Advanced Studies in Pure Mathematics 14, 1988
Representations of Lie Groups, Kyoto, Hiroshima, 1986
pp. 337–348

Some Remarks on Discrete Series Characters for Reductive p-adic Groups

Paul J. Sally, Jr.*

§ 0. Introduction

Let F be a p-adic field of characteristic zero (a finite extension of Q_p), and let G be the subgroup of F-rational points of a connected, reductive algebraic group defined over F. A complex representation (π, V) of G is *smooth* if, for each $v \in V$, there is an open subgroup K_v of G which fixes v. For any open subgroup K of G, define V^K to be the set of K fixed vectors in V. The representation (π, V) is *admissible* if (1) π is smooth; (2) dim V^K is finite for all compact open subgroups of G.

The basic theorem concerning admissible representations was proved by Bernstein.

Theorem 0.1 (Bernstein [B]). *If $(\pi, \mathscr{H})$ is an irreducible unitary representation of G, then π is admissible.*

This is to be interpreted as follows. Let V^∞ be the subspace of π-smooth vectors in the Hilbert space $\mathscr{H}$. Then V^∞ is dense in $\mathscr{H}$, and we say that $(\pi, \mathscr{H})$ is admissible if (π, V^∞) is admissible according to the above definition.

If (π, V) is an admissible representation of G, and $f \in C_c^\infty(G)$, the space of locally constant, compactly supported, complex-valued functions on G, then

$$\pi(f)=\int_G f(x)\pi(x)dx \tag{0.2}$$

is an operator of finite rank. (Here, dx is a Haar measure on G.) The map

$$f \longmapsto \hat{f}(\pi)=\text{trace } \pi(f) \tag{0.3}$$

is called the *distribution character* of π.

Received March 23, 1987.
* Partially supported by the National Science Foundation.

Motivated by the case of real reductive groups, we may ask if there exists a locally integrable function θ_π on G which is smooth (i.e. locally constant) on G', the regular set in G, such that

$$\hat{f}(\pi)=\int_G f(x)\theta_\pi(x)dx, \qquad f \in C_c^\infty(G). \tag{0.4}$$

In [HCd], Harish-Chandra gives an affirmative answer for any irreducible, admissible representation π. In addition, Harish-Chandra derives an asymptotic expansion for the character of an irreducible admissible representation in the neighborhood of a singular point $x \in G$. We content ourselves with stating Harish-Chandra's results for $x=1$. First, we need some definitions and notation.

Let χ be a non-trivial character on F^+ and B an F-valued, non-degenerate, symmetric bilinear form on $\mathfrak{g}$, the Lie algebra of G. Let dX be a Haar measure on $\mathfrak{g}$, and, for $f \in C_c^\infty(\mathfrak{g})$, set

$$\hat{f}(Y)=\int_{\mathfrak{g}} f(X)\chi(B(X, Y))dX, \qquad Y \in \mathfrak{g}. \tag{0.5}$$

The map $f \mapsto \hat{f}$ is a bijection of $C_c^\infty(\mathfrak{g})$. If T is a distribution on $\mathfrak{g}$ (i.e. a linear functional on $C_c^\infty(\mathfrak{g})$), we define the Fourier transform $\hat{T}$ by

$$\hat{T}(f)=T(\hat{f}), \qquad f \in C_c^\infty(\mathfrak{g}). \tag{0.6}$$

In [HCd], Harish-Chandra proves that, if T satisfies a certain finiteness condition, then $\hat{T}$ is a locally integable function on $\mathfrak{g}$ which is locally constant on $\mathfrak{g}'$, the regular set in $\mathfrak{g}$. It is this result which yields the properties of irreducible admissible characters which were noted above.

Now, if $\mathcal{O}$ is a G-orbit in $\mathfrak{g}$, then $\mathcal{O}$ carries a G-invariant measure which we denote by $\mu_\mathcal{O}$ (see Rao [R]). The distribution

$$f \longmapsto \mu_\mathcal{O}(f), \qquad f \in C_c^\infty(\mathfrak{g}), \tag{0.7}$$

satisfies the finiteness condition mentioned above. Hence, $\hat{\mu}_\mathcal{O}$ is a locally integrable function on $\mathfrak{g}$ which is locally constant on $\mathfrak{g}'$.

