Advanced Studies in Pure Mathematics 14, 1988
Representations of Lie Groups, Kyoto, Hiroshima, 1986
pp. 379–393

A Survey of the Generalized Geroch Conjecture

H. Doi and K. Okamoto

Dedicated to Professor M. Sugiura for his 60th birthday

Introduction

It is widely admitted that to reach the true world of unitary representations of semisimple Lie groups we had to have some impact from the outside of mathematics. That was "Quantum Mechanics". The quantum mechanics has now developed into the "Quantum Field Theory". During recent years there has been considerable activity in the study of the group theoretical approach to the quantum field theory which suggests a new "Theory of group representations". Unfortunately, however, we do not have enough mathematical structures to get an insight into the new world.

On the other hand the quantum field theory has another origin, "The Classical Field Theory". The group theoretical aspect of the recent study of field equations shows that one should consider not merely finite but also infinite dimensional Lie groups and their homogeneous spaces. To mention only two, the theorem of Sato-Sato [17] says that the space of all the local solutions of the KP equations and their hierarchy is parametrized by the closure of the infinite dimensional Grassmann manifold, and the Geroch conjecture which was proved affirmatively by I. Hauser and F. J. Ernst [8] says that the Geroch group acts transitively, up to gauge transformations, all the local solutions of the stationary axisymmetric Einstein field equations. I. Hauser and F. J. Ernst have extended their work to the case of N Abelian gauge fields interacting with the gravitational field in an astonishingly beautiful way that it contains the vacuum case ($N=0$) and the Einstein-Maxwell field equations ($N=1$) [10].

The aim of this paper is to give a more or less self-contained exposition of the mathematically beautiful work of Hauser-Ernst on the generalized Geroch conjecture [10] which, we believe, attracts many mathematicians and is promising of further developments as a new branch of mathematics.

Roughly speaking the generalized Geroch conjecture asserts, that the $SU(N+1, 1)$ "Kac-Moody Lie group" acts transitively on the "moduli

Received May 8, 1987.

space" of local solutions of the stationary axisymmetric generalized Einstein-Maxwell field equations. Strictly speaking, however, the $SU(N+1, 1)$ Kac-Moody Lie group does not act on the moduli space so that one should take a proper subgroup called the generalized Geroch group. From the field equations I. Hauser and F. J. Ernst derived a non-linear differential equation for matrix-valued functions, which we call the Hauser-Ernst equation. In this paper we shall generalize the Hauser-Ernst equation to the case of Grassmann manifold valued unknown functions. We shall prove that the $SU(N+1, 1)$ Kac-Moody Lie group acts transitively on the moduli space of local solutions of the generalized Hauser-Ernst equation.

Attempting to clarify the group theoretical aspect of the Hauser-Ernst theory we shall make the direct approach in terms of "infinite dimensional Lie groups" and their homogeneous spaces which we suspect may prove to be one fruitful and interesting direction in which the theory of group representations may move in the near future. We shall define an "infinite dimensional manifold" of functions on the upper half plane with values in the generalized flag manifold of the complexification of the $SU(N+1, 1)$ Kac-Moody Lie group. The Birkhoff-Witt decomposition gives us the canonical coordinate systems of the generalized flag manifold. We consider those functions the first coefficients (in the Laurent expansion with respect to the spectral parameter) of which satisfy the Hauser-Ernst equation. The crucial point here is that this property is stable by the canonical action of the $SU(N+1, 1)$ Kac-Moody Lie group. Thus the $SU(N+1, 1)$ Kac-Moody Lie group acts, at least as a pseudogroup of transformations, on the moduli space of local solutions of the Hauser-Ernst equation. The homogeneous Riemann-Hilbert problem corresponds in our formulation to the problem about the Birkhoff-Witt decomposition which has been settled by the first author [1].

We shall now give a short description of the contents of the paper. For the convenience of the readers with little knowledge of physics we begin with the definition of the Einstein-Maxwell field equations. We assume that the field equations are stationary axisymmetric. With this condition the fields depend only on two variables which we denote by x and y ($y\geq 0$) and the fields can be put into simple forms which we adopt as the definition of a stationary axisymmetric field. First we calculate the field equations and obtain the equations (1.1)$\sim$(1.3), which are obviously overdetermined. Theorem 1.1 shows the compatibility of the equations (1.1)$\sim$(1.3) and makes it possible to eliminate the unknown function γ. Next we express the equations (1.1)$\sim$(1.3) with the aid of closed differential forms which are given by (2.1) and (2.2). Since we consider germs of local solutions, by the Poincare's lemma every closed differential form is equivalent to an exact differential form. This fact enables us to derive the so called anti-self-duailty relation which is nothing but the Hauser-Ernst equation. Notice that our coordinates (x, y) correspond to (z, ρ) in [10] but out definition of the Hodge operator $*$ corresponds to $-*$ in [10].

One of the most striking feature here is that the generalized Ernst potentials satisfy the differential equations (2.3) which offer us an interesting mathematical model of the new notion of the "non-linear" differential equations of regular singularities along $y=0$, with which we would like to deal somewhere else. Notice that in the proof of the Helgason conjecture [12] the notion of the "linear" differential equation with regular singularities plays an important role. It is plausible that one would be able to study solutions of the Hauser-Ernst equation which have singularities along $y=0$. In this paper we consider only solutions without singularities. As is in the linear case any solution is uniquely determined by the boundary value function (see Lemma 2.2).

