree $r-l(u)$ if $q\in S(E)$ is homogeneous of degree r. Let σ be a permutation of the elements in W. Since $\sum_{w\in W_\Theta}(l(w)+l(\sigma w)-l(u))=\sum_{w\in W_\Theta}(l(w)+l(uw)-l(u))=0$, det $B(\lambda)$ is homogeneous of degree 0, that is, det $B(\lambda)$ is constant. On the other hand, since h_u is a W_Θ-harmonic skew polynomial, $\sum_{w\in W_\Theta}Ch_w=C[\partial]h_u=\sum_{w\in W_\Theta}\partial_{h_w}h_u$ (cf. [St] and (1.9)) and therefore det $B(\lambda)=\det B(0)\neq 0$ from Theorem 1.3. vi). Hence there exists $p\in\sum_{w\in W_\Theta}P(E^*)h_w$ $(\subset P(E^*)\otimes H(0))$ such that

$$\partial_{h_w}\partial_p\psi(u,\lambda;X)|_{X=0}=\partial_{h_w}\psi(v,\lambda;X)|_{X=0} \qquad \text{for any } w\in W.$$

Combining this with Theorem 1.3. v) and vi), we have $\partial_p\psi(u,\lambda;X)=\psi(v,\lambda;X)$.

Now from Lemma 1.2 we have

$$\begin{aligned}\psi(vu,u\lambda;X)&=\sum_{wv\leqq vu,w\in W}a(vu,wu;u\lambda)\exp\langle wuu\lambda,X\rangle\\&=\sum_{w\geqq v,w\in W_\Theta}a(vu,wu;u\lambda)\exp\langle w\lambda,X\rangle\end{aligned}$$

and

$$a(vu, vu; u\lambda) = (-1)^{l(vu)} \prod_{\alpha \in \Sigma(vu)} \langle u\lambda, \alpha^\vee \rangle^{-1}$$
$$= \prod_{\alpha \in \Sigma_\Theta^+ - \Sigma(v)} \langle \lambda, \alpha^\vee \rangle^{-1}$$

because

$$u^{-1}\Sigma(vu) = u(\Sigma^+ \cap (vu)^{-1}\Sigma^-) = u\Sigma_\Theta^+ \cap v^{-1}\Sigma_\Theta^- = \Sigma_\Theta^- \cap v^{-1}\Sigma_\Theta^-$$
$$= -(\Sigma_\Theta^+ \cap v^{-1}\Sigma_\Theta^+) = -(\Sigma_\Theta^+ - (\Sigma^+ \cap v^{-1}\Sigma_\Theta^-)) = -(\Sigma_\Theta^+ - \Sigma(v))$$

for $v \in W_\Theta$.

Fix $\lambda \in E_c^*$ so that $\langle \lambda, \alpha \rangle \neq 0$ for $\alpha \in \Sigma$. Since $\partial_p \psi(u, \lambda, X) = \psi(v, \lambda; X)$, we have $\partial_p a(u, v; \lambda) \exp \langle v\lambda, X \rangle = a(v, v; X) \exp \langle v\lambda, X \rangle$ and therefore Lemma 1.2 and Theorem 1.3. iv) prove (1.30). For $v' \in W_\Theta$,

$$\partial_p \psi(v'u, u\lambda; X) \in \partial_p \boldsymbol{C}[\partial]\psi(u, \lambda; X) = \boldsymbol{C}[\partial]\partial_p \psi(u, \lambda; X)$$
$$= \boldsymbol{C}[\partial]\psi(v, \lambda; X) \subset \sum_{w \leq v} \boldsymbol{C}\psi(w, \lambda; X) = \sum_{w \leq v} \boldsymbol{C} \exp \langle w\lambda, X \rangle$$

and

$$\partial_p \psi(v'u, u\lambda; X) \in \sum_{wu \leq v'u} \boldsymbol{C}\psi(wu, u\lambda; X)$$
$$= \sum_{w \geqq v', w \in W_\Theta} \boldsymbol{C}\psi(wu, u\lambda; X) = \sum_{w \geqq v', w \in W_\Theta} \boldsymbol{C} \exp \langle w\lambda, X \rangle.$$

Hence $\partial_p \psi(v'u, u\lambda; X) = 0$ if v' in W_Θ does not satisfy $v' \leqq v$. Moreover $\partial_p \psi(vu, u\lambda; X) = C(\lambda) \exp \langle v\lambda, X \rangle$ with a suitable $C(\lambda) \in P(E^*)$. Comparing (1.30) with $a(vu, vu; u\lambda)$, we have $C(\lambda) = 1$.

Let $\mu \in E(\Theta)'$. Then there exists $Y \in E_c$ so that if an element (w, w') in $W \times W$ satisfies $\langle w\mu, Y \rangle = \langle w'\mu, Y \rangle$, then $w' \in wW_\Theta$. Since $\partial_Y \exp \langle \lambda, X \rangle = \langle \lambda, Y \rangle \exp \langle \lambda, X \rangle$, Lemma 1.6 assures the existence of polynomial r of Y with coefficients in $R(E_c^*)$ so that the coefficients are holomorphic in a neighborhood $U(\mu)$ of μ and moreover $\partial_r \exp \langle w\lambda, X \rangle$ equals $\exp \langle w\lambda, X \rangle$ if $w \in W_\Theta$ and 0 otherwise. This means $\partial_r \phi(w, \lambda; X)$ equals $\phi(w, \lambda; X)$ if $w \in W_\Theta$ and 0 otherwise. Similarly $\partial_r \phi(wu, u\lambda; X)$ equals $\phi(wu, u\lambda; X)$ if $w \in W_\Theta$ and 0 otherwise. Using (1.8) and (1.9), we can choose r in $(R(E_c^*) \cap \mathcal{O}(U(\mu))) \otimes H(0)$. For any fixed $\lambda \in E_c^*$, if an element h in $H(0)$ satisfies $\partial_h \phi(w, \lambda; X) = 0$ for any $w \in W$, then $h = 0$ (cf. Theorem 1.3 iv)). This means that r does not depend on μ. Hence $r \in P_\Theta(E^*) \otimes H(0)$. Then $pr \in P_\Theta(E^*) \otimes H(0)$ is the required one. Q.E.D.

Lemma 1.6. *Let m and n be positive numbers. For* $\xi = (\xi_1, \cdots, \xi_{m+n}) \in \boldsymbol{C}^{m+n}$ *put* $P_j(z, \xi) = \prod_{i \in I(j)} (z - \xi_i)$ *with* $I(j) = \{1, \cdots, m+n\} - \{j\}$. *Moreover put* $P(z, \xi) = \sum_{j=1}^m P_j(z, \xi)/P_j(\xi_j, \xi)$. *Then* $P(z, \xi)$ *defines a holomorphic function on* $U = \{(z, \xi) \in \boldsymbol{C}^{1+m+n};\ \xi_j \neq \xi_k$ *for* $j = 1, \cdots, m$ *and* $k = m+1, \cdots, m+n\}$ *and satisfies*

$$P(\xi_j, \xi)=\begin{cases}1 & \text{if } 1\leqq j\leqq m,\\ 0 & \text{if } m<j\leqq m+n.\end{cases}$$

Proof. For positive numbers i and j with $i<j<m$, put $H_{i,j}=\{(z, \xi)\in C^{1+m+n}; \xi_i=\xi_j\}$ and $H'_{i,j}=\{(z, \xi)\in H_{i,j}; \xi_{i'}\neq\xi_{j'}$ for $1\leqq i'<j'\leqq m$ with $(i', j')\neq(i, j)\}$. Then $P(z, \xi)$ is clearly holomorphic on $U-\bigcup_{1\leqq i<j\leqq m} H_{i,j}$. Let $p\in U\cap H'_{i,j}$. Then $P_k(z, \xi)/P_k(\xi_k, \xi)$ is holomorphic at p if $k\neq i$ and $k\neq j$. Put $I(i, j)=I(i)\cap I(j)$ and

$$Q_{i,j}(z, \xi)=P_i(z, \xi)\prod_{k\in I(i,j)}(\xi_j-\xi_k)-P_j(z, \xi)\prod_{k\in I(i,j)}(\xi_i-\xi_k).$$

Since $Q_{i,j}(z, \xi)|_{\xi_i=\xi_j}=0$, there exist a polynomial $R_{i,j}(z, \xi)$ with $Q_{i,j}(z, \xi)=(\xi_i-\xi_j)R_{i,j}(z, \xi)$. Since the function

$$P_i(z, \xi)/P_i(\xi_i, \xi)+P_j(z, \xi)/P_j(\xi_j, \xi)$$

equals

$$R_{i,j}(z, \xi)/\prod_{k\in I(i,j)}(\xi_i-\xi_k)(\xi_j-\xi_k),$$

it is holomorphic in a neighborhood of p. This means $P(z, \xi)$ is holomorphic in U except a subvariety of U with codimension larger than one. Hence $P(z, \xi)$ is holomorphic in U. The other part of the lemma is clear.

Q.E.D.

For $\Theta\subset\Psi$ and $v\in W$, let $P_v^\Theta(\lambda, \partial)$ be the differential operator ∂_p in Proposition 1.4. We remark that its coefficients belong to $P_\Theta(E^*)=R(E^*)\cap\mathcal{O}(E_c^*)$. Fix a basis $\{H_1, \cdots, H_l\}$ of E_c and put $\partial_j=\partial_{H_j}$ $(j=1, \cdots, l)$. Define $a_{j,v,w}^\Theta(\lambda)\in P(E^*)$ by

$$\partial_j\phi_\Theta(v, \lambda; X)=\sum_{w\in W} a_{j,v,w}^\Theta(\lambda)\phi_\Theta(w, \lambda; X). \tag{1.31}$$

Then Theorem 1.3. v) and (1.15) say

$$a_{j,v,v}^\Theta(\lambda)=\langle v\lambda, H_j\rangle \tag{1.32}$$

and

$$a_{j,v,w}^\Theta(\lambda)\neq 0 \text{ means } v(\Theta)=w(\Theta) \text{ and } v_\Theta\geqq w_\Theta. \tag{1.33}$$

For an element $\lambda\in E_c^*$, consider a $C[\partial]$-module

$$\mathcal{N}_\Theta\colon \partial_j u_v^\Theta=\sum_{w\in W} a_{j,v,w}^\Theta(\lambda)u_w^\Theta \quad (1\leqq j\leqq l,\ v\in W) \tag{1.34}$$

with generators u_v^Θ. Then we have

Theorem 1.7. *Assume $\lambda \in E(\Theta)'$. Then the $C[\partial]$-module*

$$\mathscr{M}: \partial_p u = p(\lambda)u \qquad (p \in I(W)) \tag{1.35}$$

is isomorphic to $\mathscr{N}_\Theta$ by the map Φ of $\mathscr{M}$ to $\mathscr{N}_\Theta$ defined by $\Phi(u) = \sum_{v \in W(\Theta)} u_v^\Theta$. The inverse of Φ is the map Φ' defined by $\Phi'(u_v^\Theta) = P_v^\Theta(\lambda, \partial)u$.

Proof. First remark the following. It follows from Theorem 1.3. vi) and (1.9) that if $Q \in C[\partial]$ satisfies $Q\phi = 0$ for all $\phi \in H(\lambda)$, then $Q \in \sum_{p \in I(W)} C[\partial](\partial_p - p(\lambda))$.

Put $u_v = P_v^\Theta(\lambda, \partial)u$ for $v \in W$. For $v' \in W(\Theta)$, Proposition 1.5 implies that if we substitute u by $\phi_\Theta(v's_\Theta^*, \lambda; X)$, then the functions $u_v = P_v^\Theta(\lambda, \partial)\phi_\Theta(v's_\Theta^*, \lambda; X)$ satisfy $\mathscr{N}_\Theta$. Since $H(\lambda) = \sum_{v' \in W(\Theta)} C[\partial]\phi(v's_\Theta^*, \lambda; X)$, $u_v = P_v^\Theta(\lambda, \partial)\phi$ also satisfy $\mathscr{N}_\Theta$ for all $\phi \in H(\lambda)$ and therefore the map Φ' is a homomorphism. By the same reason, $\phi = \sum_{w \in W(\Theta)} P_w^\Theta(\lambda, \partial)\phi$ for all $\phi \in H(\lambda)$, which means Φ' is surjective. On the other hand, it is clear from the definition (1.34) that $\dim \mathscr{N}_\Theta \leqq \# W$. Since $\dim \mathscr{M} = \# W$, the homomorphism Φ' is an isomorphism and the map Φ is its inverse.

Q.E.D.

Remark 1.8. Theorem 1.3. i) is proved in [BGG, Theorem 3.4]. The proof in [BGG] is quite different from the one given here.

§ 2. Boundary value maps for riemannian symmetric spaces

In this section we will review the property of boundary value maps for eigenfunctions of invariant differential operators on riemannian symmetric spaces of the noncompact type. Before to do so, we will continue to study the system of differential equations introduced in the previous section. The strong connection between them will be revealed.

Retain the notation in § 1. Define $H_1, \cdots, H_l \in E$ so that $\alpha_i(H_j) = \delta_{i,j}$ for $1 \leqq i \leqq l$ and $1 \leqq j \leqq l$, where $\Psi = \{\alpha_1, \cdots, \alpha_l\}$ as we put in § 1. Put $t_j = \exp -2\langle \alpha_j, X \rangle$ and identify E with R_+^l by the map $R_+^l \ni t = (t_1, \cdots, t_l) \mapsto \sum -\frac{1}{2} H_j \log t_j \in E$. Then ∂_j, which is ∂_{H_j} by definition, equals $-2t_j \partial/\partial t_j$. For simplicity we put $\vartheta_j = t_j \partial/\partial t_j$ and $\vartheta = (\vartheta_1, \cdots, \vartheta_l)$. Let $p_1(H_1, \cdots, H_l), \cdots, p_l(H_1, \cdots, H_l)$ be homogeneous elements in $I(W)$ which generate $I(W)$. Fix an element ρ in E_c^* and for $\lambda \in E_c^*$ consider the system

$$\begin{aligned} \mathscr{M}: p_j(\langle \rho, H_1 \rangle - 2\vartheta_1, \cdots, \langle \rho, H_l \rangle - 2\vartheta_l)u \\ = p_j(\langle \lambda, H_1 \rangle, \cdots, \langle \lambda, H_l \rangle)u \qquad \text{for } j = 1, \cdots, l \end{aligned} \tag{2.1}$$

of differential equations on R^l. Put $Y_j = \{t \in R^l; t_j = 0\}$ and $Y =$

$Y_1 \cap \cdots \cap Y_l$. Then the system $\bar{\mathscr{M}}$ has regular singularities along the set of walls $\{Y_1, \cdots, Y_l\}$ with the edge Y. In general, we use microlocal analysis to define boundary values of solutions for the system of differential equations with regular singularities along boundaries. But the system $\bar{\mathscr{M}}$ is so simple that we can explain the method in [O2] to define boundary value maps in this case without using microlocal analysis.