Theorem 0.8 (Harish-Chandra [HCd]). *Let π be an irreducible, admissible representation of G, and let $\{\mathcal{O}\}$ be the (finite) set of nilpotent G orbits in $\mathfrak{g}$. Then, there is a neighborhood U of zero in $\mathfrak{g}$, and, for each $\mathcal{O} \in \{\mathcal{O}\}$, a constant $C_\mathcal{O}=C_\mathcal{O}(\pi) \in \mathbf{C}$ such that*

$$\theta_\pi(\exp X)=\sum_{\mathcal{O} \in \{\mathcal{O}\}} C_\mathcal{O}\hat{\mu}_\mathcal{O}(X), \qquad X \in U. \tag{0.9}$$

This remarkable theorem was first proved by Howe [Hoa] for the case $G=GL_n(F)$.

Thus, at least in the case when char $F=0$, the work of Harish-Chandra provides a qualitative analysis of admissible characters. These results have been extended to non-connected groups by Clozel [Cl]. Some of the theory of admissible characters has been worked out for positive characteristic (see [HCe], [Rod]). However, the results are not as complete as the characteristic zero case, and much remains to be done.

In this note, we focus on the discrete series characters of G. The discrete series, which is well understood for real groups, has presented one of the most vexing problems in the representation theory of p-adic groups. There has been some progress in the construction of discrete series ([Hi], [KL], [Kub]), but only a few results exist on the specific nature of their characters. It should be noted that, before Bernstein proved the admissibility of irreducible unitary representations, Harish-Chandra [HCa] gave a simple proof of the admissibility of discrete series representations.

In Section 1, we discuss the partition of the discrete series of G into supercuspidal representations and generalized special representations. It has been conjectured by many people that supercuspidal representations can be induced irreducibly from open, compact (mod center) subgroups of G. This idea goes back to Mautner [M]. Assuming that a supercuspidal representation π of G is induced in this fashion, we use a theorem of Harish-Chandra [HCa] to derive a Frobenius type formula for the character of π at any regular element of G.* This formula is the only tool currently available for determining any sort of explicit information about the value of supercuspidal characters. It has been used in [Kub], [SS], [Shm], [Shn] to compute character formulas for supercuspidal representations. For the generalized special representations, there are essentially no general techniques available for obtaining explicit character values except for the Steinberg type representations ([Casa],).

In Section 2, the particular case of $G=GL_n(F)$ is discussed. Here, we have explicit formulas for supercuspidal characters and generalized special characters on the elliptic set in G, at least in the so-called tame case, $(n, p)=1$. This is joint work of the author, L. Corwin and A. Moy ([CMS], [CS]). We also give some details about the expansion (0.9). These latter results can be obtained directly from the work of Howe [Hoa] and van Dijk [vD].

For interesting summaries of the character theory of reductive p-adic groups, the reader should consult the papers of Cartier [Car], Harish-Chandra [HCc], and the illuminating exposition of Howe [Hoc].

The author would like to thank K. Okamoto and T. Oshima for their gracious hospitality during the Taniguchi Symposium and other conferences. Thanks are also due to Mr. Taniguchi for his generous support.

* This observation emerged from a conversation with P. Kutzko.

§ 1. Discrete series characters

Let G be as in § 0. The discrete series of G consists of (equivalence classes of) irreducible unitary representations of G whose matrix coefficients are square integrable (mod Z), where Z is the center of G. The discrete series splits into two distinct classes ([HCb], [J]):

(1) *supercuspidal representations*: irreducible unitary representations whose matrix coefficients are compactly supported (mod Z);

(2) *generalized special representations*: irreducible unitary representations whose matrix coefficients are square integrable (mod Z), and which are subrepresentations of representations induced from a proper parabolic subgroup of G.