Now let us consider the Siegel domain of the second kind;

$$\operatorname{Im} z_0-\tfrac{1}{2}(|z_1|^2+|z_2|^2+\cdots+|z_N|^2)>0.$$

This domain is isomorphic to the unit open ball in C^{N+1} on which $SU(N+1, 1)$ acts transitively. It is remarkable that the boundary value functions take their values in the "outside" of the Siegel domain. The most mathematically beautiful feature in the Hauser-Ernst theory is that the action of the generalized Geroch group is realized on the space of boundary value functions by the canonical action of $SU(N+1, 1)$-valued functions on the boundary (substituting the spectral parameter by the boundary point). If the boundary value functions of the solutions of the Hauser-Ernst equation took their values, contrary to fact, in the Siegel domain itself the whole $SU(N+1, 1)$ Kac-Moody group would have acted on the moduli space of the solutions of the Hauser-Ernst equation. The fact is that the boundary value functions of the solutions of the Hauser-Ernst equation take their values in the outside of the Siegel domain. To get the action of the whole $SU(N+1, 1)$ Kac-Moody group we generalize the Hauser-Ernst equation to the case for generalized flag manifold valued solutions. The outside of the above mentioned Siegel domain in $P^{N+1}(C)$ is an affine symmetric space which is a homogeneous space of $SU(N+1, 1)$. And except for the case $N=0$ the coordinate system,

$$\operatorname{Im} z_0-\tfrac{1}{2}(|z_1|^2+|z_2|^2+\cdots+|z_N|^2)<0,$$

does not cover the whole manifold.

This shows that to get the transitivity it is not sufficient to consider the differential equations for a set of complex valued unknown functions and that we should consider the differential equations for the generalized flag manifold valued unknown functions the boundary value functions of which take their values in $P^{N+1}(C)$.

For lack of the knowledge of the authors the references in the paper are incomplete. We suggest the reader to consult the references of the relevant papers which are full of treasures for further mathematical structures.

§ 0. The Einstein-Maxwell field equations

Let $g=g_{ij}dx^i\otimes dx^j$ $(i,j=0,\cdots,3)$ be a space-time metric and $A=A_i dx^i$ an abelian gauge potential with values in $\boldsymbol{R}^N$. We use the standard notation for the Christoffel symbols, the Ricci curvature, the scalar curvature and the gauge field tensor [16].

$$\Gamma^k_{ij}=2^{-1}g^{km}(\partial_i g_{jm}+\partial_j g_{im}-\partial_m g_{ij}),$$
$$R_{ik}=\partial_m\Gamma^m_{ik}-\partial_i\Gamma^m_{km}+\Gamma^m_{ik}\Gamma^j_{mj}-\Gamma^m_{ij}\Gamma^j_{mk},$$
$$R=g^{ik}R_{ik},$$
$$F_{ij}=\partial_i A_j-\partial_j A_i.$$

Then the field equations are given as follows ([16] (95.5), (94.8), (90.6)):

$$R_{ik}-2^{-1}g_{ik}R=8\pi T_{ik},$$
$$T_{ik}=(4\pi)^{-1}(F_{im}{}^tF^m_k-4^{-1}g_{ik}F_{jm}{}^tF^{jm}),$$
$$\nabla_k F^{ik}=0.$$

Here ^{t} denotes the transpose. Since $g^{ik}T_{ik}=0$, the scalar curvature R vanishes.

§ 1. The stationary axisymmetric vacuum fields

We begin with the definition of a stationary axisymmetric field. Let sub- or superscript p, q, r, s be 0 or 1 and let a, b, c be 2 or 3. If a field (g, A) has the following special form, we call it a stationary axisymmetric field:

$$g=h_{pq}dx^p\otimes dx^q-e^{2\gamma}ds^2,$$
$$ds^2=S_{ab}dx^a\otimes dx^b,\quad \text{with}\quad S_{23}=0,\quad |S_{22}|=|S_{33}|=1,$$
$$\det h=sy^2,\quad \text{with}\quad s=\pm 1,$$
$$A=A_p dx^p,$$

where h_{pq}, γ, A_p are functions of x^2, x^3 which we denote by x, y respectively for the rest of this paper.

For a stationary axisymmetric field $(g, A)=(h, \gamma, A)$, we shall write down the field equations. Let ∂^a stand for $S^{ac}\partial_c$. Then, R_{ij}, T_{ij} are calculated as follows:

$$R_{ap}=0, \quad R_{pq}=(2y)^{-1}e^{-2\gamma}h_{ps}\partial^c(yh^{sr}\partial_c h_{rq}),$$
$$R_{ab}=-S_{ab}\partial^c\partial_c\gamma-\partial_a\partial_b\log y-S_{ab}\partial^c\gamma\partial_c\log y$$
$$+\partial_a\gamma\partial_b\log y+\partial_a\log y\,\partial_b\gamma+4^{-1}N_{ab},$$
$$\text{where}\quad N_{ab}=\partial_b h^{rs}\partial_b h_{rs}.$$
$$T_{ap}=0.$$
$$8\pi T_{pq}=e^{-2\gamma}(2S^{cc}F_{pc}{}^tF_{qc}-h_{pq}S^{cc}F_{cr}{}^tF_{cs}h^{rs}),$$
$$8\pi T_{ab}=-2F_{ar}{}^tF_{bs}h^{rs}+S_{ab}S^{cc}F_{cr}{}^tF_{cs}h^{rs}.$$

We identify A with a $2\times N$ matrix $\begin{bmatrix}A_0\\A_1\end{bmatrix}$ and hereafter an abelian gauge potential is treated as a matrix valued function. Then $R_{pq}=8\pi T_{pq}$ implies:

(1.1) $$d(yh^{-1}*dh)=2yh^{-1}dA*d^tA+2y\varepsilon dA*d^tAh^{-1}\varepsilon,$$

where $\varepsilon=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ and $*=$the Hodge operator for ds^2.

