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Closure Relations for Orbits on Affine Symmetric Spaces under the Action of Minimal Parabolic Subgroups

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§ 1. Introduction

Let G be a connected Lie group, σ an involutive automorphism of G and H a subgroup of G such that $G_0^\sigma \subset H \subset G^\sigma$ where $G^\sigma = \{x \in G \mid \sigma x = x\}$ and G_0^σ is the connected component of G^σ containing the identity. Then the factor space $H\backslash G$ is called an affine symmetric space. We assume that G is real semisimple throughout this paper.

Let P^0 be a minimal parabolic subgroup of G. Then a parametrization of the double coset decomposition $H\backslash G/P^0$ is given in [1] and [2]. In this paper we study the closure relations for the double coset decomposition.

The result of this paper can be stated as follows. Let $\mathfrak{g}$ be the Lie algebra of G and σ the automorphism of $\mathfrak{g}$ induced from the automorphism σ of G. Let θ be a Cartan involution of $\mathfrak{g}$ such that $\sigma\theta = \theta\sigma$. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ (resp. $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$) be the decomposition of $\mathfrak{g}$ into the $+1$ and -1 eigenspaces for σ (resp. θ).

Let x be an arbitrary element of G. By Theorem 1 in [1], there exists an $h \in G_0^\sigma$ such that $P = hxP^0x^{-1}h^{-1}$ can be written as

$$P = P(\mathfrak{a}, \Sigma^+) = Z_G(\mathfrak{a}) \exp \mathfrak{n}$$

where $\mathfrak{a}$ is a σ-stable maximal abelian subspace of $\mathfrak{p}$, Σ^+ is a positive system of the root system Σ of the pair $(\mathfrak{g}, \mathfrak{a})$, $Z_G(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in G and $\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}(\mathfrak{a}; \alpha)$. ($\mathfrak{g}(\mathfrak{a}; \alpha) = \{X \in \mathfrak{g} \mid [Y, X] = \alpha(Y)X$ for all $Y \in \mathfrak{a}\}$.) Since $(HxP^0)^{cl} = (HP)^{cl}hx$, we have only to study $(HP)^{cl}$.

Let K be the analytic subgroup of G for $\mathfrak{k}$ and put $H^a = (K \cap H) \cdot \exp(\mathfrak{p} \cap \mathfrak{q})$. Then $H^a\backslash G$ is called the affine symmetric space associated to $H\backslash G$ ([1]). For a subset S of G, we put $S^{op} = \{y \in G \mid (H^a yP)^{cl} \cap S \neq \emptyset\}$. Then it is clear that S^{op} is the minimal H^a-P invariant open subset of G containing S since the number of H^a-P double cosets in G is finite. For each root α in Σ, put $\mathfrak{a}^\alpha = \{Y \in \mathfrak{a} \mid \alpha(Y) = 0\}$, put $L_\alpha = Z_G(\mathfrak{a}^\alpha)$ and choose an element w_α of $N_K(\mathfrak{a})$ such that $\mathrm{Ad}(w_\alpha)|_\mathfrak{a}$ is the reflection with respect to α.

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Theorem. *Let C denote the σ-stable convex closed cone in $\mathfrak{a}$ defined by $C=\{Y \in \mathfrak{a} \mid \alpha(Y) \geq 0$ for all $\alpha \in \Sigma^{+} \cap \sigma\Sigma^{+}\}$. Fix an element Y_0 of $C \cap \mathfrak{h}$ such that $\alpha \in \Sigma$ and $\alpha(Y_0)=0$ implies $\alpha|_{\mathfrak{a}\cap\mathfrak{h}}=0$. Let w be the element of W defined by the condition*

$$w\Sigma^{+}=\{\alpha \in \Sigma^{+} \mid \alpha(Y_0) \geq 0\} \cup \{\alpha \in -\Sigma^{+} \mid \alpha(Y_0)>0\}.$$

Let $w=w_{\alpha_1} \cdots w_{\alpha_n}$ be a minimal expression of w by the reflections with respect to simple roots $\alpha_1, \cdots, \alpha_n$ in Σ^{+}. Put $w^{(i)}=w_{\alpha_1} \cdots w_{\alpha_i}$ $(i=0, \cdots, n)$, $L_1=Z_G(\mathfrak{a} \cap \mathfrak{h})$, $\mathfrak{l}_1=\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a} \cap \mathfrak{h})$ and $\mathfrak{l}=[\mathfrak{l}_1, \mathfrak{l}_1]$. Let L be the analytic subgroup of G for $\mathfrak{l}$. Then we have the followings.

(i) $$(Hw^{(i-1)}P)^{cl}=(Hw^{(i)}P)^{cl}L_{\alpha_i}$$
and $$(H^a w^{(i-1)}P)^{op}=(H^a w^{(i)}P)^{op}L_{\alpha_i} \quad \text{for } i=1, \cdots, n.$$

(ii) $$(HP)^{cl}=(HwP)^{cl}(Pw^{-1}P)^{cl}$$
and $$(H^a P)^{op}=(H^a wP)^{op}(Pw^{-1}P)^{cl}$$

(iii) $$(HwP)^{cl}=H((L \cap H)(L \cap P))^{cl}wP$$
and $$(H^a wP)^{op}=H^a((L \cap H^a)(L \cap P))^{op}wP.$$

Here

$$((L \cap H^a)(L \cap P))^{op}=\{y \in L \mid ((L \cap H^a)y(L \cap P))^{cl} \cap (L \cap H^a)(L \cap P) \neq \emptyset\}.$$

(iv) $$(HP)^{cl}=H((L \cap H)(L \cap P))^{cl}w(Pw^{-1}P)^{cl}$$
and $$(H^a P)^{op}=H^a((L \cap H^a)(L \cap P))^{op}w(Pw^{-1}P)^{cl}.$$

(v) *$(L \cap H)(L \cap P)$ is open in L and $(L \cap H^a)(L \cap P)$ is closed in L.*

(vi) *Let D (resp. D') be an arbitrary H-P double coset (resp. H^a-P double coset) contained in $(HP)^{cl}$ (resp. $(H^aP)^{op}$). Then there exist elements $y_i \in (Hw^{(i)}P)^{cl}$ (resp. $(H^a w^{(i)}P)^{op}$) for $i=0, \cdots, n$ satisfying the following four conditions.*

(a) *$\mathfrak{a}_i=\mathrm{Ad}(y_i)\mathfrak{a}$ is σ-stable and $y_i \in K$ for $i=0, \cdots, n$.*

(b) *$Hy_0P=D$ and $y_n \in ((L \cap H)(L \cap P))^{cl}w$ (resp. $H^a y_0 P=D'$ and $y_n \in ((L \cap H^a)(L \cap P))^{op}w$).*

(c) *Let α_i' be the root in $\Sigma(\mathfrak{a}_i)$ defined by $\alpha_i'=\alpha_i \circ \mathrm{Ad}(y_i)^{-1}$ for $i=1, \cdots, n$. If $\mathfrak{g}(\mathfrak{a}_i; \alpha_i') \cap \mathfrak{q}=\{0\}$, then $y_{i-1}=y_i$ or $y_i w_{\alpha_i}$. If $\mathfrak{g}(\mathfrak{a}_i; \alpha_i') \cap \mathfrak{q} \neq \{0\}$, then $y_{i-1}=y_i$, $y_i w_{\alpha_i}$, $y_i c_{\alpha_i}$ or $y_i c_{\alpha_i}^{-1}$. Here c_{α_i} is an element of L_{α_i} defined by $c_{\alpha_i}=y_i^{-1}c'_{\alpha_i}y_i$, $c'_{\alpha_i}=\exp(\pi/2)(X+\theta X)$ with an $X \in \mathfrak{g}(\mathfrak{a}_i; \alpha_i') \cap \mathfrak{q}$ satisfying $2\langle \alpha_i', \alpha_i' \rangle B(X, \theta X)=-1$. ($B(\ ,\ )$ is the Killing form on $\mathfrak{g}$ and $\langle\ ,\ \rangle$ is the inner product on $\mathfrak{a}_i^*$ induced from $B(\ ,\ )$.)*

(d) *$\dim Hy_{i-1}P \geq \dim Hy_iP$ (resp. $\dim H^a y_{i-1}P \leq \dim H^a y_i P$) for $i=1, \cdots, n$. Moreover if $y_{i-1}=y_i c_{\alpha_i}$ or $y_i c_{\alpha_i}^{-1}$ in* (c), *then* $\dim Hy_{i-1}P>$

dim Hy_iP (*resp.* dim $H^a y_{i-1}P <$ dim $H^a y_i P$).

(viii) *Let D* (*resp. D'*) *be an arbitrary closed H-P double coset* (*open H^a-P double coset*) *in G. Then*

$$D \subset (HP)^{cl} \Longleftrightarrow D \subset HRwW_{\alpha_n} \cdots W_{\alpha_1}P$$

$$(\textit{resp. } D' \subset (H^a P)^{op} \Longleftrightarrow D' \subset H^a R' w W_{\alpha_n} \cdots W_{\alpha_1}P).$$

Here R (*resp. R'*) *is the union of all the closed $L \cap H$-$L \cap P$ double cosets* (*open $L \cap H^a$-$L \cap P$ double cosets*) *in L and $W_{\alpha_i} = \{1, w_{\alpha_i}\}$ for $i = 1, \cdots, n$. Moreover let y be an element of K such that* Ad$(y)\mathfrak{a}$ *is σ-stable and that HyP is closed in G.* (*Then $H^a yP$ is open in G by Corollary of* [1] § 3.) *Then*

$$HyP \subset (HP)^{cl} \Longleftrightarrow H^a yP \subset (H^a P)^{op}.$$

(*At the end of this section, we have*

$$HyP \subset (HP)^{cl} \Longleftrightarrow H^a yP \subset (H^a P)^{op}$$

for any H-P double coset HyP in G as a corollary of Theorem. Here $y \in K$ is chosen so that Ad$(y)\mathfrak{a}$ *is α-stable.*)

Remark. (i) Since L is a connected semisimple Lie subgroup of G such that $\sigma L = \theta L = L$, we can apply Theorem to the double coset decompositions $L \cap H \backslash L / L \cap P$ and $L \cap H^a \backslash L / L \cap P$.