Fix $\mu \in E_c^*$ so that the subgroup $\{w \in W; w\mu = \mu\}$ equals W_Θ with a suitable $\Theta \subset \Psi$. By replacing μ by $w\mu$ with a suitable $w \in W$, we may assume this when we consider the system $\bar{\mathscr{M}}$ with $\lambda = \mu$. The indicial equation corresponding to $\bar{\mathscr{M}}$ is obtained by replacing ϑ_j by s_j in (2.1) and $\lambda_w = (\lambda_{w,1}, \cdots, \lambda_{w,l}) = (\frac{1}{2}\langle \rho - w\lambda, H_1\rangle, \cdots, \frac{1}{2}\langle \rho - w\lambda, H_l\rangle)$ $(w \in W)$ are the characteristic exponents. Let Ω be an open neighborhood of μ in $E(\Theta)'$ and let ${}_\Omega\mathscr{A}(E)$ be the space of real analytic function on E with the holomorphic parameter $\lambda \in \Omega$. Let ${}_\Omega\mathscr{A}(E; \bar{\mathscr{M}})$ be the space of solutions of $\bar{\mathscr{M}}$ in ${}_\Omega\mathscr{A}(E)$. Remark that the system $\bar{\mathscr{M}}$ is transformed into the system $\partial_p u' = u'$ $(p \in I(W))$ studied in § 1 through the automorphism $C[\partial] \ni Q \mapsto \exp\langle \rho, X\rangle \circ Q \circ \exp\langle -\rho, X\rangle$ of $C[\partial]$ and the correspondence $u' = \exp\langle \rho, X\rangle u$. By virtue of Theorem 1.7, the system $\bar{\mathscr{M}}$ is isomorphic to the system

$$(2.2) \qquad \bar{\mathscr{N}}_\Theta \colon \vartheta_j u_v^\Theta = \sum_{w \in W} b_{j,v,w}^\Theta(\lambda) u_w^\Theta \qquad (1 \leqq j \leqq l,\ v \in W)$$

for any $\lambda \in \Omega$. The isomorphism is given by $\bar{M} \ni u \mapsto \sum_{v \in W(\Theta)} u_v^\Theta \in \bar{\mathscr{N}}_\Theta$ and its inverse is defined by $\bar{\mathscr{N}}_\Theta \ni u_v^\Theta \mapsto Q_v^\Theta(\lambda, \vartheta)u \in \bar{\mathscr{M}}$, where

$$(2.3) \qquad b_{j,v,v}^\Theta(\lambda) = \tfrac{1}{2}\langle \rho - v\lambda, H_j\rangle,$$

$$(2.4) \qquad b_{j,v,w}^\Theta(\lambda) = -\tfrac{1}{2} a_{j,v,w}^\Theta(\lambda) \qquad \text{if } v \neq w \text{ (cf. (1.31))}$$

and

$$(2.5) \qquad Q_v^\Theta(\lambda, \vartheta) = \exp\langle -\rho, X\rangle \circ P_v^\Theta(\lambda, \partial) \circ \exp\langle \rho, X\rangle.$$

Noting

$$(2.6) \qquad \exp\langle v\lambda - \rho, X\rangle = t_1^{\lambda_{v,1}} \cdots t_l^{\lambda_{v,l}}$$

by the identification $E \simeq \boldsymbol{R}_+^l$, we put

$$(2.7) \qquad \phi_v^\Theta(\lambda, t) = \exp\langle -\rho, X\rangle \phi_\Theta(vs_\Theta^*, s_\Theta^*\lambda; X)$$

for $v \in W$. Then ${}_\Omega\mathscr{A}(E; \bar{\mathscr{M}}) = \sum_{w \in W} {}_\Omega\mathscr{A}(E)\phi_w^\Theta(\lambda, X) \simeq {}_\Omega\mathscr{A}(E)^{\# W}$ (cf. Theorem 1.3). In fact any solution $u(\lambda, t) \in {}_\Omega\mathscr{A}(E; \bar{\mathscr{M}})$ is uniquely written in the form

$$(2.8) \qquad u(\lambda, t) = \sum_{w \in W} a_w(\lambda) \phi_w^\Theta(\lambda, X)$$

with some $a_w(\lambda) \in {}_{\Omega}\mathscr{A}(E)$. By using theorem 1.7 we define boundary value maps β_w^Θ of ${}_{\Omega}\mathscr{A}(E; \mathscr{M})$ in the following way.

First remark that the solution of the system

$$\mathscr{N}_v\colon (\vartheta_j - \tfrac{1}{2}\langle \rho - v\lambda, H_j\rangle)u_v = 0 \qquad (j=1, \cdots, l) \tag{2.9}$$

with an element $v \in W$ equals $a(\lambda)t^{\lambda_v}$ with suitable function $a(\lambda)$ of λ, where t^{λ_v} equals the function (2.6). In the case when $v \in W(\Theta)$, since $Q_v^\Theta(\lambda, \vartheta)u(\lambda, t)$ is a solution of (2.9) for $u(\lambda, t) \in {}_{\Omega}\mathscr{A}(E; \mathscr{M})$, we define $\beta_v^\Theta(u)$ by

$$Q_v^\Theta(\lambda, \vartheta)u = \beta_v^\Theta(u)t^{\lambda_v}. \tag{2.10}$$

Then in this case we have

$$\beta_v^\Theta(u) = a_v(\lambda) \tag{2.11}$$

by virtue of Proposition 1.5. Given $v \in W$. Suppose $\beta_w^\Theta(u)$ are defined for $w \in W$ satisfying both $w_\Theta < v_\Theta$ and $w \in vW_\Theta$ and suppose $\beta_w^\Theta(u) = a_w(\lambda)$. Then we can define $\beta_v^\Theta(u)$ by

$$Q_v^\Theta(\lambda, \vartheta)(u(\lambda, t) - \textstyle\sum_{w<v, w \in vW_\Theta} \beta_w^\Theta(u)\phi_w(\lambda, t)) = \beta_v^\Theta(u)t^{\lambda_v} \tag{2.12}$$

because Proposition 1.5 assures that the left hand side of (2.12) is a solution of (2.9). Thus we can inductively define β_v^Θ so that (2.11) holds. We will remark some facts concerning this procedure.

For a fixed $\lambda \in U(\mu)$, we can define the boundary values for the solution of the system (2.1) by the above procedure. Then the boundary values are constants.

Put $(E_c^*)' = \{\lambda \in E_c^*; \langle \lambda, \alpha\rangle \neq 0 \text{ for any } \alpha \in \Sigma\}$. Suppose $\Theta = \varnothing$, that is, $\mu \in (E_c^*)'$. By denoting $\beta_w = \beta_w^\varnothing$, we have

$$u(\lambda, t) = \textstyle\sum_{w \in W} \beta_w(u)t^{\lambda_w}. \tag{2.13}$$

Fix an element $v \in W$. Suppose $\beta_w^\Theta(u) = 0$ for any $w \in W$ satisfying both $w_\Theta < v_\Theta$ and $w \in vW_\Theta$. Then $\beta_v^\Theta(u)$ is simply defined by (2.10). This follows from the fact that the correspondence $u = Q_v^\Theta(\lambda, \vartheta)u_v$ defines a $C[\partial]$-homomorphism of $\mathscr{N}_v$ onto the quotient of $\mathscr{M}$ defined by the relations $u_w^\Theta = 0$ ($w \in W$, $w_\Theta < v_\Theta$ and $w \in vW_\Theta$) in (1.34) through Theorem 1.7. Moreover in virtue of Proposition 1.5 we have

$$\beta_v^\Theta(u)(\lambda, t) = (\textstyle\prod_{\alpha \in \Sigma_\Theta^+ - \Sigma(v_\Theta)} \langle \lambda, \alpha^\vee\rangle)\beta_v(u)(\lambda, t) \tag{2.14}$$

for $\lambda \in U(\mu) \cap (E_c^*)'$.

Now we will review boundary value maps of eigenfunctions for invariant differential equations on riemannian symmetric spaces of the noncompact type. We will use the notation introduced in § 0. But hereafter in this section and the next section we only consider riemannian symmetric spaces. Hence we may suppose $\sigma=\theta$ and we will omit the superfix d for simplicity. For example, we will write $\mathfrak{a}_{\mathfrak{p}}$, P etc. in place of $\mathfrak{a}_{\mathfrak{p}}^d$, P^d etc.

Let $\Psi(\mathfrak{a}_{\mathfrak{p}})=\{\alpha_1, \cdots, \alpha_l\}$ be the fundamental system of $\Sigma(\mathfrak{a}_{\mathfrak{p}})$ and $\{H_1, \cdots, H_l\}$ the dual basis of $\Psi(\mathfrak{a}_{\mathfrak{p}})$. Fix homogeneous elements $p_1(H_1, \cdots, H_l), \cdots, p_l(H_1, \cdots, H_l)$ in $I(W(\mathfrak{a}_{\mathfrak{p}}))$ which generate $I(W(\mathfrak{a}_{\mathfrak{p}}))$. Then the operator $\Delta_1, \cdots, \Delta_l$ in $D(G/K)$ corresponding to $p_1, \cdots, p_l$ by the Harish-Chandra isomorphism generate $D(G/K)$. In [OS] we construct a compact G-manifold $\tilde{X}$ where the riemannian symmetric space G/K is smoothly imbedded as an open G-orbit. It has the following properties:

There exists local coordinate systems $(x, t)=(x_1, \cdots, x_n, t_1, \cdots, t_l)$ of $\tilde{X}$ so that $G/K=\{(x, t) \in \tilde{X}; t_1>0, \cdots, t_l>0\}$ and the G-orbit $B=\{(x, t) \in \tilde{X}; t_1=\cdots=t_l=0\}$ is isomorphic to G/P. For $\lambda \in (\mathfrak{a}_{\mathfrak{p}})_c^*$ the system

$$(2.15) \qquad \mathcal{M}: \Delta_j u=\chi_\lambda(\Delta_j)u \qquad \text{for } j=1, \cdots, l$$

of differential equations has an analytic extension on $\tilde{X}$ and it has regular singularities in the weak sense along the set of walls Y_j defined by $t_j=0$ with the edge B. The indicial equation for $\mathcal{M}_\lambda$ equals

$$p_j(\langle\rho, H_1\rangle-2s_1, \cdots, \langle\rho, H_l\rangle-2s_l)=p_j(\langle\lambda, H_1\rangle, \cdots, \langle\lambda, H_l\rangle) \qquad \text{for } j=1, \cdots, l$$

and therefore the characteristic exponents are

$$(2.16) \qquad \lambda_w=(\lambda_{w,1}, \cdots, \lambda_{w,l})=(\tfrac{1}{2}\langle\rho-w\lambda, H_1\rangle, \cdots, \tfrac{1}{2}\langle\rho-w\lambda, H_l\rangle)$$

which are parametrized by $w \in W(\mathfrak{a}_{\mathfrak{p}})$. Comparing the systems $\bar{\mathcal{M}}$ and $\mathcal{M}$, we can consider that the system $\mathcal{M}$ is a perturbed system from $\bar{\mathcal{M}}$ or that $\bar{\mathcal{M}}$ is the first approximation of $\mathcal{M}$.

Fix $\mu \in (\mathfrak{a}_{\mathfrak{p}})_c^*$ so that

$$(2.17) \qquad \operatorname{Re}\langle\mu, \alpha\rangle\geqq 0 \qquad \text{for all } \alpha \in \Sigma(\mathfrak{a}_{\mathfrak{p}})^+$$

and there exists a subset Θ of $\Psi(\mathfrak{a}_{\mathfrak{p}})$ satisfying

$$(2.18) \qquad \Sigma \cap \mu^\perp=\Sigma \cap \textstyle\sum_{\alpha\in\Theta} C\alpha.$$

Put $\Sigma_o=\{\alpha \in \Sigma(\mathfrak{a}_{\mathfrak{p}}); \alpha/2 \notin \Sigma(\mathfrak{a}_{\mathfrak{p}})\}$. Since Σ_o is a reduced root system in $\mathfrak{a}_\mathfrak{p}^*$ with the Weyl group $W(\mathfrak{a}_\mathfrak{p})$, we can define Σ_Θ^+, W_Θ, $W(\Theta)$ and $\Sigma(w)$ as in § 1 by replacing Σ, E and W by Σ_o, $\mathfrak{a}_\mathfrak{p}$ and $W(\mathfrak{a}_\mathfrak{p})$, respectively. Let Ω be a sufficiently small open neighborhood of μ. Let ${}_\Omega\mathscr{A}(G/K)$ denote the space of real analytic functions on G/K with the holomorphic parameter λ in Ω and let ${}_\Omega\mathscr{A}(G/K; \mathscr{M})$ denote the space of the solutions of $\mathscr{M}$ in ${}_\Omega\mathscr{A}(G/K)$. Let U be any open subset of G/P. Identifying U with a left P invariant subset of G, we put

$$\begin{aligned}\mathscr{B}(U; L_\Omega)=\{f_\lambda \in {}_\Omega\mathscr{B}(U); f_\lambda(gman)=f_\lambda(g)a^{\rho-\lambda} \\ \text{for all } (\lambda, g, m, a, n) \in \Omega\times G\times M\times A_\mathfrak{p}\times N\}.\end{aligned} \tag{2.19}$$

Here ${}_\Omega\mathscr{B}(U)$ denotes the space of hyperfunctions on U with the holomorphic parameter λ in Ω.