Also $\nabla_k F^{ik}=0$ implies:

(1.2) $$d(yh^{-1}*dA)=0.$$

We put

$$\zeta=y(8\pi T_{23}-4^{-1}N_{23})dx+y(8\pi T_{33}-1/2y^2-N_{33}/8+S_{22}S_{33}N_{22}/8)dy,$$
$$\xi=2^{-1}(S_{33}/y^2+S_{22}N_{22}/4+S_{33}N_{33}/4)dxdy.$$

Then $R_{ab}=8\pi T_{ab}$ implies:

(1.3) $$d\gamma=\zeta, \qquad d*d\gamma=\xi.$$

Theorem 1.1. *Let* (h, A) *be a solution of* (1.1) *and* (1.2). *And assume that* $\det h=sy^2$ $(s=\pm 1)$. *Then* $d\zeta=0$ *and* $d*\zeta=\xi$. *In particular*, (1.1)$\sim$(1.3) *are equivalent to* (1.1)$\sim$(1.2).

Proof. We notice that if (h, A) satisfies (1.1)$\sim$(1.2), then so does $({}^tghg, {}^tgA)$ for any $g\in SL(2, \boldsymbol{R})$. Therefore we may assume $h_{00}\neq 0$. Set $f=h_{00}$, $\omega=h_{01}/f$ and $\varphi=\log|f|$. Then we have

$$h_{11}=sy^2f^{-1}+f\omega^2.$$

In the following the subscripts x, y denote the derivatives with respect to

x, y respectively. By a simple calculation we get

$$N_{22}=-2(\varphi_x)^2-2s(f\omega_x/y)^2,$$
$$N_{23}=-2\varphi_x\varphi_y+2\varphi_x/y-2s(f/y)^2\omega_x\omega_y,$$
$$N_{33}=-4/y^2-2(\varphi_y)^2+4\varphi_y/y-2s(f\omega_y/y)^2,$$
$$\xi=4^{-1}\{2y^{-1}dy*d\varphi-d\varphi*d\varphi-s(f/y)^2d\omega*d\omega\}.$$

Now put

$$\zeta^f=2^{-1}\varphi_x(y\varphi_y-1)dx+4^{-1}\{y(\varphi_y)^2-yS_{22}S_{33}(\varphi_x)^2-2\varphi_y\}dy,$$
$$\zeta^\omega=(2y)^{-1}sf^2\omega_x\omega_y dx+(4y)^{-1}sf^2\{(\omega_y)^2-S_{22}S_{33}(\omega_x)^2\}dy,$$
$$\zeta^h=\zeta^f+\zeta^\omega,$$
$$\zeta^A=8\pi y\{T_{23}dx+T_{33}dy\}.$$

Then we obtain the following formulas:

$$\zeta=\zeta^h+\zeta^A,$$
$$d\zeta^f=-2^{-1}S_{33}\varphi_x d(y*d\varphi),$$
$$d\zeta^\omega=-2^{-1}sS_{33}\omega_x d(y^{-1}f^2*d\omega)+(2y)^{-1}sS_{33}ff_x d\omega*d\omega,$$
$$d*\zeta^f=2^{-1}\varphi_y d(y*d\varphi)-4^{-1}d\varphi*d\varphi-2^{-1}d*d\varphi,$$
$$d*\zeta^\omega=2^{-1}s\omega_y d(y^{-1}f^2*d\omega)-4^{-1}s(y^{-1}f^2)_y d\omega*d\omega.$$

We note

$$(yh^{-1}*dh)_{00}=y*d\varphi-sy^{-1}f^2\omega*d\omega,$$
$$(yh^{-1}*dh)_{10}=sy^{-1}f^2*d\omega.$$

It follows that

$$d\zeta^h=-(2f)^{-1}S_{33}\{h_{00x}d(yh^{-1}*dh)_{00}+h_{01x}d(yh^{-1}*dh)_{10}\},$$
$$d*\zeta^h=(4yf)^{-1}\{(2yh_{00y}-h_{00})d(yh^{-1}*dh)_{00}+(2yh_{01y}-h_{01})d(yh^{-1}*dh)_{10}\}$$
$$+4^{-1}\{y^{-1}dy*d\varphi-d\varphi*d\varphi-d*d\varphi\}.$$
$$d\zeta^A=2S_{33}A_{px}d(yh^{pq}*d\,{}^tA_q)-S_{33}(yh^{pq})_x dA_p*d\,{}^tA_q,$$
$$d*\zeta^A=-2A_{py}d(yh^{pq}*d\,{}^tA_q)+(yh^{pq})_y dA_p*d\,{}^tA_q.$$

Notice that

$$h_{00}h^{00}+h_{01}h^{10}=1,\ h_{10}=-sy^2h^{10}\quad\text{and}\quad f=h_{00}=sy^2h^{11}.$$

Making use of these formulas we finally get

$$d\zeta^h = -(2f)^{-1}S_{33}[h_x o]_{00} - d\zeta^A + 2S_{33}\,\mathrm{tr}\,[A_x d^t(yh^{-1}{*}dA)],$$
$$d{*}\zeta^h = (2yf)^{-1}[(yh_y - h)o]_{00} - d{*}\zeta^A - 2\,\mathrm{tr}\,[A_y d^t(yh^{-1}{*}dA)] + \xi,$$

where $o = d(yh^{-1}{*}dh) - 2yh^{-1}dA{*}d\,{}^tA - 2y\varepsilon dA{*}d\,{}^tAh^{-1}\varepsilon$. Now the theorem follows from (1.1) and (1.2).

§ 2. The Hauser-Ernst equation

In this section we assume that $ds^2 = (dx)^2 + (dy)^2$, $\det h = -y^2$. We treat h and A as germs of analytic functions at $(x, y) = (0, 0)$. Since h is a symmetric matrix we have $yh^{-1} = \varepsilon y^{-1}h\varepsilon$. Hence (1.1) and (1.2) are equivalent to the following equations:

(2.1) $$d(y^{-1}h\varepsilon{*}dA) = 0,$$

(2.2) $$d\{y^{-1}h\varepsilon{*}dh - 2(y^{-1}h\varepsilon{*}dA)\,{}^tA + 2A{*}d\,{}^tA\varepsilon y^{-1}h\} = 0.$$

We assume that $y^{-1}h\varepsilon{*}dh$ and $y^{-1}h\varepsilon{*}dA$ are analytic along $y = 0$. If $h_{00}(0, 0) \neq 0$, $h_{01}(0, 0) = 0$, then along $y = 0$ we have $d(h_{01}/h_{00}) = 0$ and $dA_1 = 0$.