(ii) If the number of the open $L \cap H$-$L \cap P$ double cosets in L is one (then the number of the closed $L \cap H^a$-$L \cap P$ double cosets in L is one by Corollary of [1] § 3), for instance when G is a complex semisimple Lie group and σ is a complex linear involution, then it is clear from Theorem (v) that

$$((L \cap H)(L \cap P))^{cl} = ((L \cap H^a)(L \cap P))^{op} = L.$$

In [3], T.A. Springer studied the double coset decomposition $H \backslash G / P$ for algebraic groups G over algebraically closed fields. He also studied closure relations in Section 6 of his paper. So the formula for $(HP)^{cl}$ in Theorem (iv) and the description of H-P double cosets contained in $(HP)^{cl}$ in Theorem (vi) are essentially the same as his results (except that $y_{i-1} = y_i$ or $y_i w_{\alpha_i}$ when $\mathfrak{g}(\mathfrak{a}_i; \alpha_i') \cap \mathfrak{q}^a \neq \{0\}$) when G is a complex Lie group and σ is a complex linear involution.

(iii) When the number of the open $L \cap H$-$L \cap P$ double cosets in L is not one, we can find by Theorem (vii) all the $L \cap H$-$L \cap P$ double cosets (resp. $L \cap H^a$-$L \cap P$ double cosets) contained in $((L \cap H)(L \cap P))^{cl}$ (resp. $((L \cap H^a)(L \cap P))^{op}$) in the following way. Let $(L \cap H)y(L \cap P)$

(resp. $(L\cap H^a)y(L\cap P)$) be an arbitrary $L\cap H$-$L\cap P$ double coset (resp. $L\cap H^a$-$L\cap P$ double coset) in L. We may assume that $\mathrm{Ad}(y)\mathfrak{a}$ is σ-stable and that $y\in K$ by [1] Theorem 1. Then considering L, $L\cap H$ and $y(L\cap P)y^{-1}$ as G, H^a and P in Theorem (vii), respectively, we can see whether $(L\cap H)(L\cap P)y^{-1}$ (resp. $(L\cap H^a)(L\cap P)y^{-1}$) is contained in $((L\cap H)y(L\cap P)y^{-1})^{op}$ (resp. $((L\cap H^a)y(L\cap P)y^{-1})^{cl}$) or not. So we can see whether $(L\cap H)y(L\cap P)$ (resp. $(L\cap H^a)y(L\cap P)$) is contained in $((L\cap H)(L\cap P))^{cl}$ (resp. $(L\cap H^a)(L\cap P)^{op}$) or not.

(iv) Let y be an element of $L\cap K$ such that $\mathrm{Ad}(y)\mathfrak{a}$ is σ-stable. Then it follows from the above consideration in (iii) and from the latter half of Theorem (vii) that

$$y\in((L\cap H)(L\cap P))^{cl}\Longleftrightarrow y\in((L\cap H^a)(L\cap P))^{op}.$$

(v) When $G=G'\times G'$, $H=\{(x, x)\mid x\in G'\}$ and $P=P'\times P'$ with a connected semisimple Lie group G' and a minimal parabolic subgroup $P'=P(\mathfrak{a}', \Sigma'^+)$ of G', the double coset decomposition $H\backslash G/P$ can be naturally identified with the Bruhat decomposition $P'\backslash G'/P'\simeq W(\mathfrak{a}')$. In this case it is known as Bruhat ordering on $W(\mathfrak{a}')$ that $(P'wP')^{cl}=P'L'_{\gamma_1}\cdots L'_{\gamma_n}P'=PW_{\gamma_1}\cdots W_{\gamma_n}P'$. Here $L'_\gamma=Z_{G_1}(\mathfrak{a}'^\gamma)$, $\mathfrak{a}'^\gamma=\{Y\in\mathfrak{a}'\mid\gamma(Y)=0\}$ for $\gamma=\Sigma'$, $w=w_{\gamma_1}\cdots w_{\gamma_n}$ is a reduced expression of $w\in W(\mathfrak{a}')$ by reflections w_{γ_1}, $\cdots$, w_{γ_n} with respect to simple roots $\gamma_1, \cdots, \gamma_n$ in Σ'^+ and $W_{\gamma_i}=\{1, w_{\gamma_i}\}$ for $i=1, \cdots, n$.

In general if the number of $K\cap H$-conjugacy classes of σ-stable maximal abelian subspaces of $\mathfrak{p}$ is one, then it follows from [1] Theorem 2 that $y_{i-1}=y_i$ or $y_iw_{\alpha_1}$ in Theorem (vi) and that $(L\cap H)(L\cap P)=(L\cap H^a)(L\cap P)=L$. Hence it follows from Theorem (iv) and Theorem (vi) that

$$(HP)^{cl}=HwPL_{\alpha_n}\cdots L_{\alpha_1}=HwW_{\alpha_n}\cdots W_{\alpha_1}P$$

and that

$$(H^aP)^{op}=H^awPL_{\alpha_n}\cdots L_{\alpha_1}=H^awW_{\alpha_n}\cdots W_{\alpha_1}P.$$

So we can say that Theorem (vi) is a generalization of Bruhat ordering.

As in Corollary 2 of [1] Theorem 1, there exists a natural one-to-one correspondence between $H\backslash G/P$ and $H^a\backslash G/P$ given by $HyP\to H^ayP$ if $\mathrm{Ad}(y)\mathfrak{a}$ is σ-stable and $y\in K$. From Remark (iv) and from Theorem (vi) we have the following.

Corollary. *Let D be an arbitrary H-P double coset and choose a $y\in D\cap K$ so that* $\mathrm{Ad}(y)\mathfrak{a}$ *is σ-stable. Then $HyP\subset(HP)^{cl}$ if and only if $H^aP\subset(H^ayP)^{cl}$.*

In the proof of the first six assertions in Theorem, a generalization (Lemma 3) of [4] Lemma 5.1 plays an essential role. The proof of Theorem (vii) is reduced to the following proposition which will be proved in Section 5.

Proposition. *For any closed H-P double coset D and for any open H-P double coset D′, we have* $D\subset(D')^{cl}$.

The author would like to thank J. Sekiguchi because the simple proof of Proposition given in Section 5 is due to him, while the original proof by the author was very complicated.

§ 2. Notations and preliminaries

Let $\boldsymbol{Z}$ denote the ring of integers and $\boldsymbol{R}$ the field of real numbers. For a set S with a map $\tau\colon S\to S$, we write $S^{\tau}=\{x\in S\mid \tau x=x\}$. For a topological group G_1, we denote by $(G_1)_0$ the connected component of G_1 containing the identity.

Let G_1 be a topological group, H_1 and H_2 be closed subgroups of G_1 and S be a subset of G_1. Then we denote by S^{cl} the closure of S in G_1 and we put $S^{op}(H_2\backslash G_1/H_1)=\{x\in G_1\mid (H_2xH_1)^{cl}\cap S\neq\emptyset\}$. If the number of H_2-H_1 double cosets in G_1 is finite, then it is clear that $S^{op}(H_2\backslash G_1/H_1)$ is the minimal H_2-H_1 invariant open subset of G_1 containing S. If S is H_2-H_1 invariant, then S^{cl} is also H_2-H_1 invariant. Since we study double coset decompositions, it is natural to use the symbol $S^{op}(H_2\backslash G/H_1)$ only when S is H_2-H_1 invariant.

The following general lemma will be used in Section 4 when $H_3=(H_2)_0$.

Lemma 1. *Let* G_1, H_1 *and* H_2 *be as above. Let* H_3 *be a normal subgroup of* H_2 *and* S *a subset of* G_1 *such that* $H_3SH_1=S$. *Suppose that the number of* H_3-H_1 *double cosets in* G_1 *is finite. Then we have the followings.*

(i) $(H_2S)^{cl}=H_2S^{cl}$.

(ii) $(H_2S)^{op}(H_2\backslash G_1/H_1)=H_2S^{op}(H_3\backslash G_1/H_1)$.

Proof. (i) Since $H_2S\subset H_2S^{cl}\subset(H_2S)^{cl}$, we have only to prove that H_2S^{cl} is closed in G_1. Since H_3 is normal in H_2, we have

$$H_2S^{cl}=\bigcup_{g\in H_2}g(H_3S)^{cl}=\bigcup_{g\in H_2}(H_3gS)^{cl}.$$

Since the number of H_3-H_1 double cosets in G_1 is finite, the right hand side of this formula is a union of a finite number of closed sets. Hence H_2S^{cl} is closed in G_1.

(ii) Since the number of H_3-H_1 double cosets in G_1 is finite, $(H_iS)^{op}(H_i\backslash G_1/H_1)$ is the minimal H_i-invariant open subset of G_1 containing H_iS for $i=2, 3$. Clearly $H_2S^{op}(H_3\backslash G_1/H_1)$ is an H_2-H_1 invariant open subset of G_1 such that $H_2S\subset H_2S^{op}(H_3\backslash G_1/H_1)\subset(H_2S)^{op}(H_2\backslash G_1/H_1)$. Hence the assertion holds. Q.E.D.

Let G be a connected real semisimple Lie group, σ an involutive automorphism of G and H a subgroup of G satisfying $G_0^\sigma\subset H\subset G^\sigma$. Then the factor space $H\backslash G$ is called an affine symmetric space.