Definition 2.1. For $\mu \in (\mathfrak{a}_\mathfrak{p})_c^*$ satisfying (2.17) and (2.18) we define an ordering $<_\mu$ in $W(\mathfrak{a}_\mathfrak{p})$:

For elements v and w in $W(\mathfrak{a}_\mathfrak{p})$ the relation $v<_\mu w$ holds if and only if one of the following two conditions holds:

i) $v\mu=w\mu$ and $v<w$ in the Bruhat ordering in $W(\mathfrak{a}_\mathfrak{p})$.

ii) $v\mu\neq w\mu$ and $\frac{1}{2}\langle v\mu-w\mu, H_j\rangle \in N$ for $j=1, \cdots, l$.

We remark that $v<w$ implies Re $\langle v\mu-w\mu, H_j\rangle\geqq 0$ for $j=1, \cdots, l$ (cf. [Di, Lemma 7.7.2]) and that the condition $-\frac{1}{2}\langle v\mu, \alpha^\vee\rangle \notin N$ for any $\alpha \in \Sigma(\mathfrak{a}_\mathfrak{p})^+$ implies the non-existence of an element w in $W(\mathfrak{a}_\mathfrak{p})$ with $w<_\mu v$ (cf. [K-, Appendix II, Proposition 2]).

Let $u(\lambda, x, t)$ be an element of ${}_\Omega\mathscr{A}(G/K; \mathscr{M})$. We will explain the definition of the boundary values of $u(\lambda, x, t)$ which is given in [O2] (cf. [KO, Section 5], [OS, § 2.2], [Sc, Chapter 5.2], [MO1, § 3] and [O3, § 3]). The system $\mathscr{M}$ has regular singularities in the sense of [KO, Definition 5.1] after the coordinate transformation $t_j\mapsto t_j^m$ with a positive integer $m>1$. Moreover after the transformation $u(\lambda, x, t)\mapsto t_1^k\cdots t_l^k u(\lambda, x, t)$ with a suitable non-negative integer k, any component of any characteristic exponents does not take a strictly negative integer. Through these transformations, the characteristic exponent $\lambda_w=(\lambda_{w,1}, \cdots, \lambda_{w,l})$ changes into $(m\lambda_{w,1}+k, \cdots, m\lambda_{w,l}+k)$. The following procedure is valid only after these transformations. But we pretend that we can assume $m=1$ and $k=0$ to make the notation simple. Let p^* be a point in $\sqrt{-1}\, S_B^*\tilde{X}-\bigcup_{j=1}^l\sqrt{-1}\, S_{Y_j}^*\tilde{X}$ with a base point p in U. Then in a neighborhood of p^*, the system $\mathscr{M}$ is microlocally isomorphic to a system

$$\tilde{\mathscr{N}}_\Theta: \vartheta_j\tilde{u}_v^\Theta=\textstyle\sum_{w\in W(\mathfrak{a}_\mathfrak{p})} B_{j,v,w}\tilde{u}_w^\Theta \qquad (1\leqq j\leqq l,\ v \in W) \tag{2.20}$$

of microdifferential equations for $\lambda \in \Omega$ by a correspondence $\tilde{u}_v^\Theta=\tilde{Q}_v^\Theta u$ and $u=\sum_{v\in W(\mathfrak{a}_\mathfrak{p})}\tilde{R}_v^\Theta\tilde{u}_v^\Theta$ with suitable microdifferential operators $\tilde{Q}_v^\Theta$ and $\tilde{R}_v^R$

([KO, Theorem 5.3]) with the holomorphic parameter λ. Here the microdifferential operators $B_{j,v,w}$ satisfy

(2.21) $$B_{j,v,w}=b_{j,v,w}(\lambda) \qquad \text{if } v\mu=w\mu$$

(cf. (2.2)–(2.4)) and

(2.22) if $B_{j,v,w}\neq 0$ and $v\mu\neq w\mu$, then $w<_{\mu}v$ and ord $B_{j,v,w}<0$.

Moreover $\sigma_*(\tilde{Q}_v^{\Theta})(\lambda, x, s)=Q_v^{\Theta}(\lambda, s)$ and $\sigma_*(\tilde{R}_w^{\Theta})(\lambda, x, s)$ equals 1 if $w\in W(\Theta)$ and 0 otherwise under the notation in [O2].

Now the condition that any $\lambda_{w,j}$ does not take a strictly positive integer for $\lambda\in\Omega$ assures the existence of the unique extension $\tilde{u}(\lambda, x, t)$ of $u(\lambda, x, t)$ such that $\tilde{u}$ is a hyperfunction on $\tilde{X}$ with the holomorphic parameter $\lambda\in\Omega$, supp $\tilde{u}\subset \mathrm{Cl}(G/K)$ in $\tilde{X}$, $\tilde{u}|_{G/K}=u$ and that $\tilde{u}$ is a solution of $\mathcal{M}$ on $\tilde{X}$ ([KO, Corollary 5.11 and Theorem 5.12]). Considering the solution $\tilde{u}$ microlocally in a neighborhood of p^*, we can define boundary values $\beta_{v,p}^{\Theta}(u)$ in a similar way as in the case for a solution of $\bar{\mathcal{M}}$. Then $\beta_{v,p}^{\Theta}(u)$ are hyperfunctions defined in a neighborhood of p in B.

If all the boundary values $\beta_{v,p}^{\Theta}(u)$ are real analytic at p, we say that the solution u is ideally analytic at p (cf. [O2, § 5]) and then u has the following form:

(2.23) $$u=\sum_{w\in W}\sum_{j=0}^{m} a_{w,j}(\lambda, x, t)q_{w,j}(\lambda, t).$$

Here m is a certain non-negative integer, $a_{w,j}(\lambda, x, t)$ are real analytic functions of (x, t) in a neighborhood of p with the holomorphic parameter $\lambda\in\Omega$ and $q_{w,j}(\lambda, t)$ are certain real analytic functions of t for $0<t_1\ll 1$, $\cdots, 0<t_l\ll 1$ with the holomorphic parameter $\lambda\in\Omega$. The functions $q_{w,j}(\lambda, t)$ do not depend on u and have the expression

(2.24) $$q_{w,j}(\lambda, t)=\sum_{v\leq_{\mu}w} q'_{v,w,j}(\lambda)t^{\lambda_v-\mu_v+\mu_w}$$

with meromorphic functions $q'_{v,w,j}(\lambda)$ which are analytic when

(2.25) $$-\tfrac{1}{2}\langle w\lambda, \alpha^{\vee}\rangle\notin \boldsymbol{N} \qquad \text{for any } \alpha\in\Sigma(\mathfrak{a}_{\mathfrak{p}})^+.$$

Furthermore we have

(2.26) $$q_{w,j}(\mu, t)=r_{w,j}(\log t)t^{\mu_w}$$

with certain polynomials $r_{w,j}(\log t)$ of $\log t=(\log t_1, \cdots, \log t_l)$. Especially if $\frac{1}{2}\langle w\mu, \alpha^{\vee}\rangle\notin \boldsymbol{Z}$ for any $\alpha\in\Sigma(\mathfrak{a}_{\mathfrak{p}})^+$, we can put $m=0$ and $q_{w,0}(\lambda, t)=t^{\lambda_w}$.

Fix an element v in $W(\mathfrak{a}_{\mathfrak{p}})$ and consider the condition that $\beta_{w,p}^{\Theta}(u)$

vanishes for any point p in U and for any w in $W(\mathfrak{a}_p)$ satisfying $w<_\mu v$. This coondition is well-defined in the sense that it does not depend on the choice of local coordinate systems and we write the condition as follows:

$$(2.27)\qquad \beta_w^\Theta(u)|_U=0 \qquad \text{for any } w\in W(\mathfrak{a}_p) \text{ with } w<_\mu v.$$

Putting

$$(2.28)\qquad {}_\Omega\mathscr{A}(G/K;\mathscr{M})_v=\{u\in {}_\Omega\mathscr{A}(G/K;\mathscr{M});\, u \text{ satisfies (2.27)}\}$$

and $v\Omega=\{v\lambda;\, \lambda\in\Omega\}$, we can define a $\mathfrak{g}$-equivariant map

$$(2.29)\qquad \beta_v^\Theta\colon {}_\Omega\mathscr{A}(G/K;\mathscr{M})_v \longrightarrow \mathscr{B}(U; L_{v\Omega})$$

by patching $\beta_{v,p}^\Theta(u)$ together for $p\in U$ (cf. [O2, § 3]). In fact, for $u\in {}_\Omega\mathscr{A}(G/K;\mathscr{M})_v$, since $(\tilde{Q}_v^\Theta\tilde{u})(\lambda, x, t)$ is a microfunction solution of the system $\mathscr{N}_v$ (cf. (2.9)), $\beta_{v,p}^\Theta(u)$ is defined by

$$(2.30)\qquad (\tilde{Q}_v^\Theta\tilde{u})(\lambda, x, t)=\beta_{v,p}^\Theta(u)(\lambda, x)t_+^{\lambda_v},$$

where the both hand side of (2.30) are regarded microfunctions defined in a neighborhood of p^*.

If $U=G/P$, then $\mathscr{B}(G/P; L_{v\Omega})$ is a G-module and β_v^Θ is a G-equivariant map. On the other hand, if $\frac{1}{2}\langle\mu, \alpha^\vee\rangle\notin \boldsymbol{Z}$ for any $\alpha\in\Sigma(\mathfrak{a}_p)$, then $\Theta=\emptyset$, ${}_\Omega\mathscr{A}(G/K;\mathscr{M})_w={}_\Omega\mathscr{A}(G/K;\mathscr{M})$ for any $w\in W(\mathfrak{a}_p)$ and in this case we write $\beta_w^\Theta=\beta_w$. For a fixed $\lambda\in\Omega$, we can similarly define boundary values for a solution u (without the holomorphic parameter) of $\mathscr{M}$ in the same way and the map β_λ in (0.7) is defined by β_e^Θ with $U=G/P$.

Let $u\in {}_\Omega\mathscr{A}(G/K;\mathscr{M})_v$. Comparing the case where the systems are $\mathscr{M}$ and $\overline{\mathscr{M}}$, we have by [O2, Theorem 4.5]

$$(2.31)\qquad \beta_v^\Theta(u)(\lambda, x)=\left(\textstyle\prod_{\alpha\in\Sigma_\Theta^+-\Sigma(v_\Theta)}\langle\lambda, \alpha^\vee\rangle\right)\beta_v(u)(\lambda, x)$$

for any $\lambda\in\Omega$ satisfying (2.25). Assume u is ideally analytic at p. Then in the expression (2.23) we have by [O2, § 4 and § 5]

$$(2.32)\qquad a_{w,j}(\lambda, x, t)=0 \qquad \text{if } w<_\mu v \text{ and } w\mu\neq v\mu$$

and

$$(2.33)\qquad \textstyle\sum_{j=0}^m a_{v,j}(\lambda, x, t)q_{v,j}(\lambda, t)=\sum_{w\in vW_\Theta} c_w(\lambda, x, t)\phi_w^\Theta(\lambda, t)$$

(cf. (2.7)) with real analytic functions $c_w(\lambda, x, t)$ satisfying

$$(2.34)\qquad c_w(\lambda, x, t)=0 \qquad \text{if } w<_\mu v \text{ and } w\mu=v\mu$$

and

$$c_v(\lambda, x, 0)=\beta_v^{\Theta}(u)(\lambda, x). \tag{2.35}$$

Lastly we remark that $\phi_v^{\Theta}(\mu, t)t^{-\mu_v}$ is a homogeneous polynomial of $\log t=(\log t_1, \cdots, \log t_l)$ with degree $l(s_{\Theta}^*)-l(v_{\Theta})$.

§ 3. Localization of intertwining operators

In this section we want to study a local property of intertwining operators between most continuous principal series of the class one with respect to K. First we will recall the intertwining operaters (cf. [Hel]).