Here we introduce a special class of solutions of the Einstein-Maxwell field equations.

Definition 2.1. Let $\mathscr{S}_{EM}$ denote the set of germs at (0, 0) of all local solutions (h, A) of the equations (2.1) and (2.2) which satisfy the following conditions:

(i) $h_{00}(x, 0) > 0$, $h_{01}(x, 0) = h_{01y}(x, 0) = 0$,

(ii) $A_1(x, 0) = A_{1y}(x, 0) = 0$.

Remark. It follows from $\det h = -y^2$ that $h_{11}(x, 0) = 0$ for $(h, A) \in \mathscr{S}_{EM}$.

Following Hauser-Ernst [10], we shall define complex potentials for members of $\mathscr{S}_{EM}$ and derive differential equations satisfied by these potentials.

Let $(h, A) \in \mathscr{S}_{EM}$ and let $\sharp$ stand for $y^{-1}h\varepsilon{*}$. Then by (2.1), there exists a $2 \times N$ real matrix valued function B such that $dB = \sharp dA$. We set $\alpha = A + iB$. Then $\sharp d\alpha = -id\alpha$. Hence $\alpha_{1x}(x, 0) = 0$, where $\alpha = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$. So we can assume $\alpha_1(x, 0) = 0$. It is easy to see that $d({}^\dagger\alpha\varepsilon d\alpha) = 0$. Here † denotes the hermitian conjugate. Hence there exists an $N \times N$ matrix function κ such that $d\kappa = {}^\dagger\alpha\varepsilon d\alpha$. Since $\kappa_x(x, 0) = 0$, we can assume $\kappa(x, 0) = 0$. It follows from $d(\kappa - {}^\dagger\kappa) = d({}^\dagger\alpha\varepsilon\alpha)$ that $\kappa - {}^\dagger\kappa = {}^\dagger\alpha\varepsilon\alpha$. By (2.2), we have a 2×2 real matrix function ψ such that $d\psi = \sharp dh - 2dB\,{}^tA - 2Ad\,{}^tB$. It is clear that $d(\psi - {}^t\psi) = \sharp dh - {}^t(\sharp dh) = y^{-1}{*}d(h\varepsilon h) = 2dx\varepsilon$. So we can assume

$\psi - {}^t\psi = 2x\varepsilon$. We set $\eta = h + \mathrm{Re}\,(\varepsilon\,\mathrm{tr}\,\kappa - \alpha\,{}^\dagger\alpha) + i(\psi + 2A\,{}^tB)$. Then $d\eta = (1 + i\sharp)(dh - 2dA\,{}^tA - 2dB\,{}^tB)$. Hence $\sharp d\eta = -id\eta$. On the other hand since $d({}^\dagger\alpha\varepsilon d\eta) = 0$, we have $\eta + {}^\dagger\eta = 2h + 2ix\varepsilon - 2\alpha\,{}^\dagger\alpha$. Thus if $y = 0$, then $\eta_{01} + {}^\dagger\eta_{10} = 2ix$, $\eta_{10} = i\psi_{10}$, $\eta_{11} = i\psi_{11}$ and $d\eta_{10} = d\eta_{11} = 0$. Therefore we can assume $\eta_{10}(x, 0) = 0$. By the above argument, we can define a potential $\theta = (\eta, \alpha)$ uniquely up to a pure imaginary valued additive constant for η_{00}. Note that $d\theta = iy^{-1}h\varepsilon * d\theta$. Then we get

$$d\theta_1 = \omega d\theta_0 + iyf^{-1} * d\theta_0,$$

where $\theta = \begin{bmatrix}\theta_0\\ \theta_1\end{bmatrix}$, $h_{00} = f$ and $h_{01}/h_{00} = \omega$. Set $\eta = \eta^R + i\eta^I$. Then clearly $-id\eta + id\eta^R = d\eta^I = y^{-1}h\varepsilon * d\eta^R$. Note that

$$d\eta^R = dh - 2\,\mathrm{Re}\,(d\alpha\,{}^\dagger\alpha),$$
$$y^{-1}h\varepsilon * d\eta^R = y^{-1}h\varepsilon * dh + 2\,\mathrm{Re}\,(id\alpha\,{}^\dagger\alpha).$$

Then we have

$$y^{-1}f^2 * d\omega = (y^{-1}h\varepsilon * dh)_{00} = -id\eta_{00} + idf - 2id\alpha_0\,{}^\dagger\alpha_0.$$

Hence

$$d\omega = yf^{-2}(i * d\eta_{00} - i * df + 2i * d\alpha_0\,{}^\dagger\alpha_0).$$

Since $*d\theta = -iy^{-1}h\varepsilon d\theta$, we have $d*d\theta + y^{-1}dy*d\theta + iy^{-1}dh\varepsilon d\theta = 0$. We note $(dh\varepsilon d\theta)_0 = iyf^{-1}df*d\theta_0 - fd\omega d\theta_0$. Thus, for $\nu = (u, \alpha_0)$ with $u = \eta_{00}/2i$, we obtain the following formula:

$$y^2 d*d\nu + ydy*d\nu - 2f^{-1}y^2(idu + d\alpha_0\,{}^\dagger\alpha_0)*d\nu = 0, \quad \text{where } f = -2\,\mathrm{Im}\,u + |\alpha_0|^2. \tag{2.3}$$

Remark that the principal part of (2.3) means

$$y^2\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right)\nu\,dxdy,$$

which is the typical form of the linear differential operator with regular singularities along $y = 0$. Moreover the lower part of (2.3) is a polynomial of the derivatives of the unknown functions with respect to $y(\partial/\partial x)$ and $y(\partial/\partial y)$.