Let $\mathfrak{g}$ be the Lie algebra of G and σ the automorphism of $\mathfrak{g}$ induced from the automorphism σ of G. Fix a Cartan involution θ of $\mathfrak{g}$ such that $\sigma\theta=\theta\sigma$. Let $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$, $\mathfrak{g}=\mathfrak{h}^a+\mathfrak{q}^a$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ denote the $+1$ and -1 eigenspace decompositions for σ, $\sigma\theta$ and θ, respectively. Let B: $\mathfrak{g}\times\mathfrak{g}\to\boldsymbol{R}$ be the Killing form on $\mathfrak{g}$.

Let K denote the analytic subgroup of G for $\mathfrak{k}$. Put $H^a=(K\cap H)\cdot\exp(\mathfrak{p}\cap\mathfrak{q})$. Then $H^a\backslash G$ is called the affine symmetric space associated to $H\backslash G$. We remark here that a property for an affine symmetric space$H\backslash G$ also holds for $H^a\backslash G$. (We can replace H, $\mathfrak{h}$, $\mathfrak{q}$ and σ by H^a, $\mathfrak{h}^a$, $\mathfrak{q}^a$ and $\sigma\theta$, respectively.) This is an important technique frequently used in this paper.

Let $\mathfrak{s}$ be a subalgebra of $\mathfrak{g}$, S a subgroup of G, $\mathfrak{t}$ an abelian subspace of $\mathfrak{p}$ and $\mathfrak{t}^*$ the space of real linear forms on $\mathfrak{t}$. Then we put $\mathfrak{s}(\mathfrak{t};\alpha)=\{X\in\mathfrak{s}\mid[Y, X]=\alpha(Y)X$ for all $Y\in\mathfrak{t}\}$ for any $\alpha\in\mathfrak{t}^*$ and put $\Sigma(\mathfrak{s};\mathfrak{t})=\{\beta\in\mathfrak{t}^*-\{0\}\mid\mathfrak{s}(\mathfrak{t};\beta)\neq\{0\}\}$. Let $Z_S(\mathfrak{t})$ (resp. $N_S(\mathfrak{t})$) denote the centralizer (normalizer) of $\mathfrak{t}$ in S and put $W_S(\mathfrak{t})=N_S(\mathfrak{t})/Z_S(\mathfrak{t})$. Write $\mathfrak{z}_\mathfrak{s}(\mathfrak{t})=\mathfrak{s}(\mathfrak{t};0)$.

When $\mathfrak{t}$ is maximal abelian in $\mathfrak{p}$, it is wellknown that $\Sigma(\mathfrak{t})=\Sigma(\mathfrak{g};\mathfrak{t})$ satisfies the axioms of a root system and that $W(\mathfrak{t})=W_K(\mathfrak{t})$ is the Weyl group of $\Sigma(\mathfrak{t})$. In this case we choose an element $w_\alpha\in N_K(\mathfrak{t})$ for each $\alpha\in\Sigma(\mathfrak{t})$ so that the restriction of $\operatorname{Ad}(w_\alpha)$ to $\mathfrak{t}$ is the reflection with respect to α. (All the statements in this paper are independent of the choice of w_α.)

When the real rank of G is one, we can describe the closure relations which we want to study in this paper as follows.

Lemma 2. *Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Suppose that* $\dim\mathfrak{a}=1$ *and that $\mathfrak{a}\subset\mathfrak{h}$. Let α be a reduced root in $\Sigma=\Sigma(\mathfrak{g};\mathfrak{a})$ and put $P=Z_G(\mathfrak{a})\exp\mathfrak{n}$ with $\mathfrak{n}=\mathfrak{g}(\mathfrak{a};\alpha)+\mathfrak{g}(\mathfrak{a};2\alpha)$. Suppose that $\mathfrak{g}(\mathfrak{a};\alpha)\cap\mathfrak{q}\neq\{0\}$ and fix an element c_α of K defined by $c_\alpha=\exp(\pi/2)(X+\theta X)$ with $X\in\mathfrak{g}(\mathfrak{a};\alpha)\cap\mathfrak{q}$ satisfying $2\langle\alpha,\alpha\rangle B(X,\theta X)=-1$. Then we have the followings.*

(i) *$G=HP\cup Hw_\alpha P\cup Hc_\alpha P\cup Hc_\alpha^{-1}P$.*

(ii) *The double cosets HP and $Hw_\alpha P$ are closed in G and the double cosets $Hc_\alpha P$ and $Hc_\alpha^{-1}P$ are open in G.*

(iii) $\dim HP = \dim Hw_\alpha P = \dim G - \dim(\mathfrak{n} \cap \mathfrak{q})$.
(iv) $H_0 P = H_0 w_\alpha P$ *if and only if* $\mathfrak{n} \cap \mathfrak{h} \neq \{0\}$.
(v) $H_0 c_\alpha P = H_0 c_\alpha^{-1} P$ *if and only if* $\dim(\mathfrak{n} \cap \mathfrak{q}) \geq 2$.
(vi) $(Hc_\alpha P)^{cl} = Hc_\alpha P \cup HP \cup Hw_\alpha P$ *and* $(Hc_\alpha^{-1} P)^{cl} = Hc_\alpha^{-1} P \cup HP \cup Hw_\alpha P$.

Proof. (i) Since $\{\phi, \{X\}\}$ is a complete set of representatives of $W(\mathfrak{a})$-conjugacy classes of $\mathfrak{q}$-orthogonal systems of Σ, the assertion follows from [1] Theorem 3.

(ii) follows from Proposition 1 and Proposition 2 in [1].

(iii) follows from Lemma 7 in Section 5. (It is easy to give a direct proof of (iii).)

(iv) If $\mathfrak{n} \subset \mathfrak{q}$, then $\bar{\mathfrak{n}} = \mathfrak{g}(\mathfrak{a}; -\alpha) + g(\mathfrak{a}; -2\alpha)$ is also contained in $\mathfrak{q}$ since $\theta\mathfrak{q} = \mathfrak{q}$. Hence $\mathfrak{h} \subset \mathfrak{l}$, $N_{K \cap H_0}(\mathfrak{a}) = Z_{K \cap H_0}(\mathfrak{a})$ and therefore $H_0 P \cap H_0 w_\alpha P = \phi$ by [1] Theorem 1. Conversely suppose that $\mathfrak{n} \cap \mathfrak{h} \neq \{0\}$. Since α or 2α is contained in $\Sigma(\mathfrak{h}; \mathfrak{a})$ and since $W_{K \cap H_0}(\mathfrak{a})$ is the Weyl group of $\Sigma(\mathfrak{h}; \mathfrak{a})$, we have $w_\alpha Z_K(\mathfrak{a}) \cap H_0 \neq \phi$. Hence $H_0 P = H_0 w_\alpha P$.

(v) Suppose that $\dim(\mathfrak{n} \cap \mathfrak{q}) = 1$. Then $\dim(\mathfrak{p} \cap \mathfrak{q}) = 1$ since $\mathfrak{p} \subset \bar{\mathfrak{n}} + \mathfrak{a} + \mathfrak{n}$ and since $\mathfrak{a} \subset \mathfrak{h}$. Hence $\mathfrak{a}' = \mathrm{Ad}(c_\alpha)\mathfrak{a} = \mathfrak{p} \cap \mathfrak{q}$ and the adjoint action of $K \cap H_0 = (K \cap H)_0$ on $\mathfrak{a}'$ is trivial. Therefore $N_{K \cap H_0}(\mathfrak{a}') = Z_{K \cap H_0}(\mathfrak{a}')$ and $H_0 c_\alpha P \cap H_0 c_\alpha^{-1} P = \emptyset$ by [1] Theorem 1. Suppose that $\dim(\mathfrak{n} \cap \mathfrak{q}) \geq 2$. Then $\dim HP = \dim Hw_\alpha P \leq \dim G - 2$ by (iii). Hence $G - HP - Hw_\alpha P$ is connected, and therefore $H_0 c_\alpha P = H_0 c_\alpha^{-1} P$.

(vi) If $HP = Hw_\alpha P$ or $Hc_\alpha P = Hc_\alpha^{-1} P$, then the assertions are trivial. So we may assume that $\dim \mathfrak{n} = 1$ by (iv) and (v). Then G/P is diffeomorphic to a circle, the two closed H_0-orbits $H_0 P$ and $H_0 w_\alpha P$ are distinct points on the circle, and the two open H_0-orbits $H_0 c_\alpha P$ and $H_0 c_\alpha^{-1} P$ are the remaining open arcs. Thus the assertions are clear. Q.E.D.

Lemma 2′. *Retain the assumptions and notations in Lemma 2. Then we have the followings.*

(i) $G = H^a P \cup H^a w_\alpha P \cup H^a c_\alpha P \cup H^a c_\alpha^{-1} P$.
(ii) *The double cosets* $H^a P$ *and* $H^a w_\alpha P$ *are open in* G *and the double cosets* $H^a c_\alpha P$ *and* $H^a c_\alpha^{-1} P$ *are closed in* G.
(iii) $\dim H^a c_\alpha P = \dim H^a c_\alpha^{-1} P = \dim G - \dim(\mathfrak{n} \cap \mathfrak{h}) - 1$.
(iv) $H_0^a = H_0^a w_\alpha P$ *if and only if* $\mathfrak{n} \cap \mathfrak{h} \neq \{0\}$.
(v) $H_0^a c_\alpha P = H_0^a c_\alpha^{-1} P$ *if and only if* $\dim(\mathfrak{n} \cap \mathfrak{q}) \geq 2$.
(vi) $(H^a P)^{cl} = H^a P \cup H^a c_\alpha P \cup H^a c_\alpha^{-1} P$ *and* $(H^a w_\alpha P)^{cl} = H^a w_\alpha P \cup H^a c_\alpha P \cup H^a c_\alpha^{-1} P$.

Proof. The assertions (i), (iv) and (v) follow from Corollary 2 of [1] Theorem 1. (ii) follows from Corollary of [1] Section 3. (vi) is proved as in the proof of Lemma 2.