The space $\mathscr{B}(G)$ of hyperfunctions on G is a left G-module by $G\times\mathscr{B}(G)\ni(g, f(x))\mapsto(\pi_g f)(x)=f(g^{-1}x)\in\mathscr{B}(G)$ and hence $\mathscr{B}(G)$ has the induced $\mathfrak{g}$-module structure. For $\lambda\in(\mathfrak{a}_{\mathfrak{p}})_c^*$ put

$$\begin{aligned}\mathscr{B}(G/P; L_\lambda)=\{f\in\mathscr{B}(G); f(gman)=f(g)a^{\lambda-\rho}\\ \text{for } (g, m, a, n)\in G\times M\times A_{\mathfrak{p}}\times N\}.\end{aligned} \tag{3.1}$$

Then any K-fixed vector of $\mathscr{B}(G/P; L_\lambda)$ is a constant multiple of the function $1_\lambda\in\mathscr{A}(G)$ defined by

$$1_\lambda(kan)=a^{\lambda-\rho}\qquad\text{for } (k, a, n)\in K\times A_{\mathfrak{p}}\times N. \tag{3.2}$$

For any $w\in W(\mathfrak{a}_{\mathfrak{p}})$ there exists a function $T_w^\lambda\in\mathscr{D}'(G)\cap\mathscr{B}(G/P; L_{w\lambda})$ with meromorphic parameter $\lambda\in(\mathfrak{a}_{\mathfrak{p}})_c^*$ so that the linear map

$$\begin{array}{cccc}\mathscr{T}_w^\lambda\colon & \mathscr{B}(G/P; L_\lambda) & \longrightarrow & \mathscr{B}(G/P; L_{w\lambda})\\ & \cup\!\!\!\!\text{ι} & & \cup\!\!\!\!\text{ι}\\ & f(x) & \longmapsto & (\mathscr{T}_w^\lambda f)(x)=\displaystyle\int_K f(k)T_w^\lambda(k^{-1}x)dk\end{array} \tag{3.3}$$

is a G-homomorphism. We fix a representative $\bar{w}$ in K for every w in $W(\mathfrak{a}_{\mathfrak{p}})$ and normalize $\mathscr{T}_w^\lambda$ so that

$$(\mathscr{T}_w^\lambda f)(x)=\int_{\bar{N}_w} f(x\bar{w}\bar{n}_w)d\bar{n}_w \tag{3.4}$$

for $f\in\mathscr{B}(G/P; L_\lambda)\cap C^\infty(G)$ and $\lambda\in -C_+$. Hence $\bar{N}_w=\bar{N}\cap\bar{w}^{-1}N\bar{w}$,

$$C_+=\{\lambda\in(\mathfrak{a}_{\mathfrak{p}})_c^*; \operatorname{Re}\langle\lambda, \alpha\rangle>0 \text{ for } \alpha\in\Sigma(\mathfrak{a}_{\mathfrak{p}})^+\} \tag{3.5}$$

and the Haar measure $d\bar{n}_w$ on $\bar{N}_w$ is normalized by

$$\int_{\bar{N}_w} 1_{-\rho}(\bar{n}_w)d\bar{n}_w=1. \tag{3.6}$$

Then T_w^λ satisfies

$$\text{(3.7)} \qquad \operatorname{supp} T_w^\lambda \subset \mathrm{Cl}(P\bar{w}^{-1}P),$$

and

$$\text{(3.8)} \qquad \begin{aligned} &T_w^\lambda(mangm'a'n') = T_w^\lambda(g)a^{\lambda+\rho}a'^{w\lambda-\rho} \\ &\quad \text{for } (m, a, n, g, m', a', n') \in M\times A_\mathfrak{p}\times N\times G\times M\times A_\mathfrak{p}\times N \end{aligned}$$

which comes from (3.4) and the G-equivariance of $\mathscr{T}_w^\lambda$. Here we remark

$$\text{(3.9)} \qquad P\bar{w}P = P\bar{w}\bar{N}_wP = \bar{w}\bar{N}_wP$$

because

$$\begin{aligned} P\bar{w}P &= NA_\mathfrak{p}M\bar{w}P = N\bar{w}P = (N\cap \bar{w}\bar{N}\bar{w}^{-1})(N\cap \bar{w}N\bar{w}^{-1})\bar{w}P \\ &= \bar{w}(\bar{w}^{-1}N\bar{w}\cap \bar{N})P = \bar{w}\bar{N}_wP. \end{aligned}$$

For an individual root α in $\Sigma(\mathfrak{a}_\mathfrak{p})$ we put

$$c_\alpha(\lambda) = \frac{\Gamma(m_\alpha+m_{2\alpha})\Gamma(\frac{1}{2}\langle\lambda, \alpha^\vee\rangle)\Gamma(\frac{1}{4}\langle\lambda, \alpha^\vee\rangle+\frac{1}{4}m_\alpha)}{\Gamma(\frac{1}{2}m_\alpha+\frac{1}{2}m_{2\alpha})\Gamma(\frac{1}{2}\langle\lambda, \alpha^\vee\rangle+\frac{1}{2}m_\alpha)\Gamma(\frac{1}{4}\langle\lambda, \alpha^\vee\rangle+\frac{1}{4}m_\alpha+\frac{1}{2}m_{2\alpha})}$$

$$d_\alpha(\lambda) = \frac{\Gamma(m_\alpha+m_{2\alpha})}{\Gamma(\frac{1}{2}m_\alpha+\frac{1}{2}m_{2\alpha})}2^{1-\frac{1}{2}\langle\lambda,\alpha^\vee\rangle-\frac{1}{2}m_\alpha}\sqrt{\pi}\,\Gamma(\tfrac{1}{2}\langle\lambda, \alpha^\vee\rangle)$$

and

$$e_\alpha(\lambda)^{-1} = \Gamma(\tfrac{1}{4}\langle\lambda, \alpha^\vee\rangle+\tfrac{1}{4}m_\alpha+\tfrac{1}{2})\Gamma(\tfrac{1}{4}\langle\lambda, \alpha^\vee\rangle+\tfrac{1}{4}m_\alpha+\tfrac{1}{2}m_{2\alpha})$$

by denoting $m_\alpha = \dim \mathfrak{g}(\mathfrak{a}_\mathfrak{p}; \alpha)$. Moreover for $w \in W(\mathfrak{a}_\mathfrak{p})$ we put

$$\text{(3.10)} \qquad \Sigma(w) = \left\{\alpha \in \Sigma(\mathfrak{a}_\mathfrak{p})^+;\ w\alpha \in \Sigma(\mathfrak{a}_\mathfrak{p})^- \text{ and } \frac{\alpha}{2} \notin \Sigma(\mathfrak{a}_\mathfrak{p})\right\},$$

$$c_w(\lambda) = \prod_{\alpha\in\Sigma(w)} c_\alpha(\lambda), \quad d_w(\lambda) = \prod_{\alpha\in\Sigma(w)} d_\alpha(\lambda) \text{ and } e_w(\lambda) = \prod_{\alpha\in\Sigma(w)} e_\alpha(\lambda).$$

By the element $s^* \in W(\mathfrak{a}_\mathfrak{p})$ with $s^*\Sigma(\mathfrak{a}_\mathfrak{p})^+ = \Sigma(\mathfrak{a}_\mathfrak{p})^-$ we define $c(\lambda) = c_{s^*}(\lambda)$, $d(\lambda) = d_{s^*}(\lambda)$ and $e(\lambda) = e_{s^*}(\lambda)$. Then $c_w(\lambda)$ equals $d_w(\lambda)e_w(\lambda)$ and $d_w(-\lambda)^{-1}T_w^\lambda$ are holomorphically extended for all $\lambda \in T_w^\lambda$ and

$$\text{(3.11)} \qquad c_w(-\lambda)1_{w\lambda} = \mathscr{T}_w^\lambda 1_\lambda.$$

If $w = w_k\cdots w_1$ is a minimal expression for $w \in W(\mathfrak{a}_\mathfrak{p})$, then we have the product formula

$$\text{(3.12)} \qquad \mathscr{T}_w^\lambda = \mathscr{T}_{w_k}^{w_{k-1}\cdots w_1\lambda}\cdots\mathscr{T}_{w_2}^{w_1\lambda}\mathscr{T}_{w_1}^\lambda.$$

For $\alpha \in \Psi(\mathfrak{a}_\mathfrak{p})$ let s_α denote the reflection with respect to α and put

$P_\alpha = \mathrm{Cl}(P\bar{s}_\alpha P)$.

Definition 3.1. For $w \in W(\mathfrak{a}_\mathfrak{p})$ and subsets S and S' of G/P we put

$$w[S] = SP_{\alpha(1)}P_{\alpha(2)}\cdots P_{\alpha(k)}, \tag{3.13}$$

$$\overline{W}(S\colon S') = \{w \in W(\mathfrak{a}_\mathfrak{p});\ w[S]\cap S' \neq \emptyset\} \tag{3.14}$$

and

$$W(S\colon S') = \{w \in \overline{W}(S\colon S');\ \{v \in \overline{W}(S\colon S');\ v < w\} = \emptyset\}, \tag{3.15}$$

where $w = s_{\alpha(k)}\cdots s_{\alpha(2)}s_{\alpha(1)}$ is a minimal expressions of w with $\alpha(1), \cdots, \alpha(k) \in \Psi(\mathfrak{a}_\mathfrak{p})$ and we identify subsets of G/P with right P-invariant subsets of G.

Here we remark

$$P_{\alpha(1)}\cdots P_{\alpha(k)} = \mathrm{Cl}(P\bar{w}^{-1}P) = \textstyle\coprod_{w' \leqq w} P\bar{w}'^{-1}P \quad \text{(cf. [MO,Lemma 8])}. \tag{3.16}$$

For an open subset U of G/P we put

$$\begin{aligned}\mathscr{B}(U; L_\lambda) = \{f \in \mathscr{B}(U);\ & f(gman) = f(g)a^{\lambda-\rho} \\ & \text{for } (g, m, a, n) \in G\times M\times A_\mathfrak{p}\times N\}\end{aligned} \tag{3.17}$$

and for a subset S of G/P we put

$$\mathscr{B}(S; L_\lambda) = \varinjlim_{U\supset S} \mathscr{B}(U; L_\lambda), \tag{3.18}$$

where U runs through open subsets of G/P containing S. Similarly for a subset V of K/M we put

$$\mathscr{B}(V) = \varinjlim_{U\supset V} \mathscr{B}(U), \tag{3.19}$$

where U runs through open subsets of K/M containing V. Then the restriction map to $K\cap S$ induces a bijection

$$\iota_{\lambda,S}\colon \mathscr{B}(S; L_\lambda) \xrightarrow{\sim} \mathscr{B}(K\cap S/M\cap S). \tag{3.20}$$

For simplicity we will denote $\iota_{\lambda,G/P}$ by ι_λ.

Since it follows from (3.7) and (3.16) that supp $T^\lambda_w \subset w[\{e\}]$, the integral transformation (3.3) induces the $\mathfrak{g}$-equivariant map of $\mathscr{B}(w^{-1}[S]; L_\lambda)$ to $\mathscr{B}(S; L_\lambda)$ for any subset S of G/P, which will be denoted by the same notation. Thus we have the commutative diagram

$$\begin{array}{ccc} \mathscr{T}^\lambda_w\colon \mathscr{B}(G/P; L_\lambda) & \longrightarrow & \mathscr{B}(G/P; L_{w\lambda}) \\ \downarrow & & \downarrow \\ \mathscr{T}^\lambda_w\colon \mathscr{B}(w^{-1}[S]; L_\lambda) & \longrightarrow & \mathscr{B}(S; L_{w\lambda}) \end{array} \tag{3.21}$$

if $-\lambda$ is not a pole of d_w. Here the vertical maps are defined by natural restrictions.

Fix an element μ in $(\mathfrak{a}_\mathfrak{p})_c^*$. Let Ω be an open neighborhood of μ. We defined the space $\mathscr{B}(U; L_\Omega)$ in § 2, which is the space of holomorphic functions f_λ on Ω with values in $\mathscr{B}(U; L_\lambda) \subset {}_\Omega\mathscr{B}(U)$. Then it is clear that the above argument is valid even if we replace L_λ and $L_{w\lambda}$ by L_Ω and $L_{w\Omega}$. Now we can state the key lemma in this paper.

Lemma 3.2. *Fix $w \in W(\mathfrak{a}_\mathfrak{p})$ and $p \in G/P$. Assume $w \in W(V: \{p\})$ by denoting $V = w^{-1}[\{p\}]$. Let $f_\lambda \in \mathscr{B}(V; L_\Omega)$. We define $\psi_\lambda \in \mathscr{B}(\{p\}; L_{w\Omega'})$ by $\psi_\lambda = \mathscr{T}_w^\lambda \circ f_\lambda$ with*

$$\Omega' = \{\lambda \in \Omega;\ d_w(-\lambda)^{-1} \neq 0\}. \tag{3.22}$$

i) *The function ψ_λ can be holomorphically extended with respect to the parameter λ and defines an element of $\mathscr{B}(\{p\}; L_{w\Omega})$. Therefore we can define $\mathscr{T}_w^\mu f_\mu = \psi_\mu \in \mathscr{B}(\{p\}; L_{w\mu})$ by the holomorphic extension because ψ_μ depends only on f_μ.*

ii) *Assume moreover*

$$e_\alpha(\mu) \neq 0 \text{ and } -\tfrac{1}{2}\langle \mu, \alpha^\vee \rangle \notin N - \{0\} \quad \text{for any } \alpha \in \Sigma(w). \tag{3.23}$$

Then $\psi_\mu \neq 0$ if $f_\mu \neq 0$.

Proof. First we remark that $T_w^\lambda(k^{-1}) \in \mathscr{D}'(K/M)$ is defined by the map

$$C^\infty(K/M) \ni \phi \mapsto (\mathscr{T}_w^\lambda \circ \iota_\lambda^{-1}\phi)(e) = \int_{\bar{N}_w} (\iota_\lambda^{-1}\phi)(\bar{w}\bar{n}_w)d\bar{n}_w \in C \tag{3.24}$$

when $\lambda \in -C_+$. If $(\operatorname{supp}\phi) \cap (K \cap \bar{w}\bar{N}_w P)$ is compact, the right hand side of (3.24) is holomorphically extended for all $\lambda \in (\mathfrak{a}_\mathfrak{p})_c^*$, which means that $T_w^\lambda(k^{-1}) \in \mathscr{D}'(K/M)$ is holomorphically extended for all λ in a neighborhood of $K \cap \bar{w}\bar{N}_w P$. Hence there exists an open neighborhood U of $P\bar{N}_w\bar{w}^{-1}P = P\bar{w}^{-1}P$ in G such that $T_w^\lambda|_U$ is holomorphically extended for all $\lambda \in (\mathfrak{a}_\mathfrak{p})_c^*$. Since the support of T_w^λ is contained in $\mathrm{Cl}(P\bar{w}^{-1}P)$, we can put $U = G - \bigcup_{w' < w} P\bar{w}'^{-1}P$. Define $T_{\lambda,t}^1 \in \mathscr{D}'(G)$ with $|t| < \varepsilon$ by

$$T_{\lambda,t}^1 = \frac{1}{2\pi\sqrt{-1}} \int_{|s|=\varepsilon} \frac{T_w^{\lambda+\rho s}}{s-t} ds$$

and put $T_{\lambda,t}^2 = T_w^{\lambda+\rho t} - T_{\lambda,t}^1$ for $0 < \varepsilon \ll 1$. Then the above argument implies that $\operatorname{supp} T_{\lambda,t}^2 \subset \bigcup_{w' < w} P\bar{w}'^{-1}P$. Define

$$\psi_{\lambda,t}^j(x) = \int_K f_\lambda(k) T_{\lambda,t}^j(k^{-1}x)dk \qquad \text{for } j = 1 \text{ and } 2.$$

Here we remark that for any neighborhood U of $w^{-1}[\{p\}]$ there exists a neighborhood U' of the point p so that the condition $x \in U'$ and $k^{-1}x \in P\bar{w}^{-1}P$ implies $k \in U$ and therefore the above integral can define a function $\psi^j_{\lambda,t} \in \mathscr{B}(\{p\}; L_{w\lambda+w\rho t})$. Since the closed set $\bigcup_{w'<w} w'[\operatorname{supp} f]$ does not contain p by the assumption, $\psi^2_{\lambda,t}=0$ as an element of $\mathscr{B}(\{p\}; L_{w\lambda+w\rho t})$. Hence ψ_λ which equals $\psi^1_{\lambda,0}$ is holomorphically extended on $w\Omega$.