Making use of these properties we can prove the following

Lemma 2.2. *Let* $\nu = \Sigma_{n\geq 0}\,\nu^{[n]}(x)y^n$ *be an analytic solution of* (2.3) *defined around* $(x, y) = (0, 0)$. *Then* $\nu^{[n]}$, $n > 0$ *is uniquely determined by*

$\nu^{[0]}$. *In particular* $\nu(x, y)=\nu(x, -y)$.

Hence we have

Corollary 2.3. *If* $(h, A) \in \mathscr{S}_{EM}$, *then* $h(x, y)=h(x, -y)$ *and* $A(x, y)=A(x, -y)$. *Moreover if* (η, α) *is the potential defined for* (h, A), *then* $\eta(x, y)=\eta(x, -y)$, $\alpha(x, y)=\alpha(x, -y)$.

Definition 2.4. Let η be a 2×2 matrix function and α a $2\times N$ matrix function. We set $\theta=(\eta, \alpha)$, $h=(\eta+{}^t\eta)/2-ix\varepsilon+\alpha\,{}^t\eta$ and $\sharp=y^{-1}h\varepsilon *$. We call the following equation the Hauser-Ernst equation:

$$\sharp d\theta=-id\theta.$$

Definition 2.5. Let $\mathscr{S}_{HE}$ denote the set of germs at (0, 0) of all local analytic solutions $\theta=(\eta, \alpha)$ of the Hauser-Ernst equation which satisfy the following conditions:

(i) $\eta_{01}(x, 0)=2ix$, $\eta_{10}(x, 0)=\eta_{11}(x, 0)=0$, $\alpha_1(x, 0)=0$,
(ii) $h_{00}(x, 0)>0$,
(iii) $\theta(x, y)=\theta(x, -y)$.

Let e_i, $i=0, \cdots, N+1$ be the standard basis of $\boldsymbol{C}^{N+2}$ (column vectors). We set $V_0=\Sigma_{1\le i\le N+1}\boldsymbol{C}e_i$. Let $Q=\{g \in GL(N+2, \boldsymbol{C}); gV_0=V_0\}$. We identify $GL(N+2, \boldsymbol{C})/Q$ with the Grassmann variety. We set $J=\begin{bmatrix} 0 & i & \\ -i & 0 & \\ & & 1_N \end{bmatrix}$. Then J defines the hermitian form ${}^t\bar{v}Jv$ for $v \in \boldsymbol{C}^{N+2}$. $J|V$ denotes the restriction of J to a subspace V. For an open subset U of $\boldsymbol{R}$, we denote by $\mathcal{O}^-(U)$ the set of analytic functions F defined on U with values in ${}^tQ\cdot V_0$ such that $\det(J|F(x))<0$. We denote by $\mathcal{O}_0^-$ the direct limit of $\{\mathcal{O}^-(U);\ 0 \in U\}$.

Proposition 2.6. *The mapping*

$$\theta=(\eta, \alpha)\longrightarrow\begin{bmatrix}1 & \nu(x, 0)\\ 0 & 1_{N+2}\end{bmatrix}\cdot Q \qquad (where\ \nu=(\eta_{00}/2i,\ \alpha_0))$$

defines an injection ρ *of* $\mathscr{S}_{HE}$ *into* $\mathcal{O}_0^-$.

Proof. Let $z=y^2$. Then the Hauser-Ernst equation implies that $z\theta_x/2=-\mu z\theta_z$ and $2z\theta_z=\mu\theta x$, where $\mu=ih\varepsilon$. We put

$$\theta=\Sigma_{n\ge 0}\theta^{[n]}(x)z^n \quad \text{and} \quad \mu=\Sigma_{n\ge 0}\mu^{[n]}(x)z^n.$$

Then

$$\theta_x^{[n-1]}/2 = -(\mu^{[0]} n\theta^{[n]} + \cdots + \mu^{[n-1]} 1\theta^{[1]}),$$
$$\mu_{00}^{[0]} = \mu_{10}^{[0]} = \mu_{11}^{[0]} = 0 \quad \text{and} \quad \mu_{01}^{[0]} \neq 0.$$

Hence $\theta_1^{[n]}$ is represented by $\theta^{[k]}$, $0 \leqq k < n$. Also we know

$$2n\theta^{[n]} = \mu^{[0]}\theta_x^{[n]} + \cdots + \mu^{[n]}\theta_x^{[0]},$$
$$\mu_{00}^{[n]} = \eta_{01}^{[n]}/2i + (\cdots),$$

where $(\cdots)$ are terms including only $\theta^{[k]}$ $(0 \leq k < n)$ and $\theta_1^{[n]}$. Hence $\theta_0^{[n]}$ is represented by $\theta^{[k]}$, $0 \leq k < n$. By induction we have the proposition.

We now give a characterization of members of $\mathscr{S}_{HE}$ which are associated with members of $\mathscr{S}_{EM}$.

Lemma 2.7. *Let* $(\eta, \alpha) \in \mathscr{S}_{HE}$. *Assume* $d({}^\dagger\alpha\varepsilon d\alpha) = 0$ *and take an analytic function* κ *such that* $d\kappa = {}^\dagger\alpha\varepsilon d\alpha$, $\kappa(x, 0) = 0$. *Put* $h = (\eta + {}^\dagger\eta)/2 - i\varepsilon x + \alpha\, {}^\dagger\alpha$ *and* $A = \operatorname{Re}\alpha$. *If* $\eta - {}^t\eta = 2ix\varepsilon + 2\varepsilon \operatorname{tr}\kappa$, *then* $(h, A) \in \mathscr{S}_{EM}$.