(iii) is proved as follows. Since $\mathfrak{p}\subset\bar{\mathfrak{n}}+\mathfrak{a}+\mathfrak{n}$ and since $\mathfrak{a}\subset\mathfrak{p}\cap\mathfrak{h}$, we have $\dim(\mathfrak{n}\cap\mathfrak{h})=\dim(\mathfrak{p}\cap\mathfrak{h})-1$. On the other hand since $\mathfrak{p}\subset \operatorname{Ad}(c_\alpha)\bar{\mathfrak{n}}+\operatorname{Ad}(c_\alpha)\mathfrak{a}+\operatorname{Ad}(c_\alpha)\mathfrak{n}$ and since $\operatorname{Ad}(c_\alpha)\mathfrak{a}\subset\mathfrak{p}\cap\mathfrak{q}=\mathfrak{p}\cap\mathfrak{h}^\alpha$, we have $\dim(\operatorname{Ad}(c_\alpha)\mathfrak{n}\cap\mathfrak{q}^\alpha)=\dim(p\cap\mathfrak{q}^\alpha)=\dim(\mathfrak{p}\cap\mathfrak{h})$. Hence it follows from Lemma 7 in Section 5 that $\dim Hc_\alpha P=\dim Hc_\alpha^{-1}P=\dim G-\dim(\operatorname{Ad}(c_\alpha)\mathfrak{n}\cap\mathfrak{q}^\alpha)=\dim G-\dim(\mathfrak{n}\cap\mathfrak{h})-1$. Q.E.D.

§ 3. Lemmas for the main theorem

We use the following notations throughout this section. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ such that $\sigma\mathfrak{a}=\mathfrak{a}$, Σ^+ a positive system of the root system $\Sigma=\Sigma(\mathfrak{a})$ and P the minimal parabolic subgroup of G defined by

$$P=Z_G(\mathfrak{a})\exp\mathfrak{n}$$

where $\mathfrak{n}=\sum_{\beta\in\Sigma^+}\mathfrak{g}(\mathfrak{a};\beta)$. Let Ψ denote the set of all the simple roots in Σ^+. Let α be a root in Ψ and put $\mathfrak{a}^\alpha=\{Y\in\mathfrak{a}\,|\,\alpha(Y)=0\}$, $L_\alpha=Z_G(\mathfrak{a}^\alpha)$, $\mathfrak{l}_\alpha=\mathfrak{z}_\mathfrak{g}(\mathfrak{a}^\alpha)$, $\mathfrak{n}_\alpha=\sum_{\beta\in\Sigma^+-\{\alpha,2\alpha\}}\mathfrak{g}(\mathfrak{a};\beta)$, $P_\alpha=L_\alpha\exp\mathfrak{n}_\alpha$, $\mathfrak{P}_\alpha=\mathfrak{l}_\alpha+\mathfrak{n}_\alpha$ and $\mathfrak{n}(\alpha)=\mathfrak{g}(\mathfrak{a};\alpha)+\mathfrak{g}(\mathfrak{a};2\alpha)$. Then P_α is a parabolic subgroup of G containing P. Let $\mathfrak{l}_\alpha^s$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{n}(\alpha)+\theta\mathfrak{n}(\alpha)$ and L_α^s the analytic subgroup of G for $\mathfrak{l}_\alpha^s$. For a subset S of G, write $S^{op}=S^{op}(H^\alpha\backslash G/P)$.

First we have the following lemma which is a generalization of [4] Lemma 5.1.

Lemma 3. *There are six cases (depending on the choice of* $\mathfrak{a}$, Σ^+ *and* α*) for the decomposition of the set* HP_α *into H-P double cosets as follows.*

(A) *If* $\sigma\alpha\neq\pm\alpha$ *and* $\sigma\alpha\notin\Sigma^+$, *then* $HP_\alpha=HP\cup Hw_\alpha P$, $\dim Hw_\alpha P=\dim HP-\dim\mathfrak{n}(\alpha)$ *and* $Hw_\alpha P\subset(HP)^{cl}$.

(B) *If* $\sigma\alpha\neq\pm\alpha$ *and* $\sigma\alpha\in\Sigma^+$, *then* $HP_\alpha=HP\cup Hw_\alpha P$, $\dim Hw_\alpha P=\dim HP+\dim\mathfrak{n}(\alpha)$ *and* $HP\subset(Hw_\alpha P)^{cl}$.

(C) *If* $\sigma\alpha=\alpha$ *and* $\mathfrak{g}(\mathfrak{a};\alpha)\cap\mathfrak{q}=\{0\}$, *then* $HP_\alpha=HP$.

(D) *The case when* $\sigma\alpha=\alpha$ *and* $\mathfrak{g}(\mathfrak{a};\alpha)\cap\mathfrak{q}\neq\{0\}$. *Define an element* $c_\alpha\in L_\alpha^s$ *by* $c_\alpha=\exp(\pi/2)(X+\theta X)$ *with an* $X\in\mathfrak{g}(\mathfrak{a};\alpha)\cap\mathfrak{q}$ *satisfying* $2\langle\alpha,\alpha\rangle\cdot B(X,\theta X)=-1$. *Then* $\operatorname{Ad}(c_\alpha)\mathfrak{a}=\operatorname{Ad}(c_\alpha^{-1})\mathfrak{a}$ *is* σ*-stable,*

$$HP_\alpha=HP\cup Hw_\alpha P\cup Hc_\alpha P\cup Hc_\alpha^{-1}P,$$

$$\dim Hc_\alpha P=\dim Hc_\alpha^{-1}P=\dim HP+\dim(\mathfrak{n}(\alpha)\cap\mathfrak{q})$$
$$=\dim Hw_\alpha P+\dim(\mathfrak{n}(\alpha)\cap\mathfrak{q}),$$

$$(Hc_\alpha P)^{cl}\supset HP\cup Hw_\alpha P,\qquad (Hc_\alpha^{-1}P)^{cl}\supset HP\cup Hw_\alpha P,$$

$$HP=Hw_\alpha P\qquad \textit{if } \mathfrak{n}(\alpha)\cap\mathfrak{h}\neq\{0\},$$

and $\qquad Hc_\alpha P = Hc_\alpha^{-1}P \qquad$ *if* $\dim(\mathfrak{n}(\alpha)\cap\mathfrak{q})\geq 2$.

(E) *If* $\sigma\alpha=-\alpha$ *and* $\mathfrak{g}(\mathfrak{a};\alpha)\cap\mathfrak{q}^a=\{0\}$, *then* $HP_\alpha=HP$.

(F) *The case when* $\sigma\alpha=-\alpha$ *and* $\mathfrak{g}(\mathfrak{a};\alpha)\cap\mathfrak{q}^a\neq\{0\}$. *Define an element* $c_\alpha\in L_\alpha^s$ *by* $c_\alpha=\exp(\pi/2)(X+\theta X)$ *with an* $X\in\mathfrak{g}(\mathfrak{a};\alpha)\cap\mathfrak{q}^a$ *satisfying* $2\langle\alpha,\alpha\rangle\cdot B(X,\theta X)=-1$. *Then* $\mathrm{Ad}(c_\alpha)\mathfrak{a}=\mathrm{Ad}(c_\alpha^{-1})\mathfrak{a}$ *is* σ*-stable*,

$$HP_\alpha=HP\cup Hw_\alpha P\cup Hc_\alpha P\cup Hc_\alpha^{-1}P,$$

$$\dim Hc_\alpha P=\dim Hc_\alpha^{-1}P=\dim HP-\dim(\mathfrak{n}(\alpha)\cap\mathfrak{h}^a)-1$$

$$=\dim Hw_\alpha P-\dim(\mathfrak{n}(\alpha)\cap\mathfrak{h}^a)-1,$$

$$(HP)^{cl}\supset Hc_\alpha P\cup Hc_\alpha^{-1}P,\qquad (Hw_\alpha P)^{cl}\supset Hc_\alpha P\cup Hc_\alpha^{-1}P,$$

$$HP=Hw_\alpha P\qquad \textit{if}\ \mathfrak{n}(\alpha)\cap\mathfrak{h}^a\neq\{0\},$$

and $\qquad Hc_\alpha P=Hc_\alpha^{-1}P \qquad$ *if* $\dim(\mathfrak{n}(\alpha)\cap\mathfrak{q}^a)\geq 2$.

Proof. Since the statements are independent of the choice of w_α in $N_K(\mathfrak{a})$, we may assume that $w_\alpha\in L_\alpha^s$. Let p be the projection of P_α onto L_α with respect to the Langlands decomposition $P_\alpha=L_\alpha\exp\mathfrak{n}_\alpha$. Then we have natural bijections

$$(3.1)\qquad H\backslash HP_\alpha/P\xleftarrow{\sim}P_\alpha\cap H\backslash P_\alpha/P\xrightarrow[p]{\sim}J\backslash L_\alpha/L_\alpha\cap P$$

where $J=p(P_\alpha\cap H)$. Since $(L_\alpha)_0=L_\alpha^sZ_{(L_\alpha)_0}(\mathfrak{a})$ and since $Z_{(L_\alpha)_0}(\mathfrak{a})\subset P$, we have $L_\alpha^s/L_\alpha^s\cap P\simeq(L_\alpha)_0/(L_\alpha)_0\cap P$. Since $L_\alpha\cap P$ intersects with every connected component of L_α, we have $(L_\alpha)/(L_\alpha)_0\cap P\simeq L_\alpha/L_\alpha\cap P$. Hence we have a natural surjection

$$(3.2)\qquad L_\alpha^s\cap J\backslash L_\alpha^s/L_\alpha^s\cap P\longrightarrow J\backslash L_\alpha/L_\alpha\cap P.$$

Let $\mathfrak{j}$ be the Lie algebra of J.