Suppose an element $f'_\lambda \in \mathscr{B}(V; L_\Omega)$ satisfies $f'_\mu = f_\mu$. Then there exists $f''_t \in \mathscr{B}(V; L_{\mu+\rho t})$ with the holomorphic parameter t ($|t| \ll 1$) which satisfies $f_{\mu+\rho t} - f'_{\mu+\rho t} = tf''_t$. Hence $\mathscr{T}^\mu_w f_\mu = (\mathscr{T}^{\mu+\rho t}_w f'_{\mu+\rho t} + t\mathscr{T}^{\mu+\rho t}_w f''_t)|_{t=0} = \mathscr{T}^\mu_w f'_\mu$ by the holomorphic extension and therefore we can replace f_λ by $\iota_\lambda^{-1} \circ \iota_\mu f_\mu$ to prove the second part of the lemma. Moreover since the restriction map of $\mathscr{B}(G/P; L_\mu)$ to $\mathscr{B}(V; L_\mu)$ which contains $\iota_\lambda f_0$ is surjective we may also assume $f_\lambda \in \mathscr{B}(G/P; L_\Omega)$.

The idea of the rest part of the proof is same as in the proof of [MO, Lemma 2]. Hereafter we can assume $e_w(\lambda) \neq 0$ and $w[\operatorname{supp} f_t] \ni p$ for $t \in \Omega$. We will prove the last statement of the theorem by the induction on $l(w)$. Hence assume $w = s_\alpha$ with $\alpha \in \Psi(\mathfrak{a}_\mathfrak{p})$. Let $P_\alpha = M_\alpha A_\alpha N_\alpha$ be the Langlands decomposition of P_α with $A_\alpha \subset A_\mathfrak{p}$ and $N_\alpha \subset N$ and let $\mathfrak{a}(\alpha)$ be the Lie algebra of $M_\alpha \cap A_\mathfrak{p}$. If λ and λ' belong to $-C_+$ and their restrictions on $\mathfrak{a}(\alpha)$ coincide, it follows from (3.4) that $(\mathscr{T}^\lambda_w \iota_\lambda^{-1}\phi)(k) = (\mathscr{T}^{\lambda'}_w \iota_{\lambda'}^{-1}\phi)(k)$ for $k \in K$ and $\phi \in C^\infty(K/M)$ because $\bar{w}\bar{N}_w \subset M_\alpha$. This implies that $\iota_\lambda T^\lambda_w$ only depends on $\lambda|_{\mathfrak{a}(\alpha)}$. Therefore to prove the last statement of the theorem, we may assume

$$\operatorname{Re} \langle \lambda, \beta \rangle > 0 \text{ for any } \beta \in \Sigma(\mathfrak{a}_\mathfrak{p})^+ \text{ which is not parallel to } \alpha.$$

Then the Poisson transform

$$u_\lambda(g) = \int_K f_\lambda(gk)dk$$

of f_λ is a simultaneous eigenfunctions of invariant differential operators on G/K. Suppose $\lambda \in \Omega'$. Then we can define boundary values $\beta_e u_\lambda \in \mathscr{B}(G/P; L_{\Omega'})$ and $\beta_w u_\lambda \in \mathscr{B}(G/P; L_{w\Omega'})$ because $\Theta = \emptyset$ under the notation in § 2. They satisfy

$$\beta_e u_\lambda = c(\lambda) f_\lambda$$

and

$$\beta_w u_\lambda = c(w\lambda) c_w(-\lambda)^{-1} \mathscr{T}^\lambda_w f_\lambda$$

as was shown in [K-] Proposition 6.1. We remark that the function $c(w\lambda)c_w(-\lambda)^{-1}$ can be holomorphically extended to $\lambda = \mu$ and the value at $\lambda = \mu$ is not zero.

First consider the case

(3.25) $$\operatorname{Re}\langle\mu,\alpha\rangle\geqq 0 \quad\text{and}\quad \langle\mu,\alpha\rangle\neq 0.$$

Then $c(\lambda)$ and $\beta_e u_\lambda$ are also holomorphically extended to $\lambda=\mu$ and their values at $\lambda=\mu$ are not zero. Since $\beta_e u_\lambda$ vanishes in a certain neighborhood U of p, $\beta_w u_\lambda|_U$ is holomorphically extended to $\lambda=\mu$ and it follows from [MO, Lemma 1] that the support of the value at $\lambda=\mu$ contains p. This means $\psi_\mu\neq 0$.

Next consider the case

(3.26) $$\operatorname{Re}\langle\mu,\alpha\rangle<0,\quad e_\alpha(\mu)\neq 0 \text{ and } -\tfrac{1}{2}\langle\mu,\alpha^\vee\rangle\notin \boldsymbol{N}-\{0\}.$$

Then T_w^λ and $T_w^{w\lambda}$ are analytic at $\lambda=\mu$. Put $f'_\lambda=\mathscr{T}_w^\lambda f_\lambda$ and suppose $\operatorname{supp} f'_\mu\not\ni p$. We will lead a contradiction. Note that $\mathscr{T}_w^{w\lambda}f'_\lambda=\mathscr{T}_w^{w\lambda}\mathscr{T}_w^\lambda f_\lambda$ $=c_\alpha(-\lambda)c_\alpha(\lambda)f_\lambda$ and $c_\alpha(-\mu)c_\alpha(\mu)\neq 0$. Therefore by replacing S and λ by V and $w\lambda$, respectively, in the diagram (3.21), we can see $V\cap(\operatorname{supp} f'_\lambda)\neq\emptyset$ because $w[V]=V$ and $V\cap(\operatorname{supp} f_\lambda)\neq\emptyset$. Since $\operatorname{Re}\langle w\mu,\alpha\rangle>0$, the condition $\operatorname{supp} f'_\lambda\not\ni p$ and $V\cap(\operatorname{supp} f'_\lambda)\neq\emptyset$ implies $\operatorname{supp}\mathscr{T}_w^{w\mu}f'_\mu\ni p$ as we have just proved. This contradicts to the fact $\operatorname{supp}\mathscr{T}_w^{w\mu}f'_\mu=\operatorname{supp} f_\mu\not\ni p$.

Now consider the case

(3.27) $$\langle\mu,\alpha\rangle=0.$$

By shrinking Ω if necessary, we can define boundary values $\beta_e^\Theta u_\lambda\in\mathscr{B}(G/P;L_\Omega)$ and $\beta_w^\Theta u_\lambda\in\mathscr{B}(U;L_{w\Omega})$ with $\Theta=\{\alpha\}$, where U is a open subset of G/P satisfying $U\cap\operatorname{supp}\beta_e^\Theta u_\lambda=\emptyset$. They satisfy

$$\beta_e^\Theta u_\lambda=\langle\lambda,\alpha^\vee\rangle\beta_e u_\lambda \quad\text{and}\quad \beta_w^\Theta u_\lambda=\beta_w u_\lambda|_U$$

for $\lambda\in\Omega'$ (cf. (2.29)). Since $\beta_e^\Theta u_\mu$ is a non-zero constant multiple of f_μ, [MO, Lemma 2] says $\operatorname{supp}\beta_w^\Theta u_\mu\ni p$ as in the case (3.25) and therefore $\psi_\mu\neq 0$.

Lastly assume $l(w)>1$. Let $w=s_{\alpha(k)}\cdots s_{\alpha(1)}$ be a minimal expression with $\alpha(j)\in\Psi(\mathfrak{a}_\mathfrak{p})$. Put $v=s_{\alpha(k-1)}\cdots s_{\alpha(1)}$ and $V'=s_{\alpha(k)}[\{p\}]$. For any $q\in V'$ we have $v^{-1}[\{q\}]\subset V$ for $v\in W(\operatorname{supp} f_\lambda\colon V')$. Hence $\psi'_\lambda=T_v^\lambda f_\lambda\in\mathscr{B}(V';L_{v\Omega'})$ can be holomorphically extended to $\lambda=\mu$. Since $\Sigma(w)=\Sigma(v)\cup v^{-1}\alpha(k)$ and since there exists $q\in V'$ satisfying $v[\operatorname{supp} f_\lambda]\ni q$, we have $\psi'_\lambda\neq 0$ for any $\lambda\in\Omega$ by the hypothesis of the induction. Then applying the result in the case $l(w)=1$ to $\psi_\lambda=\mathscr{T}_{s_{\alpha(k)}}^{v\lambda}\psi'_\lambda$, we can conclude $\psi_\mu\neq 0$. Q.E.D.

Remark 3.3. The above proof implies that Lemma 3.2 is also valid in the distribution category. But considering that we have reduced the proof of Lemma 2.2. ii) to [MO, Lemma 2] which is proved by using Holmgren's theorems ([SKK, Chap. III, Proposition 2.1.3] and [O2, Theorem 4.4]) and Sato's fundamental theorem ([SKK, Chap. III, Corollary 2.1.2]), we can prove Lemma 2.2. ii) by using [BG, Theorem 4]. The precise argument will be left to readers.

Let S be a subset of G/P and f an element in $\mathscr{B}(w^{-1}[S]: L_\mu)$. Then applying Lemma 3.2 to the function $f_\lambda = \iota_\lambda^{-1} \circ \iota_\mu(f)$, we can define $\mathscr{T}_w^\mu f \in \mathscr{B}(\{p\}; L_{w\mu})$ for any point $p \in S$ under the notation in Lemma 3.2 and therefore we have $\mathscr{T}_w^\mu f \in \mathscr{B}(S; L_{w\mu})$. Replacing μ by λ, we have

Theorem 3.4. *Let w be an element of $W(\mathfrak{a}_\mathfrak{p})$, λ an element of $(\mathfrak{a}_\mathfrak{p})_c^*$, S a subset of G/P, U an open subset of G/P with $U \supset w^{-1}[S]$ and V a closed subset of U with $w \in W(V: S)$. Putting*

$$\mathscr{B}([V]; L_\lambda) = \{f \in \mathscr{B}(U; L_\lambda); \operatorname{supp} f \subset V\}, \tag{3.29}$$

we can define a $\mathfrak{g}$-homomorphism

$$\mathscr{T}_w^\lambda: \mathscr{B}([V]; L_\lambda) \longrightarrow \mathscr{B}(S; L_{w\lambda}) \tag{3.30}$$

in the way mentioned above. Let $f \in \mathscr{B}([V]; L_\lambda)$ with $w^{-1}[S] \cap \operatorname{supp} f \neq \emptyset$. Then $\mathscr{T}_w^\lambda(f) \neq 0$ if the following condition holds:

$$e_\alpha(-\lambda) \neq 0 \quad \textit{and} \quad -\tfrac{1}{2}\langle \lambda, \alpha \rangle \notin N - \{0\} \textit{ for any } \alpha \in \Sigma(w). \tag{3.31}$$

Proof. Note that for the function f in the theorem there exists a point $p \in S$ with $w^{-1}[\{p\}] \cap \operatorname{supp} f \neq \emptyset$. Hence Theorem 3.4 easily follows from Lemma 3.2. Q.E.D.

Definition 3.5. Let λ be an element of $(\mathfrak{a}_\mathfrak{p})_c^*$ satisfying (0.2) and (0.3). For subsets S and S' of G/P we put

$$\bar{\Xi}(S: S'; \lambda) = \{w\lambda;\ w \in \bar{W}(S: S')\},$$

$$\Xi(S: S'; \lambda) = \{\mu \in \bar{\Xi}(S: S'; \lambda);$$
$$\{\nu \in \bar{\Xi}(S: S'; \lambda);\ (\operatorname{Re}\langle \mu-\nu, H_1\rangle, \cdots, \operatorname{Re}\langle \mu-\nu, H_l\rangle) \in [0, \infty)^l - \{0\}\} = \emptyset\},$$

$$W(S: S'; \lambda) = \{w \in W(S: S');\ \{v \in W(\mathfrak{a}_\mathfrak{p});\ v <_\lambda w\} \cap \bar{W}(S: S') = \emptyset\}$$

and

$$\bar{W}(S: S'; \lambda) = \{w \in \bar{W}(S: S');\ w\lambda \in \Xi(S: S'; \lambda)\}.$$

Theorem 3.4 has several applications. One of them is the following:

Theorem 3.6. *Use the notation in* § 1 *and* § 3. *Let λ be an element* *in* $(\mathfrak{a}_\mathfrak{p})^*_c$ *satisfying* (0.2) *and* (0.3) *and let* β_λ *be the boundary value map*

$$\beta_\lambda\colon \mathscr{A}(G/K; \mathscr{M}_\lambda) \longrightarrow \mathscr{B}(G/P; L_\lambda) \tag{3.32}$$

which corresponds to β^Θ_e *(cf.* § 2). *Here* $\Theta=\{\alpha\in\Psi(\mathfrak{a}_\mathfrak{p}); \langle\lambda,\alpha\rangle=0\}$. *Let* u *be a non-zero function in* $\mathscr{A}(G/K; \mathscr{M}_\lambda)$ *and* p *a point in* G/P. *Put* $S=$ supp $\beta_\lambda u$. *Then for an element* v *in* $w(\mathfrak{a}_\mathfrak{p})$

$$\beta^\Theta_{v,p}(u)=0 \qquad \textit{if } v\notin \overline{W}(S\colon\{p\}) \tag{3.33}$$

and

$$\beta^\Theta_{v,p}(u)\neq 0 \qquad \textit{if } v\in W(S\colon\{p\};\lambda). \tag{3.34}$$

Especially when u *is ideally analytic at* p, *then there exist a positive number* ε *and a neighborhood* U *of* p *such that*

$$u(x,t)=\sum_{w\in\overline{W}(S:\{p\};\lambda)} a_w(x)\phi^\Theta_w(\lambda,t)+r(x,t)\sum_{j=1}^{l}\sum_{w\in\overline{W}(S:\{p\};\lambda)} t^{\lambda w}t_j^{\varepsilon} \tag{3.35}$$

with certain real analytic functions $a_w(x)$ *on* $U\cap(G/P)$ *and a certain bounded real analytic function* $r(x,t)$ *on* $U\cap(G/K)$. *Moreover*

$$a_v(x)=\beta^\Theta_{v,p}(u) \qquad \textit{if } v\in W(S\colon\{p\};\lambda)\cap\overline{W}(S\colon\{p\};\lambda). \tag{3.36}$$

Remark 3.7. The functions $\phi^\Theta_w(\lambda,t)$ $(w\in W(\mathfrak{a}_\mathfrak{p}))$ are given in (2.7). They are linearly independent and

$$\phi^\Theta_w(\lambda,t)=h(w;t)t^{\lambda w}, \tag{3.37}$$

where $h(w;t)$ are homogeneous polynomials of $(\log t_1,\cdots,\log t_l)$ with degree $l(s^*_\Theta)-l(w_\Theta)$ and correspond to W_Θ-harmonic polynomials on $\mathfrak{a}_\mathfrak{p}$ (cf. Theorem 1.3). Especially if $\langle\lambda,\alpha\rangle\neq 0$ for any $\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})$, then $h(w;t)=1$. Moreover if $\operatorname{Re}\langle\lambda,\alpha\rangle\neq 0$ for any $\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})$, then $\overline{W}(S\colon\{p\};\lambda)=W(S\colon\{p\};\lambda)$ in Theorem 3.6.