Proof. We set $B = \operatorname{Im}\alpha$. Then $\sharp dA = B$. So $A_1(x, 0) = 0$. We know $h_{01}(x, 0) = h_{10}(x, 0) = h_{11}(x, 0) = 0$ and $h = \operatorname{Re}\eta + \alpha\, {}^\dagger\alpha - \varepsilon \operatorname{tr} {}^\dagger\kappa$. Therefore $dh = \operatorname{Re} d\eta + d\alpha\, {}^\dagger\alpha + d\bar{\alpha}\, {}^t\alpha$. It follows that h is a real matrix. Also $\sharp dh = \operatorname{Re}(-id\eta) - id\alpha\, {}^\dagger\alpha + id\bar{\alpha}\, {}^t\alpha$. Hence we have $d(\sharp dh) = 2dAd\,{}^tB - 2dBd\,{}^tA$. It is easy to see $\sharp dh - {}^t(\sharp dh) = 2\varepsilon dx$. So $d(h\varepsilon h + y^2\varepsilon) = 0$. Therefore we have $\det h = -y^2$.

§ 3. The generalized Geroch conjecture

In this section we keep the notation Q, J and $\mathcal{O}_0^-$ defined in the previous section.

For an open subset U of $P^1(\boldsymbol{C}) \times \boldsymbol{R}^2$, we denote by $\mathscr{G}^c(U)$ the set of $GL(N+2, \boldsymbol{C})$-valued analytic functions $Z = Z(s, x, y)$ defined on U which are holomorphic in s. For an open neighborhood U of $(\infty, 0) \in P^1(\boldsymbol{C}) \times \boldsymbol{R}^2$, we put $\mathscr{N}(U) = \{Y \in \mathscr{G}^c(U);\ Y(\infty, x, y) = 1\}$. Let $\mathscr{N}$ denote the direct limit of $\{\mathscr{N}(U);\ (\infty, 0) \in U\}$ and let $\mathscr{P}$ denote the direct limit of $\{\mathscr{G}^c(U);\ (0, 0) \in U\}$. For $Z \in \mathscr{G}^c(U)$, we define $\sigma Z \in \mathscr{G}^c(U)$ by $(\sigma Z)(s, x, y) = J\, {}^\dagger(Z(\bar{s}, x, y))^{-1} J$.

Let $\delta(s)$ stand for $\begin{bmatrix} s & \\ & 1_{N+1} \end{bmatrix}$. For $Y \in \mathscr{N}$, we put $Z(s, x, y) = \delta(s) Y(s, x, y)$ and $\Omega = Y^{-1} dY$. Here d denotes the exterior differentiation on $\boldsymbol{R}^2$. We shall consider the following conditions:

(3.1) $\Omega + i{*}\Omega$ and $\Omega - i{*}\Omega$ are meromorphically extended to $P^1(\boldsymbol{C})$. They have poles possibly only at $s = x + iy$ and $x - iy$. The orders of the poles are at most one.

(3.2) $(\sigma Z)^{-1}Z$ and $(\sigma Z)^{-1}dZ$ are polynomials of s.

The following result is due to I. Hauser and F. J. Ernst [10].

Lemma 3.1. *For $Y \in \mathcal{N}$, let $Y = 1 + \Sigma_{n\geq 1} Y_n(x, y)s^{-n}$ be the Laurent expansion. We put $Y_1 = \begin{bmatrix} H & \alpha \\ L & K \end{bmatrix}$, where H is a 2×2 matrix. We set $\eta = -2iH\varepsilon$, $\lambda = 2L\varepsilon$ and $\kappa = iK$. If Y satisfies* (3.1) *and* (3.2), *then (η, α) is a solution of the Hauser-Ernst equation and $d\lambda = {}^{t}\alpha\varepsilon d\eta$, $d\kappa = {}^{t}\alpha\varepsilon d\alpha$.*

Proof. In (3.2), compute the coefficients of s^{-1}.

Definition 3.2. Let $\mathcal{M}^-$ denote the set of $Y \in \mathcal{N}$ which satisfies (3.1), (3.2) and the following conditions:

(i) $Y(s, x, y) = Y(s, x, -y)$,

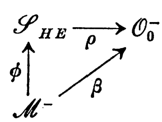
(ii) $Y(s, x, 0) = \begin{bmatrix} 1 - x/s & \nu(x)/s \\ 0 & 1_{N+1} \end{bmatrix}$,

(iii) $\delta(s)Y(s, x, 0)\delta(s-x)^{-1} \in \mathcal{O}_0^-$.

From Lemma 3.1 we obtain the mapping

$$\phi\colon \mathcal{M}^- \ni Y \longrightarrow (-2iH\varepsilon, \alpha) \in \mathcal{S}_{HE}.$$

Making use of the mapping ρ defined in Proposition 2.6 we define a mapping β by the following commutative diagram.



We know that the mapping β is injective (cf. [1] Theorem 3.2). Let $\mathcal{G}_0^c$ denote the set of all germs at $0 \in \boldsymbol{C}$ of $GL(N+2, \boldsymbol{C})$-valued holomorphic functions of $s \in \boldsymbol{C}$. We regard $\mathcal{G}_0^c$ as a subset of $\mathcal{P}$. We set $\mathcal{G} = \{g \in \mathcal{G}_0^c;\ \sigma g = g\}$ and $\mathcal{H} = \{g \in \mathcal{G};\ g(s) \in {}^{t}Q\}$. If $Y \in \mathcal{N}$ satisfies (3.1), then $Y(s, x, y)$ is defined for $|s|^2 > x^2 + y^2$ (cf. [1] Section 2). Put

$$Y^0 = \begin{bmatrix} (s-x+\tau)/2s & -i/2s & 0 \\ i(s-x-\tau) & 1 & 0 \\ 0 & 0 & 1_N \end{bmatrix},$$

where $\tau = s\{(1-x/s)^2 + (y/s)^2\}^{1/2}$. Then $Y^0 \in \mathcal{M}^-$.