We denote by $\hat{G}_d$, $\hat{G}_{sc}$, $\hat{G}_{gsp}$, the discrete series, the supercuspidal representations, and the generalized special representations of G respectively. Thus,

$$\hat{G}_d=\hat{G}_{sc}\cup\hat{G}_{gsp}, \qquad \hat{G}_{sc}\cap\hat{G}_{gsp}=\varnothing. \tag{1.1}$$

Remark 1.2. As of this writing, all supercuspidal representations which have been constructed explicitly may be induced irreducibly from open, compact (mod center) subgroups of G. The paper of Hijikata [Hi] indicates a general procedure for constructing supercuspidal representations by induction. For more background, see [Kub] and [Ms].

Now, suppose that (π, V) is an irreducible, (smooth) supercuspidal representation of G, and assume, for simplicity, that the center of G is compact. Let K be an open, compact subgroup of G, and let (σ, W) be an irreducible representation of K such that

$$\pi=\mathrm{Ind}_K^G\,\sigma \text{ (compact induction).} \tag{1.3}$$

Lemma 1.4. $[\pi|_K:\sigma]=1$ and

$$V=\tilde{W}\oplus\sum_{\substack{x\in G/K\\ x\neq 1}}\oplus\pi(x)\tilde{W}$$

where $\tilde{W}\simeq W$ is the σ-space for $\pi|_K$.

Proof. This is a standard situation. We set $V_\sigma=\{f\in V\,|\,\mathrm{supp}\,f\subset K\}$ and define T: $V_\sigma\to W$ by $T: f\mapsto f(1)$. The result is then clear.

Corollary 1.5. *Take $v\in W$, $\|v\|=1$, and set*

$$\theta(x)=\begin{cases}(\sigma(x)v\,|\,v), & x\in K\\ \quad 0\quad, & x\in G\setminus K.\end{cases}$$

Then θ is a matrix coefficient of π.

Corollary 1.6. *If Haar measure on G is normalized so that* $\mathrm{vol}\,(K)=1$. *then* $\deg\,(\sigma)=\deg\,(\pi)$, *where* $\deg\,(\pi)$ *denotes the formal degree of* π.

We now turn to a result of Harish-Chandra which gives an integral formula for the character of a supercuspidal representation at any regular element of G.

Theorem 1.7 (Harish-Chandra [HCa]). *Let K be an open, compact subgroup of G. Let (π, V) be a supercuspidal representation of G, and θ a K-finite matrix coefficient of π. Assume that Haar measure dx on G is normalized so that* $\mathrm{vol}\,(K)=1$. *Then, if $\gamma \in G'$, the set of regular elements in G, we have*

$$\theta_\pi(\gamma)=\deg\,(\pi)\int_G\int_K \theta(xk\gamma k^{-1}x^{-1})dkdx, \tag{1.8}$$

where dk is the restriction of dx to K.

Note. We are still assuming that the center of G is compact. Theorem 1.7 makes no reference to the possibility that π may be induced. However, Corollary 1.5 and (1.8) can be combined to provide a Frobenius type formula for θ_π at any regular element of G.

Theorem 1.9. *Let (π, V) be an irreducible supercuspidal representation of G such that $\pi=\mathrm{Ind}_K^G\,\sigma$ as in* (1.3). *Let χ_σ be the character of σ, and set*

$$\dot{\chi}_\sigma(x)=\begin{cases}\chi_\sigma(x), & x \in K,\\ 0\ , & x \in G\setminus K.\end{cases}$$

Then, if $\gamma \in G'$,

$$\theta_\pi(\gamma)=\sum_{x\in K\backslash G/K}\ \sum_{y\in K\backslash KxK}\dot{\chi}_\sigma(y\gamma y^{-1}). \tag{1.10}$$

This is the Frobenius formula for the induced character θ_π.