Proof of Theorem 3.6. Let Ω be a small open neighborhood of λ. Put $f_\nu=\iota_\nu^{-1}\circ\iota_\lambda\circ\beta_\lambda(u)$ and $u_\nu=\mathscr{P}_\nu(f_\nu)$ for $\nu\in\Omega$ and moreover put $\Omega'=\{\nu\in\Omega; \frac{1}{2}\langle\lambda,\alpha^\vee\rangle\notin \boldsymbol{Z}$ for any $\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})\}$. By the induction on $l(v)$ we will prove $\beta^\Theta_{v,p}(u_\nu)=0$ (resp. $\beta^\Theta_{v,p}(u_\nu)\neq 0$) for any $\nu\in\Omega$ if $v\notin\overline{W}(S\colon\{p\})$ (resp. $v\in W(S\colon\{p\};\lambda)$). The hypothesis of the induction means $\beta^\Theta_{w,p}(u_\nu)=0$ if $w<_\lambda v$. Hence by (2.29) we have

$$\beta^\Theta_{v,p}(u_\nu)=\Big(\prod_{\alpha\in\Sigma^+_\Theta-\Sigma(v_\Theta)}\langle\nu,\alpha^\vee\rangle\Big)\beta_{v\nu}(u_\nu) \qquad \text{for } \nu\in\Omega'$$

in a neighborhood of p. By virture of [K-, Proposition 6.1] we have $\beta_{v\nu}(u_\nu)=c(v\nu)c_v(-\nu)^{-1}\mathscr{T}^\nu_v f_\nu$ for $\nu\in\Omega'$. Note that

$$c(v\nu)c_v(-\nu)^{-1}\prod_{\alpha\in\Sigma_\Theta^+-\Sigma(v_\Theta)}\langle\nu,\alpha^\vee\rangle$$
$$=\Big(\prod_{\alpha\in\Sigma_0^+-\Sigma(v)}c_\alpha(\nu)\Big)\Big(\prod_{\alpha\in\Sigma_\Theta^+-\Sigma(v_\Theta)}\langle\nu,\alpha^\vee\rangle\Big)$$

and therefore that this function is holomorphically extended to the point $\nu=\lambda$ with a non-zero value. Hence it follows from (3.21) and Lemma 2.2 that $\beta_{v,p}^{\Theta}(u_\nu)$ has the required property. Thus we have the theorem because $u_\lambda=Cu$ with a non-zero $C\in\boldsymbol{C}$.

The second part of Theorem 3.6 is clear from § 2. Q.E.D.

§ 4. Asymptotic behavior of spherical functions

In this section we will use the notation defined in § 0. As in the argument in [MO, § 3], we can study the asymptotic behavior of a K-finite spherical function ψ at infinity on G/H through the boundary value problem on the riemannian form G^d/K^d of G/H by using Flensted-Jensen's duality. Applying Theorem 3.6 to our situation we have a precise result for the asymptotic behavior in terms of the set $\mathrm{FBI}_\lambda(\psi)$. To state the result we prepare some notation.

We put $\Psi(\mathfrak{a})=\{\alpha_1,\cdots,\alpha_l\}$ and $\Psi(\mathfrak{a}_\mathfrak{p}^d)=\{\tilde{\alpha}_1,\cdots,\tilde{\alpha}_{l'}\}$. They are compatible fundamental systems of the root systems $\Sigma(\mathfrak{a})$ and $\Sigma(\mathfrak{a}_\mathfrak{p}^d)$, respectively. Let $\{\omega_1,\cdots,\omega_l\}$ and $\{\tilde{\omega}_1,\cdots,\tilde{\omega}_{l'}\}$ be the dual basis (i.e. $\alpha_i(\omega_j)=\delta_{ij}$, $\tilde{\alpha}_i(\tilde{\omega}_j)=\delta_{ij}$, $\omega_i\in\mathfrak{a}$ and $\tilde{\omega}_i\in\mathfrak{a}_\mathfrak{p}^d$). As we defined in § 3, for subsets S and S' of G^d/P^d we put

$$\overline{W}(S\colon S')=\{w\in W(\mathfrak{a}_\mathfrak{p}^d);\ S\cdot\mathrm{Cl}(P^dw^{-1}P^d)\cap S'\neq\emptyset\}$$

and

$$W(S\colon S')=\{w\in\overline{W}(S\colon S');\ \{v\in\overline{W}(S\colon S');\ v<w\}=\emptyset\},$$

where the ordering in $W(\mathfrak{a}_\mathfrak{p}^d)$ is the Bruhat ordering. For a subset $I=\{\alpha_{j(1)},\cdots,\alpha_{j(m)}\}$ of $\Psi(\mathfrak{a})$ with a positive number $m\leq l$ we define a map

$$(4.1)\qquad \begin{array}{ccc}(0,\infty)^m & \longrightarrow & A\\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid\\ y=(y_1,\cdots,y_m) & \longmapsto & a_I(y)=\exp\big(-\sum_{i=1}^{l}\omega_{j(i)}\log y_i\big).\end{array}$$

We put $a(y)=a_{\Psi(\mathfrak{a})}(y)$ for simplicity. For $\nu=(\nu_1,\cdots,\nu_m)\in\boldsymbol{C}^m$ and $y\in(0,\infty)^m$ we put

$$y^\nu=y_1^{\nu_1}\cdots y_m^{\nu_m}.$$

For the above S, S', I and an element λ in $(\mathfrak{a}_\mathfrak{p}^d)_c^*$ we put

$$\nu_I(\lambda)=(\nu_I(\lambda)_1,\cdots,\nu_I(\lambda)_m)=(\langle\rho-\lambda,\omega_{j(1)}\rangle,\cdots,\langle\rho-\lambda,\omega_{j(m)}\rangle)\in\boldsymbol{C}^m,$$

$$\bar{\Xi}_I(S\colon S';\lambda)=\{\nu_I(w\lambda);\ w\in\bar{W}(S\colon S')\},$$

$$\Xi_I(S\colon S';\lambda)=\{\nu\in\bar{\Xi}_I(S\colon S';\lambda);\ \{\nu'\in\bar{\Xi}_I(S\colon S';\lambda);$$
$$(\mathrm{Re}\,(\nu_1-\nu_1'),\cdots,\mathrm{Re}\,(\nu_m-\nu_m'))\in[0,\infty)^m-\{0\}\}=\varnothing\}$$

and

$$\Xi_I'(S\colon S';\lambda)=\Xi_I(S\colon S';\lambda)\cap\{\nu_I(w\lambda);\ w\in W(S\colon S')\}.$$

For a subset Θ of $\Psi(\mathfrak{a}_\mathfrak{p}^d)$ we define

$$\Sigma_\Theta^+=\left\{\beta\in\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+\cap\sum_{\alpha\in\Theta}\boldsymbol{R}\alpha;\ \frac{\beta}{2}\notin\Sigma(\mathfrak{a}_\mathfrak{p}^d)\right\}$$

and

$$w(\Theta)=\{w\in W(\mathfrak{a}_\mathfrak{p}^d);\ w\Sigma_\Theta^+\subset\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+\}.$$

Let W_Θ be the subgroup of $W(\mathfrak{a}_\mathfrak{p}^d)$ generated by the reflections with respect to the roots in Θ. Then for $w\in W(\Theta)$ we define unique elements $w(\Theta)\in W(\Theta)$ and $w_\Theta\in W_\Theta$ such that $w=w(\Theta)w_\Theta$.

Theorem 4.1. *Let λ be an element of $(\mathfrak{a}_\mathfrak{p}^d)_c^*$ satisfying* (0.2) *and* (0.3) *and let Ψ be a non-zero element of $\mathscr{A}_K(G/H;\mathscr{M}_\lambda)$. Put $\Theta=\Psi(\mathfrak{a}_\mathfrak{p}^d)\cap\lambda^\perp$ and $\Theta'=\{\alpha\in\Psi(\mathfrak{a}_\mathfrak{p}^d);\ \mathrm{Re}\,\langle\alpha,\lambda\rangle=0\}$. For an element w_0 of $W(\mathfrak{a})$ and a non-void subset $I=\{\alpha_{j(1)},\cdots,\alpha_{j(m)}\}$ of $\Psi(\mathfrak{a})$ we fix a representative of w_0 in K^d, denote it by the same symbol for simplicity and put*

$$\Lambda=\Xi_I(\mathrm{FBI}_\lambda(\Psi)\colon w_0P^d;\lambda)\quad\textit{and}\quad\Lambda'=\Xi_I'(\mathrm{FBI}_\lambda(\Psi)\colon w_0P^d;\lambda)$$

(*cf. Definition* 0.1 *for* $\mathrm{FBI}_\lambda(\Psi)$). *Then there exists a positive number ε such that*

$$\Psi(gw_0a_I(y)w_0^{-1}H)=\sum_{\nu\in\Lambda}\sum_{k=1}^{\kappa(\nu)}c_{\nu,k}(g)\phi_{\nu,k}(\log y_1,\cdots,\log y_m)y^\nu+r(g,y)\sum_{\nu\in\Lambda}\sum_{i=1}^{m}y^\nu y_i^\varepsilon \tag{4.2}$$

for $(g,y)\in G\times(0,\infty)^m$. Here $c_{\nu,k}(g)$ are real analytic functions on G, $r(g,y)$ is a continuous functions on $G\times[0,\infty)^m$ and $\phi_{\nu,k}$ are homogeneous polynomials with m-variables whose degree$\leq\#\Sigma_\Theta$. Moreover

$$\sum_{k=1}^{\kappa(\nu)}c_{\nu,k}(g)\phi_{\nu,k}(\log y_1,\cdots,\log y_m)\not\equiv0\qquad\textit{if }\nu\in\Lambda'. \tag{4.3}$$

Assume $I=\Psi(\mathfrak{a})$ and fix $\nu\in\Lambda'$. Putting

$$N(w_0,\nu)=\max\{N;\ \sum_{\{k;\deg\phi_{\nu,k}=N\}}c_{\nu,k}(g)\phi_{\nu,k}(\log y_1,\cdots,\log y_m)\not\equiv0\}. \tag{4.4}$$

we have

(4.5) $N(w_0, \nu) \geq \max\{\# \Sigma_\Theta^+ - l(w_\Theta);\ w \in W(\mathrm{FBI}_\lambda(\psi): w_0 P^d)\ \textit{and}\ \nu_I(w\lambda) = \nu\}$.

Especially the equality holds in (4.5) *when the following conditions is satisfied.*

(4.6) *If* $w \in \overline{W}(\mathrm{FBI}_\lambda(\psi): w_0 P^d) \cap W(\mathrm{FBI}_\lambda(\psi): w_0 P^d) W_{\Theta'}$ *and* $\nu_I(w\lambda) = \nu$, *then* $w \in W(\mathrm{FBI}_\lambda(\psi): w_0 P^d) W_\Theta$.

Remark 4.2. i) Since $\mathrm{FBI}_\lambda(\Psi)$ is an H^d-invariant subset of G^d/P^d, the condition $\mathrm{FBI}_\lambda(\Psi) \cdot \mathrm{Cl}(P^d w^{-1} P^d) \cap w_0 P^d \neq \varnothing$ is equivalent to $\mathrm{FBI}_\lambda(\Psi) \cdot \mathrm{Cl}(P^d w^{-1} P^d) \supset H^d w_0 P^d$. Any open H^d-orbit in G^d/P^d is of the form $H^d w_0 P^d$ with a suitable $w_0 \in W(\mathfrak{a})$ and conversely $H^d w_0 P^d$ is open for any $w_0 \in W(\mathfrak{a})$. Moreover the number of the cosets of $H^d \backslash G^d / P^d$ is finite and the cosets are parametrized in [Ma1].

ii) If λ satisfies

$$\langle \lambda, \alpha \rangle \neq 0 \qquad \text{for all } \alpha \in \Sigma(\mathfrak{a}_\mathfrak{p}^d), \tag{4.7}$$

then in the above theorem $\Theta = \varnothing$ and degree $\phi_{\nu,k} = 0$ and therefore we can put $\phi_{\nu,k} \equiv 1$ and $\kappa(\nu) = 1$ and the equality holds in (4.5).

iii) If

$$\lambda^\perp \cap \Sigma(\mathfrak{a}) = \{\alpha \in \Sigma(\mathfrak{a}^d);\ \mathrm{Re}\langle \lambda, \alpha \rangle = 0\}, \tag{4.8}$$

then $\Theta = \Theta'$ $\Lambda = \Lambda'$ and (4.6) is valid.

iv) If $\mathfrak{a}_\mathfrak{p}^d = \mathfrak{a}$, then (4.6) is also valid because $\nu_I(w\lambda) \in \Lambda'$ implies $w \in W(\mathrm{FBI}_\lambda(\psi): w_0 P^d) W_\Theta$.

v) If there exists $\nu \in \Lambda - \Lambda'$ or the condition (4.6) is not valid, then the term in (4.2) corresponding to ν is studied by considering $\mathrm{FBI}_{w\lambda}(\psi)$ for $w \in W_{\Theta'}$. The precise argument will be discussed elsewhere.