Now the generalized Geroch conjecture can be stated as follows.

Theorem 3.3. $\mathcal{H}\delta Y^0\mathcal{P}/\mathcal{P} = \delta\mathcal{M}^-\mathcal{P}/\mathcal{P} \simeq \mathcal{M}^- \simeq \mathcal{S}_{HE}$.

Corollary 3.4. *We have a natural surjection* $\mathscr{S}_{HE} \to \mathscr{S}_{EM}$.

Proof. We prove that $\eta - {}^t\eta = 2i\varepsilon x + 2\varepsilon \operatorname{tr} \kappa$. Note that ${}^tH - \varepsilon H \varepsilon = \operatorname{tr} H$ for a 2×2 matrix H. By Lemma 3.1, it is enough to show that $\operatorname{tr} Y_1 + x = 0$ for any $Y \in \mathscr{M}^-$. From Theorem 3.3 it follows that $\det Y = \det Y^0$. Therefore $\operatorname{tr} Y_1 = \operatorname{tr} Y_1^0 = -x$.

We define $\mathscr{O}_0^+$ by exchanging the condition $\det(J|(F(x)) < 0$ for $\det(J|F(x)) > 0$ in the definition of $\mathscr{O}_0^+$. Similarly we define $\mathscr{S}_{HE}^+$, $\mathscr{S}_{EM}^+$ and $\mathscr{M}^+$. Put

$$Y_+^0 = \begin{bmatrix} (s-x+\tau)/2s & i/2s & 0 \\ i(x-s+\tau) & 1 & 0 \\ 0 & 0 & 1_N \end{bmatrix}.$$

Then $Y_+^0 \in \mathscr{M}^+$ and we have an analogue of Theorem 3.3 as follows.

Theorem 3.3′. $\mathscr{G}\delta Y_+^0 \mathscr{P}/\mathscr{P} = \delta \mathscr{M}^+ \mathscr{P}/\mathscr{P} \simeq \mathscr{M}^+ \simeq \mathscr{S}_{HE}^+$.

For each $g \in \mathscr{G}$ we put $V_g = g\delta\mathscr{N}\mathscr{P}/\mathscr{P}$. Then we have an "open" covering of the "infinite dimensional manifold";

$$\mathscr{G}\delta\mathscr{N}\mathscr{P}/\mathscr{P} = \bigcup_{g\in\mathscr{G}} V_g.$$

The mapping $g\delta Y\mathscr{P}/\mathscr{P} \to Y$ defines a local coordinate system. Let Y, $Y' \in \mathscr{N}$ and assume that Y satisfies (3.1) and (3.2). Suppose that $g\delta Y\mathscr{P}/\mathscr{P} = g'\delta Y'\mathscr{P}/\mathscr{P}$ for some g and $g' \in \mathscr{G}$. Then it is easy to see that Y' also satsifies (3.1) and (3.2). Hence the property that Y satisfies (3.1) and (3.2) can be regarded as "invariants" of the $\mathscr{G}$-manifold.

For each $g \in \mathscr{G}$ we put $W_g = g\delta\mathscr{M}^-\mathscr{P}/\mathscr{P}$. Then we have an "open" covering of the "submanifold";

$$\mathscr{G}\delta\mathscr{M}^-\mathscr{P}/\mathscr{P} = \bigcup_{g\in\mathscr{G}} W_g.$$

The mapping $g\delta Y\mathscr{P}/\mathscr{P} \to Y$ defines a local coordinate system of this submanifold. For a row vector $z \in \boldsymbol{C}^{N+1}$, we set $V_z = \begin{bmatrix} 1 & z \\ 0 & 1_{N+1} \end{bmatrix} \cdot Q$. Then the mapping $z \to V_z$ defines an affine coordinate system of the Grassmann manifold $GL(N+2, \boldsymbol{C})/Q$. The action of $g \in GL(N+2, \boldsymbol{C})$ on this coordinate system is given by the following linear fractional transformation:

$$g\cdot z := (g_{00}z + g_{01})(g_{10}z + g_{11})^{-1} \longrightarrow g\cdot V_z, \quad \text{where } g = \begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix}.$$

We note that $\det(J|V_z) = 2\operatorname{Im} z_0 - |z_1|^2 - \cdots - |z_N|^2$ for $z = (z_0, \cdots, z_N)$. Put

$$G=\{g \in SL(N+2, \boldsymbol{C});\ {}^{t}\bar{g}Jg=J\},$$
$$M^{-}=\{z \in \boldsymbol{C}^{N+1};\ \det(J \mid V_z)<0\},$$
$$M^{+}=\{z \in \boldsymbol{C}^{N+1};\ \det(J \mid V_z)>0\}.$$

Then G is isomorphic to $SU(N+1, 1)$ by an inner automorphism and M^{+} is a homogeneous space of G which is called a Siegel domain of the second kind. We notice that M^{-} is not stable under the action of G. But M^{-} is a homogeneous space of ${}^{t}Q \cap G$.