Proof. Let $\{v_i\}$ be an orthonormal basis for W, and set $\theta_i(x)=(\sigma(x)v_i\,|\,v_i)$, $x \in K$. Then, using (1.8) and summing, we get

$$\theta_\pi(\gamma)=\int_G\int_K \dot{\chi}_\sigma(xk\gamma k^{-1}x^{-1})dkdx.$$

Simple manipulation with double cosets gives the desired result.

Remarks 1.11. (a) The condition of compactness for $Z(G)$ can be removed easily to yield a formula similar to (1.10).

(b) The expression (1.10) has been used to compute supercuspidal characters on the elliptic set in several cases [CS], [Kua], [SS]. The formula has not yet proved to be an effective tool for analyzing characters off the elliptic set.

(c) It was observed by Deligne [D] that the character of a supercuspidal representation is supported on the union of the maximal compact subgroups of G. This may also be inferred from the paper of Casselman [Casb].

§ 2. Discrete series characters for GL_n

We now take $G=GL_n(F)$. In this case, we can use the results in [CS], [CMS] and [Kua] to give some detailed information about the discrete series characters on the elliptic set. We assume, as above, that char $F=0$, and, in addition, that $(n,p)=1$ (the tame case). Let D_n be a division algebra of dimension n^2 over F, and let $G'=D_n^x$, the multiplicative group of D_n. Much of our information about the discrete series characters of G is derived through the use of the abstract matching theorem.

The abstract matching theorem was proved by Rogawski [Rog] and Deligne-Kazhdan-Vingeras [DKV]. Recall that, if E/F is an extension of degree n, then E^x can be embedded in both G and G'. In fact, any compact (mod center) Cartan subgroup of G (and G') is isomorphic to E^x for some extension of degree n.

Abstract Matching Theorem. *There is a bijection $\pi'\leftrightarrow\pi$ between irreducible representations of G' and the discrete series of representations of G with the following properties*:

(1) *if $\theta_{\pi'}$ and θ_π are the characters of π' and π respectively, and γ is a regular element in a compact* (mod *center*) *Cartan subgroup E^x, then*

$$\theta_{\pi'}(\gamma)=(-1)^{n-1}\theta_\pi(\gamma);$$

(2) *If the formal degree of the Steinberg representation* [B] *is normalized to be equal to one, then*

$$\deg(\pi')=\deg(\pi),$$

where $\deg(\pi')$ *is the ordinary degree of the finite-dimensional representation π', and* $\deg(\pi)$ *is the formal degree of the infinite-diemsional representation π.*

The notion of admissible character, which is due to Howe [Hob], is the key to the construction of the representations of $G'=D_n^x$ and the supercuspidal representations of $G=GL_n(F)$.

Definition 2.1 (Howe). *Let E/F be an extension of degree n with* $(n, p)=1$. A character θ of E^x is *admissible* if

(1) θ does not come via the norm from a subfield of E containing F,

(2) if $\theta|_{1+\mathscr{P}_E}$ comes via the norm from a subfield $E \supset L \supset F$, then E/L is unramified.

Remark 2.2. (1) In this note, the term character as used in Definition 2.1 refers to unitary character, that is, a continuous homomorphism into the complex numbers of modulus one.

(2) If E/F is an extension of degree m, $m|n$, $m<n$ and θ is an admissible character of E^x, we say that θ is a *subadmissible* character (for n).

Definition 2.3. Let θ_1 and θ_2 be admissible characters of E_1/F and E_2/F respectively. We say that θ_1 and θ_2 are *conjugate* if there is an F-isomorphism ϕ: $E_1 \to E_2$ such that $\theta_1 = \theta_2 \circ \phi$.

We are now in a position to give a classification of the irreducible representations of $G' = D_n^x$ and the discrete series representations of $G = GL_n(F)$.