Proof of Theorem 4.1. First note that the proof here will go similar as in [MO, § 3]. We remark that we have only to prove (4.2) for $(g, y) \in G \times (0, \delta)^m$. Let $\tilde{X}$ be a compact G-manifold constructed [O3, § 1] where G/H is embedded as an open G-orbit X. We will identify G/H with X. Then for $g_0 \in G$ the point $g_0 w_0 a_I(y) w_0^{-1} H \in X$ converges to a point $p(g_0) \in \partial X$ in $\tilde{X}$ when $y \in (0, \infty)^m$ converges to 0. As in [O3, § 3], we can define boundary values of Ψ on the G-orbit $G \cdot p(g_0)$. Since ψ is ideally analytic at $p(g_0)$, we have an estimate (4.2) for $(g, y) \in U(g_0) \times (0, \delta)^m$ with a suitable finite subset Λ of $\boldsymbol{C}^m$ and polynomials $\phi_{\nu,k}$. Here $U(g_0)$ is a neighborhood of g_0 in G. Moreover the condition (4.3) and the number $N(w_0, \lambda)$ ar described by the vanishing or the nonvanishing of the corresponding boundary values of ψ. Since the boundary values are real analytic, if we prove (4.2) and (4.3) for $(g, y) \in U(g_0)\times(0, \delta)^m$ with only one fixed $g_0 \in G$, they are also valid for any $(g, y) \in G\times(0, \infty)^m$. It is the same for the number $N(w_0, \lambda)$.

Let $p \in \partial X$ to which $w_0 a(y) w_0^{-1}$ converges when $y \in (0, \delta)^l$ converges to 0. Let V be a neighborhood of p in $\tilde{X}$. Since there exists $g_0 \in G$ with $p(g_0) \in V$, the estimate (4.2) for $(g, y) \in U(g_0)\times(0, \delta)^m$ follows from (4.2) for $(g, y) \in U(e)\times(0, \delta)^l$ with $I=\Psi(\alpha)$. Hence to prove Theorem 4.1 we have only to consider (4.2) for $(g, y) \in U(e)\times(0, \delta)^l$ with $I=\Psi(\alpha)$. Moreover since $G\cdot p=K\cdot p$ we may replace $U(e)$ by a neighborhood U of e in K.

Now we apply the Flensted-Jensen isomorphism η to ψ. Let K_c be a complexification of K. Then $\psi(ka)$ ($k \in K$, $a \in A$) is extended to a function $\psi(ka)$ on $K_c\times A$ so that $\psi(ka)$ is holomorphic in $k \in K_c$. And $\eta(\psi)(haK^d)=\psi(ha)$ for $(h, a) \in H^d\times A$.

Let $\tilde{X}^r$ be a compact G^d-manifold constructed in [O1] where the riemannian symmetric space G^d/K^d is smoothly embedded as an open G^d-orbit X^r. We put $\tilde{a}(t)=\exp\sum -\tilde{\omega}_i \log t_i \in A_{\mathfrak{p}}^d$ for $t=(t_1, \cdots, t_{l'}) \in (0, \infty)^{l'}$. Then $\tilde{a}(t)=a(y)$ means $t_i=y_j$ if $\tilde{\alpha}_i|_{\mathfrak{a}}=\alpha_j$ and $t_i=1$ if $\tilde{\alpha}_i|_{\mathfrak{a}}=0$. Let $q(h)$ (resp. q) be a points in ∂X^r to which $hw_0a(y)K^d$ (resp. $w_0\tilde{a}(t)K^d$) converge when y (resp. t) converge 0. Note that $w_0 \in K^d$, $G\cdot q\simeq G^d/P^d$ and q corresponds to w_0P^d. Moreover we remark that $\eta(\psi)$ is ideally analytic at $q(h)$ and also at q. Since $\eta(\psi)$ is H^d-finite and any neighborhood of p in $\tilde{X}^r$ contains a point $q(h_0)$ with a suitable $h_0 \in H^d$, the asymptotic behavior of $\eta(\psi)(hw_0a(y)K^d)$ for $y\to 0$ can be reduced to that of $\eta(\psi)(hw_0\tilde{a}(t)K^d)$ for $t\to 0$. Then applying Theorem 3.6 and Remark 3.7 to $\eta(\psi)$, we exactly obtain (4.2) and (4.3).

Now suppose $I=\Psi(\alpha)$. Since $\eta(\Psi)$ is H^d-finite, we may assume that the functions $c_{\nu,k}(h)$ on H^d are H^d-finite. For $a \in A_{\mathfrak{p}}^d$ define $a_0 \in H^d\cap A_{\mathfrak{p}}^d$ and $a_1 \in A$ so that $a=a_0a_1$. Then for each $c_{\nu,k}(h)$ there exist a finite subset $\{\mu_1, \cdots, \mu_N\}$ of $(\mathfrak{h}^d\cap\mathfrak{a}^d)^*$ and non-zero real analytic functions $c_{\nu,k,i}(h)$ on H^d such that

$$(4.9)\qquad c_{\nu,k}(hw_0a_0w_0^{-1})=\sum_{i=1}^N c_{\nu,k,i}(h)a_0^{\mu_i} \qquad \text{for } h \in H^d \text{ and } a \in A_{\mathfrak{p}}^d.$$

We remark that μ_i are real valued on $\mathfrak{h}^d\cap\mathfrak{a}^d$. Combining (4.2) with (4.9) we have an asymptotic behavior of $\eta(\psi)(x)$ when x tends to p and therefore the rest part of the theorem follows from Theorem 3.6 because (4.6) implies the following:

If $w \in \overline{W}(\mathrm{FBI}_\lambda(\psi)\colon w_0P^d)$ and $w' \in W(\mathrm{FBI}_\lambda(\psi);\ w_0P^d)$ satisfy $\nu_I(w'\lambda)=\nu$, $\mathrm{Re}\langle w\lambda-w'\lambda, \omega_i\rangle\geq 0$ for $i=1, \cdots, l'$ and $\langle w\lambda-w'\lambda, \omega_j\rangle=0$ for $j=1, \cdots, l$, then there exists $w'' \in W(\mathrm{FBI}_\lambda(\psi)\colon w_0P^d)$ so that $w\lambda=w''\lambda$.

Q.E.D.

Corollary 4.3. *Let p be a positive number. Then for a function* $\psi \in \mathscr{A}_K(G/H; \mathscr{M}_\lambda)$ *the following conditions are equivalent.*

(4.10) $\psi \in L^p(G/H)$ $(=\{\psi$; ψ *is a measurable function on* G/H *and* ψ^p *is integrable modulo the center of* G *with respect to the invarinat measure.*$\})$.

(4.11) $\mathrm{FBI}_\lambda(\psi)\cdot \mathrm{Cl}(P^d w^{-1} P^d)$ *has no inner point for any* $w \in W(\mathfrak{a}_\mathfrak{p}^d)$ *satisfying*

$$\left(\mathrm{Re}\left\langle w\lambda+\left(\frac{2}{p}-1\right)\rho, \omega_1\right\rangle, \cdots, \mathrm{Re}\left\langle w\lambda+\left(\frac{2}{p}-1\right)\rho, \omega_l\right\rangle\right) \notin (-\infty, 0)^l.$$

(4.12) *There exist positive numbers* ε *and* C *such that*

$$|\psi(kwa(y)w^{-1}H)| \leq C \prod_{j=1}^{l} y_j^{(2/p)\langle\rho,\omega_j\rangle+\varepsilon}$$
$$\text{for } (k, y) \in K\times(0, \infty)^l \text{ and } w \in W(\mathfrak{a}).$$

(4.13) *There exist positive numbers* ε, δ *and* C *such that*

$$|\psi(kwa(y)w^{-1}H)| \leq C \prod_{j=1}^{l} y_j^{(2/p)\langle\rho,\omega_j\rangle+\varepsilon}$$
$$\text{for } (k, y) \in K\times(0, \delta)^l \text{ and } w \in W(\mathfrak{a}).$$

(4.14) *There exist positive numbers* ε, δ *and* C *such that the following holds for* $w \in W(\mathfrak{a})$ *and* $k=1, \cdots, l$:

$$|\psi(kwa(y)w^{-1}H)| \leq C y_k^{(2/p)\langle\rho,\omega_k\rangle+\varepsilon}$$
$$\text{for } k \in K \text{ and } y \in (1-\delta, 1)^{k-1}\times(0, \delta)\times(1-\delta, 1)^{l-k}.$$

Proof. The equivalence of the conditions (4.11), (4.12), (4.13) and (4.14) is a direct consequence of Theorem 4.2 and its proof. On the other hand, the invariant measure $d\mu$ on G/H satisfies

$$\int_{G/H} \phi d\mu = C \sum_{w\in W(\mathfrak{a})} \int_{K\times(0,1)^l} \phi(kwa(y)w^{-1}H)D(y)\frac{dy_1}{y_1}\cdots\frac{dy_l}{y_l}$$

for compactly supported continuous functions ϕ on G/H and $D(y)$ satisfies

$$C_1 \prod_{j=1}^{l} y_j^{-2\langle\rho,\omega_j\rangle} \leq 1+D(y) \leq C_2 \prod_{j=1}^{l} y_j^{-2\langle\rho,\omega_j\rangle} \qquad \text{for } y \in (0, 2)^l.$$

Here C, C_1 and C_2 are positive constant number. Hence the condition (4.10) follows from (4.12). The proof of the fact that (4.10) implies (4.13) is the same as the proof of [MO, Proposition 2]. Q.E.D.

Remark 4.4. The condition (4.12) for $p=2$ is better than [MO, Lemma 1 and Proposition 2]. This enables us to simplify the proof of

[MO, Theorem 1]. A simpler proof is given in [Ma2].

Corollary 4.5. *For a function* $\psi \in \mathscr{A}_K(G/H; \mathscr{M}_\lambda)$ *the following conditions are equivalent. If* ψ *satisfies the equivalent conditions, we say that* ψ *is tempered.*

(4.15) $\psi \in L^{2+\delta}(G/H)$ *for any* $\delta>0$.

(4.16) $\mathrm{FBI}_\lambda(\psi)\cdot \mathrm{Cl}(P^d w^{-1} P^d)$ *has an inner point for any* $w \in W(\mathfrak{a}_\mathfrak{p}^d)$ *satisfying*

$$(\mathrm{Re}\langle w\lambda, \omega_1\rangle, \cdots, \mathrm{Re}\langle w\lambda, \omega_l\rangle) \notin (-\infty, 0]^l.$$

(4.17) *There exist positive numbers* C_ε *and* δ *such that*

$$|\psi(kwa(y)w^{-1}H)| \leq C_\varepsilon \prod_{j=1}^{l} y_j^{\langle\rho,\omega_j\rangle+\varepsilon}$$
$$\text{for } \varepsilon>0,\ w \in W(\mathfrak{a}_\mathfrak{p}^d) \text{ and } (k, y) \in K\times(0, \delta)^l.$$

(4.18) *There exist positive numbers* C *and* N *such that*

$$|\psi(kwa(y)w^{-1}H)| \leq C(1+\langle\log a(y), \log a(y)\rangle)^N \prod_{j=1}^{l} y_j^{\langle\rho,\omega_j\rangle}$$
$$\text{for } w \in W(\mathfrak{a}_\mathfrak{p}^d) \text{ and } (k, y) \in K\times(0, \infty)^l.$$

Proof. The equivalence of (4.15) and (4.17) follows from Corollary 4.3. The equivalence of (4.16), (4.17) and (4.18) follows from Theorem 4.1. Q.E.D.