(A) Let X be an element of $\theta\mathfrak{n}(\alpha)$. Then $X+\sigma X\in\mathfrak{P}_\alpha\cap\mathfrak{h}$ since $-\sigma\alpha\in\Sigma^+$. Hence $X=p(X+\sigma X)\subset\mathfrak{j}$ since $-\sigma\alpha\in\Sigma^+-\{\alpha,2\alpha\}$. Thus we have

$$\theta\mathfrak{n}(\alpha)\subset\mathfrak{l}_\alpha^s\cap\mathfrak{j}.$$

By the Bruhat decomposition of L_α^s, we have

$$L_\alpha^s=D(1)\cup D(w_\alpha)\quad\text{and}\quad D(w)\subset D(1)^{cl}$$

where $D(x)=(L_\alpha^s\cap J)x(L_\alpha^s\cap P)$ for $x\in L_\alpha^s$. Hence by (3.1) and (3.2),

$$HP_\alpha=HP\cup Hw_\alpha P\quad\text{and}\quad Hw_\alpha P\subset(HP)^{cl}.$$

Since $\sigma\alpha\neq\pm\alpha$, we have $w_\alpha\notin W_{K\cap H}(\mathfrak{a})$ and therefore $HP\neq Hw_\alpha P$ by [1]

Theorem 1. Hence $D(1)\neq D(w_\alpha)$ and it follows from the naturality of (3.1) and (3.2) that

$$\begin{aligned}\dim HP-\dim Hw_\alpha P&=\dim D(1)-\dim D(w_\alpha)\\&=\dim \mathfrak{n}(\alpha).\end{aligned}$$

(B) By a similar argument as in (A), we have

$$\mathfrak{n}(\alpha)\subset \mathfrak{l}_\alpha^s\cap \mathfrak{j}.$$

By the Bruhat decomposition of L_α^s, we have

$$L_\alpha^s=D(1)\cup D(w_\alpha)\quad\text{and}\quad D(1)\subset D(w_\alpha)^{cl}$$

where $D(x)=(L_\alpha^s\cap J)x(L_\alpha^s\cap P)$ for $x\in L_\alpha^s$. Hence by (3.1) and (3.2),

$$HP_\alpha=HP\cup Hw_\alpha P\quad\text{and}\quad HP\subset(Hw_\alpha P)^{cl}.$$

Since $HP\neq Hw_\alpha P$ and $D(1)\neq D(w_\alpha)$ as in (A),

$$\begin{aligned}\dim Hw_\alpha P-\dim HP&=\dim D(w_\alpha)-\dim D(1)\\&=\dim \mathfrak{n}(\alpha).\end{aligned}$$

(C) Since $\mathfrak{l}_\alpha^s$ is generated by $\mathfrak{g}(\mathfrak{a};\alpha)+\mathfrak{g}(\mathfrak{a};-\alpha)$, $\mathfrak{l}_\alpha^s$ is contained in $\mathfrak{h}$. Hence $HP_\alpha=HP$ by (3.1) and (3.2).

(D) Since $L_\alpha^s\cap H\subset L_\alpha^s\cap J$, we have a natural surjection

$$L_\alpha^s\cap H\backslash L_\alpha^s/L_\alpha^s\cap P\longrightarrow J\backslash L_\alpha/L_\alpha\cap P \tag{3.3}$$

by (3.2). Since $\dim(\mathfrak{l}_\alpha^s\cap\mathfrak{a})=1$ and $\mathfrak{l}_\alpha^s\cap\mathfrak{a}\subset\mathfrak{h}$, it follows from Lemma 2 (i) and (vi) that $L_\alpha^s=D(1)\cup D(w_\alpha)\cup D(c_\alpha)\cup D(c_\alpha^{-1})$, $D(c_\alpha)^{cl}=D(c_\alpha)\cup D(1)\cup D(w_\alpha)$ and $D(c_\alpha^{-1})^{cl}=D(c_\alpha^{-1})\cup D(1)\cup D(w_\alpha)$. Here $D(x)=(L_\alpha^s\cap H)x(L_\alpha^s\cap P)$ for $x\in L_\alpha^s$. Hence by (3.1) and (3.3),

$$\begin{aligned}&HP_\alpha=HP\cup Hw_\alpha P\cup Hc_\alpha P\cup Hc_\alpha^{-1}P,\quad (Hc_\alpha P)^{cl}\supset HP\cup Hw_\alpha P\\&\text{and}\quad (Hc_\alpha^{-1}P)^{cl}\supset HP\cup Hw_\alpha P.\end{aligned}$$

Since $\mathrm{Ad}(c_\alpha)\mathfrak{a}$ is not $K\cap H$-conjugate to $\mathfrak{a}$, $(HP\cup Hw_\alpha P)\cap(Hc_\alpha P\cup Hc_\alpha^{-1}P)=\varnothing$ by [1] Theorem 1. Thus we have

$$\begin{aligned}\dim Hc_\alpha P=\dim Hc_\alpha^{-1}P&=\dim HP+\dim(\mathfrak{n}(\alpha)\cap\mathfrak{q})\\&=\dim Hw_\alpha P+\dim(\mathfrak{n}(\alpha)\cap\mathfrak{q})\end{aligned}$$

since $\dim D(1)=\dim D(w_\alpha)=\dim D(c_\alpha)-\dim(\mathfrak{n}(\alpha)\cap\mathfrak{q})=\dim D(c_\alpha^{-1})-\dim(\mathfrak{n}(\alpha)\cap\mathfrak{q})$ by Lemma 2 (iii). The remaining assertions are clear from

Lemma 2 (iv) and (v).

(E) Since $\mathfrak{l}_\alpha^s$ is generated by $\mathfrak{g}(\mathfrak{a};\alpha)+\mathfrak{g}(\mathfrak{a};-\alpha)$, $\mathfrak{l}_\alpha^s$ is contained in $\mathfrak{h}^\alpha$. Hence $\mathfrak{l}_\alpha^s\cap\mathfrak{k}\subset\mathfrak{l}_\alpha^s\cap\mathfrak{h}$ and $L_\alpha^s=(L_\alpha^s\cap H)(L_\alpha^s\cap P)$ by the Iwasawa decomposition of L_α^s. Therefore $HP_\alpha=HP$ by (3.1) and (3.2).

(F) Clearly (3.3) is also valid in this case. Note that $\dim(\mathfrak{l}_\alpha^s\cap\mathfrak{a})=1$ and that $\mathfrak{l}_\alpha^s\cap\mathfrak{a}\subset\mathfrak{q}$. Consider L_α^s, $L_\alpha^s\cap H$ and σ as G, H^α and $\sigma\theta$ in Lemma 2′, respectively. Then we have $L_\alpha^s=D(1)\cup D(w_\alpha)\cup D(c_\alpha)\cup D(c_\alpha^{-1})$, $D(1)^{cl}=D(1)\cup D(c_\alpha)\cup D(c_\alpha^{-1})$ and $D(w_\alpha)^{cl}=D(w_\alpha)\cup D(c_\alpha)\cup D(c_\alpha^{-1})$ by Lemma 2′ (i) and (vi). Here $D(x)=(L_\alpha^s\cap H)x(L_\alpha^s\cap P)$ for $x\in L_\alpha^s$ and c_α is defined in the statement of (F). Hence

$$HP_\alpha=HP\cup Hw_\alpha P\cup Hc_\alpha P\cup Hc_\alpha^{-1}P,\quad (HP)^{cl}\supset Hc_\alpha P\cup Hc_\alpha^{-1}P$$
$$\text{and}\quad (Hw_\alpha P)^{cl}\supset Hc_\alpha P\cup Hc_\alpha^{-1}P$$

by (3.1) and (3.3). We have $(HP\cup Hw_\alpha P)\cap(Hc_\alpha P\cup Hc_\alpha^{-1}P)=\varnothing$ by the same reason as in (D). Hence

$$\begin{aligned}\dim Hc_\alpha P=\dim Hc_\alpha^{-1}P&=\dim HP-\dim(\mathfrak{n}(\alpha)\cap\mathfrak{h}^\alpha)-1\\&=\dim Hw_\alpha P-\dim(\mathfrak{n}(\alpha)\cap\mathfrak{h}^\alpha)-1\end{aligned}$$

since $\dim D(c_\alpha)=\dim D(c_\alpha^{-1})=\dim D(1)-\dim(\mathfrak{n}(\mathfrak{a})\cap\mathfrak{h}^\alpha)-1=\dim D(w_\alpha)-\dim(\mathfrak{n}(\mathfrak{a})\cap\mathfrak{h}^\alpha)-1$ by Lemma 2′ (iii). The remaining assertions are clear from Lemma 2′ (iv) and (v). Q.E.D.

Lemma 4. *The following three conditions on Σ^+ are equivalent.*

(i) *If $\alpha\in\Sigma^+$ and $\sigma\alpha\neq-\alpha$, then $\sigma\alpha\in\Sigma^+$.*

(ii) *If $\alpha\in\Psi$ and $\sigma\alpha\neq-\alpha$, then $\sigma\alpha\in\Sigma^+$.*

(iii) *There exists a $Y\in\mathfrak{a}\cap\mathfrak{h}$ such that $\alpha(Y)>0$ for all $\alpha\in\Sigma^+$ satisfying $\sigma\alpha\neq-\alpha$.*

Proof. (i)⇒(ii) is trivial.

(ii)⇒(i). Every root β in Σ^+ can be written as $\beta=\sum_{\alpha\in\Psi}n_\alpha\alpha$ with some nonnegative integers n_α. Put $\Psi_-=\{\alpha\in\Psi\mid\sigma\alpha=-\alpha\}$ and $\Psi_0=\{\alpha\in\Psi\mid\sigma\alpha\neq-\alpha\}$. Then we have

$$\sigma\beta=-\sum_{\alpha\in\Psi_-}n_\alpha\alpha+\sum_{\alpha\in\Psi_0}n_\alpha\sigma\alpha. \tag{3.4}$$

It follows from the assumption that

$$\sum_{\alpha\in\Psi_0}n_\alpha\sigma\alpha\in\sum_{\alpha\in\Psi}\boldsymbol{Z}_+\alpha \tag{3.5}$$

where $\boldsymbol{Z}_+=\{n\in\boldsymbol{Z}\mid n\geq0\}$. Suppose that $\sigma\beta\neq-\beta$. Then

$$\sigma\beta\notin\sum_{\alpha\in\Psi_-}\boldsymbol{Z}\alpha. \tag{3.6}$$

Write $\sigma\beta=\sum_{\alpha\in\Psi} n'_\alpha \alpha$ ($n'_\alpha \in \boldsymbol{Z}$). Then it follows from (3.4), (3.5) and (3.6) that $n'_\alpha>0$ for some $\alpha \in \Psi_0$. If $\sigma\beta$ is a negative root then $n'_\alpha \leq 0$ for all $\alpha \in \Psi$. Hence $\sigma\beta \in \Sigma^+$.