Theorem 2.4 (Corwin [Co]). *The irreducible representations of G' may be parametrized by (conjugacy classes of) admissible and subadmissible characters of extensions of degree m over F $(m|n)$. More specially, if θ is admissible or subadmissible, there is an open subgroup H'_θ of G' and an irreducible representation σ'_θ of H'_θ such that the induced representation*

$$\pi'_\theta = \mathrm{Ind}_{H'}^{G'} \sigma'_\theta$$

is an irreducible representation of G'. Moreover, π'_{θ_1} is equivalent to π'_{θ_2} if and only if θ_1 is conjugate to θ_2. The collection of equivalence classes $\{\pi'_\theta\}$ is a complete set of irreducible representations for G', that is, $(G')^\wedge = \{\pi'_\theta \mid \theta$ is admissible or subadmissible$\}$.

Theorem 2.5 (Howe [Hob]). *If θ is an admissible character of E^x/F $([E: F]=n)$, then there is an open, compact subgroup H_θ of G and an irreducible representation σ_θ of $E^x H_\theta$ such that induced representation*

$$\pi_\theta = \mathrm{Ind}_{E^x H_\theta}^{G} \sigma_\theta$$

is an irreducible supercuspidal representation of G. Moreover, π_{θ_1} is equivalent to π_{θ_2} if and only if θ_1 is conjugate to θ_2.

Before we state a theorem about generalized special representations, we need a little preparation, Let θ be a subadmissible character for E^x/F, $[E: \mathrm{F}]=m$, $m|n$, $m<n$. Then, by Theorem 2.5, there is an irreducible

supercuspidal representation ρ_θ of $GL_m(F)$ corresponding to θ. Consider the parabolic subgroup P of G defined by

$$P=\begin{pmatrix} GL_m(F) & & & * \\ & GL_m(F) & & \\ & & \ddots & \\ 0 & & & GL_m(F) \end{pmatrix}.$$

We take the representation of P which is defined on the Levi component $M=\prod GL_m(F)$ ($r=n/m$ copies) by $\otimes_{j=0}^{r-1}\rho_\theta|\cdot|^j$, where $|g|=|\det g|$, and extend trivially to the unipotent radical to obtain $(\otimes_{j=0}^{r-1}\rho_\theta|\cdot|^j)\otimes 1$.

Theorem 2.6 (Bernsteion-Zelevinsky [BZ], [Z]). *The induced representation*

$$\mathrm{Ind}_P^G\,[(\otimes_{j=0}^{r-1}\rho_\theta|\cdot|^j)\otimes 1]$$

contains a unique irreducible, square integrable (mod Z) *subquotient* π_θ. *Moreover,* π_{θ_1} *is equivalent to* π_{θ_2} *if and only if* θ_1 *is conjugate to* θ_2.

Remark 2.7. Theorem 2.6 is a slight rewriting of history. In fact, Bernstein-Zelevinsky show that *all* non-supercuspidal (i.e. generalized special) discrete series for G are obtained by choosing a supercuspidal representation of $GL_m(F)$ and proceeding as above. They make no reference to the explicit construction of the supercuspidal representations of $GL_m(F)$, nor do they impose any restrictions on the residual characteristic of F. Only after the exhaustion theorem of Moy [Mo] do we know (in the tame case) that all supercuspidal representations of $GL_m(F)$ can be constructed via subadmissible characters. This is a consequence of the following theorem.

Theorem 2.8 (Moy [Mo]). *If* $(n,p)=1$, *then the irreducible supercuspidal representations of* $GL_n(F)$ *are parametrized by conjugacy classes of admissible characters of* $E^\times/F$, *where* E *is an extension of degree* n. *In particular, all the irreducible supercuspidal representations of* $GL_n(F)$ *are induced from open, compact* (mod Z) *subgroups.*

Not surprisingly, the degrees and formal degrees of the representations π'_θ and π_θ depend on n, E and specific data related to the admissible or subadmissible character θ of $E^\times$. The specific data related to θ comes from the Howe factorization of θ ([Hob], [Mo]). For present purposes, we shall surpress the specific data and combine it into an expression $\alpha(\theta)$. The explicit form of $\alpha(\theta)$ is given in [CMS]. We normalize measures as in [CMS].