§ 5. An imbedding theorem

In this section we also use the notation defined in § 0. For simplicity we assume G has a finite center. First we review principal series for G/H (cf. [O3, § 4]). Let P_σ denote the parabolic subgroup of G with the Langlands decomposition $P_\sigma = M_\sigma A_\sigma N_\sigma$ such that $M_\sigma A_\sigma$ is the centralizer of $\mathfrak{a}$ in G and the Lie algebra $\mathfrak{n}_\sigma$ of N_σ is spanned by the root spaces in $\mathfrak{g}$ corresponding to $\Sigma(\mathfrak{a})^+$. Let $\mathfrak{m}_\sigma$ and $\mathfrak{a}_\sigma$ be the Lie algebras of M_σ and A_σ, respectively. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}_c$, $\mathfrak{a}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}$, $\Sigma(\mathfrak{a}_\mathfrak{p})$ be the root system for the pair $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$, $\Sigma(\mathfrak{a}_\mathfrak{p})^+$ be a positive system of $\Sigma(\mathfrak{a}_\mathfrak{p})$ compatible to $\Sigma(\mathfrak{a})^+$, $\mathfrak{g}(\sigma)$ be the Lie algebra spanned by the root spaces $g(\mathfrak{a}_\mathfrak{p}; \lambda)$ in $\mathfrak{g}$ for the roots $\lambda \in \Sigma(\mathfrak{a}_\mathfrak{p})$ with $\lambda|_\mathfrak{a}=0$ and $\mathfrak{m}(\sigma)$ be the centralizer of $\mathfrak{g}(\sigma)$ in $\mathfrak{m}_\sigma$. Let $G(\sigma)$ and $M(\sigma)_0$ be the analytic subgroups of G with Lie algebras $\mathfrak{g}(\sigma)$ and $\mathfrak{m}(\sigma)$, respectively, and put $M(\sigma)=M(\sigma)_0 \mathrm{Ad}_G^{-1}(\mathrm{Ad}(K)\cap \exp(\sqrt{-1}\,\mathfrak{a}_\mathfrak{p}))$. Then $\mathfrak{m}_\sigma$ is the direct sum of $\mathfrak{m}(\sigma)$ and $\mathfrak{g}(\sigma)$ and moreover we have $[\mathfrak{a}_\mathfrak{p}, \mathfrak{a}_\mathfrak{p}^d]=0$, $M_\sigma \subset M$, $G(\sigma)\subset H$ and $M_\sigma = M(\sigma)G(\sigma)$. Let $W(\mathfrak{a}; H)$ be the subgroup of $W(\mathfrak{a})$ whose elements have representatives in $K\cap H$. For an element w of $W(\mathfrak{a};H)\backslash W(\mathfrak{a})$, we fix a representative of w in $W(\mathfrak{a})$ and also that of w in K and denote them by the same symbol w. We can choose the representative w so that $\mathrm{Ad}(w)\mathfrak{a}_{\mathfrak{p}}=\mathfrak{a}_{\mathfrak{p}}$, $\mathrm{Ad}(w)\mathfrak{j}=\mathfrak{j}$, $\mathfrak{m}(\sigma)\cap\mathrm{Ad}(w)^{-1}\mathfrak{h}=\mathfrak{m}(\sigma)\cap\mathfrak{h}$ and $w(\Sigma(\mathfrak{j})_{\theta}^{+})=\Sigma(\mathfrak{j})_{\theta}^{+}$. Here we put

$$\mathfrak{t}=\sqrt{-1}(\mathfrak{h}\cap\mathfrak{a}_{\mathfrak{p}}^{d}),\ \mathfrak{j}=\mathfrak{t}+\mathfrak{a},\ \Sigma(\mathfrak{j})=\Sigma(\mathfrak{a}_{\mathfrak{p}}^{d}),\ \Sigma(\mathfrak{j})^{+}=\Sigma(\mathfrak{a}_{\mathfrak{p}}^{d})^{+},$$
$$\Sigma(\mathfrak{j})_{\theta}=\{\alpha\in\Sigma(\mathfrak{j});\ \alpha|_{\mathfrak{a}}=0\}\quad\text{and}\quad\Sigma(\mathfrak{j})_{\theta}^{+}=\Sigma(\mathfrak{j})_{\theta}\cap\Sigma(\mathfrak{j})^{+}.$$

We identify $\mathfrak{j}_c^*$ with $(\mathfrak{a}_{\mathfrak{p}}^{d})_c^*$. The restricted root system for the reductive symmetric pair $(\mathfrak{m}(\sigma),\mathfrak{m}(\sigma)\cap\mathfrak{h})$ is naturally identified with $\Sigma(\mathfrak{j})_{\theta}$.

Let (τ, E_τ) be a finite dimensional irreducible representation of P_σ which has a non-zero $(P_\sigma\cap w^{-1}Hw)$-fixed vector with a suitable $w\in W(\mathfrak{a})$ and let V_τ be a vector bundle over G/P_σ associated to τ. Then the space of (hyperfunction, C^∞ or K-finite etc.) sections of V_τ is called to belong to the (most continuous) principal series for G/H (cf. [O3, Definition 4.3]). In this section we consider principal series in the category of Harish-Chandra modules and so we denote by U_τ the Harish-Chandra module of K-finite sections of V_τ. For the above τ, there exist an element $\mu\in\mathfrak{a}_c^*\subset\mathfrak{j}_c^*$ and a finite dimensional irreducible representation ξ of $M(\sigma)$ with a non-zero $(M(\sigma)\cap w^{-1}Hw)$-fixed vector satisfying

(5.1) $\tau(mxan)=a^{\rho-\mu}\xi(m)\qquad$ for $(m,x,a,n)\in M(\sigma)\times G(\sigma)\times A_\sigma\times N_\sigma$.

Here ρ is the half sum of the roots in $\Sigma(\mathfrak{a}_{\mathfrak{p}})^{+}$ counting the multiplicities. In this case we put $U_\tau=U_{\xi,\mu}$. Let $d\xi$ (resp. $\overline{d\xi}$) be the highest weight (resp. lowest weight) of the representation ξ with respect to $\Sigma(\mathfrak{j})_{\theta}^{+}$. Then $d\xi\in\sqrt{-1}\,\mathfrak{t}^*$ and $\overline{d\xi}\in\sqrt{-1}\,\mathfrak{t}^*$ because ξ has a non-zero $(M(\sigma)\cap w^{-1}Hw)$-fixed vector.

Theorem 5.1. *Let λ be an element of $(\mathfrak{a}_{\mathfrak{p}}^{d})_c^*$ satisfying* (0.2) *and* (0.3), *ψ be a non-zero element of $\mathscr{A}_K(G/H;\mathscr{M}_\lambda)$ and $U(\psi)$ be the Harish-Chandra module generated by ψ. Fix an element $w\in W(\mathfrak{a})$ and choose any element $v\in W(\mathrm{FBI}_\lambda(\psi);wP^d)$ (cf. the first part of* § 4). *Then there exist a Harish-Chandra module $U_{\xi,\mu}$ belonging to the principal series for G/H and a linear map*

$$\iota\colon U(\psi)\longrightarrow U_{\xi,\mu}$$

which satisfies the following conditions:

(5.2) $\iota(\psi)\neq0$.

(5.3) $v\lambda|_{\mathfrak{a}}=\mu$ *and* $(v\lambda-\rho)|_{\mathfrak{t}}=-\overline{d\xi}$.

(5.4) *ξ has a non-zero $(M(\sigma)\cap w^{-1}Hw)$-fixed vector.*

(5.5) $\pi(X)\circ\iota=\iota\circ\pi(X)$ *and* $\pi(k)\circ\iota=\iota\circ\pi(k)$ *for* $X\in\mathfrak{g}$ *and* $k\in K$.

Here $\pi(X)$ *and* $\pi(k)$ *are linear maps induced by the left translations which define the structure of Harish-Chandra modules.*

Remark 5.2. If $U(\psi)$ is irreducible, then $\mathrm{FBI}_\lambda(\phi)=\mathrm{FBI}_\lambda(\psi)$ for any non-zero $\phi\in U(\psi)$. This is clear from the $\mathfrak{g}$-equivariance of the boundary value map β_λ.

Proof. Use the notation in § 0 and § 4. Put $U_1=\beta_\lambda\circ\eta(U(\psi))$ and $S=H^dwP^d$. Since the support of any element of U_1 is contained in $\mathrm{FBI}_\lambda(\psi)$, Theorem 3.4 assures an existence of a non-trivial $\mathfrak{g}_c$-homomorphism

$$\mathcal{T}_v^\lambda\colon U_1\longrightarrow\mathcal{B}(S;L_{v\lambda}).$$

Put $U_2=\mathrm{Im}\,\mathcal{T}_v^\lambda$. Since S is an open H^d-orbit in G^d/P^d and any element ϕ of U_2 is H^d-finite, ϕ is real analytic on S. Let K_c be the complexification of K and put $\phi_1(k)=\phi(kwP^d)$ for $k\in H^d$. Since, η, β_λ and $\mathcal{T}_v^\lambda$ are $\mathfrak{k}$-equivariant, the germ of ϕ_1 at e has a unique holomorphic extension ϕ_2 on K_c. Then we can define a function ϕ_3 on G by

$$\phi_3(kxan)=\phi_2(kw^{-1})a^{v\lambda-\rho}\qquad\text{for }(k,x,a,n)\in K\times G(\sigma)\times A_\sigma\times N_\sigma$$

and an injective map $r\colon U_2\ni\phi_1\mapsto\phi_3\in\mathcal{A}(G)$. We note that the Lie algebra of $G(\sigma)$ is contained in the complexification of the Lie algebra of M^d and so is $\mathfrak{a}_\sigma\cap\mathfrak{h}$. Also $\mathfrak{n}_\sigma$ is contained in the complexification of the Lie algebra of N^d. Moreover we have $\pi(X)\circ r=r\circ(X)$ because ϕ_3 is defined through the analytic continuation.

Put $\mathfrak{g}'=(\mathfrak{m}(\sigma)\cap\mathfrak{h})+\sqrt{-1}\,(\mathfrak{m}(\sigma)\cap\mathfrak{q})$ and let G' be the analytic subgroup of H^d with the Lie algebra $\mathfrak{g}'$. Fix a non-zero $\phi\in U_2$ and also fix $k\in H^d$ with $\phi(wk)\neq0$. Consider the function $u(g)=\phi(wkg)$ $(g\in G')$ on G'. The group $G'\cap P^d$ is a minimal parabolic subgroup of G' with the Langlands decomposition $(G'\cap M^d)(G'\cap A_\mathfrak{p}^d)(G'\cap N^d)$ and the Lie algebra of $G'\cap A_\mathfrak{p}^d$ equals $\sqrt{-1}\,\mathfrak{t}$. Since

$$u(gman)=u(g)a^{v\lambda-\rho}\qquad\text{for }(m,a,n)\in(G'\cap M^d)\times(G'\cap A_\mathfrak{p}^d)\times(G'\cap N^d)$$

and the linear span of left translations of u by the element of G' is a finite dimensional vector space, $(v\lambda-\rho)|_\mathfrak{t}$ is a highest weight of an irreducible finite dimensional representation δ of G' with a non-zero $(G'\cap K^d)$-fixed vector. Moreover

$$\int_{G'\cap K^d}u(gm)dm\not\equiv0.$$

Now remark that $\mathfrak{g}' \cap \mathfrak{k}^d = \mathfrak{m}(\sigma) \cap \mathfrak{h}$ and $\mathrm{Ad}(w^{-1})(\mathfrak{m}(\sigma) \cap \mathfrak{h}) = \mathfrak{m}(\sigma) \cap \mathfrak{h}$. Hence we have the $\mathfrak{g}_c$-isomorphism

$$\begin{array}{ccc} p\colon \operatorname{Im} r & \longrightarrow & \mathscr{A}(G) \\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ \phi_3 & \longmapsto & \phi_4(g) = \int_{(M(\sigma)\cap H)_0} \phi_3(gm)dm. \end{array}$$

Here $(M(\sigma) \cap H)_0$ is the identity component of $M(\sigma) \cap H$.

Let $\tilde{J}$ be a Cartan subgroup of G whose Lie algebra contains both $\mathfrak{j}$ and $\mathfrak{a}_{\mathfrak{p}}$. Then [O3, Lemma 4.6] claims $M(\sigma) \cap H = (M(\sigma) \cap H)_0(M(\sigma) \cap H \cap \tilde{J})$. When we apply the Flensted-Jensen isomorphism to $U(\psi)$, we may consider that the group $M(\sigma) \cap H \cap \tilde{J}$ $(\subset K \cap H)$ is contained in M^d and therefore $\phi_3(g)$ is right $w^{-1}(M(\sigma) \cap H \cap \tilde{J})w$-invariant. Since $M(\sigma) \cap w^{-1}Hw = (M(\sigma) \cap H)_0 w^{-1}(M(\sigma) \cap H \cap \tilde{J})w$, $\phi_4(g)$ is right $(M(\sigma) \cap w^{-1}Hw)$-invariant.

Combining the above arguments, we conclude that the image of p is contained in the following space:

$$\begin{aligned} U_4 = \Big\{ & f \in \mathscr{A}(G); f \text{ is left } K\text{-finite}, \\ & f(gmxan) = f(g)a^{\nu\lambda - \rho} \text{ for } (g, m, x, a, n) \\ & \qquad \in G \times (M(\sigma) \cap w^{-1}Hw) \times G(\sigma) \times A_\sigma \times N_\sigma, \\ & f(g) = \chi_{\delta^*}(e) \int_{M(\sigma)_0} f(gm)\chi_{\delta^*}(m)dm \Big\}, \end{aligned}$$

where χ_{δ^*} is the character of the representation δ^* of $M(\sigma)_0$ contragradient to δ by identifying δ with a representation of $M(\sigma)_0$ through the complexification of G' in K_c. Then the highest weight of δ is equal to the negative of the lowest weight of δ^*. Since $M(\sigma)/M(\sigma)_0$ is a finite group, U_4 decomposes into a finite direct sum of Harish-Chandra modules $U_{\xi,\mu}$ with multiplicity free which satisfy the conditions in Theorem 5.1 (cf. [O3, Theorem 4.10 and Theorem 4.11]). Thus we have the theorem.

Q.E.D.

Remark 5.3. Let G' be a connected real semisimple Lie group. Put $G = G' \times G'$, $\sigma(g_1, g_2) = (g_2, g_1)$ and $H = \{(g, g) \in G; g \in G'\}$. Then the symmetric space G/H is naturally identified with the group manifold G'. We call this case a group case.

In this case a sphereical function ψ on G/H means a right and left K'-finite function on G' with an infinitesimal character and the Harish-Chandra module $U(\psi)$ means a $(\mathfrak{g}', K')$-bimodule whose structure is induced from the left and right G'-actions on G'. A principal series for G/H means the direct tensor product of the usual most continuous (non-unitary) principal series of G' and its contragradient. In this case $W(\mathfrak{a}; H) = W(\mathfrak{a}) \simeq W(\mathfrak{a}'_\mathfrak{p})$ and we can always assume $w = e$ in Theorem 5.1. If we apply Theorem 5.1 to the discrete series of G', by the study of the structure of $H^d \backslash G^d / P^d$ (cf. [Ma1]) we have the same result as in [KW] for the imbedding of the discrete series into the principal series.

Using a similar technique as in [FOS], we can prove the following claim in the group case, which is not true in a general case. The precise argument will be given elsewhere.

Let v be an element of $W(\mathfrak{a}_\mathfrak{p})$ which satisfies

$$(\mathrm{Re}\langle v\lambda - w\lambda, \omega_1\rangle, \cdots, \mathrm{Re}\langle v\lambda - w\lambda, \omega_{l'}\rangle) \in [0, \infty)^{l'} - \{0\}$$
$$\text{for any } w \in W(\mathrm{FBI}_\lambda(\psi); P^d),$$

then for any Harish-Chandra module which satisfies (5.3) there exists no non-zero homomorphism of $U(\psi)$ to $U_{\xi,\mu}$.

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