(i)$\Rightarrow$(iii). Let X be an element of $\mathfrak{a}$ such that $\alpha(X)>0$ for all $\alpha \in \Sigma^+$. Then $Y=X+\sigma X$ is a desired element.

(iii)$\Rightarrow$(i). If $\alpha \in \Sigma^+$ and $\sigma\alpha \neq -\alpha$, then $\sigma\alpha(Y)=\alpha(Y)>0$ by (iii). Hence $-\sigma\alpha$ is not contained in Σ^+ by (iii) and therefore $\sigma\alpha \in \Sigma^+$.

Q.E.D.

Definition. A positive system Σ^+ of the root system $\Sigma=\Sigma(\mathfrak{a})$ is said to be σ-compatible if one of the equivalent three conditions in Lemma 4 is satisfied.

Suppose that Σ^+ is not σ-compatible. Then by the above definition, there exists a simple root α of Σ^+ such that $\sigma\alpha \notin \Sigma^+$ and that $\sigma\alpha \neq -\alpha$.

Lemma 5. *Let α be a simple root of Σ^+ such that $\sigma\alpha \notin \Sigma^+$ and that $\sigma\alpha \neq -\alpha$. Then* (i) $(HP)^{cl}=(Hw_\alpha P)^{cl}L_\alpha$ *and* (ii) $(H^a P)^{op}=(H^a w_\alpha P)^{op}L_\alpha$.

Proof. (i) By Lemma 3 (A), $HP_\alpha=HP \cup Hw_\alpha P$ and $Hw_\alpha P \subset (HP)^{cl}$. Hence $(HP)^{cl}=(HP_\alpha)^{cl}$. Since $HP_\alpha \subset (Hw_\alpha P)^{cl}P_\alpha \subset (HP_\alpha)^{cl}$, we have only to prove that $(Hw_\alpha P)^{cl}P_\alpha=(Hw_\alpha P)^{cl}L_\alpha$ is closed in G. Since G/P is compact, $(Hw_\alpha P)^{cl}/P$ is a compact subset of G/P. Consider the natural map of G/P onto G/P_α. Then the image $(Hw_\alpha P)^{cl}P_\alpha/P_\alpha$ of $(Hw_\alpha P)^{cl}/P$ by this map is compact. Hence $(Hw_\alpha P)^{cl}P_\alpha$ is closed in G.

(ii) By Lemma 3 (B), $H^a P_\alpha=H^a P \cup H^a w_\alpha P$ and $H^a P \subset (H^a w_\alpha P)^{cl}$. Hence $H^a w_\alpha P \subset (H^a P)^{op}$ and so $(H^a P)^{op}=(H^a P_\alpha)^{op}$. Since $H^a P_\alpha \subset (H^a w_\alpha P)^{op}L_\alpha \subset (H^a P_\alpha)^{op}$ and since $(H^a w_\alpha P)^{op}L_\alpha$ is open in G, we have $(H^a P)^{op}=(H^a P_\alpha)^{op}=(H^a w_\alpha P)^{op}L_\alpha$. Q.E.D.

§ 4. Proof of Theorem

In this section we prove Theorem in Section 1.

Proof. (i) Put $\beta_i=w^{(i-1)}\alpha_i$ for $i=1, \cdots, n$. Then we will first prove that

$$\sigma\beta_i \neq \pm\beta_i \quad \text{and} \quad \sigma\beta_i \notin w^{(i-1)}\Sigma^+. \tag{4.1}$$

Put $\Sigma_0^+=\{\alpha \in \Sigma^+ \mid \frac{1}{2}\alpha \notin \Sigma\}$ (the set of reduced roots in Σ^+). Then $w^{(i)}\Sigma_0^+ =(\Sigma_0^+-\{\beta_1, \cdots, \beta_i\}) \cup \{-\beta_1, \cdots, -\beta_i\}$ for $i=1, \cdots, n$. We also have $\beta_i(Y_0)<0$ for $i=1, \cdots, n$ by the definition of w. Hence by the choice of Y_0, we have $\sigma\beta_i \notin \Sigma^+$ (which implies $\sigma\beta_i \neq \beta_i$). On the other hand, we have $\sigma\beta_i \notin \{-\beta_1, \cdots, -\beta_{i-1}\}$ since $\beta_i(Y_0)=(\sigma\beta_i)(Y_0)<0$ for any $i=1, \cdots,$

n. Thus we have proved that $\sigma\beta_i \notin w^{(i-1)}\Sigma_0^+$ which clearly implies that $\sigma\beta_i \notin w^{(i-1)}\Sigma^+$. The remaining assertion $\sigma\beta_i \neq -\beta_i$ is clear from $\sigma Y_0 = Y_0$ and $\beta_i(Y_0) < 0$.

Put $P^{(i)} = w^{(i)}P(w^{(i)})^{-1}$ and define L_{β_i} as in Section 3 for $i=1, \cdots, n$. (For any $\beta \in \Sigma$, put $\mathfrak{a}^\beta = \{Y \in \mathfrak{a} \mid \beta(Y)=0\}$ and $L_\beta = Z_G(\mathfrak{a}^\beta)$.) Then by (4.1) and Lemma 5 (i), we have

$$(HP^{(i-1)})^{cl} = (Hw_{\beta_i}P^{(i-1)})^{cl}L_{\beta_i}$$

and therefore $(Hw^{(i-1)}P)^{cl} = (Hw^{(i)}P)^{cl}L_{\alpha_i}$ for $i=1, \cdots, n$ since $L_{\beta_i} = w^{(i-1)}L_{\alpha_i}(w^{(i-1)})^{-1}$.

The latter formula can be proved by Lemma 5 (ii) in a similar way.

(ii) follows directly from (i) because $(Pw^{-1}P)^{cl} = PL_{\alpha_n} \cdots L_{\alpha_1}$.

(iii) Since $L_1 = Z_G(Y_0)$ by the choice of Y_0, we can define a parabolic subgroup P_1 of G containing $P^{(n)}$ by $P_1 = L_1 \exp \mathfrak{n}_1$, $\mathfrak{n}_1 = \sum_{\gamma \in \Sigma, \gamma(Y_0) > 0} \mathfrak{g}(\mathfrak{a}; \gamma)$. Since L_1 and $\mathfrak{n}_1$ are σ-stable, it is easy to show that $P_1 \cap H = (L_1 \cap H) \cdot \exp(\mathfrak{n}_1 \cap \mathfrak{h})$. Since $P_1 \cap H_0$ is the parabolic subgroup of H_0 defined by $Y_0 \in \mathfrak{a} \cap \mathfrak{h}$, $H_0/P_1 \cap H_0$ is compact. Hence H_0P_1 is closed in G and so HP_1 is also closed in G by Lemma 1.

Let p be the projection of P_1 onto L_1 with respect to the Langlands decomposition $P_1 = L_1 \exp \mathfrak{n}_1$. Considering the natural bijections

$$H\backslash HP_1/P^{(n)} \xleftarrow{\sim} P_1 \cap H\backslash P_1/P^{(n)} \xrightarrow[p]{\sim} L_1 \cap H\backslash L_1/L_1 \cap P^{(n)},$$

we have

$$(HP^{(n)})^{cl} = H((L_1 \cap H)(L_1 \cap P^{(n)}))^{cl}P^{(n)} \tag{4.2}$$

since HP_1 is closed in G.

Let Z be the center of $(L_1)_0$. Since $(L_1)_0 = LZ$ and since $Z \subset P^{(n)}$, we have $L/L \cap P^{(n)} \simeq (L_1)_0/(L_1)_0 \cap P^{(n)}$. Since $L_1 \cap P^{(n)}$ intersects with every connected component of L_1, we have $(L_1)_0/(L_1)_0 \cap P^{(n)} \simeq L_1/L_1 \cap P^{(n)}$. So we have natural bijections

$$L/L \cap P^{(n)} \xrightarrow{\sim} L_1/L_1 \cap P^{(n)}$$

and

$$(L \cap H)_0\backslash L/L \cap P^{(n)} \xrightarrow{\sim} (L_1 \cap H)_0\backslash L_1/L_1 \cap P^{(n)}.$$

since $(L_1 \cap H)_0 = (L \cap H)_0(Z \cap H)_0$ and since $Z \subset P^{(n)}$. Hence we have

$$((L_1 \cap H)(L_1 \cap P^{(n)}))^{cl} \tag{4.3}$$

$$=(L_1\cap H)((L_1\cap H)_0(L_1\cap P^{(n)}))^{cl}$$
$$=(L_1\cap H)((L\cap H)_0(L\cap P^{(n)}))^{cl}(L_1\cap P^{(n)})$$
$$=(L_1\cap H)((L\cap H)(L\cap P^{(n)}))^{cl}(L_1\cap P^{(n)})$$

by Lemma 1.

From (4.2) and (4.3) we get the formula for $(HwP)^{cl}$. (Note that $L\cap P^{(n)}=L\cap P$ since $w\Sigma^+\cap\Sigma(\mathfrak{l};\mathfrak{a})=\Sigma^+\cap\Sigma(\mathfrak{l};\mathfrak{a})$.)