Theorem 2.9 ([CMS]). *Let θ be an admissible character of E^x/F ($[E:F]=n$). Let $e=e(E/F)$ and $f=f(E/F)$. Let π'_θ be the irreducible representation of G' given by Theorem* 2.4, *and π_θ the irreducible supercuspidal representation of G given by Theorem* 2.5. *Then,*

$$\deg(\pi'_\theta)=\deg(\pi_\theta)=f\frac{q^n-1}{q^{n/e}-1}q^{(f/2)(\alpha(\theta)+2-n-e)},$$

where $\alpha(\theta)$ is an expression depending only on data in the Howe factorization of θ.

If θ is a subadmissible character (for n), then the degree of the representation π'_θ of G' given by Theorem 2.4 is computed in a fashion entrirely similar to that for π'_θ, θ admissible. On the other hand, the generalized special representations of G are not induced in the same manner as the supercuspidal representations, and very different techniques must be used to compute the formal degrees of these representations. These techniques are based on the theory of Hecke algebra isomorphisms developed in Howe-Moy [HM] and further refined by the same authors.

Theorem 2.10. *Let θ be a subadmissible character (for n) of E^x/F where $[E:F]=m$, $m|n$, $m<n$. Write $n=ma$. Let $e=e(E/F)$, $f=f(E/F)$ so that $ef=m$. Let π'_θ be the irreducible representation of G' given by Theorem* 2.4, *and π_θ the irreducible generalized special representation given by Theorem* 2.6. *Then,*

$$\deg(\pi'_\theta)=\deg(\pi_\theta)=f\frac{q^n-1}{q^{n/e}-1}q^{(af/2)(a\alpha(\theta)+a+1-an-e)},$$

where $\alpha(\theta)$ is the same expression for the present subadmissible θ as that for the admissible θ in Theorem 2.9.

We now consider the following sets:

$$(2.11)\qquad \begin{aligned} A'_1&=\{\pi'_\theta\in(G')^\wedge\,|\,\theta \text{ is admissible}\};\\ A'_2&=\{\pi'_\theta\in(G')^\wedge\,|\,\theta \text{ is subadmissible}\}.\end{aligned}$$

Here π'_θ is the representation of G' constructed from θ by Corwin, and $(G')^\wedge$ is the unitary dual of G'. In a similar fashion, we define

$$(2.12)\qquad \begin{aligned} A_1&=\{\pi_\theta\in\hat{G}_d\,|\,\theta \text{ is admissible}\};\\ A_2&=\{\pi_\theta\in\hat{G}_d\,|\,\theta \text{ is subadmissible}\}.\end{aligned}$$

In this case, we have the supercuspidal representations (resp. generalized special representations) constructed by Howe (resp. Bernstein-Zelevinsky), and $\hat{G}_d$ denotes the discrete series in the unitary dual of G.

Now, letting $\deg(\pi)$ denote the ordinary or formal degree of a representation, we set

$$\begin{aligned} &\Delta_1'=\{\deg(\pi_\theta')\,|\,\pi_\theta'\in A_1'\};\\ &\Delta_2'=\{\deg(\pi_\theta')\,|\,\pi_\theta'\in A_2'\};\end{aligned}\tag{2.13}$$

$$\begin{aligned} &\Delta_1=\{\deg(\pi_\theta)\,|\,\pi_\theta\in A_1\};\\ &\Delta_2=\{\deg(\pi_\theta)\,|\,\pi_\theta\in A_2\}.\end{aligned}\tag{2.14}$$

If we assume that $\deg(\text{Steinberg})=1$, then (2) in the abstract matching theorem implies that $\Delta_1'\cup\Delta_2'=\Delta_1\cup\Delta_2$.