Clearly, the natural map $\mathscr{M}^{-} \rightarrow \delta\mathscr{M}^{-}\mathscr{P}/\mathscr{P}$ is bijective. So we can consider β as a mapping $\delta\mathscr{M}^{-}\mathscr{P}/\mathscr{P} \rightarrow \mathscr{O}_0^{-}$. The crucial point here is that for $Y, Y' \in \mathscr{M}^{-}$ and $g \in \mathscr{G}$, if $g\delta Y\mathscr{P}/\mathscr{P}=\delta Y'\mathscr{P}/\mathscr{P}$, then the following beautiful formula holds;

$$\beta(g\delta Y\mathscr{P}/\mathscr{P})(x)=g(x)\cdot\beta(\delta Y\mathscr{P}/\mathscr{P})(x),$$

(as seen above, this action is a linear fractional transformation). Remark that

$$\beta(\delta Y^{0}\mathscr{P}/\mathscr{P})=\begin{bmatrix} 1 & -i/2 & 0\cdots0 \\ 0 & & 1_{N+1} \end{bmatrix}\cdot Q,$$
$$\beta(\delta Y_{+}^{0}\mathscr{P}/\mathscr{P})=\begin{bmatrix} 1 & i/2 & 0\cdots0 \\ 0 & & 1_{N+1} \end{bmatrix}\cdot Q.$$

It is clear now that

$$\beta(\mathscr{H}\delta Y^{0}\mathscr{P}/\mathscr{P})=\mathscr{O}_0^{-},$$
$$\beta(\mathscr{G}\delta Y_{+}^{0}\mathscr{P}/\mathscr{P})=\mathscr{O}_0^{+},$$

Since ρ is injective it follows that ϕ is surjective. Hence three mappings ρ, ϕ and β are bijective. Now Theorem 3.3 and Theorem 3.3′ are obvious.

We put

$$g_1=1,\ g_2=J \quad \text{and} \quad g_{k+2}=\begin{bmatrix} 1 & 0 & 0 \\ i/2 & 1 & e_k \\ i\,{}^{t}e_k & 0 & 1_N \end{bmatrix} \quad (1 \leq k \leq N),$$

where e_k $(1 \leq k \leq N)$ denotes the standard basis of $\boldsymbol{C}^N$ (row vectors). Then it is easy to check that

$$\{V \in GL(N+2, \boldsymbol{C})/Q;\ \det J \mid V<0\}=G\cdot M^{-}=\bigcup\nolimits_{1\leq k\leq N+2} g_k M^{-}.$$

Making use of this formula, finally we obtain the following

Theorem 3.5.

(1) $\mathcal{G}\delta Y^0\mathcal{P}/\mathcal{P}=\bigcup_{1\leq k\leq N+2} g_k\delta\mathcal{M}^-\mathcal{P}/\mathcal{P}$.

(2) *For each* k $(1\leq k\leq N+2)$, *we have*

$$g_k\delta\mathcal{M}^-\mathcal{P}/\mathcal{P}\simeq\mathcal{M}^-\simeq\mathcal{S}_{HE}.$$

Remark. All $g_k\delta\mathcal{M}^-\mathcal{P}/\mathcal{P}$ $(1\leq k\leq N+2)$ look alike through the coordinate system. However $\delta\mathcal{M}^-\mathcal{P}/\mathcal{P}$ is not stable by the whole group $\mathcal{G}$ so that, to get the well-defined action of $\mathcal{G}$, it is essential that we should consider the covering of those sets. This is clearly observed by the following fact. The group action on the space of boundary value functions of solutions is realized by the linear fractional transformation so that the boundary value functions may take the value "infinity". This trouble has been resolved by introducing a notion of the projective space valued functions; the "infinity" turns out to be a certain subset of ordinary points in another coordinate system.

References

[1] H. Doi, Nonlinear equations on a Lie group, Hiroshima Math. J., **17** (1987).

[2] F. J. Ernst, New formulation of the axially symmetric gravitational field problem II, Phys. Rev., **168**, 1415 (1968).

[3] ——, Complex potential formulation of the axially symmetric gravitational field problem, J. Math. Phys., **15**, 1409 (1974).

[4] R. Geroch, A method for generating solutions of Einstein's equations, J. Math. Phys., **12**, 918 (1971).

[5] ——, A method for generating new solutions of Einstein's equations, J. Math. Phys., **13**, 394 (1972).

[6] I. Hauser and F. J. Ernst, Integral equation method for effecting Kinnersley-Chitre transformations II, Phys. Rev., **20**, 1783 (1979).

[7] ——, A homogeneous Hilbert problem for the Kinnersley-Chitre transformations of electrovac spacetimes, J. Math. Phys., **21**, 1418 (1980).

[8] ——, Proof of a Geroch conjecture, J. Math. Phys., **22**, 1051 (1981).

[9] ——, New proof of an old conjecture, In the Robinson Festschrift. Naples: Bibliopolis (1985).

[10] ——, Proof of a generalized Geroch conjecture, Galaxies axisymmetric systems and relativity, edited by M. A. H. MacCallum, Cambridge University Press 115 (1985).

[11] I. Hauser, On the homogeneous Hilbert problem for effecting Kinnersley-Chitre transformations, Proceedings of the international seminar on exact solutions of Einstein's equations at Retzbach, Germany. Berlin: Springer-Verlag (1984).

[12] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima and M. Tanaka, Eigenfunctions of invariant differential operators on a symmetric space, Ann. of Math., **107**, 1 (1978).

[13] W. Kinnersley and D. Chitre, Symmetries of the stationary Einstein-Maxwell field equations II, J. Math. Phys., **18**, 1538 (1977).

[14] ——, Symmetries of the stationary Einstein-Maxwell field equations III, J. Math. Phys., **19**, 1926 (1978).

[15] ——, Symmetries of the stationary Einstein-Maxwell equations IV, Transformations which preserve asymptotic flatness, J. Math. Phys., **19**, 2037 (1978).

[16] L. D. Landau and E. M. Lifshitz, The classical theory of fields, Fourth revised English edition, Pergamon press (1975).

[17] M. Sato and Y. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifold, Proc. U. S.-Japan Seminar "Nonlinear partial differential equations in applied science", H. Fujita, P. D. Lax and G. Strang ed., Kinokuniya/North-Holland 259 (1982).

Department of Mathematics,
Faculty of Science,
Hiroshima University
Hiroshima, 730
Japan