The formula for $(H^a wP)^{op}$ is proved as follows. First we have $P_1\cap H^a=L_1\cap H^a$ since $P_1\cap\sigma\theta P_1=L_1$. Next we will prove that $H^a P_1$ is open in G. We have only to prove that $\mathfrak{h}^a+\mathfrak{P}_1=\mathfrak{g}$. ($\mathfrak{P}_1$ is the the Lie algebra of P_1.) Let γ be a root in Σ such that $\gamma(Y_0)<0$ and X an element of $\mathfrak{g}(\mathfrak{a};\gamma)$. Then

$$X=(X+\sigma\theta X)-\sigma\theta X\in\mathfrak{h}^a+\mathfrak{g}(\mathfrak{a};\sigma\theta\gamma)\subset\mathfrak{h}^a+\mathfrak{P}_1$$

since $(\sigma\theta\gamma)(Y_0)=-\gamma(Y_0)>0$. Since $\mathfrak{g}=\mathfrak{P}_1+\sum_{\gamma\in\Sigma,\gamma(Y_0)<0}\mathfrak{g}(\mathfrak{a};\gamma)$, we have $\mathfrak{g}=\mathfrak{h}^a+\mathfrak{P}_1$. Considering the natural bijections

$$H^a\backslash H^aP_1/P^{(n)}\xleftarrow{\sim}P_1\cap H^a\backslash P_1/P^{(n)}\xrightarrow[p]{\sim}L_1\cap H^a\backslash L_1/L_1\cap P^{(n)},$$

we have

$$(4.4)\qquad (H^aP^{(n)})^{op}(H^a\backslash G/P^{(n)})$$
$$=H^a(((L_1\cap H^a)(L_1\cap P^{(n)}))^{op}(L_1\cap H^a\backslash L_1/L_1\cap P^{(n)}))P^{(n)}$$

since H^aP_1 is open in G.

By a similar argument as that for (4.3), we have

$$(4.5)\qquad ((L_1\cap H^a)(L_1\cap P^{(n)}))^{op}(L_1\cap H^a\backslash L_1/L_1\cap P^{(n)})$$
$$=(L_1\cap H^a)(((L\cap H^a)(L\cap P^{(n)}))^{op}(L\cap H^a\backslash L/L\cap P^{(n)}))(L_1\cap P^{(n)}).$$

From (4.4) and (4.5) we get the desired formula for $(H^awP)^{op}$ since $L\cap P^{(n)}=L\cap P$.

(iv) follows from (ii) and (iii).

(v) Since $\mathfrak{l}\cap\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{l}\cap\mathfrak{p}$ contained in $\mathfrak{l}\cap\mathfrak{p}\cap\mathfrak{q}$, it follows from Proposition 1 and Proposition 2 in [3] that $(L\cap H)(L\cap P)$ is open in L and that $(L\cap H^a)(L\cap P)$ is closed in L.

(vi) By (ii) we can choose a sequence $D_0,\cdots,D_n$ of H-P double cosets (resp. $D'_0,\cdots,D'_n$ of H^a-P double cosets) satisfying the following four conditions.

(1) $D_0=D$ (resp. $D'_0=D'$).

(2) $D_i\subset(Hw^{(i)}P)^{cl}$ (resp. $D'_i\subset(H^aw^{(i)}P)^{op}$).

(3) $D_iL_{\alpha_i}\supset D_{i-1}$ (resp. $D'_iL_{\alpha_i}\supset D'_{i-1}$).

(4) If $D_{i-1}\subset(Hw^{(i)}P)^{cl}$, then $D_i=D_{i-1}$. (resp. If $D'_{i-1}\subset(H^a w^{(i)}P)^{op}$, then $D'_i=D'_{i-1}$.)

We choose representatives y_i of D_i (resp. D'_i) for $i=0, \cdots, n$ in the following inductive procedure.

We can choose $y_n\in D_n\cap((L\cap H)(L\cap P))^{cl}w$ (resp. $D'_n\cap((L\cap H^a)(L\cap P))^{op}w$) so that $\mathfrak{a}_n=\mathrm{Ad}\,(y_n)\mathfrak{a}$ is σ-stable by [1] Theorem 1. Suppose that we have chosen $y_n\in D_n$ (resp. D'_n), $\cdots$, $y_i\in D_i$ (resp. D'_i). Then we choose $y_{i-1}\in D_{i-1}$ as follows. If $D_{i-1}=D_i$ (resp. $D'_{i-1}=D'_i$), then we put $y_{i-1}=y_i$. So we may assume that $D_{i-1}\not\subset(Hw^{(i)}P)^{cl}$ (resp. $D'_{i-1}\not\subset(H^a w^{(i)}P)^{op}$). Put $P'=y_iPy_i^{-1}$, $P'_{\alpha_i}=y_iPL_{\alpha_i}y_i^{-1}$ and $w'_{\alpha_i}=y_iw_{\alpha_i}y_i^{-1}$. Then

$$D_{i-1}\subset D_iL_{\alpha_i}=Hy_iPL_{\alpha_i}=HP'_{\alpha_i}y_i$$
$$(\text{resp. } D'_{i-1}\subset D'_iL_{\alpha_i}=H^a y_iPL_{\alpha_i}=H^aP'_{\alpha_i}y_i).$$

Since $D_{i-1}\cap(Hw^{(i)}P)^{cl}=\varnothing$ (resp. $D'_{i-1}\cap(H^a w^{(i)}P)=\varnothing$) and since $D_i\subset(Hw^{(i)}P)^{cl}$ (resp. $D'_i\subset(H^a w^{(i)}P)^{op}$), we have

$$D_{i-1}y_i^{-1}\subset HP'_{\alpha_i}-(HP')^{cl} \tag{4.6}$$
$$(\text{resp. } D'_{i-1}y_i^{-1}\subset H^aP'_{\alpha_i}-(H^aP')^{op}(H^a\backslash G/P')).$$

Now we apply Lemma 3 to $(H\backslash G, P', P'_{\alpha_i})$ (resp. $(H^a\backslash G, P', P'_{\alpha_i})$).

First suppose that $\mathfrak{g}(\mathfrak{a}_i;\alpha'_i)\cap\mathfrak{q}=\{0\}$. Then it follows from (4.6) and from the five cases except (D) in Lemma 3 (resp. from the five cases except (F) in Lemma 3) that

$$D_{i-1}y_i^{-1}=Hw'_{\alpha_i}P' \qquad (\text{resp. } D'_{i-1}y_i^{-1}=H^a w'_{\alpha_i}P')$$

and

$$\dim Hw'_{\alpha_i}P'\geq\dim HP' \qquad (\text{resp. } \dim H^a w'_{\alpha_i}P'\leq\dim H^aP').$$

(In the cases (B), (C) and (E) (resp. (A), (C) and (E)), we get $HP'_{\alpha_i}\subset(HP')^{cl}$ (resp. $H^aP'_{\alpha_i}\subset(H^aP')^{op}(H^a\backslash G/P')$), a contradiction to (4.6).) Hence $D_{i-1}=Hy_iw_{\alpha_i}P$ (resp. $D'_{i-1}=H^a y_iw_{\alpha_i}P$) and $\dim D_{i-1}\geq\dim D_i$ (resp. $\dim D'_{i-1}\leq\dim D'_i$). We put $y_{i-1}=y_iw_{\alpha_i}$. (Then $\mathfrak{a}_{i-1}=\mathfrak{a}_i$.)

Next suppose that $\mathfrak{g}(\mathfrak{a}_i;\alpha'_i)\cap\mathfrak{q}\neq\{0\}$. Then it follows from (4.6) and from Lemma 3 (D) (resp. Lemma 3 (F)) that

$$D_{i-1}y_i^{-1}=Hw'_{\alpha_i}P',\quad Hc'_{\alpha_i}P' \quad\text{or}\quad Hc'^{-1}_{\alpha_i}P'$$
$$(\text{resp. } D'_{i-1}y_i^{-1}=H^a w'_{\alpha_i}P',\quad H^a c'_{\alpha_i}P' \quad\text{or}\quad H^a c'^{-1}_{\alpha_i}P')$$

and that $\dim HP'=\dim Hw'_{\alpha_i}P'<\dim Hc'_{\alpha_i}P'=\dim Hc'^{-1}_{\alpha_i}P'$ (resp. $\dim H^aP'=\dim H^a w'_{\alpha_i}P'>\dim H^a c'_{\alpha_i}P'=\dim H^a c'^{-1}_{\alpha_i}P'$). Hence

$$D_{i-1}=Hy_iw_{\alpha_i}P, \quad Hy_ic_{\alpha_i}P \quad \text{or} \quad Hy_ic_{\alpha_i}^{-1}P$$
$$(\text{resp. } D'_{i-1}=H^ay_iw_{\alpha_i}P, \ H^ay_ic_{\alpha_i}P \ \text{or} \ H^ay_ic_{\alpha_i}^{-1}P)$$

and $\dim Hy_iP=\dim Hy_iw_{\alpha_i}P<\dim Hy_ic_{\alpha_i}P=\dim Hy_ic_{\alpha_i}^{-1}P$ (resp. $\dim H^ay_iP=\dim H^ay_iw_{\alpha_i}P>\dim H^ay_ic_{\alpha_i}P=\dim H^ay_ic_{\alpha_i}^{-1}P$). Thus we can choose a representative y_{i-1} of D_{i-1} (resp. D'_{i-1}) such that $y_{i-1}=y_iw_{\alpha_i}$, $y_ic_{\alpha_i}$ or $y_ic_{\alpha_i}^{-1}$. It is clear from the choice of c_{α_i} that $\alpha_{i-1}=\mathrm{Ad}(y_{i-1})\alpha$ is σ-stable.

(vii) Let D(resp. D') be a closed H-P double coset (resp. an open H^a-P double coset) contained in $HRwW_{\alpha_n}\cdots W_{\alpha_1}P$ (resp. $H^aR'wW_{\alpha_n}\cdots W_{\alpha_1}P$). We have $R\subset((L\cap H)(L\cap P))^{cl}$ (resp. $R'\subset((L\cap H^a)(L\cap P))^{op}$) by (v) and Proposition in Section 1. Hence we have $D\subset(HP)^{cl}$ (resp. $D'\subset(H^aP)^{op}$) by (iv).