Theorem 2.15 [CMS]. *Let Δ_1', Δ_2', Δ_1, Δ_2 be the sets defined by* (2.13) *and* (2.14). *Then $\Delta_1'\cap\Delta_2'=\Delta_1\cap\Delta_2=\Delta_1'\cap\Delta_2=\Delta_2'\cap\Delta_1=\varnothing$. Thus $\Delta_1'=\Delta_1$ and $\Delta_2'=\Delta_2$.*

Corollary 2.16. (1) *If θ is an admissible character of $E^\times/F$ ($[E:F]=n$) and π_θ' is the representation of G' given by Theorem* 2.4, *then the representation π of G corresponding to π_θ' under the abstract matching theorem is supercuspidal. Moreover, if $\theta_{\pi_\theta'}$ is the character of π_θ', then $\theta_\pi=(-1)^{n-1}\theta_{\pi_\theta'}$ is a supercuspidal character on the elliptic set in G.*

(2) *If θ is a subadmissible character (for n) of $E^\times/F$, $[E:F]=m$, $m\,|\,n$, $m<n$, and π_θ' is the representation of G' given by Theorem* 2.4, *then the representation π of G corresponding to π_θ' under the abstract matching theorem is a generalized special representation of G. Moreover, if $\theta_{\pi_\theta'}$ is the character of π_θ', then $\theta_\pi=(-1)^{n-1}\theta_{\pi_\theta'}$ is a generalized special character on the elliptic set in G.*

The Corollary follows from the fact that the trivial representation of G' is in A_2' and the Steinberg representation of G is in A_2.

Remark 2.17. (a) From the matching theorem, it follows that the character of a discrete series representation π of G is equal to $\pm\deg(\pi)$ near the identity on any compact (mod Z) Cartan subgroup of G. Thus, the characters of the supercuspidal representations and generalized special representations on compact (mod Z) Cartan subgroups of G are given (up to a sign) near the identity by the formulas in Theorems 2.9 and 2.10 respectively. Moreover, it follows from Theorem 2.15 that supercuspidal characters can be distinguished from generalized special characters by their values near the identity.

(b) From the results in [CS] and [Kua] one sees that a supercuspidal character picks out a particular torus in G. This torus is isomorphic to the extension of degree n which gave rise to the supercuspidal representation as in Theorem 2.5. On the other hand, a generalized special character picks out several tori which have a subtorus isomorphic to the extension of degree m which gave rise to the generalized special representation as in Theorem 2.6. For more details and explicit formulas see [CS].

(c) The irreducible characters of G' and the supercuspidal characters of G can be computed on the elliptic set by formulas similar to (1.10). However, the generalized special character for G can only be obtained from the appropriate characters of G' by using the matching theorem and Corollary 2.16 (2).

References

[B] I. N. Bernstein, All reductive p-adic groups are type I, Funct. Anal. Appl., **8** (1974), 3–6.

[BZ] I. Bernstein and A. Zelevinsky, Representations of the groups $GL_n(F)$ where F is a non-archimedean local field, Russian Math. Surveys, **31** (1976), 1–68.

[Bo] A. Borel, Admissible representations of a semisimple group over a local field with vectors fixed under an Iwahori subgroup, Invent. Math., **35** (1976), 233–259.

[Car] P. Cartier, Les représentations des groupes réductifs p-adiques et leurs caractères, Sem. Bourbaki, n° 471, 1975/76.

[Casa] W. Casselman, The Steinberg character as a true character, in Harmonic Analysis on Homogeneous Spaces, AMS PSPM 26, 1973, 413–417.

[Casb] ——, Characters and Jacquet modules, Math. Ann., **230** (1977), 101–105.

[Cl] L. Clozel, Characters of non-connected reductive p-adic groups, Canad. J. Math., to appear.

[Co] L. Corwin, Representations of division algebras over local fields, Adv. in Math., **13** (1974), 259–267.

[CMS] L. Corwin, A. Moy and P. J. Sally, Jr., Degrees and formal degrees for division algebras and GL_n over a local field, to appear in Pacific J. Math.

[CS] L. Corwin and P. J. Sally, Jr., Discrete series characters on the elliptic set in GL_n, to appear.

[D] P. Deligne, Le support du caractère d'une representation supercuspidale, CRAS, Paris, Series A, **283** (1976), 155–157.

[DKV] P. Deligne, D. Kazhdan and M.-F. Vigneras, Représentations des algèbres centrales simples p-adiques, in Répresentations des groups réductifs sur un corps local, Hermann, Paris, 1984, pp. 33–117.

[HCa] Harish-Chandra, Harmonic Analysis on Reductive p-adic Groups, SLN 162, Springer, Berlin, 1970, (Notes by G. van Dijk).

[HCb] ——, Harmonic analysis on reductive p-adic groups. in Harmonic Analysis on Homogeneous Spaces, AMS PSPM **26**, 1973, 167–192.

[HCc] ——, The characters of reductive p-adic groups, in Contributions to Algebra, Academic Press, New York, 1977, pp. 175–182.

[HCd] ——, Admissible invariant distributions on reductive p-adic groups, in Lie Theories and their Applications, Queen's papers in pure and applied mathematics No. 48, Queen's University, Kingston, Ontario, 1978, pp. 281–347.

[HCe] ——, A submersion principle and its applications, Proc. Indian Acad.

Sci., **90** (1981), 95–102.

[Hi] H. Hijikata, Some supercuspidal representations induced from parahoric subgroups, in Automorphic Forms in Several Variables, I. Satake and Y. Morita, Eds. Birkhäuser, 1984, pp. 160–178.

[Hoa] R. Howe, The Fourier transform and germs of characters (case of GL_n over a p-adic field), Math. Ann., **208** (1974), 305–322.

[Hob] ——, Tamely ramified supercuspidal representations of GL_n, Pacific J. Math., **73** (1977), 437–460.

[Hoc] ——, The work of Harish-Chandra on reductive p-adic groups, in Harish-Chandra's Collected Papers, v.I, V. S. Varadarajan, Ed., Springer, 1984, pp. lix–lxxiii.

[HM] R. Howe and A. Moy, Harish-Chandra Homorphisms for p-adic Groups, CBMS Regional Conference Series in Mathematics, n° 49, AMS, Providence, 1985.

[J] H. Jacquet, Représentations des groupes linéaires p-adique, in Theory of group representations and harmonic analysis, CIME, Rome, 1971, pp. 119–220.

[KL] D. Kazhdan and G. Lusztig, Proof of the Deligne-Langlands conjecture for Hecke algebras, preprint.

[Kua] P. Kutzko, Character formulas for supercuspidal representations of GL_l, l a prime, Amer. J. Math., **109** (1987), 201–222.

[Kub] ——, Towards a classification of supercuspidal representations of GL_N, preprint.

[M] F. Mautner, Spherical functions over p-adic fields, II, Amer. J. Math., **86** (1964), 171–200.

[Mo] A. Moy, Local constants and the tame Langlands correspondence, Amer. J. Math., **108** (1986), 863–930.

[Ms] L. Morris, Filtrations and supercuspidal representations, preprint.

[R] R. R. Rao, Orbital integrals on reductive groups, Ann. of Math., **96** (1972), 505–510.

[Rod] F. Rodier, Intégrabilité locale des caractères du groupe $GL(n, k)$ où k est un corps local de caracteristique positive, preprint.

[Rog] J. Rogawski, Representations of $GL(n)$ and division algebras over a p-adic field, Duke Math. J., **50** (1983), 161–196.

[SS] P. J. Sally, Jr. and J. A. Shalika, Characters of the discrete series of representations of $SL(2)$ over a local field, Proc. NAS, **61** (1968), 1231–1237.

[Shm] H. Shimizu, Some examples of new forms, J. Fac. Sci. U. Tokyo, **24** (1977), 97–113.

[Shn] T. Shintani, On certain square-integrable irreducible unitary representations of some p-adic linear groups, J. Math. Soc. Japan, **20** (1968), 522–565.

[vD] G. van Dijk, Computation of certain induced characters of p-adic groups, Math. Ann., **199** (1972), 229–240.

[Z] A. Zelevinsky, Induced representations of reductive p-adic groups II. On irreducible representations of $GL(n)$, Ann. Sci. ENS, **13** (1980), 165–210.

Department of Mathematics
University of Chicago
Chicago, IL 60637
U.S.A.