Conversely let D (resp. D') be a closed H-P double coset (resp. an open H^a-P double coset) contained in $(HP)^{cl}$ (resp. $(H^aP)^{op}$). Let $y_0, \cdots, y_n$ be as in (vi). Since all the closed H-P double cosets in G have the same dimension by Lemma 7 in Section 5 (resp. since all the open H^a-P double cosets in G have the same dimension), it follows from (vi) (d) that Hy_iP is closed (resp. H^ay_iP is open) in G for $i=0, \cdots, n$ and that $y_{i-1}=y_i$ or $y_iw_{\alpha_i}$ for $i=1, \cdots, n$. Clearly $(L\cap H)y_nw^{-1}(L\cap P)$ is closed (resp. $(L\cap H^a)y_nw^{-1}(L\cap P)$ is open) in L. Hence we have

$$D=Hy_0P\subset HRwW_{\alpha_n}\cdots W_{\alpha_1}P$$
$$(\text{resp. } D'=H^ay_0P\subset H^aR'wW_{\alpha_n}\cdots W_{\alpha_1}P).$$

Put $U=\{y\in K\,|\,\mathrm{Ad}(y)\alpha \text{ is } \sigma\text{-stable}\}$ and $U_0=\{y\in U\,|\,HyP \text{ is closed in } G\}$. ($U_0=\{y\in U\,|\,H^ayP \text{ is open in } G\}$ by Colollary of [1] § 3). Then by the above result, we have the followings for $y\in U_0$.

(4.7) $$HyP\subset(HP)^{cl}\Longleftrightarrow \text{There exists a } y_0\in(R\cap U)wW_{\alpha_n}\cdots W_{\alpha_1} \text{ such that } HyP=Hy_0P.$$

(4.8) $$H^ayP\subset(H^aP)^{op}\Longleftrightarrow \text{There exists a } y_0\in(R'\cap U)wW_{\alpha_n}\cdots W_{\alpha_1} \text{ such that } H^ayP=H^ay_0P.$$

On the other hand it follows from Corollary 2 of [1] Theorem 1 and Corollary of [1] Section 3 that

(4.9) $$R\cap U=R'\cap U$$

and that if y, $y_0\in U$, then

(4.10) $$HyP=Hy_0P\Longleftrightarrow H^ayP=H^ay_0P.$$

Hence for $y \in U_0$, we have

$$HyP \subset (HP)^{cl} \Longleftrightarrow H^a yP \subset (H^a P)^{op}$$

by (4.7), (4.8), (4.9) and (4.10). Q.E.D.

§ 5. Proof of Proposition

Let HxP^0 be an arbitrary closed H-P_0 double coset in G. Then by [1] Proposition 2, there exists an $h \in H$ such that $P = hxP^0x^{-1}h^{-1}$ can be written as

$$P = P(\mathfrak{a}_0, \Sigma^+).$$

Here $\mathfrak{a}_0$ is a σ-stable maximal abelian subspace of $\mathfrak{p}$ such that $\mathfrak{a}_0^\sigma = \mathfrak{a}_0 \cap \mathfrak{h}$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{h}$ and Σ^+ is a σ-compatible positive system (Definition following Lemma 4) of $\Sigma = \Sigma(\mathfrak{g}; \mathfrak{a}_0)$. Then we have only to prove that $D^{cl} \supset HP$ for any open H-P double coset D in G. Put $\Sigma^{\sigma\theta} = \{\alpha \in \Sigma \mid \sigma\theta\alpha = \alpha\}$. Let $\mathfrak{l}$ be the subalgebra of $\mathfrak{g}$ generated by $\{\mathfrak{g}(\mathfrak{a}_0; \alpha) \mid \alpha \in \Sigma^{\sigma\theta}\}$ and L the corresponding analytic subgroup in G. Let $\mathfrak{P}$ denote the Lie algebra of P.

Lemma 6. (i) $\mathfrak{g}(\mathfrak{a}_0; \alpha) \subset \mathfrak{h}^a$ *for all* $\alpha \in \Sigma^{\sigma\theta}$. (*Hence* $\mathfrak{l} \subset \mathfrak{h}^a$ *and* $L \subset H^a$.)

(ii) $\mathfrak{l} \subset \mathfrak{h} + \mathfrak{P}$ *and* $L \subset HP$.

Proof. Since $\mathfrak{g}(\mathfrak{a}_0; \alpha)$ is $\sigma\theta$-stable, we have only to prove that $\mathfrak{g}(\mathfrak{a}_0; \alpha) \cap \mathfrak{q}^a = \{0\}$. Suppose that there exists a nonzero element X of $\mathfrak{g}(\mathfrak{a}_0; \alpha) \cap \mathfrak{q}^a$. Then $X - \theta X$ is an element of $\mathfrak{p} \cap \mathfrak{q}^a = \mathfrak{p} \cap \mathfrak{h}$ commuting with $\mathfrak{a}_0^\sigma$. But this contradicts to the assumption that $\mathfrak{a}_0^\sigma$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{h}$.

(ii) We have only to prove that $L \subset HP$. Since $\theta|_L$ is a Cartan involution of L and since $L \cap P$ is a minimal parabolic subgroup of L, we have

$$L = (L \cap K)(L \cap P)$$

by the Iwasawa decomposition of L. On the other hand, we have $L \cap K = L \cap K \cap H^a = L \cap K \cap H$ since $L \subset H^a$ by (i). Hence $L \subset HP$. Q.E.D.

Next we will prove the following lemma which we used in Section 2 and Section 4.

Lemma 7. *Put* $\bar{\Sigma} = \{\bar{\alpha} \mid \alpha \in \Sigma,\ \bar{\alpha} \neq 0\}$ *and* $\bar{\Sigma}^+ = \{\bar{\alpha} \mid \alpha \in \Sigma^+,\ \bar{\alpha} \neq 0\}$ *where* $\bar{\alpha}$ *is the restriction of* α *to* $\mathfrak{a}_0^\sigma$. *Then*

$$\dim HP = \dim G - \Sigma_{\lambda \in \bar{\Sigma}^+} \dim (\mathfrak{g}(\mathfrak{a}_0^\sigma; \alpha) \cap \mathfrak{q})$$
$$= \dim G - \tfrac{1}{2} \sum_{\lambda \in \bar{\Sigma}} \dim (\mathfrak{g}(\mathfrak{a}_0^\sigma; \alpha) \cap \mathfrak{q}).$$

Especially all the closed H-P double cosets in G have the same dimension.

Proof. By Lemma 6 (ii), we have

$$\mathfrak{h} + \mathfrak{P} = \mathfrak{P} + \mathfrak{l} + \sum_{\lambda \in -\bar{\Sigma}^+} (\mathfrak{g}(\mathfrak{a}_0^\sigma; \alpha) \cap \mathfrak{h})$$

and therefore

$$\dim H_0 P = \dim (\mathfrak{h} + \mathfrak{P}) = \dim G - \sum_{\lambda \in \bar{\Sigma}^+} \dim (\mathfrak{g}(\mathfrak{a}_0^\sigma; \alpha) \cap \mathfrak{q})$$
$$= \dim G - \tfrac{1}{2} \sum_{\lambda \in \bar{\Sigma}} \dim (\mathfrak{g}(\mathfrak{a}_0^\sigma; \alpha) \cap \mathfrak{q})$$

since $\dim (\mathfrak{g}(\mathfrak{a}_0^\sigma; \alpha) \cap \mathfrak{q}) = \dim (\mathfrak{g}(\mathfrak{a}_0^\sigma; -\alpha) \cap \mathfrak{q})$ for $\lambda \in \bar{\Sigma}$. Since $HP = \bigcup_{y \in H} y H_0 P = \bigcup_{y \in H} H_0 y P$ is a finite union of H_0-P double cosets having the same dimension, we have the desired formula for $\dim HP$. Q.E.D.

Lemma 8 (J. Sekiguchi). *Put* $\bar{N} = \exp (\sum_{\alpha \in \Sigma^+} \mathfrak{g}(\mathfrak{a}_0; -\alpha))$. *Let D be an arbitrary H-P double coset in G. Then*

$$D^{cl} \supset HP \Longleftrightarrow D \cap \bar{N}P \neq \emptyset.$$

(**Remark.** Proposition follows from this lemma since $\bar{N}P$ is dense in G.)

Proof. $\Rightarrow$ is clear since $\bar{N}P$ is open in G. Suppose that $D \cap \bar{N}P \neq \emptyset$. Then $D \cap \bar{N} \neq \emptyset$. Let $x \in D \cap \bar{N}$ and write $x = \exp \sum_{\alpha \in \Sigma^+} X_{-\alpha}$ with $X_{-\alpha} \in \mathfrak{g}(\mathfrak{a}_0; -\alpha)$. By Lemma 4, we can choose an element $Y \in \mathfrak{a}_0^\sigma$ so that $\alpha(Y) > 0$ for all $\alpha \in \Sigma^+ - \Sigma^{\sigma\theta}$. Put $a_t = \exp tY$ for $t \in \boldsymbol{R}$. Then

$$a_t x a_t^{-1} = \exp \sum_{\alpha \in \Sigma^+} e^{-\alpha(Y)t} X_{-\alpha} \in D \cap \bar{N}$$

(since $a_t \in H \cap P$) and it follows from the choice of Y that

$$x_\infty = \lim_{t \to \infty} a_t x a_t^{-1} = \exp \sum_{\alpha \in \Sigma^+ \cap \Sigma^{\sigma\theta}} X_{-\alpha} \in L.$$

Hence $D^{cl} \cap L \ni x_\infty$ and therefore $D^{cl} \cap HP \neq \emptyset$ by Lemma 6 (ii). Since $HD^{cl}P = D^{cl}$, we have $D^{cl} \supset HP$. Q.E.D.

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